PI MU EPSILON JOURNAL

VOLUME 7 FALL 1980 NUMBER 3

CONTENTS

PI MU EPSILON JOURNAL

VOLUME 7 FALL 1980 NUMBER 3

CONTENTS

PI MU EPSILON JOURNAL
THE OFFICIAL PUBLICATION
OF THE HONORARY MATHEMATICAL FRATERNITY

David Ballew, Editor

ASSOCIATE EDITORS

Roger Opp   Leon Bankoff

Clayton Dodge

OFFICERS OF THE FRATERNITY

President: R.V. Andree, University of Oklahoma

President-Elect: E. Maurice Beasley, University of Nevada

Secretary-Treasurer: R.A. Good, University of Maryland

Past-President: E. Allan Davis, University of Utah

COUNCILORS

A.D. Stewart, Prairie View A & M University

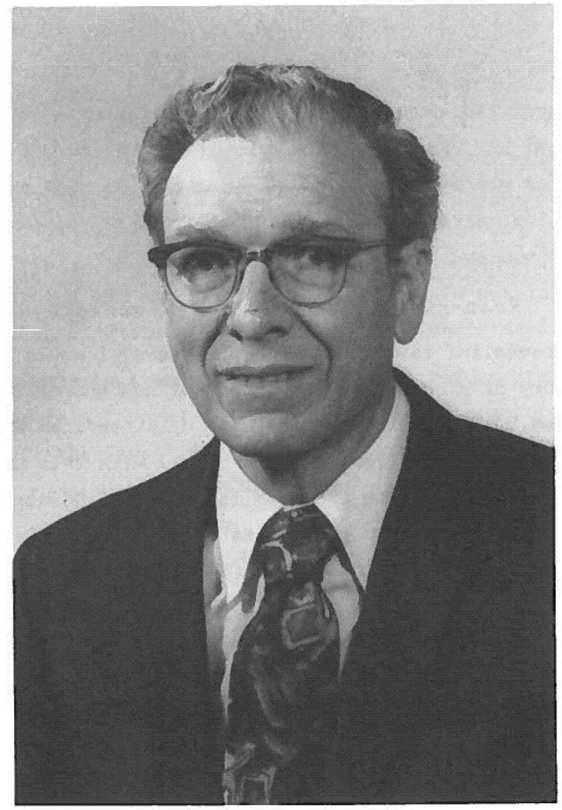
Milton D. Cox, Miami University, Ohio

Eileen L. Poiani, St. Peter's College

Fredric Zerla, University of South Florida

Chapter reports, books for review, problems for solution and solutions to problems, should be mailed directly to the special editors found in this issue under the various sections. Editorial correspondence, including manuscripts (two copies) and news items should be mailed to THE EDITOR OF THE PI MU EPSILON JOURNAL. South Dakota School of Mines and Technology. Rapid City, SD 57701. For manuscripts, authors are requested to identify themselves as to their class or year if they are undergraduates or graduates, and the college or university they are attending, and as to position if they are faculty members or in a non-academic profession.

PI MU EPSILON JOURNAL is published at the South Dakota School of Mines and Technology twice a year—Fall and Spring. One volume consists of five years (10 issues) beginning with the Fall 19x4 or Fall 19x9 issue, starting in 1949. For rates, see inside back cover.



Dr. Richard Good

Pi Mu Epsilon's highest award, the C. C. *MacDuffee Distinguished Service Award,* was presented to *Dr. Richard Good* at the Summer **1980** meetings at Ann Arbor, Michigan. This is the first time since **1975** that this award has been presented.

Professor Good, the National Secretary-Treasurer of **Pi Mu Epsilon,** is an outstanding teacher, and exceptional scholar and a diligent worker on behalf of mathematical excellence in general and **Pi Mu Epsilon** in particular. In addition to his well-known efforts to promote scholarship in mathematics, Dr. Good is the personality who introduced television teaching of mathematics so successfully at the University of Maryland. He is also an accomplished organist.

With humble pride, we add Dr. Good's name to the list of recipients of the C. C. *MacDuffee Distinguished Service Award.*

1964	Dr. J. Sutherland Frame
1966	Dr. Richard Andree
1967	Dr. John S. Gold
1970	Dr. Francis Regan
1972	Dr. J. C. Eaves
1975	Dr. Houston Karnes
1980	Dr. Richard Good

A TEST FOR PRIMENESS USING STERN'S DIATOMIC SERIES

Michael L. Call
Rose-Hulman Institute of Technology

Before revealing this "test" for primeness, I would like to discuss the history of a rather curious array of numbers called "Stern's Diatomic Series." The history begins with a letter from G. Eisenstein to Stern in 1850 and ends (for the time being) with this article. The rest is outlined below in the order of its order of "discovery."

The latest reference to the series occurs in the first volume of the number theory abstracts: B-84-2, "A Family of Integers and a Theorem on Circles," by T. T. Williams and D. H. Browne. This article appeared in the American *Mathematical* Monthly, volume 59, pages 534-536 (1947).

The article had as a reference a 1938 article, "Fractions," by L. R. Ford [2] which touched only briefly on the geometric interpretation of the array. The 1947 article also had an editor's reference to a Monthly problem #4236 proposed by H. D. Grossman in the advanced problems section of the American Mathematical Monthly, volume 59, page 112.

Although of no merit in itself, the problem led the investigation to the solution. The solution appeared in the American Mathematical Monthly [6]. The problem was, incidentally, solved by a number of prominent mathematicians including Howard Eves, William Gustin, P. T. Bateman, Y. S. Luan, L. M. Kelly, Ivan Niven, Leo Moser, and R. Steinberg. In a lengthy editorial note, references were given not only to the 1947 article, but also to a 1929 article by D. H. Lehmer, "On *Stern's* Diatomic Series" [3]. This article contains a wealth of information about the array (including a tie-in to the Fibonacci sequence) and is strongly recommended for anyone who is interested in studying the properties of the array. This article referenced Stern's original paper, *"Ueber eine zahlentheoretische Funktion"* [4].

Although Stern was the first to publish results about the array, and to investigate its properties thoroughly, Eisenstein "discovered" the array and did some preliminary work with it. In his 1858 paper, Stern says that there was published in 1850 "a paper by Eisenstein on a number theoretic function to which he *was* led in his investigation of the higher reciprocity *tea}* [*reciprocitätsgesetz*]. He [*Eisenstein*] further remarked that the study of this function led him to a *noteworthy* number sequence." Stern quotes Eisenstein as having said (in his 1850 letter to Stern), "*my proofs* of the sentences [*theorems*] *are* rather *complicated*, perhaps you can find *simpler* ones." In writing his 1858 paper, Stern fulfilled Eisenstein's request of eight years earlier.

In the next section, I would like to introduce Stern's Diatomic series and prove a theorem due to Stern. From the nature of the theorem and its proof, the "test" for primeness should be obvious to the reader.

Theorem. In the following array

$$
\begin{array}{ccccccccc}
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & 3 & & 2 & & 3 & 1 & \\
1 & 4 & 3 & 5 & 2 & 5 & 3 & 4 & 1 \\
 & & & & \vdots & & & &
\end{array}
$$

where the next row is obtained by inserting between any 2 numbers their sum, the integer n is prime if and only if n appears $n - 1$ times in the nth row.

Proof. First, we introduce the following notation: let $[a,b]$ denote the presence of a and b as integers having consecutive positions in some row of the array and be called a partition of n (where $n = a + b$). Note also that $[a,b] \neq [b,a]$ for distinct a and b. If a and b have no common factor other than 1, $[a,b]$ shall be called a relatively prime partition of n.

I. Suppose some partition of n as $[a,b]$ occurs in the mth row of the array. Because of the method of construction of the array, entries in the mth row alternate between sums of entries in the $(m-1)$th row and entries in the $(m-1)$th row. Since the first row contains $[1,1]$, closure guarantees that all entries are positive integers. Let $[x,y]$ be the partition in the $(m-1)$th row from which a,b is derived. Thus, a; = a and a; + $y = b$ or a; + y = a and y = b. From this reset a; =a, y = b - a, or x = a - b, y = b. If a > b, then the latter must be the case since. b - a < 0; if a < b, the former must be the case since now, a - b < 0. Since the presence of [a - b,b] or [a,b - a] in the (m -1)th row implies the presence of [a ,b] in the mth row, it follows that given a > b, [a ,b] occurs in the mth row if and only if $[a - b, b]$ occurs in the $(m-1)$th row. Given $b > a$, $[a, b]$ occurs in the mth row if and only if $[a, b - a]$ occurs in the $(m-1)$th row. This argument and its conclusion shall be called the reduction argument.

II. All partitions $[a,b]$ which occur in the array are relatively prime. To prove this, assume $[a,b]$ occurs in the mth row where a and b have a common factor k; $a = kx$, $b = ky$ and $k > 1$. By the reduction argument, either $[a - b, b]$ or $[a, b - a]$ occur in the $(m-1)$th row. But $a - b = k(x - y)$, $b - a = k(y - x)$, so that the partition in the $(m-1)$th row also has a common factor k. Inducting upon this, there must be a partition in the first row which has k as a common factor. Since the first row is just $[1,1]$, and $k > 1$ by assumption, we have the contradiction necessary to complete the proof.

III. The integer n is prime if and only if there are $n - 1$ relatively partitions of n. First, note that there are $n - 1$ partitiions of n, as may be seen by listing them: $[1, n - 1]$, $[2, n - 2]$ ..... $[n - 1, 1]$. Assume n is not prime; thus n has a factor k, where $kx = n$, $x > 1$ and $1 < k < n$. If $x = y + z$, then $[ky, kz]$ is one of the $n - 1$ partitions of n and is not relatively prime. Conversely, let n be prime and assume there are not $n - 1$ relatively prime partitions of n. Since there must be less than $n - 1$ relatively prime partitions of n, at least one of the $n - 1$ partitions of n is not relatively prime. Thus, $[kx, ky]$ is a partition of n with $k > 1$. Since $n = k(x + y)$, n has a factor k, contrary to n being prime. This contradiction shows that there must be $n - 1$ relatively prime partitions of n. Hence, n is prime if and only if there are $n - 1$ relatively prime partitions of n.

IV. All relatively prime partitions of n occur exactly once in the array, and occur within the first $n - 1$ rows. To prove this, assume the hypothesis for all integers from 2 up to $n - 1$.

Let $[a,b]$ be a relatively prime partition of n, $a > b$. From the reductionargument, $[a,b]$ occurs (exactly once) within the first $n - 1$ rows if and only if $[a - b, b]$ occurs (exactly once) within the first $n - 2$ rows. Clearly, $[a - b, b]$ is a relatively prime partition of $n - b$, (if not, $a - b = kx$, $b = ky$, $k > 1$ so $[a,b] = [k(x + y), ky]$ contrary to the choice of relatively prime $[a,b]$). Since $n - b \le n - 1$, $[a - b, b]$ does occur exactly once within the first $n - 2$ rows (by assumption), then $[a,b]$ will occur exactly once within the first $n - 1$ rows. Arguments for the case $b > a$ are nearly identical.

The proof is completed by noting that $[a,b]$ is an arbitrary relatively prime partition of n and noting that the induction hypothesis is true for $n = 2$ (i.e., $[1,1]$ occurs in the first row).

Since each partition of n causes n to appear in every row below that in which the partition appears, and (from IV) every relatively prime partition of n occurs exactly once, within the first $n - 1$ rows, and (from II) every "non-relatively prime" partition of n does not occur, we conclude that there are as many occurrences of n in the nth row as there are (distinct) relatively prime partitions of n. Combining this with the result of III yields the desired implication: n is prime if and only if n occurs $n - 1$ times in the nth row.

Corollary. n occurs exactly $\phi(n)$ times in the nth row, where ϕ is Euler's phi function.

Proof. We simply need to replace (III) from above with the statement, "There are exactly $\phi(n)$ relatively prime partitions of n." Let x_1 be one of the $\phi(n)$ numbers such that $x_1 < n$ and x_1 and n are relatively prime. Now, assume that x_1 and $n - x_1$ are not relatively prime. Thus, $x_1 = k_q$ and $n - x_1 = kr$, for some integers r, q, k with $k > 1$. Hence, $n = k(q + r)$, and n and x_1 have k as a common factor. This contradiction shows that x_1 and $r - x_1$ are relatively prime, so that $[x_1, n - x_1]$ is a relatively prime partition of n.

Conversely, if x_o and n have a common factor $k(k > 1, x > n)$, then $n = kq$ and $x_o = kr$ for some integers q, r. Hence, $n - x_o = k(q - r)$, x_o and $n - x$ have k as a common factor, and $[x_o, n - x]$ is not a relatively prime partition of n. So, all the numbers of the form of x_1 and only those numbers generate a relatively prime partition of n. Since there are $\phi(n)$ numbers of the form x_1, there are $\phi(n)$ relatively prime partitions of n, and the proof is complete.

Stern's Diatomic Series has a multitude of properties, involving such widely varying concepts as the Fibonacci numbers, the Farey series, and continued fractions. Below are listed a few of these which the reader may enjoy proving for himself.

1) the number of terms in the nth line is $2^{n-1} + 1$, and the sum of these

terms is $3^{n-1} + 1$

2) the number of terms in all rows up to and including the nth is $2^n + n - 1$ and the sume of these terms is $1/2(3^n + 1) + n - 1$

3) the largest number in the nth row is the $(n + 1)$st Fibonacci number (1, 1, 2, 3, 5, 8, 1.1, 21 ...). The average value of the numbers in the nth row is nearly $(3/2)^{n-1}$; the average value of the numbers in all rows up to and including the nth row is about $1/2(3/2)^{n-1}$

4) If a, b, and c appear consecutively in some row of the array, then b is a divisor of $(a + c)$

5) If $[a,b]$ is a relatively prime partition, it appears in the row whose number is equal to the sum of the quotients in the continued fraction expansion of a/b.

In closing, I would like to thank Dr. Roger Lautzenheiser, who showed me the initial problem and gave me constant encouragement; Mrs. Mary Lou McCullough, who has spent hours typing and retyping the manuscript; and Dan Hatten (also of Rose-Hulman) who was kind enough to translate Stern's 27-page article from German to English.

REFERENCES

1. Williams, G. and Browne, D., "A Family of Integers and a Theorem on *Circles*," American Mathematical Monthly, 54(1947), 534-536.

2. Ford, L., "Fractions," American Mathematical Monthly, 45(1938), 586-601.

3. Lehmer, D., "On Stern's Diatomic *Series*," American Mathematical Monthly, 36(1929), 59-67.

4. Stern, *"Uber eine zahlentheoretische Funktion,"* Journal für die reine und angewandte Mathematik, 55(1858), 193-220.

5. Eisenstein, F., *"Über ein einfaches Mittel zur Auffindung der höheren Reciprocitätsgesetze und der mit ihnen zu* verbindenden *Ergänzungssätze."* Journal fur die reine und angewandte Mathematik, 39(1850), 351-364.

5. Grossman, D., Moser, L.; and Steinberg, R., "Advanced Problems and Solutions," American Mathematical Monthly, 55(1948), 259.

THE INTEGRATION OF $\sec^3 x\, dx$

by *Michael Lee Walloga*
University of, *South Florida*

The following is an integration of $\int \sec^3 x\, dx$ using trigonometric substitutions. The author has searched various introductory or first year calculus textbooks for this type of integration of $\int \sec^3 x\, dx$. Some of these texts are listed in the bibliography. In all cases, when shown or mentioned, the "integration by parts" method was used, suggested, or said to be the only method to use.

The integration is as follows:

$$\begin{aligned}\int \sec^3 x\, dx &= \tfrac{1}{2}\int \sec x\,(2\sec^2 x)\, dx\\ &= \tfrac{1}{2}\int \sec x\,(\sec^2 x + 1 + \tan^2 x)\, dx\\ &= \tfrac{1}{2}\int \sec x\, dx + \tfrac{1}{2}\int(\sec^3 x + \sec x \tan^2 x)\, dx\\ &= \tfrac{1}{2}\int(\sec x \tan x + \sec^2 x)/(\sec x + \tan x)\, dx\\ &\qquad + \tfrac{1}{2}\int(\sec^3 x + \sec x \tan^2 x)\, dx.\end{aligned}$$

We now use the substitutions:

$$v = \sec x \tan x$$
$$dv = (\sec^3 x + \sec x \tan^2 x)\, dx$$

and

$$u = \sec x + \tan x$$
$$du = (\sec^2 x + \sec x \tan x)\, dx.$$

Now it follows that

$$\begin{aligned}\int \sec^3 x\, dx &= \tfrac{1}{2}\int \frac{du}{u} + \tfrac{1}{2}\int dv\\ &= \tfrac{1}{2}\ln|u| + \tfrac{1}{2}v + C\\ &= \tfrac{1}{2}\ln|\sec x + \tan x| + \tfrac{1}{2}(\sec x \tan x) + C.\end{aligned}$$

Therefore

$$\int \sec^3 x\, dx = \tfrac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) + C.$$

REFERENCES

1. Cambell, H. E. and Dierker, P. F., *Calculus with Analytic Geometq, 2nd Edition,* Prindle, Weber & Schmidt, Inc., Boston, 1978.
2. Goodman, A. W., *Analytic Geometry and the Calculus, 3rd Edition,* Macmillan Publishing Co., Inc., New York, 1974.
3. Leithold, L., *The Calculus with Analytic Geometry,* 3rd *Edition,* Harper & Row, San Francisco, 1976.
4. Thomas, G. B., *Elements of Calculus and Analytic Geometry, 2nd Edition,* Addison-Wesley Publishing Co., Reading, MA., 1972.
5. Riddle, D. F., *Calculus and Analytic Geometq, 2nd Edition,* Wadsworth Publishing Co., Belmont, CA., 1974.

1979-80 STUDENT PAPER COMPETITION

The papm for the. 1979-80 Student Papa Competition have been judged and the winners are:

First Prize ($200)	*Ziad Haddad, UCLA, "Two Remarks On the Quaternions",* (To Appear in the next issue of the Journal)
Second Prize ($100)	*Robert Smith, University of Arkansas, "Uniform Algebras and Scattered Spaces",* (To appear in the next issue of the Journal)
Third Prize ($50)	*Alma Posey, Hendrix College, "Rolling Cones"* (appearing as the next article in this issue of the Journal)

This is an annual Student Paper Competition open to students who have not received their Master's degiee at the time of submission. Further, each Chaptm which submits five papm creates a mini-contest amoung just those papm. The best will receive $20 and all such papm ate also eligible for the National Content. Two copies of submitted papm should be sent to the Editor.

