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First Prize Paper
National Student Paper Competition
1979-80

TWO REMARKS ON THE QUATERNIONS

by Z. Haddad
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We will discuss two questions in this paper. In the first section, we will consider a possible definition of differentiability of quaternion - valued functions analogous to the definition of differentiability of complex - valued functions. According to this definition, the differentiable functions will have to satisfy partial differential equations analogous to the Cauchy–Riemann equations of complex analysis However, in contrast to complex analysis, the greater complexity of the quaternions gives more partial differential equations which force the functions to be linear. In the second section, we will establish a formula for the $n^{\underline{th}}$ power of a quaternion, analogous to Moivre's formula for complex numbers.

§1. Since quaternions don't commute, we start with the following definition:

> Let Q be the ring of real quaternions. A function $f : Q \to Q$ is "left-differentiable" at $w_0 \varepsilon Q$ if
>
> $$\lim_{w \to w_0} (w - w_0)^{-1}(f(w) - f(w_0))$$
>
> exists and is finite, the metric being the standard norm on Q.

"Right-differentiability" is defined in the analogous way. When they exist, $\lim\limits_{w \to w_0} (w - w_0)^{-1}(f(w) - f(w_0))$ is denoted by $L(f)(w_0)$, and $\lim\limits_{w \to w_0} f(w) - f(w_0))(w - w_0)^{-1}$ by $R(f)(w_0)$.

Let $f(x + yi + zj + tk) = a(x,y,z,t) + b(x,y,z,t)i + c(x,y,z,t)j + d(x,y,z,t)k$ be a function on Q, so that a,b,c,d are real - valued with continuous partial derivatives. We will look for conditions which make $f$left-differentiable on an open domain D in Q.

We proceed exactly as in the case of complex functions. Assume that for some $w_0 \varepsilon D$, $L(f)(w_0)$ exists.

Denote w_0 by $x_0 + y_0 i + z_0 j + t_0 k$, and w by $x + yi + zj + tk$.

Then we know that

$$\lim_{w \to w_0} ((x - x_0) + (y - y_0)i + (z - z_0)j + (t - t_0)k)^{-1}(f(w) - f(w_0)) = L(f)(w_0).$$

Hence

$$L_1 = \lim_{x \to x_0} (x - x_0)^{-1}((a(x,y_0,z_0,t_0) - a(x_0,y_0,z_0,t_0))$$
$$+ (b(x,y_0,z_0,t_0) - b(x_0,y_0,z_0,t_0))i + (c(x,y_0,z_0,t_0) - c(x_0,y_0,z_0,t_0))j$$
$$+ (d(x,y_0,z_0,t_0) - d(x_0,y_0,z_0,t_0))k)$$

exists also and is equal to $L(f)(w_0)$.

But

$$L_1 = \lim_{x \to x_0} \left(\frac{a(x,y_0,z_0,t_0) - a(x_0,y_0,z_0,t_0)}{x - x_0} \right)$$
$$+ \lim_{x \to x_0} \left(\frac{b(x,y_0,z_0,t_0) - b(x_0,y_0,z_0,t_0)}{x - x_0} \right) i$$
$$+ \lim_{x \to x_0} \left(\frac{c(x,y_0,z_0,t_0) - c(x_0,y_0,z_0,t_0)}{x - x_0} \right) j$$
$$+ \lim_{x \to x_0} \left(\frac{d(x,y_0,z_0,t_0) - d(x_0,y_0,z_0,t)}{x - x_0} \right) k$$
$$= \frac{\partial a}{\partial x}(x_0,y_0,z_0,t_0) + \frac{\partial b}{\partial x}(x_0,y_0,z_0,t_0)i$$
$$= \frac{\partial c}{\partial x}(x_0,y_0,z_0,t_0)j + \frac{\partial d}{\partial x}(x_0,y_0,z_0,t_0)k$$

(we assumed that these partials exist).