Third Prize Paper
National Student Paper Competition
1979-SO

ROLLING CONES

by Alma E. Posey
Hendrix College

The cycloid is a curve generated by rolling a circle on a line and observing the curve traced out by a fixed point on the circle. The origins of this observation are unknown, but Bouelles is given credit for its study in 1500. He felt the arc was a part of a circle with a radius of 5/4 that of the generating circle. In 1599 Galileo was concerned with the quadrature of the cycloidal arc. Galileo gave the curve its name, however it is also called a roulette or a trochoid. Gilles Persone de Roberval effected the quadrature in 1634. After this, discussion began on the construction of the tangent to the curve. Torricelli published the results of the quadrature and tangent in 1644. Blaise Pascal, in 1658, after eight days of intense work, composed a full account of the geometry of the cycloid. Later in 1686, Leibniz wrote the equation of the curve in this form:

$$y = \sqrt{2x-xx} + \int \frac{dx}{\sqrt{2x-xx}} .$$

Presently the equation appears in parametric form with $y = a(1-\cos\alpha)$ and $x = a(\alpha-\sin\alpha)$ (Figure 1).

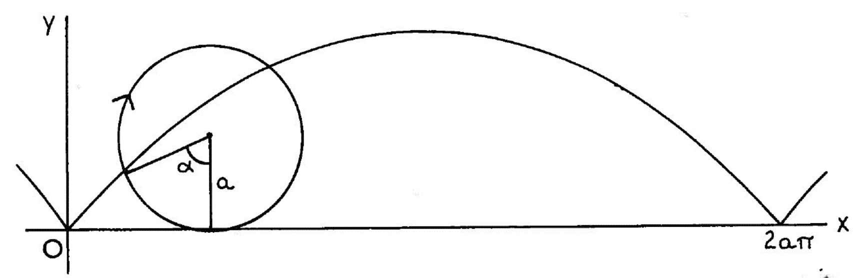
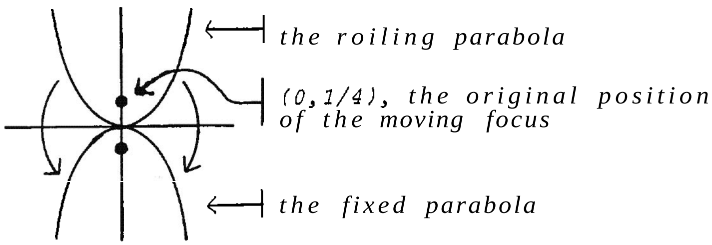


FIGURE 1

After the work of Leibniz, there began to appear general methods for determining the nature of curves generated by rolling other conic sections [3]. In this paper we examine curves generated by such rolling conics.

An example of these rolling cones appeared as a problem on the William Lowell **Putnam** Mathematical Competition [1, Problem A-5], which states:

> Consider the two mutually tangent parabolas $y = x^2$ and $y = -x^2$. [These have foci at (0,1/4) and (0,-1/4), and directrices $y = -1/4$ and $y = 1/4$, respectively.] The upper parabola rolls without slipping around the fixed lower parabola. Find the locus of the focus of the moving parabola.



In order to solve this problem we need to use the reflection property of the parabola.

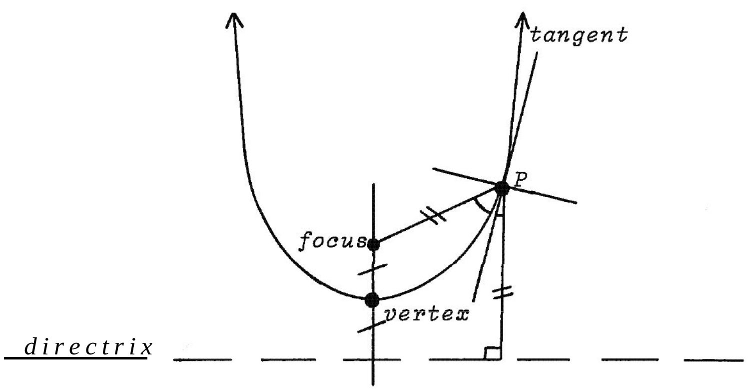


FIGURE 2

Consider Figure 2. For any point P on the parabola, the tangent line to the parabola at P bisects the angle between the line from the focus to P and the line through P perpendicular to the directrix.

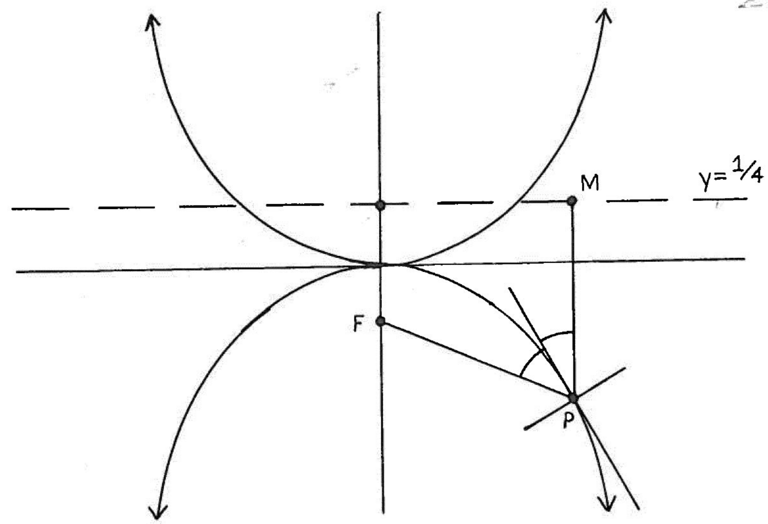


FIGURE 3

Now in solving the **Putnam** problem let F be the focus of the fixed parabola and M be the moving focus (Figure 3). If P is the point of mutual tangency, then using the reflection property along with the congruence of the parabolas, the line segment from M to P, denoted $\overline{MP}$, is perpendicular to the directrix of the fixed parabola and $\overline{MP}$ and $\overline{FP}$ have equal length, denoted UP = FP. Thus M is on the directrix of the fixed parabola. So the locus of the focus of the rolling parabola is the directrix of the fixed parabola.

Now let us consider the two mutually tangent hyperbolas $y = \sqrt{1+x^2} - 1$ and $y = 1 - \sqrt{1+x^2}$ and the curve traced out by the focus of the upper hyperbola as it rolls without slipping along the lower hyperbola. (These have foci at $(0,\sqrt{2})$, $(0,-2-\sqrt{2})$ and $(0, -\sqrt{2})$, $(0,2+\sqrt{2})$ respectively, with slope of the asymptotes equal to ± 1).

We first note that a hyperbola has a reflection property similar to that of the parabola.

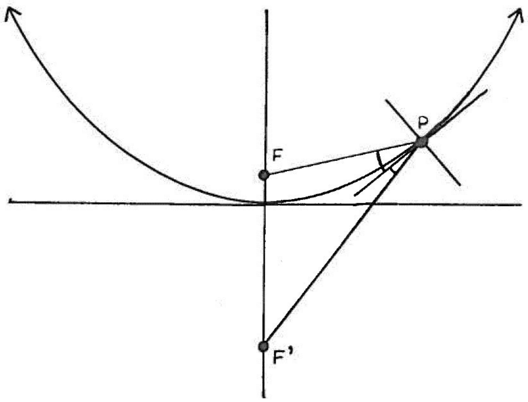


FIGURE 4

Consider Figure 4. For any point P on the hyperbola, the tangent line to the hyperbola bisects the angle between $\overline{FP}$ and $\overline{F'P}$ where F and F' are the foci of the hyperbola. Another property of the hyperbola is that the difference between the lengths of these two lines is a constant, and for the specific hyperbolas above, this constant is 2.

Let F and F' be the foci of the fixed hyperbola, and M and M' the foci of the moving hyperbola (Figure 5). If P is the point of mutual tangency, then using the reflection property along with the congruence of the hyperbolas, $\overline{FP}$ and $\overline{M'P}$ correspond in direction, and so do $\overline{F'P}$ and $\overline{MP}$. Since FP = UP, $F'P = M'P$, and $F'P$ - FP = 2, $F'P - MP = 2$ which implies M is 2 units away from F'. Therefore an arc of a circle of radius 2 is traced out by the focus M.

As P moves along the hyperbolas, the line segment $\overline{F'P}$ approaches a position parallel to the asymptote of the fixed hyperbola, and $\angle OF'M$ approaches $\pi/4$ radians. Therefore the locus of the focus of the rolling hyperbola is an arc, π units inlengthwithout endpoints, of the circle centered at F' with radius 2.

Now consider the curve traced out by the focus of the parabola $y = x^2$ when it rolls without slipping along the x-axis.

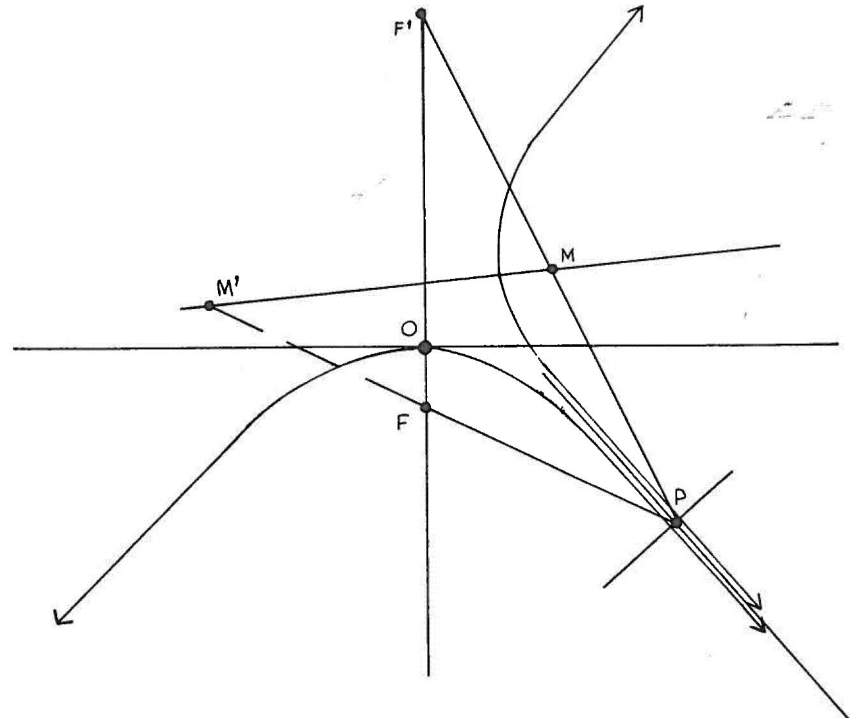


FIGURE 5

Figure 6 shows the original position of the parabola and Figure 7 shows the position of the parabola after it has rolled into a position tangent to the x-axis at point B'. Point B corresponds to B' in Figure 7 and the other points correspond similarly so that distances and angles are congruent. For example, the length of $\overline{FB}$ where F is the focus in Figure 6 is equal to thelengthof $\overline{F'B'}$ in Figure 7

In order to find the curve traced out by the focus F', we determine its coordinates in terms of x, the first coordinate of B. Construct the line $C'F'$ perpendicular to the horizontal axis.

Construct the tangent at B and extend it until it intersects the y-axis at point A. It follows from the reflection property of the parabola that AABF is an isosceles triangle so ?%?bisects $\overline{AB}$. For the &a-bola $y = x^2$, the distance from the vertex to the focus is 1/4, thus FB $= x^2 + 1/4$. Since $FA = x^2 + 1/4$ also, A has coordinates $(0, -x^2)$.

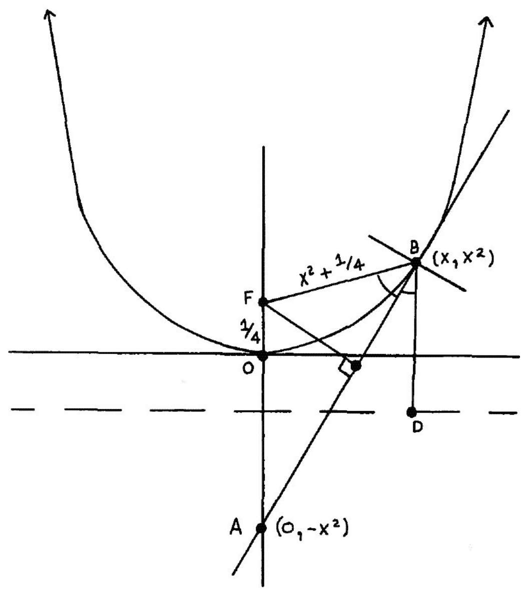


FIGURE 6

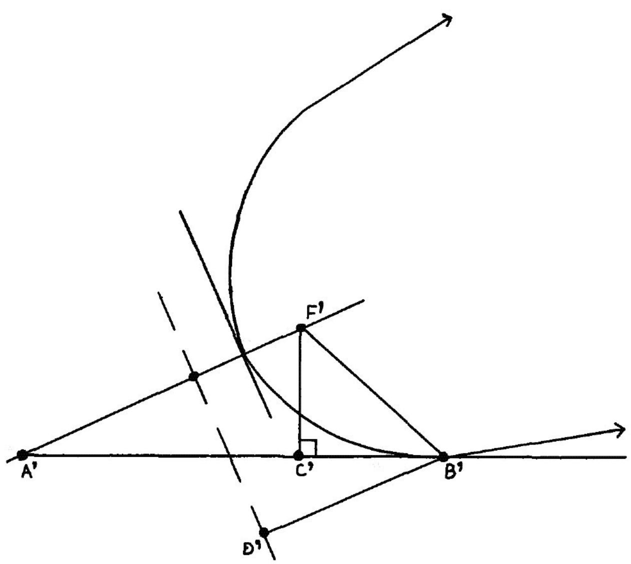


FIGURE 7

Therefore $AB = |x|\sqrt{1+4x^2}$ and $BC = |x|\sqrt{\frac{1+4x^2}{\ }}$. Using the Pythagorean Theorem, $FC = \frac{\sqrt{1+4x^2}}{4}$ which is also the second coordinate of F'.

To determine the first coordinate of F' we must subtract $B'C' = |x|\ \sqrt{1+4x^2/2}$ from the arc length of the parabola from O to B. This arc length is given by $\int_0^{|x|} \sqrt{1+4t^2}\, dt$. Using the substitution $\sinh s = 2t$ we get:

$$\int_0^{|x|} \sqrt{1+4t^2}\, dt = |x| \frac{\sqrt{4x^2+1}}{2} + 1/4 \text{ Arc sinh } 2|x|.$$

Therefore, F' has coordinates $(1/4 \text{ Arc sinh } 2\ |x|, \frac{\sqrt{1+4x^2}}{4}$. Substituting $z = 1/4 \text{ Arc sinh } 2|x|$, the coordinates become $(z, \frac{\cosh 4z}{4})$. Thus, the equation of the curve generated by the focus of the parabola $y = x^2$ rolling on the x-axis is the catenary, $y = 1/4 \cosh 4x$. (This result is mentioned in [2]).

These are just a few curves generated by cones. Many more may be formed with other conics.

REFERENCES

1. Hillman, A., *The William Lowell Putnam Mathematical Competition,* The American Mathematical Monthly 82 (Nov. 1975), 905-912.
2. Lockwood, E., *A Book of Curves,* Cambridge University Press, Cambridge, 1961.
3. Whitman, E., *Sow Historical Notes on the Cycloid,* The American Mathematical Monthly 50 (May 1943) 309-315.

This paper written under the direction of *Professor Robert Eslinger.*

Alma Posey was an undergraduate at Hendrix College at the time of the writing of this paper.

IN MEMORIUM

*It is with deep feeling that we report that Pi Mu Epsilon's Past President, **Houston T. Karnes**, was buried early in March, 1980. Dr. Kames was well known for his many activities in mathematics and mathematics education. He will be sorely missed by his many friends who stilt think of him as the epitome of a gracious, kindly gentleman who was a friend to all who knew him.*

CONFERENCE ON UNDERGRADUATE MATHEMATICS

The Sixth Annual Conference on Undergraduate Mathematics will be held at Hendrix College, Conway, Arkansas, *Arkansas Beta*, on April 10-11, 1981. The program of the meeting will include papers written by undergraduates together with papers presented by the following faculty:

Professor R. H. Bing, University of Texas,
Professor Paul R. Halmos, Indiana University,
Professor Button W. Jones, University of Colorado,
Professor M. Z. Nashed, University of Delaware,
Professor John W. Newberger, North Texas State University.

NEW CHAPTERS

The Fraternity welcomes the following four nw chapters installed during 1979-SO.

St. Mary's College
Loyola of Chicago
North Carolina Central University
University of Arkansas at Pinebluff

There are now 226 chapters and 2504 members were initiated during the past year. Congratulations to you all.

AWARD CERTIFICATES

Your Chapter can make use of the Pi Mu Epsilon award certificates available to help you recognize student's mathematics achievements. To obtain these, write to:

Dr. Richard Good
Department of Mathematics
University of Maryland
College Park, Maryland 20742

ON EVALUATING THE LEGENDRE SYMBOL

by Michael Filaseta
University of Arizona

To evaluate the Legendre symbol $(\frac{n}{p})$ where n is an integer and p a prime, one could take up the lengthy job of listing the quadratic residues mod p and then checking directly to see if n, reduced mod p, is in the list. To do this is a tedious task which is usually made simple by noting that $(\frac{nm}{p}) = (\frac{n}{p})(\frac{m}{p})$ and by proving the following three properties of the Legendre symbol:

$$\text{(i)} \qquad (\frac{-1}{p}) = \begin{cases} 1 \text{ if } p \equiv 1 \pmod 4 \\ -1 \text{ if } p \equiv -1 \pmod 4, \end{cases}$$

$$\text{(ii)} \qquad (\frac{2}{p}) = \begin{cases} 1 \text{ if } p \equiv \pm 1 \pmod 8 \\ -1 \text{ if } p \equiv \pm 3 \pmod 8, \text{ and} \end{cases}$$

$$\text{(iii)} \qquad (\frac{p}{q}) = \begin{cases} (\frac{q}{p}) \text{ if } p \text{ or } q \equiv 1 \pmod 4 \\ -(\frac{q}{p}) \text{ if } p \text{ and } q \equiv -1 \pmod 4 \end{cases}$$

where p and q are odd primes. The purpose of this paper is to show that an arbitrary Legendre symbol can be evaluated fairly quickly by using only the third of the three properties above, the so called Quadratic Reciprocity Law. To show this, it is sufficient to give the procedure to use in the cases when $(\frac{-1}{p})$ or $(\frac{2}{p})$ arises.

For convenience p will now designate a prime greater than three. If $(\frac{-1}{p})$ should occur in the process of evaluating $(\frac{n}{p})$, then the procedure being introduced is to rewrite the Legendre symbol thusly

$$(*) \qquad (\frac{-1}{p}) = (\frac{-4}{p}) = (\frac{p-4}{p}) .$$

How consider what happens in the two cases $p \equiv 1 \pmod 4$ and $p \equiv -1 \pmod 4$:

Case 1. $p \equiv 1 \pmod 4$.

Write $p - 4 = \prod_{i=1}^{r} p_i$, where the p_i are odd (not necessarily distinct) primes. Since $p \equiv 1 \pmod 4$, $(\frac{p_i}{p}) = (\frac{p}{p_i})$ and

$$(\frac{p-4}{p}) = \prod_{i=1}^{r} (\frac{p_i}{p}) = \prod_{i=1}^{r} (\frac{p}{p_i}) = \prod_{i=1}^{r} (\frac{4}{p_i}) = 1$$

as was to be expected.

Case 2. $p \equiv -1 \pmod 4$.

Again write $p - 4 = \prod_{i=1}^{r} p_i$. Since $p \equiv -1 \pmod 4$, $p - 4 \equiv -1 \pmod 4$ so that there are an odd number of $p_i \equiv -1 \pmod 4$, giving

$$(\frac{p-4}{p}) = \prod_{i=1}^{r} (\frac{p_i}{p}) = - \prod_{i=1}^{r} (\frac{p}{p_i}) = - \prod_{i=1}^{r} (\frac{4}{p_i}) = -1,$$

again as was to be expected.

The two cases together show that (*) will always readily aid in evaluating $(\frac{-1}{p})$ if we don't have properties (i) and (ii) of Legendre symbols. In essence, property (i) has been proven via property (iii).

To evaluate $(\frac{2}{p})$, the procedure being introduced is to rewrite the Legendre symbol thusly

$$(**) \qquad (\frac{2}{p}) = (\frac{2p+2}{p}) \text{ if } p \equiv 1 \pmod 4$$
$$(\frac{2}{p}) = (\frac{-2p+2}{p}) \text{ if } p \equiv -1 \pmod 4.$$

To show (**) leads quickly to the desired conclusion, it will be convenient to use both property (i) and (iii) which, since (i) follows from (iii), insures that property (iii) need only be used. Consider the cases as before:

Case 1. $p \equiv 1 \pmod 4$.

Write $p + 1 = 2 \prod_{i=1}^{r} p_i$, where the p_i are odd primes. If $p \equiv 1 \pmod 8$, then $\prod_{i=1}^{r} p_i = (p+1)/2 \equiv 1 \pmod 4$ so that there are an even number of $p_i \equiv -1 \pmod 4$; therefore

$$\prod_{i=1}^{r} (\frac{p_i}{p}) = \prod_{i=1}^{r} (\frac{p}{p_i}) = \prod_{i=1}^{r} (\frac{-1}{p_i}) = 1$$

giving $(\frac{2}{p}) = 1$. For $p \equiv -3 \pmod 8$, $\prod_{i=1}^{r} p_i \equiv -1 \pmod 4$ so that there are an odd number of $p_i \equiv -1 \pmod 4$; hence

$$\prod_{i=1}^{r} (\frac{p_i}{p}) = \prod_{i=1}^{r} (\frac{p}{p_i}) = \prod_{i=1}^{r} (\frac{-1}{p_i}) = -1$$

giving $(\frac{2}{p}) = -1$.

Case 2. $p \equiv -1 \pmod 4$.

Write $p - 1 = 2 \prod_{i=1}^{r} p_i$, where the p_i are odd primes. In a similar manner to Case 1, if $p \equiv 3 \pmod 8$, then there are an even number of $p_i \equiv -1 \pmod 4$ so that

$$(\frac{2}{p}) = (\frac{-2p+2}{p}) = (\frac{-4}{p}) \prod_{i=1}^{r} (\frac{p_i}{p}) = - \prod_{i=1}^{r} (\frac{p_i}{p})$$
$$= - \prod_{i=1}^{r} (\frac{p}{p_i}) = - \prod_{i=1}^{r} (\frac{1}{p_i}) = -1.$$

For $p \equiv -1 \pmod 8$, there are an odd number of $p_i \equiv -1 \pmod 4$ so that

$$\left(\frac{2}{p}\right) = -\prod_{i=1}^{r}\left(\frac{p_i}{p}\right) = -\left[-\prod_{i=1}^{r}\left(\frac{p}{p_i}\right)\right] = \prod_{i=1}^{r}\left(\frac{1}{p_i}\right) = 1.$$

Combining the results of the two cases proves property (ii) via property (iii).

Finally, as an example of how the technique described above would be used, the following shows the steps for evaluating $\left(\frac{61}{307}\right)$:

$$\left(\frac{61}{307}\right) = \left(\frac{307}{61}\right) = \left(\frac{2}{61}\right) = \left(\frac{2\cdot 61+2}{61}\right) = \left(\frac{124}{61}\right) = \left(\frac{31}{61}\right)$$
$$= \left(\frac{61}{31}\right) = \left(\frac{-1}{31}\right) = \left(\frac{31-4}{31}\right) = \left(\frac{27}{31}\right) = \left(\frac{3}{31}\right)$$
$$= -\left(\frac{31}{3}\right) = -\left(\frac{4}{3}\right) = -1.$$

Michael Filaseta was an undergraduate at the University of Arizona when this paper was written.

A COMPARISON OF COMPUTER ALGORITHMS TO SOLVE FOR KNIGHT'S TOUR

by Gary Ricard
Sooth Dakota School of Mines and Technology

Probably some of the most common chess puzzles are those where one is required to move a chess piece in such a manner that it shall move successively to every possible cell on a chessboard once and only once [1]. The task is ridiculously easy for a king, queen or rook and obviously impossible for a bishop. However, a knight, with its L-shaped moves, offers a unique and rather difficult puzzle.

A great deal of research has gone into the knight's tour puzzle. The earliest systematic solution is attributed to Euler in 1759 [3]. This method involved moving a knight at random around the chessboard until only a few cells were left open. Euler then gave a set of rules by which the open cells could be incorporated into the knight's path [1]. The method works but the rules are very involved and consequently the method is extremely tedious and is not easily applicable as either a computer or mental algorithm.

In the 18th and 19th centuries different methods were given by Vandermonde in 1771 [2], by Warnsdorff in 1823[5], and by Roget in 1840 [4]. Of these the most remarkable and easiest to use was the method proposed by Warnsdorff [1]. In this method the knight is initially placed somewhere on the board and it is always moved to the cell where it will command the least number of open cells. Warnsdorff also stated that when, by the previous rule, the knight has two or more cells to which it can move, it may be moved to either or any of them indifferently. Another somewhat unique aspect of this algorithm is that if a bad move is made it will not usually affect the result except in the last three or four moves. Warnsdorff's method has never been proven to give results for any size or shape board, but no board, which does have a solution, has ever been found where the method fails. Of all the systematic algorithms reviewed in this paper, Warnsdorff's is the simplest and most easily programmable.

The actual solutions to knight's tour can be classed into two groups; re-entrant and non-re-entrant paths. Re-entrant paths are those where the knight's final move allows it to jump back into the starting cell and, obviously, non-re-entrant paths are those where the knight's final move does not allow it to jump back into the starting cell (see Figure 1). Non-re-entrant paths occur much more frequently than re-entrant paths but, with a hypothesized 122,802,512 total solutions [1] to knight's tour on a standard chessboard, there are a large number of both types of paths.

1	50	3	26	63	14	17	40
4	27	64	13	48	41	62	15
51	2	49	44	25	16	39	18
28	5	12	47	42	45	24	61
11	52	43	32	23	36	19	38
6	29	8	35	46	33	60	57
53	10	31	22	55	58	37	20
30	7	54	9	34	21	56	59

RE-ENTRANT SOLUTION TO 8 BY 8 BOARD

1	56	13	28	3	50	23	26
14	29	2	61	24	27	4	49
55	12	57	38	51	62	25	22
30	15	64	41	60	37	48	5
11	54	39	58	63	52	21	36
16	31	42	53	40	59	6	47
43	10	33	18	45	8	35	20
32	17	44	9	34	19	46	7

NON-RE-ENTRANT SOLUTION TO 8 BY 8 BOARD

FIGURE 1

The remainder of this paper will discuss four computer algorithms written by the author to solve for knight's tour. Of the four only the Warnsdorff method gave results for an 8 by 8 board within an 80 minute time allocation on a CDC-6400. All the methods gave solutions for a 5 by 5 board which is the smallest solvable square board.

The first method tried will be called the "Naive Method". In this method the knight is initially positioned in the upper left hand corner of the board. The knight's available moves are then calculated starting with north one cell, two cells to the east and proceeding as shown in Figure 2. The knight is always moved to the first open cell that is found. If the knight does not have an available cell to move into, it is backed up into the cell it previously occupied and moves are calculated from the last move the knight made from this cell. Again, the

			7		2		
		8				1	
				K			
		5				4	
			6		3		

THE ORDER IN WHICH THE KNIGHT'S POSSIBLE MOVES ARE CALCULATED IN THE "NAIVE METHOD."

FIGURE 2

knight takes the first open cell that is found. This process is repeated until a solution is found. The "Naive Method" is actually an exhaustive tree search of all the knight's possible moves. This method found a solution to a 5 by 5 board in about 20 cpu seconds on a CDC-6400 but failed to find a solution to an 8 by 8 board in 80 minutes. This is not very surprising since the tree for an 8 by 8 board has an estimated 7.84×10^{56} leaves; an exhaustive search could take years (approx. 10^{29}). The problem is that the "Naive Method" is creating open cells which command no open cells. The knight could never reach these "islands" since it could never be in a cell where it could jump into the "island" cell (see Figure 3). The knight would then go merrily on its way trying to find a path but not realizing that it is impossible with the "islands" it was creating. One might decide to write an optimization routine for this program that would eliminate the "islands" and thus cut down the tree search time.

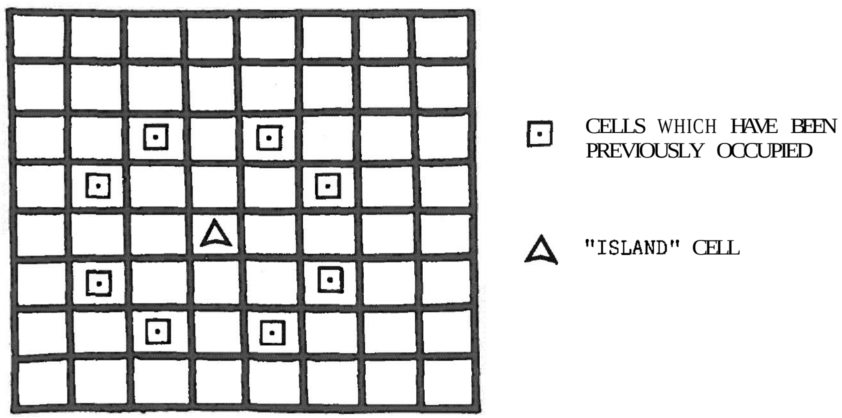


FIGURE 3

Such an optimization routine was written and incorporated into the "Naive Method" program. For every move this routine scanned the eight cells commanded by the present position of the knight and assured that the last move had not created any "islands". With this routine the program found a solution to a 5 by 5 board in about 5 seconds. Again the program exceeded the time allocation and failed to find a solution for an 8 by 8 board. The problem was that instead of isolating just one cell; a "one cell island"; the program was creating two and three cell "islands" (see Figure 4). Thus, the knight could not get to large groups of cells without an extreme amount of backtracking. At this point it was decided to write an optimization routine that would eliminate all "islands" without regard to the number of cells they contained.

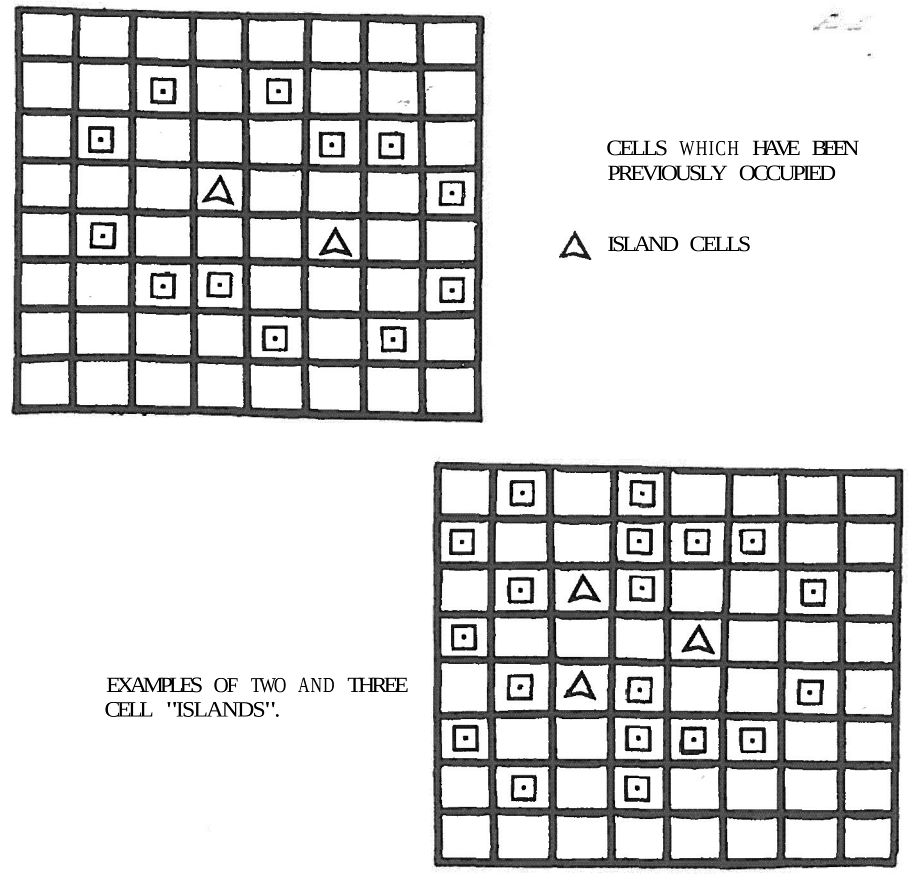


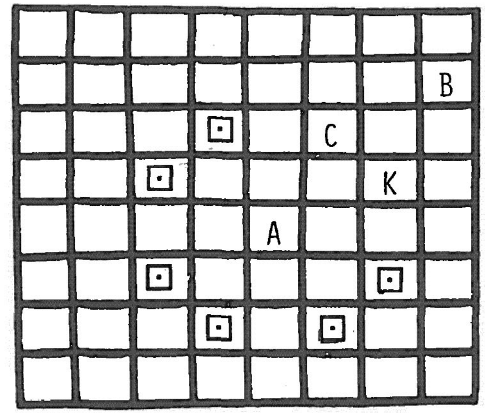
FIGURE 4

The second optimization routine was written such that it scanned the entire board starting with the cell in the upper left corner and ending with the cell in the lower right corner. This routine checked the eight cells that each open cell commands and assured that there were at least two open cells among these; one open cell to enter the cell in question and another to exit this cell. This assured that there were no "islands" of any size since each open cell had to have a path into it and a path out of it. With this routine appended to the program, the "Naive Method" generated a solution to a 5 by 5 board in a little more than one second but it still failed to find a solution for an 8 by 8 board.

It seems that the "Naive Method", even with the optimization routines, is just not an acceptable algorithm when applied to large boards. As mentioned before, there are a hypothesized 122,802,512 solutions to knight'stour on an 8 by 8 board, but there are also approximately 7.84 x 10^{56} erroneous paths. The odds of finding a solution using an exhaustive tree search are just too small. Therefore, the next logical step might be to search for completely different algorithms which solve the knight's tour puzzle.

The algorithm chosen was the Warnsdorff method. This was chosen mainly because of the ease with which this algorithm can be programmed. The method, as stated earlier, always requires the knight to move into the cell which commands the least number of open cells. It is interesting to note how this method prevents "islands" from occurring. In order to create an "island" the knight must have already moved into the eight cells that a particular open cell commands without moving into this cell. As the knight moves into these eight cells it decreases the number of cells that the cell in question commands. As the number of these cells decrease, the cell in question has a higher probability of being chosen. Thus, the knight will always move into the cell that is in danger of being isolated before it will move into a cell which commands a larger number of open cells (See Figure 5). For every case tried, the author has found that if the Warnsdorff algorithm does not find a solution for a particular board with the knight initially positioned in the upper left cell, then that board has no solution. No proof of this could be found in the literature but it does hold true for 3 by 3, 4 by 4, and 7 by 7 boards. This method will solve an 8 by 8 board in less than point five seconds and will find a solution to a 30 by 30 board, whose tree has 7.55 x 10^{811} leaves, in approximately one second. The method has also been tried for many rectangular boards and it finds solutions to these boards just as rapidly.

As a curiosity, the solution to a 100 by 100 board took 38 cpu seconds on a CDC-6400.



Assume the knight is in the cell marked "K", and that the shaded cells have previously been occupied by the knight. Cell "A" is in danger of being isolated. If the knight moves into cell "B" and then into cell "C", cell "A" becomes and "island". However, this will never happen since cell "B" commands two cells and cell "A" commands only cell "C". The knight will move into cell "A" since it commands less open cells than cell "B", and the "island" is prevented.

FIGURE 5

This paper was written while Gary Ricard was an undergraduate at the South Dakota School of Mines and Technology under the supervision of Dr. David Ballew. Copies of the FORTRAN program discussed in this artiale can be obtained from the author or Dr. Ballew.