The existence of $L(f)(w_0)$ also implies that

$$L_2 = \lim_{y \to y_0} ((y - y_0)i)^{-1}((a(x_0,y,z_0,t_0) - a(x_0,y_0,z_0,t_0))$$
$$+ (b(x_0,y,z_0,t_0) - b(x_0,y_0,z_0,t_0))i + (c(x_0,y,z_0,t_0) - c(x_0,y_0,z_0,t_0))j$$
$$+ (d(x_0,y,z_0,t_0) - d(x_0,y_0,z_0,t_0))k)$$

exists also and is equal to $L(f)(w_0)$.

And

$$L_2 = \lim_{y \to y_0} \left(\frac{a(x_0,y,z_0,t_0) - a(x_0,y_0,z_0,t_0)}{y - y_0} \right) \frac{1}{i}$$
$$+ \lim_{y \to y_0} \left(\frac{b(x_0,y,z_0,t_0) - b(x_0,y_0,z_0,t_0)}{y - y_0} \right) \frac{1}{i} i$$
$$+ \lim_{y \to y_0} \left(\frac{c(x_0,y,z_0,t_0) - c(x_0,y_0,z_0,t_0)}{y - y_0} \right) \frac{1}{i} j$$
$$+ \lim_{y \to y_0} \left(\frac{d(x_0,y,z_0,t_0) - d(x_0,y_0,z_0,t_0)}{y - y_0} \right) \frac{1}{i} k$$
$$= -\frac{\partial a}{\partial y}(x_0,y_0,z_0,t_0)i + \frac{\partial b}{\partial y}(x_0,y_0,z_0,t_0)$$
$$- \frac{\partial c}{\partial y}(x_0,y_0,z_0,t_0)k + \frac{\partial d}{\partial y}(x_0,y_0,z_0,t_0)j$$

(as $\frac{1}{i} = -i, \frac{1}{i} j = -k, \frac{1}{i} k = j$).

Approaching w_0 similarly in the j and k directions, we get two more expressions for $L(f)(w_0)$, so that if $L(f)(w)$ exists for all $w \varepsilon D$, then, on D, we must have

$$L(f) = a_x + b_x i + c_x j + d_x k = -a_y i + b_y - c_y k + d_y j$$
$$= -a_z j + b_z k + c_z - d_z i = -a_t k - b_t j + c_t i + d_t.$$

Hence we get the following Cauchy-Riemann equations:

(1) $a_x = b_y = c_z = d_t$

(2) $b_x = -a_y = -d_z = c_t$

(3) $c_x = d_y = -a_z = -b_t$

(4) $d_x = -c_y = b_z = -a_t$

Notice that if f were a function on C, these equations reduce to $a_x = b_y$, $b_x = -a_y$, the familiar Cauchy-Riemann equations for complex functions. However, we have here twelve equations instead of two, and we can attempt to solve them. Indeed, since the component functions were assumed to have continuous partials, and since we restricted ourselves to open domains, we can solve the system (1), (2), (3), (4) by taking mixed second partials, e.g.

$$a_{xy} \underset{(1)}{=} c_{zy} \underset{(4)}{=} a_{tz} \text{ and } a_{xy} \underset{(1)}{=} d_{ty} \underset{(3)}{=} -a_{zt},$$

giving

$$a_{xy} = a_{zt} = 0 \text{ on } D.$$

Also,

$$a_{xz} \underset{(1)}{=} b_{yz} \underset{(4)}{=} -a_{ty} \text{ and } a_{xz} \underset{(1)}{=} d_{tz} \underset{(2)}{=} a_{yt},$$

giving

$$a_{xz} = a_{yt} = 0 \text{ on } D.$$

Continuing in this way, we get all mixed second partials of a equal to 0 on D. Therefore $a_x = a_1(x)$, $a_y = a_2(y)$, $a_z = a_3(z)$, $a_t = a_4(t)$, i.e. a_x is a function of x alone, a_y is a function of y alone... and therefore $a(x,y,z,t) = \overline{a}_1(x) + \overline{a}_2(y) + \overline{a}_3(z) + \overline{a}_4(t) + a$, where $\overline{a}_i$ is an antiderivative of a_i, a ε R.