REFERENCES

1. Ball, W. W. Rouse, *Mathematical Recreations and Essays*, rev. H.S.M. Coxeter, 11th ed., New York, 1939.
2. *L'Histoire de l'Académie des Sciences* for 1771, Paris, (1774), 566-574.
3. *Memoires de Berlin* for 1759, Berlin, (1766), 310-337.
4. Roget, P., *Philisophical Magazine*, series 3, vol. XVI, April, 1840.
5. Warnsdorff, H., *Des Rösselsprungs Einfachste Und Allegemienste Lösung*, Schmalkalden, 1823.

ONE-WAY ORIENTATIONS OF GRAPHS

by *David Burns*, *S.F. Kapoor*
Ferris State College and Western Michigan University

One way to produce a directed graph (also commonly called a 'digraph') is to start with a graph and assign directions to each of its edges. A digraph created from a graph G in this way is called an <u>orientation</u> of G. For $i = 1, 2, 3$ the digraph D_i of Figure 1 is an orientation of the graph G_i.

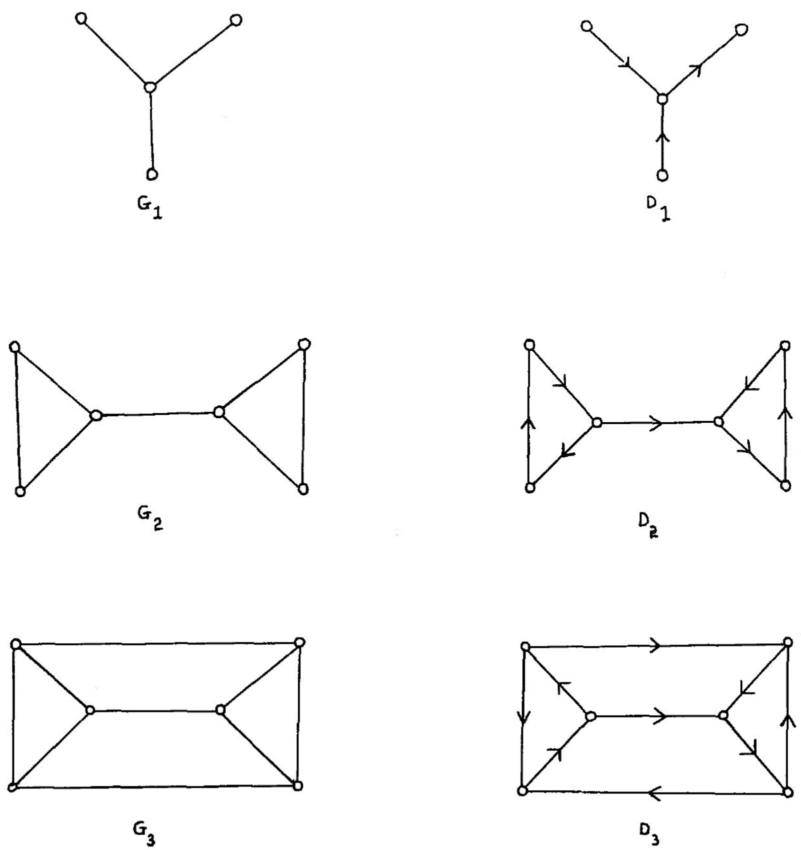


Figure 1

Many structural properties of digraphs are dependent on related properties of their underlying graphs. For example, if a city planner whose municipal road system is represented by the graph G were instructed to convert all of the roads in the city into one-way thoroughfares, then the orientation D_3 would represent one of many possible such conversions. Clearly any such one-way network of roads must allow travel from any one vertex to any other vertex. Following the terminology of [1], we say that a graph is <u>strongly orientable</u> if G admits an orientation in which every two vertices are mutually reachable. The graph G_3 is strongly orientable, and it is not hard to decide that neither G_1 nor G_2 are strongly orientable.

The question of which graphs are strongly orientable was settled by Robbins [2] who proved that a graph G is strongly orientable if and only if G is connected and bridgeless.

Another use for digraphs is to model communication networks. Here a vertex might represent a station in the network, and a directed edge from vertex u to vertex v would symbolize the ability of station u to directly communicate with station v. Two-way interchanges are not always possible in a communications network. In a military organization, for example, the general is able to communicate with the private, however, there may be no way for the private to communicate with the general. As in [1] we call a digraph <u>unilateral</u> if for any pair of its vertices, at least one of them is reachable from the other. Also, we say that a graph G is <u>unilaterally orientable</u> if G admits a unilateral orientation. Of the graphs in Figure 1, only G_1 fails to be unilaterally orientable. It is known [1] that a digraph is unilateral if and only if it contains a spanning directed walk. Using this result and terminology of [1] we present a characterization of unilaterally orientable graphs.

<u>*Theorem.*</u> A graph G is unilaterally orientable if and only if G is connected and there exists a path in G containing all the bridges of G.

Proof. Assume first that G is unilaterally orientable. Then clearly G is connected. Let D be a unilateral orientation of G so that D is a unilateral digraph. Then D contains a spanning directed walk W_D: $u_1, u_2, \ldots, u_n$. (It is possible that a vertex of G receives more than one label.) Let $W_G : u_1, u_2, \ldots, u_n$ be the corresponding spanning walk in G. Let b be a bridge of G. If b is not an edge in W_G, then since both vertices of b occur in W_G, there exists a cycle in G containing b, contradicting the fact that b is a bridge. Hence all bridges of G belong to the walk W_G. Among all walks in G which contain the bridges of G let W^* be one of minimal length. Then W^* is a path in G which contains all the bridges of G.

Conversely, assume that G is a connected graph with a path $P : u_1, u_2, \ldots, u_n$ in G containing all the bridges of G. If $u_i u_{i+1}$ is a bridge of G, orient this edge from u_i to u_{i+1}. Every block of G not yet oriented must have order at least three, and by [2], can be strongly oriented. Let u and v be any pair of vertices in this orientation of G. If u and v lie in the same block of G then clearly there exists either a directed u-v path or a directed v-u path in the digraph. If u and v lie in different blocks of G where, say, the block or collection of blocks containing u is encountered on the path P following its orientation prior to the block or collection of blocks containing v, then that portion of the path P between these distinguished block structures together with any intervening blocks guarantees the existence of a directed u-v path in the digraph. Therefore this orientation of G is unilateral.

REFERENCES

1. Behzad, M., Chartrand, G., and Lesniak-Foster, L., *Graphs and Digraphs.* Prindle, Weber and Schmidt, Inc., Boston (1979).
2. Robbins, H., *A theorem on graphs with an application to a problem of traffic control,* Amer. Math. Monthly 46 (1939), 281-283.

There will be a Pi Mu Epsilon Regional Mathematics Conference at St. John's University in Collegeville Minnesota on April 30 and May 1, 1981. The Student Conference will be held in conjunction with a seminar entitled "Mathematics and the Humanities". Doris Schattschneider will speak on Mathematics and Art, Leonard Gillman wilt. speak on Mathematics and Music, and, hopefully, Don Koehler wilt. lecture on Mathematics and Literature. Student papers are not restricted to the "Humanities" theme and should be sent to Jerry Lenz, St. John's College, Collegeville, Minn.

PUZZLE SECTION

David Ballew

This department is for the enjoyment of those readers who are addicted to working crossword puzzles or who find an occasional mathematical puzzle attractive. We consider mathematical puzzles to be problems whose solutions consist of answers immediately recognizable as correct by simple observation and requiring little formal proof. Material submitted and not used here will be sent to the Problem Editor if deemed appropriate for that department.

Address all proposed puzzles and puzzle solutions to David Ballew, Editor of the Pi Mu Epsilon Journal, Department of Mathematical Sciences, South Dakota School of Mines and Technology, Rapid City, South Dakota, 57701. Deadlines for puzzles appearing in the Fall issue will be the next February 15, and puzzles appearing in the Spring issue will be due on the next September 15.

Mathacrostic No. 11

submitted by Joseph D. E. Konhauser
Macalester College, St. Paul, Minnesota

Like the proceeding puzzles, this puzzle (on the next page) is a keyed anagram. The 213 letters to be entered in the diagram in the numbered spaces will be identical with those in the 27 keyed words at matching numbers, and the key letters have been entered in the diagram to assist in constructing your solution. When completed, the initial letters will give a famous author and the title of his book; the diagram will be a quotation from that book.

Cross-Number Puzzles

submitted by Mark Isaak
Went, University of California, Berkeley

In the cross-number puzzles (starting two pages hence), each of the letters stands for a positive, nonzero integer. The algebraic expressions evaluate out to two to five digit numbers which fit in the squares as in a normal crossword puzzle. None of the numbers in the squares have any lead-zeros; i. e., if there is room for a four digit number, that number will be at least 1000, never, for example, 0999.

1 J	2 Z	3 H	4 L	5 E	6 P	7 Y		8 O	9 G	10 R		11 I	12 N
13 H	14 V	15 Y		16 A	17 D		18 C	19 Q	20 R	21 Y	22 G	23 K	24 a
25 E	26 B		27 R	28 T	29 P	30 Y	31 G	32 Q	33 E	34 X	35 S		36 D
37 J		38 K	39 M	40 O		41 R	42 Y	43 U	44 C		45 E	46 J	47 I
48 X	49 D	50 G	51 M	52 K	53 R	54 F		55 J	56 a	57 W	58 V		59 R
60 Y		61 W	62 C	63 G	64 B	65 T	66 P	67 X	68 F		69 Z	70 S	71 V
	72 J	73 L	74 F	75 Y	76 D	77 Q	78 E	79 Z		80 O	81 L		82 U
83 A	84 P	85 Y	86 T	87 a		88 N	89 L	90 Q	91 O	92 S	93 U	94 J	95 D
96 a	97 I		98 E	99 A	100 Y	101 L	102 H	103 a	104 J	105 F	106 O	107 W	
108 E	109 G	110 U	111 V		112 J	113 K	114 F		115 R	116 D	117 T	118 C	
119 B	120 G	121 H	122 P	123 Y	124 A		125 L	126 E		127 X	128 F	129 Z	130 M
131 A		132 D	133 N	134 P	135 G	136 E	137 H	138 V	139 K	140 A	141 Z	142 M	
143 C	144 N	145 T		146 F	147 I	148 O		149 X	150 W	151 H	152 Q	153 P	154 V
	155 D	156 A		157 S	158 W	159 J		160 B	161 J	162 E	163 M	164 G	165 X
	166 R	167 U	168 O		169 C	170 J	171 G	172 L	173 B	174 R	175 Z		176 a
177 W	178 K	179 N	180 O		181 Y	182 P	183 X	184 G		185 D	186 I	187 a	188 Q
	189 Z	190 W	191 J		192 H	193 N	194 D	195 T	196 C	197 L	198 A	199 K	200 S
201 O	202 Q		203 N	204 D	205 Z	206 I	207 X	208 H	209 A	210 C	211 B	212 T	213 M

KEY WORD AND PHRASES

A. harmonically unresolved — 131 83 209 124 99 156 198 140 16

B. off-color — 173 211 119 160 64 26

C. to make blunt — 118 196 169 18 210 143 44 62

D. concealment of one heavenly body by another — 155 49 132 116 185 76 36 194 95 17 204

E. self-correspondent (2 wds.) — 98 162 33 45 5 108 78 136 126 25

F. rhythmically recurrent contraction — 68 54 105 146 114 128 74

G. radio's juvenile answer to <u>Information, Please</u> (3 wds.) — 120 9 184 63 109 50 171 164 22 31 135

H. star-shaped optical phenomenon — 102 137 151 208 121 13 3 192

I. source-sink combination — 11 186 47 147 97 206

J. meets every plane in three points, real or imaginary, distinct or coincident (2 wds.) — 37 55 104 46 112 159 191 72 161 1 170 94

K. perspective collineation with center and axis incident — 139 52 38 199 178 113 23

L. branched — 73 172 197 4 81 125 89 101

M. a knot <u>not</u> for sailors — 213 163 51 130 39 142

N. simple wind instrument — 133 179 88 12 203 144 193

O. division into two — 168 91 201 180 80 8 148 106 40

P. Klein's Programm to codify geometries — 122 66 153 182 134 84 6 29

Q. affirmative expressed by the negative of the contrary — 19 77 152 32 90 188 202

R. based on observation or experience — 10 115 27 59 174 20 41 166 53

S. translate — 35 70 200 92 157

T. first modern acoustician (1756-1827) — 117 195 86 28 145 212 65

U. sometimes found at ends of problems — 82 93 167 110 43

V. subsets of sample spaces — 71 14 154 58 138 111

W. Robert Schumann's third — 61 177 57 190 150 107 158

X. lampooner of infinitesimals as ghosts of departed quantities" — 127 34 207 183 67 149 48 165

Y. accelerator (2 wds.) — 75 60 42 21 123 181 30 7 85 15 100

Z. connect in a series — 141 129 205 175 79 189 69 2

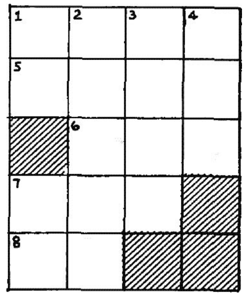
a. at the midpoint — 56 24 103 187 176 96 87

ACROSS

1. AB
5. A^2
6. C
7. $3A - 18$
8. $(1/9)A^2 - D$

DOWN

1. A/E
2. B^3
3. $E^6 + 1000$
4. $BC/89$
7. $E2$

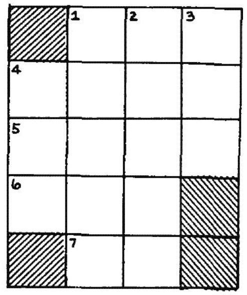


ACROSS

1. $5AB/91$
4. $66C + 82$
5. $(\sqrt{118} + 22)^2$
6. D
7. $(7D + 208)/20$

DOWN

1. $5AB$
2. $619(B + C + E) - 30$
3. $C + 7E$
4. $6C + 3$

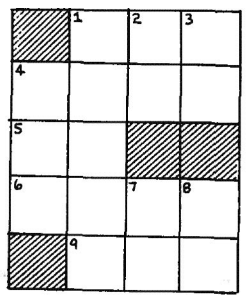


ACROSS

1. A^2
4. BC
5. $63C/5$
6. $3D\sqrt{5E}$
7. B

DOWN

1. $10(B + F) - 2$
2. E
3. G
4. F
7. $\sqrt{E} + \sqrt{G} + H$
8. D

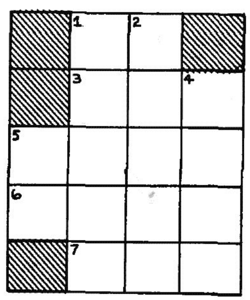


ACROSS

1. $A^4 + BC + C$
2. D
5. $(E + 1)^4F$
6. $((C/B) + A)^G$
7. $H(A + G)(C - 2A)$

DOWN

1. $J-DK$
2. $2J + 20$
4. GHK
5. $B + C$



Who Stole the Candy?

submitted by Wayne M. Delia and Bernadette D. Barnes
Clarkson College, Potsdam NY

A group of five children entered a candy store; soon afterward, the store owner noticed a box of candy missing from the shelf. The thief was one of the following: Ivan, Sylvia, Ernie. Dennis or Linda. Each child made three statements.

Ivan:
1) I didn't take the box of candy.
2) I have never stolen anything.
3) Dennis did it.

Sylvia:
4) I didn't take the box of candy.
5) I'm rich and I can buy my own candy.
6) Linda knows who the crook is.

Ernie:
7) I didn't take the box of candy.
8) I didn't know Linda until this year.
9) Dennis did it.

Dennis:
10) I didn't take the box of candy.
11) Linda did it.
12) Ivan is lying when he says I stole the candy.

Linda:
13) I didn't take the box of candy.
14) Syliva is guilty.
15) Ernie can vouch for me, because he has known me since I was a baby eight years ago.

Each child later admitted that two of the statements they made were true and one was false. If this is the case, then who stole the candy?

SOLUTIONS

Mathacrostic No. 10 (See Spring 1980 issue) (*proposed by J. D. E. Konhauser*)

Definitions and Key:

A. Bode's Law	H. Rhubarb	O. Synthetic	V. Caliban
B. Rolypoly	I. Logogriph	P. Idempotent	W. Nephroid
C. Enantiomorph	J. Identity	Q. Roberval	X. Eyelash
D. Wolfskehl	K. Fortran	R. Involution	Y. Wad
E. Senet	L. Eratosthenes	S. Sheath	Z. Tomahawk
F. Tweedledee	M. Oval	T. Affinity	a. Oughtred
G. Elements	N. Freedom	U. Antinomy	b. Negabinary

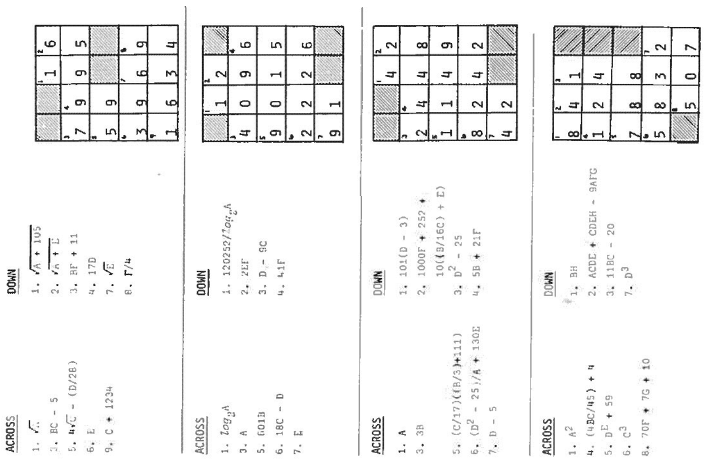
First Letters: Brewster Life of Sir Isaac Newton

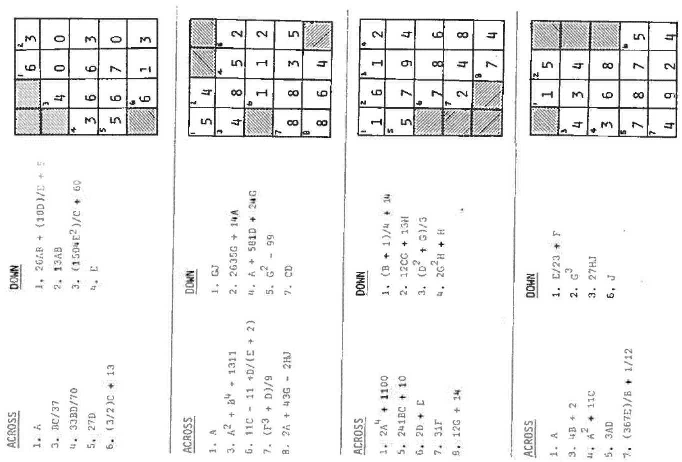
Quotation: *I do not know what I may appear to the world, but to myself I seem to have been only like a boy playing on the seashore and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me.*

Solved by: *Jeanette Bickley*, Webster Groves High School, Missouri; *Louis H. Cairoli*, Kansas State University; *Robert C. Gebhardt*, Hopdtcong, NJ; *Henry S. Lieberman*, John Hancock Mutual Life Ins. Co.; *Robert* **Prielipp**, University of Wisconsin-Oshkosh; *Sister Stephanie* **Sloyan**, Georgian Court College; *Robert Forsberg*, Lexington, Mass.; *The Proposer* and *The Editor*.

Cross Number Puzzles (See *Spring* 1980 issue; *(proposed by Mark Isaak)*

Solved by: *Eleanor Joyner*, University of Tennessee; *Randall J. Scheer*, Boonville, NY; *Mark Walker*, Horace Mann Ins. Co.; *The Proposer* and *The Editor*.





Mathacrostic No. 9 (See *Fall* 1979 issue) *The Editor did not include the solution by Robert Prelipp*, The University of Wisconsin-Oshkosh, *with the solutions in the Spring* 1980 *issue*.

REFEREES

The following mathematicians have served as referees since the publication of the Spring 1980 *issue; without their help, the Journal would not be possible.* **Bruce Peterson**, Middlebury College; **Warren Loud**, University of Minnesota; **Fredrick Zerla**, University of South Florida; **Sabra Anderson**; University of Minnesota at Duluth; **Charles Ziegenfus**; James Madison University; **Joseph Konhauser**, Macalester College; **Daniel Sterling**, Colorado College and Martin-Maretta; **Clayton Dodge**, University of Maine; **William Fishback**, Earlham College; **Herbert Taylor**; **Frank Harary**, University of Michigan; **Augusto Salvatore**, Union College; **Peter Lindstrom**, Genesee Community College; **Paul Campbell**, Beloit College; **Mabel Szeto**; **Richard Andree**, University of Oklahoma; *John* **Schumaker**, Northern Illinois University; *and my colleagues at the South Dakota School of Mines and Technology--Dean C.* **Benson**, **Bonita Leonhardt**, **Harold Carda**, **Al Grimm**, **James Patterson**, **Roger** *Opp*, **Dale Rognlie**, **Salias Sengupta** *and* **Ronald Weger**. *The Editor assumes responsibility for all errors and misprints.*

High School Mathematics Contests

Many Pi Mu Epsilon Chapters either sponsor or contribute services to contests, competitions and "Math Days" amoung high school students. The Editor's office can act as an information source and a clearing house to swap examinations and ideas. If your Chapter is involved, please send the Editor an outline of how your examinations is conducted and copies of your materials; about 25 copies of advertisements, brouchures, and if possible, your examinations for swaping purposes. The Editor will send you copies of the materials on hand. If you are starting such events, let the Editor know and materials can be sent to you. This could be a great help to us all!

PROBLEM DEPARTMENT

Edited by Leon Bankoff
Los Angeles, California
and
Clayton W. Dodge,
University of, Maine

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also acceptable. The choice of proposals for publication will be based on the editor's evaluation of their anticipated reader response and also on their intrinsic interest. Proposals should be accompanied by solutions if available and by any information that will assist the editor. Challenging conjectures and problem proposals not accompanied by solutions will be designated by an asterisk (*).

Problem proposals offered for publication should be sent to Dr. Leon Bankoff, 6360 Wilshire Boulevard, Los Angeles, California 90048.

To facilitate consideration of solutions for publication, solvers should submit each solution on separate sheets (one side only) properly identified with name and address and mailed before June 30, 1981 to Professor Clayton W. Dodge, Mathematics Department, University of Maine, Orono, Maine 04469.

Contributors desiring acknowledgment of their proposals and solutions are requested to enclose a stamped and self-addressed postcard or, for those outside the U.S.A., an unstamped card or mailing label.

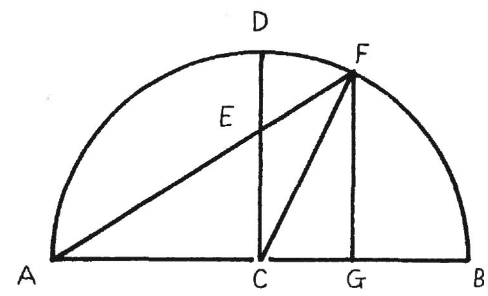
PROBLEMS FOR SOLUTION

474. *Proposed by Scott Kim, Artificial Intelligence Laboratory, Stanford University.*

Knotted path: Consider a 2 by 3 by 7 block of unit cubical cells. Your task is to find a path moving from cell to adjacent cell, returning to the original cell so that the path traced is a 3-dimensional knot. Each cell *must* be visited exactly once; two cells are adjacent only if they share a face.

475. *Proposed by Zelda Katz, Beverly Hills, California.*

In the accompanying diagram DC is the radius perpendicular to the diameter AB of the semicircle ADB; FG is a half-chord parallel to DC; AF cuts DC in E. Show that the sides of triangle FCG are integers if and only if DE/EC or its reciprocal is an integer.



476. *Proposed by Jack Garfunkel, Queens College, Flushing, N.Y.*

If A, B, C, D are the internal angles of a quadrilateral, that is, if $A + B + C + D = 360°$, then $\sqrt{2}\,[\cos(A/2) + \cos(B/2) + \cos(C/2)] \le [\cot(A/2 + \cot(B/2) + \cot(C/2)]$, with equality when $A = B = C = D = 90°$.

477. *Proposed by Solomon W. Golomb, University of Southern California.*

In the eleventh row of Pascal's Triangle, the first five terms (1, 11, 55, 165, 330) have the property that each is an integral multiple of its predecessor. Is there a row of Pascal's triangle where there are eleven consecutive terms with this property?

478. *Proposed by Charles W. Trigg, San Diego, California*

$$PIGS = ROOT + ROOT + ROOT,$$

but can only dig up a single solution when each different letter represents a distinct digit, and PIGS contains three consecutive odd digits. What is the unique representation of the addition?

479. *Proposed by Herbert Taylor, South Pasadena, California.*

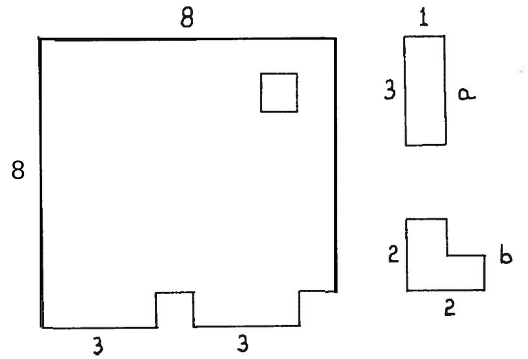
Prove that the following statement is true whenever $0 < r \le n$, or else find a counterexample:

Given a $m \times n$ matrix of 0's and 1's, with each column sum equal to $2r$ and each row sum equal to r, it is always possible to mark $2n$ of the 1's in such a way that one 1 is marked in each row and two 1's are marked in each column.

480. *Proposed by Richard I. Hess, Palos Verdes, California.*

a) Cut the large piece at right into two pieces which can be reassembled with piece a into an 8 X 8 square.

b) Do the same, using piece b.



481. *Proposed by Clayton W. Dodge, University of Maine at Orono.*

Find all roots of the polynomial equation

$$x^6 - x^5 - 4x^4 + 5x^3 - 41x^2 + 36x - 36 = 0.$$

given that it has two roots whose sum is zero.

482. *Proposed by Ronald E. Shiffler, Georgia State University.*

Let X be a continuous random variable having a uniform distribution with domain $[a,b]$ and mean and standard deviation represented by μ and σ, respectively. Verify that $P(\mu - 2\sigma \le X \le \mu + 2\sigma) = 1$.

483. *Proposed by Paul Erdos, Spaceship Earth.*

Let μ_n be the smallest integer for which $\mu_n(\mu_n + 1) \equiv 0 (\mathrm{mod}\ n)$.

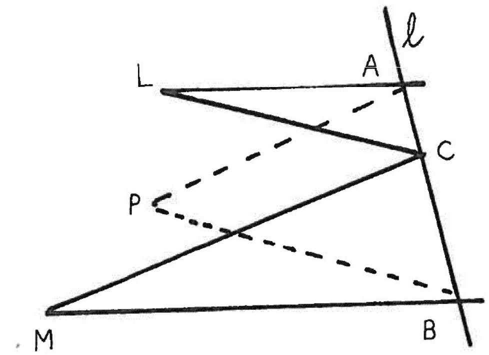
Prove $\displaystyle\sum \frac{1}{\mu_n(\mu_n + 1)} < \infty$.

484. *Proposed by the late R. Robinson Rome.*

In a triangle with base AB and vertex C, secants from A and B to points D and E on BC and AC divide the area into four subareas S, T, U and V. In some order of S, T, U, V, the points D and E can be located so that the subareas are in increasing arithmetical progression, or so that they are in decreasing aritmetical progression. Find that order and evaluate the subareas.

485. *Proposed by R. S. Luthar, University of Wisconsin, Janesville.*

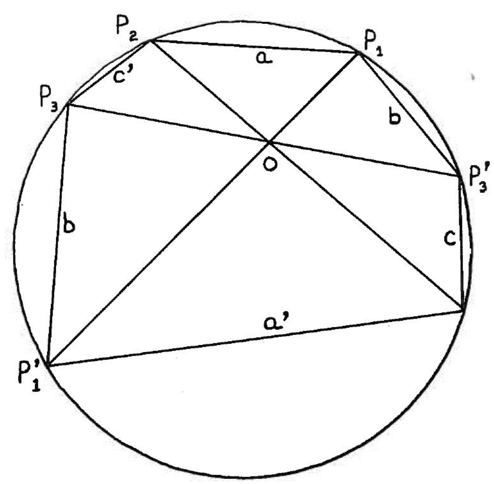
A line l cuts two parallel rays emanating from L and M in A and B respectively. A point C is taken anywhere on l. Lines through A and B respectively parallel to MC and LC intersect in P. Find the locus of P.



Solutions

438. [Spring 1979; Spring 1980] *Proposed by Ernst Straus, University of California at Lob Angeles.*

Prove that the sum of the lengths of alternate sides of a hexagon with concurrent major diagonals inscribed in the unit circle is less than 4.



Solution by the Proposer.

Referring to the diagram, we have $OP_i \cdot OP_i'$ equal for $i = 1, 2, 3$ so that $abc = a'b'c'$. The perimeter of an inscribed hexagon (in the unit circle) is ≤ 6, with equality only for the regular hexagon. Now assume $a + b + c \ge 4$. Then $a' + b' + c' < 2$ and therefore $abc = a'b'c' < (2/3)^3 = 8/27$.

If $2 \ge a \ge b \ge c$ then $a + b \ge 4 - c$ and hence $ab \ge 2(2-c)$. Thus

$$2(2-c)c \le abc < 8/27$$

implies

$$1 - (1-c)^2 = (2-c)c < 4/27$$

and hence

$$(1-c)^2 > 23/27,\ c < 1 - \sqrt{23/27} < .08.$$

Thus

$$2 \ge a \ge b > 1.92.$$

We have thus shown that two of the sides of the hexagon are nearly diameters of the circle so that the total arc subtended by the other four sides is small. We are now going to derive a contradiction from this fact.

Let a subtend an arc of length $\pi - \alpha$ and b subtend an arc of length $\pi - \beta$. Then $3.92 < a + b = 2\left(\cos\frac{\alpha}{2} + \cos\frac{\beta}{2}\right)$ and $\alpha + \beta$ is maximal when

$$\delta = \alpha = \beta < 2 \arccos .98.$$

Set $c = 2\sin\gamma$. Then $4 \le a + b + c < 4\cos\frac{\delta}{2} + 2\sin\gamma$, or

(1) $\sin\gamma > 2\left(1 - \cos\frac{\delta}{2}\right) = 4\sin^2\frac{\delta}{4}$.

The total arc length of the arcs subtended by a', b', c' is therefore $\le 2\delta - 2\gamma$ and

$$a'b'c' \le 8\sin^3\frac{\delta-\gamma}{3} < 8\sin^3\frac{\delta}{3}$$

while

$$abc \ge 2\left(4\cos\frac{\delta}{2} - 2\right)\cdot 2\sin\gamma$$

$$\ge 8(1 - \sin\gamma)\sin\gamma$$

$$\ge 8(1 - .04)\cdot 4\sin^2\frac{\delta}{4}$$

leads to the inequality

(2) $$4\sin^2\frac{\delta}{4}\left(1 - 4\sin^2\frac{\delta}{4}\right) < \sin^3\frac{\delta}{3}.$$

Now (1) implies $\sin^2\frac{\delta}{4} < \sin\frac{\gamma}{4} < .01$; $\sin\frac{\delta}{4} < .1$.

But (2) does not hold when $\sin\frac{\delta}{4} < 0.1$.

This contradicts the assumption $a + b + c \ge 4$.

449. [Fall 1979] *Proposed by Richard I. Hess, Palos Verdes, California.*

A fairly young man was married at the beginning of the month. At the end of the month his wife gave him a chess set for his birthday. If he was married and received the chess set on the same day of the week he was born, how old was he when he got married?

Solution by Charles W. Trigg, San Diego, California.

The number of days in the month must be of the form $7k + 1$. Therefore the young man was born on February 29th in a leap year. Leap years are divisible by 4 but not by 400. In the span 1604-1996 (and comparable spans), February 29th occurs in a repeating cycle of Sunday, Friday, Wednesday, Monday, Saturday, Thursday, and Tuesday. That is, February 29th occurs on the same day of the week every 28 years. The young man was married at the beginning of the month in which his 28th birthday fell. Consequently, he was married at an age of 27 years, 338 days, even though he could have celebrated only six birthdays up to that time.

Also solved by CHUCK ALLISON, CLAYTON W. DODGE, MIKE CALL, MARK EVANS, VICTOR G. FESER, ROBERT C. GEBHARDT, DOUGLAS JUNGREIS, R. E. KING, ERIC J. KROHN, ROGER E. KUEHL, HENRY S. LIEBERMAN, MICHAEL MAY, AVI LOSICE, DARRELL PITTMAN and ELISA HOLT (jointly), GEORGE W. RAINEY, E. SHERMAN GRABLE, JACK MILLER, THE SANTA CLARA PROBLEM SOLVING RING, and the Proposer.

450. [Fall 1979] *Proposed by Clayton W. Dodge, University of Maine at Orono.*

In triangle ABC, let $\angle A \le \angle B \le \angle C$. Then

$$s \begin{smallmatrix}\ge\\=\\<\end{smallmatrix} (R+r)\sqrt{3} \quad \text{if and only if } \angle B \begin{smallmatrix}\ge\\=\\<\end{smallmatrix} \pi/3$$

is a well-known theorem, where s is the triangle's semiperimeter, r its inradius, and R its circumradius. Prove it.

Solution by Bob Prielipp, The University of Wisconsin-Oshkosh.

The following relations are known:

$$\Sigma \cos \angle A = \frac{R+r}{R},$$

$$\Sigma \cos \angle A \cos \angle B = \frac{s^2-4R^2+r^2}{4R^2}, \text{ and}$$

$$\pi \cos \angle A = \frac{s^2-4R^2-4Rr-r^2}{4R^2}$$

Multiplying out the left-hand side and using the above, it can be shown that

$$(2\cos \angle A - 1)(2\cos \angle B - 1)(2\cos \angle C - 1) = \frac{s^2-3(R+r)^2}{R^2}$$

When $s \gtreqless (R+r)\sqrt{3}$ $(2\cos \angle A - 1)(2\cos \angle B - 1)(2\cos \angle C - 1) \gtreqless 0$. If triangle ABC is not equilateral, $\angle A$ must be less than $\pi/3$ and $\angle C$ must be greater than $\pi/3$ because $\angle A \le \angle B \le \angle C$ so $2\cos \angle A - 1 > 0$ and $2\cos \angle C - 1 < 0$. Hence $2\cos \angle B - 1 \lesseqgtr 0$ so $\angle B \gtreqless \pi/3$.

If $\angle B \gtreqless \pi/3$ then $2\cos \angle B - 1 \lesseqgtr 0$. If triangle ABC is not equilateral, $2\cos \angle A - 1 > 0$ and $2\cos \angle C - 1 < 0$. Hence $(2\cos \angle A - 1)(2\cos \angle B - 1)(2\cos \angle C - 1) \gtreqless 0$ so $s \gtreqless (R+r)\sqrt{3}$

The case where triangle ABC is equilateral is easily handled.

Also solved by Walter Blumberg, Henry S. Lieberman *and the* Proposer.

451. [Fall 1979] *Proposed by Solomon W. Golomb, University of Southern California, Los Angeles, California.*

Find all instances of three consecutive terms in a row of Pascal's triangles in the ratio 1:2:3.

Solution by Charles W. Trigg, San Diego, California

The given ratio requires that the binomial coefficients which are the terms in Pascal's triangle have the following relationships: $2C(n,k) = C(n,k+1)$ and $3C(n,k) = C(n,k+2)$. These equations simplify to $n = 3k+2$ and $3(k+1)(k+2) = (n-k)(n-k-1)$, respectively. Solving simultaneously and discarding the negative root, $k = 4$, $n = 14$. That is, in the binomial expansion to the fourteenth power, the fifth, sixth, and seventh coefficients are 1001, 2002, and 3003. The solution is unique.

Since the eighth coefficient is $3432 \neq 4004$, there are no four consecutive terms in a row of Pascal's triangle in the ratio 1:2:3:4.

This is the solution given in Charles W. Trigg, Mathematical Quickies, McGraw-Hill, 1967, Problem 126, pages 36, 134-135.

Another solution by Lawrence A. Ringenberg appears in Mathematics Magazine, 29 (January-February 1956), p. 164.

Also solved by CHUCK ALLISON, JEANETTE BICKLEY, WALTER BLUMBERG, MIKE CALL, CLAYTON W. DODGE, MARK EVANS, MICHAEL W. ECKER, ROBERT C. GEBHARDT, JOHN M. HOWELL, ERIC J. KROHN, KEITH A. LIVINGSTON, MICHAEL MAY, BOB PRIELIPP, SAHIB SINGH, JOSEPH C. TESTEN, E. SHERMAN GRABLE *and the* Proposer. *One unidentified solution was received.*

452. [Fall 1979] *Proposed by Tom M. Apostol, California Institute of Technology.*

Given integers $m > n > 0$, let

$$a = \sqrt{m} + \sqrt{n}, \qquad b = \sqrt{m} - \sqrt{n}.$$

If $m - n$ is twice an odd integer, prove that both a and b are irrational.

I. Solution by the Proposer.

We consider the polynomial

$$Q(x) = (x-a)(x+a)(x-b)(x+b) = x^4 - 2(m+n)x^2 + (m-n)^2$$

and show it has no rational roots.

If $Q(x)$ has a rational root r then r is an integer, and $Q(r) = 0$ implies r^4 is even so r is even, say $r = 2t$. Hence

$$16t^4 - 8(m+n)t^2 + (m-n)^2 = 0.$$

This implies $(m-n)^2$ is a multiple of 8, contradicting the fact that $m-n$ is twice an odd integer. Hence both a and b are irrational.

Alternate solution by the Proposer.

First we note that since $ab = m - n$ is rational, both a and b are rational or both are irrational. Next, it is easy to verify that both a and b are rational if, and only if, $m = r^2$ and $n = s^2$ for some pair of integers $r > s > 0$. Therefore, we need only prove that if $m = r^2$ and $n = s^2$ then $m - n$ is never twice an odd integer.

Now $m - n = r^2 - s^2 = (r-s)(r+s)$. If r, s have the same parity (both even or both odd), $m - n$ is a multiple of 4. If r, s have opposite parity, both $r-s$ and $r+s$ are odd so $m-n$ is odd. Hence, in either case $m-n$ is not twice an odd integer.

Examples. If t is an odd integer, then for every integer $n > t$ both $\sqrt{n+t} + \sqrt{n-t}$ and $\sqrt{n+t} - \sqrt{n-t}$ are irrational.

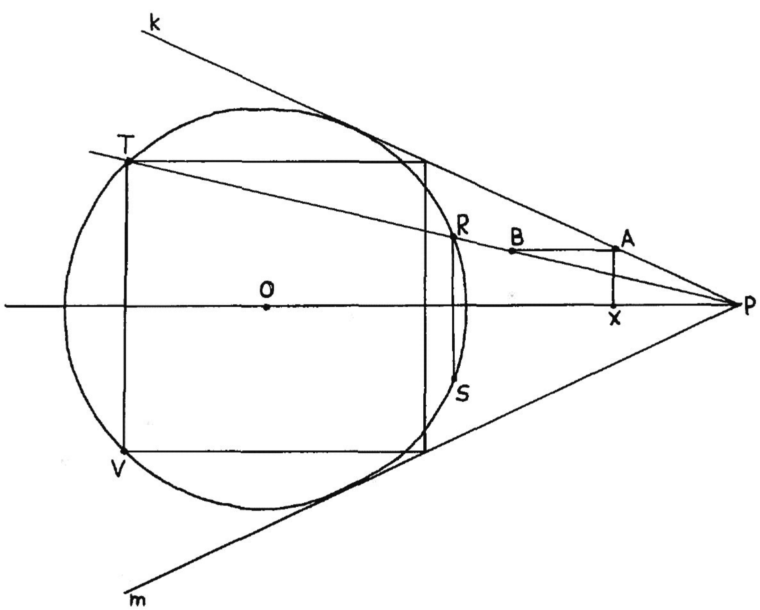
11. *Solution by Kenneth M. Wilke, Topeka, Kansas.*

Since a and b are conjugate roots of the quadratic equation $x^2 - (a+b)x + ab = 0$, if one of a and b is rational then so is the other. Now suppose a and b are both rational. Then $\sqrt{m}$ and $\sqrt{n}$ are both integers. Hence $m = A^2$ and $n = B^2$ for some integers A and B and $m - n = A^2 - B^2 = 2(2k+1)$ for some odd integer $2k+1$. But the equation $A^2 - B^2 = 2(2k+1)$ has no solutions because $(A - B)$ and $(A+B)$ have the same parity, so that either $A^2 - B^2$ is odd or it is divisible by 4. This contradiction establishes the irrationality of both a and b.

Also solved by WALTER BLUMBERG, MIKE CALL, MICHAEL W. ECKER, MARK EVANS, VICTOR G. FESER, HENRY S. LIEBERMAN, KEITH A. LIVINGSTON, MIKE MAY, BOB PRIELIPP, SANTA CLARA PROBLEM SOLVING RING, SAHIB SINGH, AL WHITE, SAMUEL F. TUMOLO AND LEON MACDUFF.

453. [Fall 1979] *Proposed by Jack Garfunkel, Queens College, Flushing, New York.*

Given two intersecting lines and a circle tangent to each of them, construct a square having two of its vertices on the circumference of the circle and the other two on the intersecting lines.



Solution by Jeanette Bickley, St. Louis, Missouri.

Let P be the intersection of lines k and m, which are tangent to the circle (0). Choose an arbitrary point X on OP. Construct AX perpendicular to OP, as shown in the diagram. Construct AB perpendicular to XA with $AB = 2\,(XA)$. PB intersects circle (0) at R and again at T. At R construct n perpendicular to OP, cutting circle (0) in S. Similarly construct p perpendicular to OP cutting the circle again in V. Then RS and TV are the sides of the required squares since PB is the locus of all points such that their distances to k (measured horizontally) is twice the distance from OP.

Also solved by JOSEPH C. TESTEN, CHARLES W. TRIGG, DONALD CANARD, BARBARA SEVILLE, *and the* Proposer.

454. [Fall 1979] *Proposed by Marian Haste, Reno, Nevada.*

The point within the triangle whose combined distances to the vertices is a minimum is (or should be) known as the Fermat-Torricelli point, designated by T. In a triangle ABC, if AT, BT, CT form a geometric progression with a common ratio of 2, find the angles of the triangle.

Solution by Sister Stephanie Sloyan, Georgian Court College, Lakewood, New Jersey.

Angles BTC, CTA and ATB are each 120° if T is the Fermat-Torricelli point. Let $AT : BT : CT = x : 2x : 4x$. Applying the law of cosines to triangle ABT we get

$$c^2 = x^2 + 4x^2 - 4x^2 \cos 120^\circ = 7x^2.$$

By two more applications of the law, we find

$$a^2 = 28x^2 \quad \text{and} \quad b^2 = 21x^2.$$

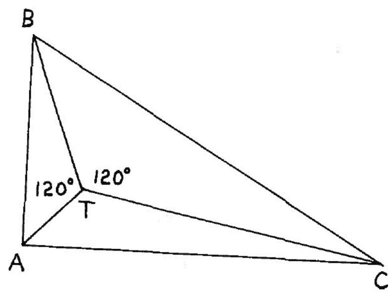
Now we apply the law of cosines to triangle ABC where a^2, b^2, and c^2 known. For example:

$$\cos B = \frac{7x^2 + 28x^2 - 21x^2}{28x^2} = 1/2 \text{ and}$$

angle $B = 60^\circ$. In a similar fashion we find $\cos C = \sqrt{3}/2$ and angle $C = 30^\circ$; $\cos A = 0$ so angle $A = 90^\circ$.

Diagram follows.

Also solved by WALTER BLUMBERG, MICHAEL W. ECKER, JACK GARFUNKEL, HENRY S. LIEBERMAN, ALAN WAYNE, *and the* Proposer.



455. [Fall 1979] *Proposed by Kenneth M. Wilke, Topeka, Kansas.*

Young Leslie Morley noticed that the perimeter of a 6 x 4 rectangle equals the area of a 2 x 10 rectangle while the area of the 6 x 4 rectangle equals the perimeter of the 2 x 10 rectangle also. Show that there are an infinite number of pairs of rectangles related in the same way and find all pairs of such rectangles whose sides are integers.

Solution by Charles W. Trigg, San Diego, California.

If the dimensions of the related rectangles a x b and x x y are such that the area of each is numerically equal to the perimeter of the other, then

$$2(a+b) = xy \quad \text{and} \quad 2(x+y) = ab.$$

Thus

$$x = ab/2 - y,\quad 2(a+b) = aby/2 - y^2, \text{ and}$$

$$y = [ab \pm \sqrt{a^2b^2 - 32(a+b)}]/4,$$

which gives both dimensions of the x-by-y rectangle.

Any arbitrarily chosen pair of positive numbers, for which the expression under the radical sign is positive, will lead to a pair of related rectangles.

Five pairs of related rectangles are: 1 x 34 and 7 x 10, 1 x 38 and 6 x 13, 1 x 54 and 5 x 22, 2 x 10 and 4 x 6, and 2 x 13 and 3 x 10. There are two self-related rectangles with perimeter and area numerically equal, namely, 3 x 6 and 4 x 4.

Also solved by MIKE CALL, MARK EVANS, VICTOR G. FESER, BOB PRIELIPP, *and the* Proposer.

456. [Fall 1979] *Proposed by Paul Erdős, Spaceship Earth.*

Is there an infinite path on visible lattice points avoiding all (u,v) where both u and v are primes?

(The Proposer offers twenty-five dollars for a solution).

Comment by the Editor. Two solutions were offered, one by Charles W. Trigg and the other by the Santa Clara Problem Solving Ring. These will be shown to Dr. Erdős, who is presently in orbit, when he touches down at Los Angeles, sometime before the end of 1980. His comments will be published in the next issue of the Pi Mu Epsilon *Journal*.

457. [Fall 1979] *Proposed by the late R. Robinson Rowe.*

Defining the last n digits of a square as its n-tail, what is the longest n-tail consisting of some part of the cardinal sequence 0, 1, 2, 3, ...9? What is the smallest square with that n-tail?

Solution by the Proposer.

Obviously the last digit of the tail cannot be 2, 3, 7, or 8, since no squares end with those digits. No squares end in 34 or 45, limiting n to 1 for tails ending in 4 or 5, as it is also for 0. For n = 2 we can have tails ending in 01, 56 and 89. But 01 is limited to n = 2 and so is 89, since no squares end in 789. Thus the answer is limited to tails ending in 456.

If a certain sequence is desired for a tail and a F_m has been found such that F_m^2 has a tail with the first m digits of the sequence, that is, T_m, and if x is the next digit of F_m, making the tail T_{m+1}, then

$$\frac{F_m^2 - T_{m+1}}{10^m} + 2F_m x \equiv 10 \qquad (1)$$

An example will show that this is a simple way to derive the next digit of the tail, if there is a solution. Suppose we have found that $384^2 = 147456$. Here $m = 3$, $T_3 = 456$, $T_4 = 3456$, $F_3 = 384$, $F_3^2 = 147456$,

whence

$$\frac{147456 - 3456}{1000} + 2 \cdot 384x \equiv 10$$

$$144 + 768x \equiv 10$$

$x = 2$ or 7 $\qquad 2384^2 = 5683456; \quad 7384^2 = 54523456.$

Using (1) as an algorithm, digit by digit, we reach

$457384^2 = 20920\ 0123456$ $\qquad$ for $n = 7$.

Comment. *There will always be either 0 or 2 solutions to (1), differing by 5. It can be used for any desired tail. Round-the-corner tails were considered, but there is no solution for ....901 nor ....90123456.*

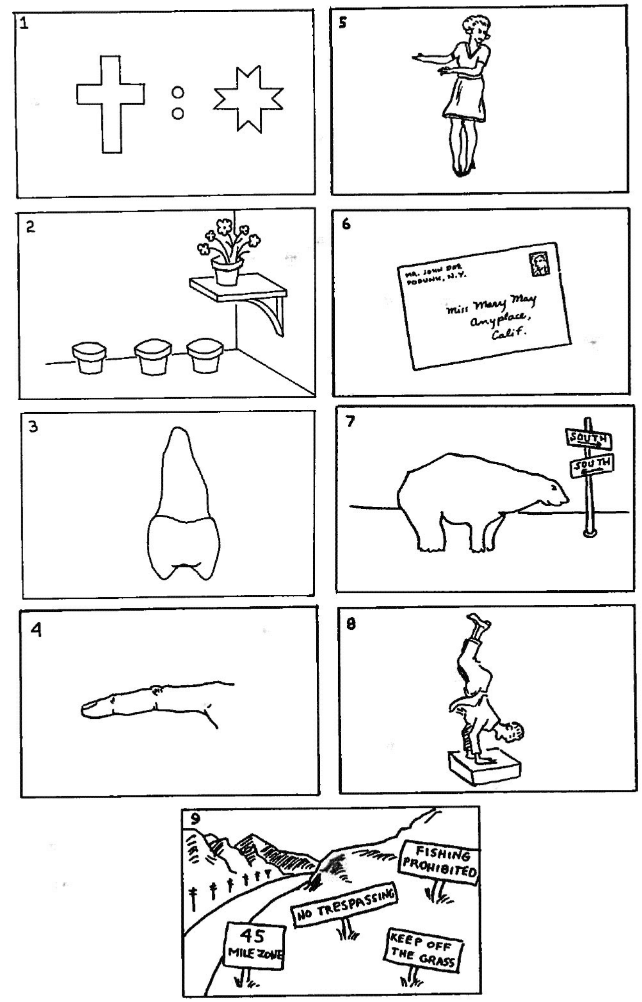
Also solved by MIKE CALL, VICTOR G. FESER, JAMES A. PARSLY, MIKE MAY, CHARLES W. TRIGG, CLAYTON W. DODGE, BOB PRIELIPP, KENNETH M. WILKE, *and* AMANDA B. REKUNDWITH.

PRIELIPP found the results shown above by using the following computer program, written in *BASIC:*

```
10   FOR I = 1 TO 100000
20   LET Y + 10 ** 7 * I + 123456.
30   LET Z = SQR(Y)
40   IF Z = INT(Z) THEN 60
50   GO TO 70
60   PRINT "I = "; I; "Z = "; Z
70   NEXT I
80   END
```

458. *Proposed by Charles W. Trigg, San Diego, California and Lion Bankoff, Los Angeles, California.*

Translate each of the following sketches into a mathematical term.



Contributions by WALTER BLUMBERG, MICHAEL E. ECKER, ROGER E. KUELL, ALFRED E. NEUMAN, BARBARA SEVILLE, ALAN WAYNE, *and the* Proposers.

1. Ratio, cross ratio, mapping, transformation
2. Disjoint sets, multiply, derivative, hypotenuse (high pot in use).
3. Radix, root, double cusp, extracted root.
4. Digit, unit digit, point, index, directed segment.

5. Step function, double contact, rotation. asymmetric, harmonic motion, acute (a cute) angle.

6. Envelope.

7. Pole, pole and polar, polar coordinates.

8. Inversion, inverse function. equilibrium. inverse cotangent (coat and gent). square of the inverse cotangent.

9. Signs. law of sines. rule of signs. arc signs, necessary conditions. negative signs, restrictions.

459. [Fall 19791 *Proposed by Bob Prielipp, The University of Wisconsin-Oshkosh.*

Let x, y, and z be positive integers. Then (x, y, z) is a Pythagorean triangle if and only if $x^2 + y^2 = z^2$. Prove that every Pythagorean triangle where both x and z are prime numbers and $x \geq 11$ is such that 60 divides y.

Solution by Charles W. Trigg, San Diego, California.

It is well-known that all primitive Pythagorean triangles are given by

$$x = m^2 - n^2, \quad y = 2mn, \quad z = m^2 + n^2.$$

If x and z are prime, the triangle is primitive. For $x = (m+n)(m-n)$ to be prime, $m - n = 1$. Consequently,

$$x = 2n + 1, \quad y = 2n(n+1), \quad z = 2n^2 + 2n + 1.$$

All primes have one of the forms $6k \pm 1$. If $x = 2n + 1 = 6k + 1$, then $n = 3k$, and $y = 6k(3k + 1)$. If $x = 2n + 1 = 6k - 1$, then $n = 3k - 1$, and $y = 6k(3k - 1)$.

Waclaw Sierpinski [Pythagorean *Triangles*, Yeshiva University, New *York*, 1962, pp. 24-25] has shown that at least one side of a Pythagorean triangle is divisible by 5. If x and z are primes > 5, then 5 is a factor of y. For all values of k for which either of the products $k(3k + 1)$ and $k(3k - 1)$ is divisible by 5, the product is even. Hence, $y = 60p$.

The five smallest triangles in the class are:

n	x	y	z
5	11	60	61
9	19	180	181
14	29	420	421
29	59	1740	1741
30	61	1860	1861

Note that the last x and the smallest z are the same.

Also solved by WALTER BLUMBERG, DAVID DEL SESTO, CLAYTON W. DODGE, MICHAEL W. ECKER, VICTOR G. FESER, MICHAEL MAY, SAHIB SINGH, KENNETH M. WILKE, *and the* Proposer.

460. *Proposed by Barbara Seville, University of, Bologna, Italy. Dedicated to: Jean 3. Pedersen, University of Santa Clara.*

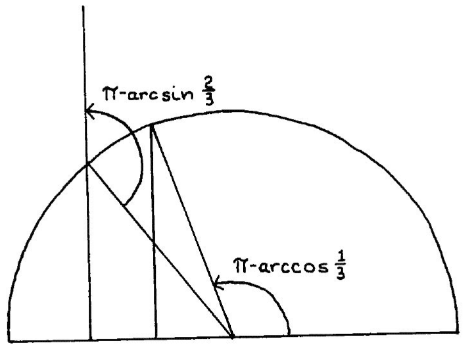
The dihedral angle of a cube is 90°. The other four Platonic solids have dihedral angles which are approximately 70°31'43.60", 109°28'16.3956", 116°33'54.18", and 138°11'22.866". How closely can these angles be constructed with straightedge and compasses? Can good approximations be accomplished by paper folding? If so, how?

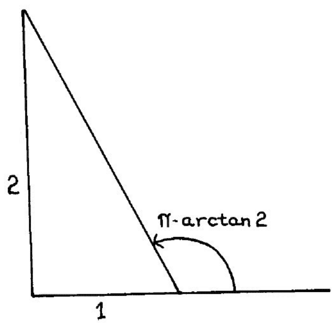
Solution by Zelda Katz, Beverly Hills, California.

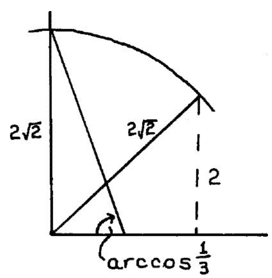
The trigonometric expressions for the dihedral angles of the tetrahedron, octahedron, dodecahedron and the icosahedron are respectively arc cos (1/3), π - arc cos (1/3), π - arc tan 2, and π - arc sin (2/3). These values are given in *Mathematical Models*, by Cundy and Rollett, Oxford 1952, pages 78-82, and their derivations are not required by the statement of this proposal. The constructions suggested by these simple ratios become trivial and, in answer to the posed question, they are precise. Good approximations by paper folding can be obtained by first constructing a rectangular coordinate system with integer abscissae and ordinates and using the resulting lattice points in an appropriate manner.

Sample constructions are shown in the annexed diagrams and the great variety of related paper-folding constructions are left to the ingenuity of the reader.

Very thorough and most commendable contributions to the solution of thus problem were submitted by MIKE CALL, HERB TAYLOR, CHARLES W. TRIGG, *and the* Proposer.







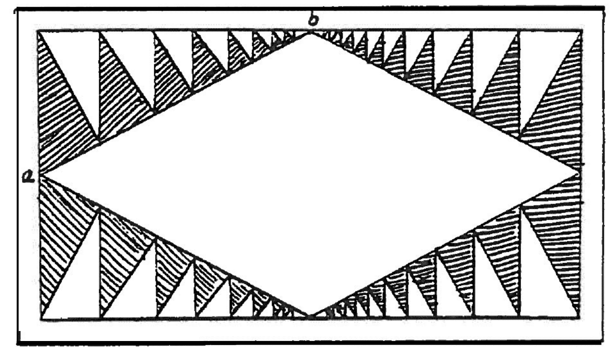
461. *Proposed by David C. Kay, University of Oklahoma, Norman, Oklahoma.*

(a) A right triangle with unit hypotenuse and legs r and s is used to form a sequence of similar right triangles T_1, T_2, T , ... where the sides of T_1 are r times those of the given triangle, and for $n \geq 1$ the sides of T_{n+1} are s times those of T_n. Prove that the sequence T_n will

tile the given triangle.

(b) What happens if the multipliers r and s are reversed?

(c) The art of the Hopi American Indians is known for its zigzag patterns. The blanket illustrated below is made from a rectangle of (inside) dimensions a x b, and the zigzag is formed by dropping perpendiculars to alternating sides of the triangle in the design. Show that the area of the design (shaded portion) is given by the formula $(a^3b + ab^3)/(2a^2 + 4b^2)$.



Solution by Sister Stephanie Sloyan, Georgian Court College, Lakewood, Nw Jersey.

(a) T_1 has legs rs and r^2, hypotenuse r; T_2 has lees rs^2 and r^2s, hypotenuse rs; T_3 has legs rs^3 and r^2s^2, hypotenuse rs^2; T_4 has legs rs^4 and r^2s^3, hypotenuse rs^3; etc. Then adding the areas of T_1, T_2, ... we get

$$\frac{1}{2}\left[(rs)r^2 + (rs^2)r^2s + (r^2s^2)rs^3 + (r^2s^3)(rs^4 + \ldots\right] =$$

$$\frac{1}{2}\left[r^3s\ (1 + s^2 + s^4 + s^6 + \ldots)\right] = \frac{1}{2} r^3s \frac{(1)}{(1 - s^2)} = \frac{1}{2} rs$$

which is the area of the original triangle, and the sequence of triangles will tile the original.

(b) If the multipliers $\underline{r}$ and $\underline{s}$ are reversed the sum of the areas of the small triangles becomes

$$\frac{1}{2} rs^3(1 + r^2 + r^4 + r^6 + \ldots\ldots) = \frac{1}{2} rs^3 \frac{(1)}{(1 - r^2)} = \frac{1}{2} rs$$

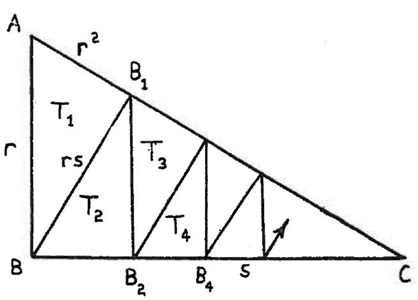
as before.

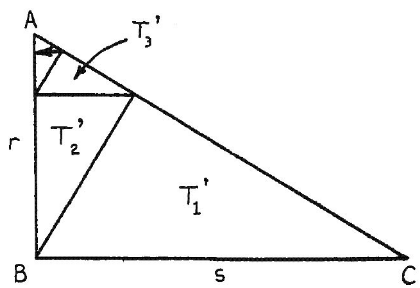
(c) The original triangle has legs $a/2$ and $b/2$ and hypotenuse $(a^2+b^2)^{1/2}/2$. T_1 can be found from the original by multiplying all dimensions by $a/(a^2+b^2)^{1/2}$ (or by using right triangle relations concerning the altitude on the hypotenuse). T_2 and all subsequent triangles can be found by multiplying the dimensions of T_2 by $b/(a^2+b^2)^{1/2}$ to get T_3, T_3 by the same multiplier to get T_4, etc. The shaded portion consists of the area of

$$T_1 + T_3 + T_5 + \dots = [\frac{a^3}{8}(\frac{b}{a^2+b^2} + \frac{b^5}{(a^2+b^2)^3} + \frac{b^9}{(a^2+b^2)^5} + \dots\dots].$$

The quantity within parentheses is a geometric series with ratio $b^4/(a^2+b^2)^2$ which is less than one. Its sum is $\dfrac{b(a^2+b^2)}{a^2(a^2+2b^2)}$.

Multiplying this sum by $a^3/8$ and by four, because there are four portions of the rug, we get $(a^3b+ab^3)/(2a^2+4b^2)$ as the area of the design.





Also solved by MIKE CALL and by *the* Proposer, *whose diagrams appear here.*

PROBLEMATICAL POSTSCRIPTS

Devotees of this (and other) problem departments will be heartened by the article entitled "The Heart of Mathematics" by P. R. Halmos, published in the October 1980 issue of the American Mathematical Monthly. I heartily recommend this article to my readers and am delighted to see how nicely it reinforces the thesis of my own article entitled "Reflections of a Problem Editor", which appeared in the Fall 1975 issue of the Pi Mu Epsilon Journal and was later republished in the December 1976 issue of Eureka, now known as Crux *Mathematicorum.* The Spring 1977 issue of this Journal contains a list of periodicals that cater to problemists, a list that could be expanded by the inclusion of Pentagon and the SIAM Review. Back issues of these journals are available for a trifling sum, and subscription information will gladly be furnished on request.

The articles mentioned here should provide insights helpful to problem proposers and solvers who welcome hints and principles that encourage acceptance by problem editors.

* * * * * * * *

Show me a publication without errata and I will show you alert, conscientious and competent authors, typists, proofreaders and editors. The Pi *Mu* Epsilon Journal boasts of all of these; yet, for some unknown reason, misprints, omissions and typographical errors do somehow creep in. For example, the Spring 1980 issue contains about a half-dozen trivial, insignificant misprints, too inconsequential to mention. However, it would be helpful if readers would change the "O" in the diagram on page 134 to a "*P*" so as to have it conform with the text. Furthermore, the statement of problem 470 (page 133) is marred by the omission of two plus signs. For that reason, the problem is restated here in its entirety and the deadline for submission of solutions is extended to January 1, 1981

* * * * * * * *

470. [Spring 1980] *Proposed by Tom Apostol, California Institute of Technology.*

Given integers $m > n > 0$. Let

$$\alpha = a\sqrt{m} + b\sqrt{n}$$
$$\beta = c\sqrt{m} + d\sqrt{n}$$

where a, b, c, d are rational numbers.

(a) If $ad + bc = 0$ or if mn is a square, prove that both α and β are rational or both are irrational.

(b) If $m = r^2$ and $n = s^2$ for some pair of integers $r > s > 0$ then α and β are both rational. Prove that the converse is also true if $ad \neq bc$.

* * * * * * * *

Mr. Roger E. Kuehl, the traffic engineer who submitted the road-construction problem published in the Spring 1973 issue (problem 297) writes:

It may be of some passing interest that although the road project that originally gave birth to the problem of the tangent circles connecting non-parallel lines was not built because of lack of funds (a common occurrence in municipal work), there are two other intersections in Kansas City that have since been constructed using the techniques of the solved problem to produce portions of the design plans.

This is another striking example of how dependent our practical, everyday world is on the emanations from our ivory towers.

* * * * * * * *

A newcomer to our problem department, *Miss Amanda B. Reckundwith,* has this comment to make:

In one of our mathematical texts, it is stated that it was a standard part of Euclid's "Elements" that the three altitudes of any triangle (the perpendiculars from vertices to their opposite sides) intersect in a point called the orthocenter. Even Einstein's "Autobiographical Notes" contains a passage where he rapturously describes the unforgettable impression this theorem made on him. Would someone please help me find the numbers of the Book and the Proposition that proves this theorem?

I join Miss Rekundwith in her quest for the answer to this question and I personally offer a prize of six gold-plated buttonholes to the reader who can tell where the first mention of this theorem appears in mathematical literature.

EDITORIAL POLICY

The Journal was founded in 1949 and dedicated to undergraduate students interested in mathematics. I believe the articles, features and departments of the Journal should be directed to this group. Undergraduates are strongly encouraged to submit their papers to the Journal for consideration and possible publication. Expository articles in all fields of mathematics are actively sought.

Manuscripts (in duplicate) should be sent to the editor at:

Department of Mathematical Sciences
South Dakota School of Mines and Technology
Rapid City, South Dakota 57701

David Ballew
Editor, Pi Mu Epsilon

SUMMER MEETING AT ANN ARBOR

The Summer Meeting of The Pi Mu Epsilon Fraternity was held on the Campus of the University of Michigan, August 18 through 20, 1980. The following papers were presented:

Ann Zabinski College of St. Benedict Minnesota Delta	*Some Sums of Sums*
James McElheny Hope College Michigan Delta	*Simulating Transformations in the Plane Symmetry Patterns Using the Tektronix 4051*
My Graham University of Arizona Arizona Gamma	*A New Numeric Algorithm To Solve Convolution Kernel Volterra Integral Equations*
Kathlyn Nolan University of Arkansas a* Pine Bluff Arkansas Gamma	*Characterization of Primes*
Christopher J. Roesmer University of 'Dayton. Ohio zate	*Resolution of Russell's Paradox*
Wood-Wai Lee Lamar University Texas Bate	*Equivalent Statements*
Daniel Pollak Miami University Ohio Delta	*Applications of the Programmable Calculator in Mathematics and Statistics*
Larry N. Stroud Eat Carolina University North Carolina Delta	*Mean Value Theorem Revisited*
Nersi Nazari Southern Illinois University Illinois Delta	*Numerical Recovery of Noisy Exponential Sums, using Generalized Differential Approximation*
Michael L. Orrick Macalester College Minnesota Gamma	*The Area of a Triangle Formed by Three Lines*
Sandra Cousins Hendrix College Arkansas Beta	*Singular Functions*

Mark Walker South Dakota School of Mines and Technology South Dakota Bate	*Unpacking the Knapsack Algorithm*
Stephen Semmes Washington University Missouri Beta	*Properties of the Infinite Symmetric Group*
Marie Spetseris College of Charleston South Carolina Gamma	*Self Taught Calculus*
Kevin Saylors Pomona College California Epsilon	*English Soccer Predictions*
Beverly J. Skeans Marshall University West Virginia Bate	*Correctness Proofs For Flow Chart Programs*
Michael L. Call Rose-Hulman Institute of Technology Indiana Gamma	*Discrete Versus Continuous: Or, What is the Calculus of Finite Differences?*
Douglas Eugene Jewett Texas A&M University Texas Eta	*The Centralizer of a Linear Transformation*
Martha Blackwelder Appalachian State University North Carolina Eta.	*Finite State Machines*
Kurt L. Wiese University of Arkansas Arkansas Alpha	*Nested Interval Theorem With Two Applications*
William Terkeurst Hope College Michigan Delta	*Relativity in Perspectivity*
Michael K. May, S.J. St. Louis University Missouri Gamma	*Problems With Infinity*
THE J. SUTHERLAND FRAME LECTURER Prof. Richard A. Askey University of Wisconsin	*Ramanujan and Some Extensions of the Gamma and Beta Functions*

» «

LOCAL CHAPTER AWARDS WINNERS

ALABAMA BETA (AUBURN UNIVERSITY). The winners of the two Mathematics Achievement Contests are as follows:

Calculus Contest	Open Contest
Lee Mahavier	*Steve Stringfellow*
David Scollard	*Ralph Quigley*
Randall Smith	*Raymond Quigley*
	Ken Goodman

ALABAMA DELTA (UNIVERSITY OF SOUTH ALABAMA). The Pi Mu Epsilon award for outstanding achievement in the field of mathematics was given to

Daniel Pix

ARKANSAS BETA (HENDRIX COLLEGE). The following awards were given at the annual Honors Convocation. The McHenry-Lane Freshman Mathematics award-

Karen Shirley
Hike McClurkan.

The Hogan Senior Mathematics Award-

Sandy Scrimshire.

The Phillip Parker Undergraduate Research Award-

Sandra Cousins.

CALIFORNIA ETA (UNIVERSITY OF SANTA CLARA). The Evans Prize for the highest score on the Putnam Examination was given to

Edward Dunne and *Lisa Townsley.*

The Evans Prize for Research was given to

Edward Dunne;

The Freshman Mathematics Contest Grand Prize was awarded to

Chris Eich;

and The Faculty Recognition Awards went to

James Foster and *Alice Kelly.*

CONNECTICUT BETA (UNIVERSITY OF HARTFORD). The Senior Book Awards were given to

Edward Hackett and *Mahir Dugentas;*

The Stanley Klock Jr. Memorial Awards were presented to

William DiCecca and *Trevor French;*

and the Junior Award for Excellence in Mathematics went to

Mark White.

FLORIDA DELTA (UNIVERSITY OF FLORIDA). The award for Best Graduating Senior was presented to

Robert Christianson.

The Award for the Highest Placement on the Putnam Examination was given to

John Mattox,

and The Best Mathematics Paper Award was presented to

Mark Bateh.

GEORGIA BETA (GEORGIA INSTITUTE OF TECHNOLOGY). The Outstanding Graduates in Mathematics Awards were presented to

Donald A. Hawley, Jr. and *Patricia E. Olcott.*

GEORGIA GAMMA (ARMSTRONG STATE COLLEGE). The Award for the Outstanding Senior in Mathematics was given to

Stephen Semmes.

IOWA ALPHA (IOWA STATE UNIVERSITY). The Pi Mu Epsilon Scholarship Awards were presented to

Paul McAvoy and *William Somsky.*

The Dio Lewis Holl Award to the Outstanding Graduating Senior Mathematics Major was earned by

Steven Wigmann.

The Gertrude Herr Adamson Awards for Demonstrated Ingenuity in Mathematics were presented to

Jeffrey Strang	*Lee Roberts*
Norrae Merkel	*Wai Nam Wong*
Gregory Anderson	*Gary McGraw*
Barbara Rus	*Steven Seda.*

MICHIGAN DELTA (HOPE COLLEGE). The Hope College Teams took First and Fifth places in the Fourth Annual Lower Michigan Mathematics Competition.

MISSOURI GAMMA (SAINT LOUIS UNIVERSITY, FONTBONNE COLLEGE, MARYVILLE COLLEGE). The James W. Garneau Mathematics Award was presented to

Kathleen Drebes.

The Francis Regan Scholarship was earned by

Carol Ann Mayer.

The Missouri Gamma Undergraduate Award was presented to

Becky Kirkpatrick

while the Missouri Gamma Graduate Award was won by

Robert Russell.

In the **Pi Mu Epsilon** Contests, the Senior Winner was

Michael May,

and the Junior Contest Winners were

Kim Wicks and *Karen Keefer.*

The John J. Andrews Graduate Service Award was presented to

Mark Hopfinger,

and the Beradino Family Fraternityship Award was given to

Maryanne Bieg.

NEW JERSEY BETA (DOUGLASS COLLEGE).

Patricia O'Brien

received the **Pi Mu Epsilon** Junior Award in Mathematics.

Pi. Katherine Hazard

who had been the Chapter Advisor for many years was honored on her retirement.

NEW JERSEY EPSILON (SAINT PETER'S COLLEGE). The **Pi Mu Epsilon** Seniors instituted an annual award named the Francis A. Varrichio Memorial Award to be given for excellence in teaching. The first recipient was

Dr. Francis A. Rink.

NEW YORK OMEGA (ST. BONAVENTURE). The **Pi Mu Epsilon** Award winners were

Mark P. Sleggs and *Thomas M. Marra.*

NEW YORK OMICRON (CLARKSON COLLEGE). The **Pi Mu Epsilon** Outstanding Sophomore Award was won by

Jeffrey Stec.

NEW YORK ALPHA ALPHA (QUEENS COLLEGE OF CUNY).

Robbin Bura

was presented with the first **Pi Mu Epsilon** prize.

OHIO THETA (XAVIER UNIVERSITY). The annual **Pi Mu Epsilon** award was given to

Judith Cates.

OHIO NU (THE UNIVERSITY OF AKRON).

Jerry Young

received the Samuel Selby Mathematics Scholarship Award.

$ $ $ $ $

POSTERS AVAILABLE FOR LOCAL ANNOUNCEMENTS

We have a supply of 10 x 14-inch Fraternity Crests available. One in each color *will* be sent free to each local chapter on request. Additional posters *may* be ordered at the following rates:

(1) Purple on goldenrod stock------------------$1.50/dozen,

(2) Purple on Lavendar on goldenrod------------$2.00/dozen.

FRATERNITY KEY-PINS

Gold Clad key-pins are available at the National Office at the special price of $8.00 each. Write to Dr. Richard Good, Dept. of Math., University of Maryland, College Park, Maryland, 20742.

Be sure to indicate the Chapter into which you were initiated and the approximate date of initiation.

MATCHING PRIZE FUND

If your Chapter presents awards for Outstanding *Mathematical* papers or Student Achievement in Mathematics, you may apply to the National Office to match the amount spent by your Chapter. For example, $30 of awards can result in Chapter receiving $15 reimbursement from the National Office. These funds may also be used for the rental of Mathematics films. Write to:

Dr. Richard Good
Secretary-Treasurer, **Pi Mu Epsilon**
Dept. of Mathematics
The University of Maryland
College Park, Maryland 20742

BE SURE AND SEND THE NAMES OF LOCAL AWARD WINNERS TO THE JOURNAL FOR PUBLICATION IN THE NEXT POSSIBLE ISSUE.

GLEANINGS FROM CHAPTER REPORTS

ALABAMA BETA (AUBURN UNIVERSITY). The Chapter held two contests during the year. One contest was restricted to first year calculus students and the second was open to any student enrolled at the University.

ARKANSAS BETA (HENDRIX COLLEGE). The Chapter had a variety of programs which included visits to other colleges and universities. The following papers were presented at Chapter meetings: *Sandy Scrimshire*, *"Mathematical Games"*, *John Merrill, Louis Rolleigh, Richard Thornton*, *"Student Applications of Mathematics and/or Physics Skills in Summer Work"*; *Dr. Robert Eslinger, Dr. Cecil Mc Dermott*, *"Life After Graduation--Graduate and Employment Opportunities in Mathematics"*: *Dr. Richard Rolleigh*, *"Space Time Games"*, *Dr. Steve Smith* (Harding College), *"Mathematics as a Creative Art"*; *Kurt Wise* (Univ. of Arkansas), *"The Nested Interval Theorem"*; *Mickey Holsey* (Univ. of Arkansas), *"Sets of Points of Discontinuity"*; *Dennis Davenport* (Univ. of Arkansas), *"Taylor's Theorem: An Application of the Mean Valve Theorem"*; *Robert Smith* (Univ. of Arkansas), *"Ordinal Numbers and Transfinite Induction"*: *Dr. Jeanne Agnew* (Oklahoma State University), *"Some Applications of Undergraduate Mathematics"*; *Sandra Cousins*, *"Singular Functions"*; *David Sutherland*, *"Mathematics in 19^{th} Century Russia"*; *Sandy Scrimshire*, *"Women in Mathematics"*; *John Merrill*, *"Measures of Convexity on R^n"*; *Dwayne Johnson*, *"Optimal Time for Broadcasting"*; *Marj Threet*, *"Secret Coding With Matrices"*; *Dr. David Larson*, *"Correspondence of a German Peasant in the 1870's"*.

CALIFORNIA ETA (UNIVERSITY OF SANTA CLARA). The Chapter presented the following programs: *Professor Paul Erdos* (Hungarian Academy of Science), *"Child Prodigies I have Known"*; *Professor William Firey* (Oregon State University), *"The First Global Theorem of Differential Geometry"*.

CONNECTICUT BETA (UNIVERSITY OF HARTFORD). *Professor Richard Dolliver* spoke on *"Mathematical Recreations and Recreational Mathematics"*.

FLORIDA DELTA (UNIVERSITY OF FLORIDA). There were nine meetings during the year at which the following programs were heard: *Dr. Frank May*, *"Mathematics and Modeling"*. *Dr. Bruce Edwards*, *"The Four Color Problem"*; *Dr. Grant Ritter*, *"Random Walks"*; *Dr. Chuck Medlin*, (Pratt and Whitney);

N. A. Drews (IBM), *"Opportunities in the Computer Industry for Mathematics Majors"*.

GEORGIA GAMMA (ARMSTRONG STATE COLLEGE). The Chapter presented several film programs and heard the following addresses: *Dr. John Neff* (Georgia Tech), *"Beyond the Binomial"*; *Dr. Robert Taylor* (University of South Carolina), *"Games of Chance: Problems in Probability and Statistics"*; *Dr. Charles Curt Lindner*, *"Steiner Triples"*.

ILLINOIS ZETA (LOYOLA UNIVERSITY OF CHICAGO). *Dr. Milton Cox* of Miami University installed the Chapter and spoke on *"A Calculus Solution to a Geometry Problem"*.

IOWA ALPHA (IOWA STATE UNIVERSITY). The Chapter presented a variety of programs including films and picnics. The following programs were heard: *Professor Elgin Johnston*, *"Some History and Elementary Results in Map Coloring"*; *Professor G. McNulty* (University of South Carolina), *"Avoiding Patterns in the Plane"*, *Professor Joseph Vauben* (CUNY and Columbia University), *"Hypotheses non fingo: Georg Cantor's Set Theory and His Philosophy of the Infinite"*; *Professor Richard Epstein*, *"Mathematics as the Art of Analogy"*.

KENTUCKY GAMMA (MURRAY STATE UNIVERSITY). The following talks were presented this year: *Dr. Louis Beqm*, *"Nuclear Power and the Brown's Ferry Nuclear Plant"*; and *Dr. Mel Scott*, *"What is an Industrial Mathematician?"*.

MASSACHUSETTS DELTA (UNIVERSITY OF LOWELL). The Chapter sponsored a Mathematics Day Contest in which over 900 students participated.

MICHIGAN DELTA (HOPE COLLEGE). The following programs were heard: *Professor Elliot Tanis*, *"M. C. Escher, Artist and Mathematician"*; *Dr. Richard Feldman* (Texas A&M University), *"Operations Research"*; *Dr. Philip Tuchinsky* (Ford Motor Company) gave three talks: *"In Calculus and Linear Algebra - Why So Much Interest in Functions?"*, *"Word Frequency Research Connected with Joyce's 'Ulysses'"*, *"Management of a Buffalo Herd"*; *Dr. Thomas Tucker* (Colgate University), *"Symmetry Groups From Alhambra to Escher"*; *Gary Immink* and *Bill Terkeurst*, *"Applications and the Use of Models in Geometry"*; *Barbara Koeppe* and *Dee Holly*, *"A Semester at Oak Ridge"*; *David Boundy*, *"An APL Tutorial for Math Students"*; *Professor Paul Enigenburg* (Western Michigan University), *"Circle Preserving Trans-

formations of the Plane", Professor John Whittle, *"How to Use the TRS-80"*; Paul Hospers, *"Can You Save Time and Effort Programming in GPDS?"*; Hugh Bartels, *"Forecasting for COL Today"*; Jamie McElheny, *"Simulating Transformation in the Plane Symmetry Patterns Using the Tektronix 4051"*; Powell Quirling, *"Square Limit - An Approach to Infinity"*.

MINNESOTA DELTA (ST. JOHN'S UNIVERSITY) hosted its Second Undergraduate Mathematics Seminar. Featured Speaker was Professor Mary Ellen Ruden of the University of Wisconsin who gave three talks: *"What Does a Topologist Do?"*, *"What Does Set Theory Have to Do With Mathematics?"*, and *"Crazy Spaces"*. Other talks included Edwin Strickland (St. Cloud State), *"Why Study Mathematics - A View of Mathematics and Mathematics Education From the Historical Prespective"*; Nancy Schurrer (St. Cloud State), *"The Split Method For Factoring Trinomials"*, Dean Shea (St. John's), *"Information Theory - An Expository Overview"*; Anne Zabinski (College of St. Benedict), *"Some Sums of Sums"*; Eric Mohr (Macalester College), *"Sequences of Touching Circles"*; Jim '&& (Macalester College), *"Symmetry Elements and Optically Active Molecules"*; Mike Orrick (Macalester College), *"The Area of a Triangle Bounded by Three Lines"*; Terry Langen (St. John's), *"Mathematics and Economics"*.

MINNESOTA EPSILON (ST. CLOUD STATE UNIVERSITY) presented the following programs: Ed Strickland, *"Historical Trivia About Mathematicians"*; Walter Larson, *"Job Opportunities"*; Ricardo Reinking, *"Mayan Abacus"*; Dr. Ralph Carr, *"Interest Related Problems?* Dr. John Mellby, *"A Computer Science Topic"*; Jennie Hansen, *"The British Mathematics Education System"*; Monte Johnson, *"The Apple II Computer"*.

MISSOURI GAMMA (SAINT LOUIS UNIVERSITY, FONTBONNE COLLEGE, MARYVILLE) gave the following programs: Dr. Albert Bartlette (University of Colorado), *"The Forgotten Fundamentals of the Energy Crisis"*; Professor Joseph Kennedy, *"Kaleidoscopes"*; Dr. Charles Ford, *"Crystals and Symmetry"*; and the James E. Case, S. J. Memorial Lecture was given by Dr. Charles Diminnie of St. Bonaventure University entitled *"Orthogonality in Normed Vector Spaces"*.

NEW JERSEY DELTA (SETON HALL UNIVERSITY) presented a film program and Dr. Tom Marlowe, *"Stars, Bars and Number Theory"*.

NEW YORK EPSILON (ST. LAWRENCE UNIVERSITY) sponsored the 36th Annual Pi Mu Epsilon Interscholastic Mathematics Contest.

NEW JERSEY EPSILON (SAINT PETER'S COLLEGE) presented the Annual-Collins Lecture by Dr. Edward A. Boyno (Montclair State college), *"Intersection Graphs"*. Other programs were: Dr. John J. Santa Pietro (Lockheed Electronics Company), *"Recent Algebraic Trends in Systems Engineering"*; Dr. Evan Maletsky (Montclair State College), *"Some Cubical Curiosities"*; Professor Mitchell P. Preiss, *"Optimization Problems in Micro-Economics"*; Richard Kempinski, *"A Shopping System Model and Simulation*; Dr. Max Sobel (President of the National Council of Teachers of Mathematics), *"The Beauty and Magic of Mathematics"*; Roberto Rodriquez, *"A Mathematical Model for the Detection of Diabetes"*.

NEW YORK OMEGA (SAINT BONAVENTURE UNIVERSITY) hosted a film program and the following presentations: Dr. Myra Reed and Dr. Steven Andrianoff, *"Applications of Computer Science"*; Dr. Charles Diminnie, *"Careers in Mathematics"*; Professor Henry Caruso, *"Careers in Mathematics Education"*; Allen Butterworth (General Motors Research Labs), *"Policy Analysis: Making the Most of Alternatives"*; Walter Doherty (IBM Research Division), *"Man - Computer Interactions"*.

NEW YORK OMICRON (CLARKSON COLLEGE). Dr. Graham Holmes of Schenectady General Electric spoke on *"Large Scale Industrial Computing Problems"*.

NEW YORK ALPHA ALPHA (QUEENS COLLEGE OF CUNY) hosted several film programs and a Faculty-Student softball game.

NORTH CAROLINA GAMMA presented the following programs: Dr. Nicholas J. Robe, *"Curves of Pursuit"*; Dr. Robert Silber, *"Red-Slue Hackenbush Strings: An Introduction to the Conway Theory of Combinatorial Games"*; Dr. John W. Bishir, *"Dualists, Sex of Children, Wachovia Customer Service and Miracles"*.

OHIO EPSILON gave the Annual Arts and Sciences Lecture on *"Rosenthal's Dichotomy on Convergence"* by Joseph Diestel. Other programs included John C. Kane (ACM), *"Job Opportunities"*; Dr. Charles Cleaver, *"Packing Problem"* and *"The Golden Section"*; Dr. John Fridy, *"Bachet's Weighing Problem and Bases and Number Systems"*.

OHIO NU (THE UNIVERSITY OF AKRON) hosted informational programs, the All Campus Activities Fair, helped coordinate and judge two science fairs, and a Panel Discussion on *"Mathematics Beyond the Classroom"*. In addition, two talks were heard: *Charles Cummings*, *"Computer Graphics"*; *Dale Borowiak*, *"A Mathematical Model to Represent a Physical Occurence"*.

OHIO ZETA had the following programs: *Gerald Shaughnessy*, *"The Beer Can Project"*: *Brother Bernard Ploeger*, *"Project Tapis Verte"*; *Dr. Joseph Diestal* (Kent State University), *"The Rosenthal Dichotomy"*.

SOUTH CAROLINA GAMMA (THE COLLEGE OF CHARLESTON) presented programs devoted to Career Opportunities in Mathematics, and helped present programs on computers in an elementary school. Further, the Chapter sponsored the Third Annual Mathematics Meet for State high schools.

SOUTH CAROLINA DELTA (FURMAN UNIVERSITY) presented *Dr. Robert D. Fray*, *"Biostatistics"*; *Dr. Clinton Miller*, *"Estimation and Simple Regression and Applications in Biometry"*; *Dr. Gil Proctor* (Clemson University), *"Job Opportunities for Mathematics Majors and Graduate Students"*; *Ernestine Bailey* (Southern Bell Telephone), *"Job Opportunities"*; *Dr. Mary Neff* (Emory University), *"Lattices"*; *Ron Morgan*, *"Affine Geometry"*.

SOUTH DAKOTA BETA (SOUTH DAKOTA SCHOOL OF MINES AND TECHNOLOGY) hosted talks by *Mark Walker*, *"Unpacking the Knapsack"*, *Gary Ricard*, *"The Knight's Tow"*; and *Dr. David Ballew*, *"Job Opportunities in Mathematical Sciences"*. Also the Chapter sponsored a film program and assisted with the 31st Annual West River Mathematics Contest.

TEXAS DELTA (STEPHEN F. AUSTIN STATE UNIVERSITY) heard the following presentations: *Thomas A. Atchison*, *"Paradoxes of Mathematics"*; *Roy Alston*, *"Continued Rational Expressions"*, *Robert Yeagy*, *"The New Trends in Cryptography"*; *Harold Bunch*, *"The Number 'e' in Financial Formulas"*; *Dr. J. W. Drane* (SMU), *"Statistics and Employment Discrimination"*.

A

1981 NATIONAL PI MU EPSILON MEETING

IT IS TIME TO BE MAKING PLANS TO SEND AN UNDERGRADUATE DELEGATE OR SPEAKER FROM YOUR CHAPTER TO THE ANNUAL MEETING OF Pi Mu Epsilon AT THE UNIVERSITY OF PITTSBURGH IN AUGUST OF 1981. EACH SPEAKER WHO PRESENTS A PAPER WILL RECEIVE TRAVEL UP TO $500 AND EACH DELEGATE, UP TO $250 (ONLY ONE SPEAKER OR DELEGATE *CAN* BE FUNDED FROM A SINGLE CHAPTER, BUT OTHERS CAN ATTEND).

SOME CHALLENGE PROBLEMS

Richard Andree
University of Oklahoma

These problems are designed for your investigation and exploration. Some of them might be explored first by generating example on a computer. The complete answers to some of these problems and their extensions is still (1980) unknown; others have been solved completely, but usually do not appear in the standard texts. Papers or solutions to these problems should be sent to the Editor for possible publication in future issues of this **Journal**.

Problem 1. Let N_0 be any postive integer and let

$$N_{k+1} = \begin{cases} 3N_{k+1} & \text{if } N_k \text{ is odd,} \\ N_k/2 & \text{if } N_k \text{ is even,} \\ \text{Halt} & \text{if } N_{k+1} = 1 \end{cases}$$

Thus if $N_0 = 11$, the corresponding chain is 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1. We call the number of stops from N_0 to 1 the <u>length</u> of the chain from N_0 to 1. Thus in the above, $N_0 = 11$, $N_{14} = 1$, and the chain is 14 links.

A. Do all of the starting values $N_0 > 1$ converge to 1?

B. If given a starting value N which does converge to 1, can you find the length of the chain without actually constructing the entire chain?

C. Given $\alpha > 0$, can you find an N_0 such that the chain beginning with N_0 has chain lenght a?

D. What can you say about the last half dozen elements of a chain that converges to 1?

Problem 2. There are integers such as 13, 31 or 1021, 1201 in which the second number of the pair is obtained by <u>reversing</u> the order of the digits in the first number and such that both of the numbers are prime. Investigate this phenomenon.

Problem 3. Investigate the following:
Given a starting integer N_0 of several digits, let BIG be the largest integer that can be made using the digits of N_k, let SMALL be the smallest integer that can be made using the digits of N_k, and let N_{k+1} = BIG - SMALL. Repeat the sequence. For example,

$$N_0 = 7380$$
$$N_1 = 8730 - 0378 = 8352$$
$$N_2 = 8532 - 2358 = 6174$$
$$N_3 = 7641 - 1467 = 6174 \quad \text{which obviously repeats.}$$

Problem 4. Knowing that our calendar repeats on a 400-year cycle, determine on which day of the week the 13th day of the month is most apt to occur. (Is it really Friday?)

Problem 5. Two hundred years ago, Euler conjectured that for $k>2$, the sum of $(k-1)$ kth powers of positive integers could not equal the kth power of a positive integer. I.e.,

$k = 3$, $I^3 + J^3 = K^3$ has no solution in positive integers,

$k = 4$, $I^4 + J^4 + K^4 = L^4$ has no solution in positive integers,

$k = 5$, $I + J^5 + K^5 + L^5 = M^5$ has no solution in positive integers.

Actually, for k = 3 Euler was correct; for k = 5, there is a solution for which M has only 3 digits (found in 1960), so Euler was wrong for k = 5. We still do not know about k = 4. Curiously, there are infinitely many counterexamples for the case of k = 5. Investigate the case of 4.

Problem 6. A Palindrome is a phrase or sentence which reads the same forward as backward (the punctuation and the spaces are ignored). "Ma is as selfless as I am". Numbers such as 12321, 498894, or 333 may also be considered as palindromes. It is fairly easy to find perfect squares which are also palindromes-- $(202)^2 = 40804$, $(264)^2 = 69696$. However, palindromic squares having an even number of digits such as $(836)^2 = 698896$ are fairly rare. Investigate perfect squares that are also palindromes. An extension of this problem might concern itself with which palindromes have squares that are also palindromes.

PI MU EPSILON JOURNAL PRICES

PAID IN ADVANCE ORDERS:

Members:	**$ 8.00 for 2 years** **$20.00 for 5 years**
Non-Members:	**$12.00 for 2 years** **$30.00 for 5 years**
Libraries:	**$30.00 for 5 years (same as non-members)**
Back Issues	**$ 4.00 per issue (paid in advance)**
Complete volume	**$30.00 (5 years, 10 issues)**
All issues	**$150.00 5 complete back volumes plus current volume subscription (7 volumes — 30 years)**