Proceeding similarly for b, c, and d, we get

$$b(x,y,z,t) = \overline{b}_1(x) + \overline{b}_2(y) + \overline{b}_3(z) + \overline{b}_4(t) + \beta_0$$

$$c(x,y,z,t) = \overline{c}_1(x) + \overline{c}_2(y) + \overline{c}_3(z) + \overline{c}_4(t) + \gamma_0$$

$$d(x,y,z,t) = \overline{d}_1(x) + \overline{c}_2(y) + \overline{d}_3(z) + \overline{d}_4(t) + \delta_0 .$$

Now we return to the Cauchy-Riemann equations and substitute the above expressions for the component functions of f. Partial derivatives become one variable real derivatives and we have

$$\overline{a}_1'(x) = \overline{b}_2'(y) = \overline{c}_3'(z) = \overline{d}_4'(t) \qquad (1)$$

$$\overline{b}_1'(x) = -\overline{a}_2'(y) = -\overline{d}_3'(z) = \overline{c}_4'(t) \qquad (2)$$

$$\overline{c}_1'(x) = \overline{d}_2'(y) = -\overline{a}_3'(z) = -\overline{b}_4'(t) \qquad (3)$$

$$\overline{d}_1'(x) = -\overline{c}_2'(y) = \overline{b}_3'(z) = -\overline{a}_4'(t) \qquad (4)$$

The variables being all different in every set of equations, we must have

(1) $\overline{a}_1'(x) = \ldots = \overline{d}_4'(t) = \alpha$, some real constant,

(2) $\overline{b}_1'(x) = \ldots = \overline{c}_4'(t) = \beta$, some real constant,

(3) $\overline{c}_1'(x) = \ldots = -\overline{b}_4'(t) = \gamma$, some real constant,

(4) $\overline{d}_1'(x) = \ldots = -\overline{a}_4'(t) = \delta$, some real constant,

for all x,y,z,t such that $x + yi + zj + tk \in D$.

We finally get

$$a(x,y,z,t) = \alpha x - \beta y - \gamma z - \delta t + \alpha_0$$

$$b(x,y,z,t) = \beta x + \alpha y + \delta z - \gamma t + \beta_0$$

$$c(x,y,z,t) = \gamma x - \delta y + \alpha z + \beta t + \gamma_0$$

$$d(x,y,z,t) = \delta x + \gamma y - \beta z + \alpha t + \delta_0$$

which yields ■

$$f(x+yi+zj+tk) = (x+yi+zj+tk)(\alpha+\beta i+\gamma j+\delta k) + (\alpha_0+\beta_0 i+\gamma_0 j+\delta_0 k).$$

Conversely, such a function is indeed left-differentiable on Q, with $L(f)(w) = \alpha + \beta i + \gamma j + \delta k$, for all $w \in Q$.

We have proved

f: Q+Q is left-differentiable if and only if for some A, $\mu \in Q$,

$$f(w) = w\lambda + \mu \quad \text{for all } w \in Q$$

(and $L(f)(w) = \lambda$).

Similarly, one can prove

f: $Q \to Q$ is right-differentiable if and only if for some $\lambda, \mu \in Q$,

$$f(w) = \lambda w + \mu \quad \text{for all } w \in Q.$$

(and $R(f)(w)$ = A).

5 2. Let

$$A = \alpha + \beta i_0 + \gamma j_0 + \delta k_0, \; X = x + y i_0 + z j_0 + t k_0,$$

$$X' = x' + y' i_0 + z' j_0 + t' k_0$$

be three quaternions, and assume $X' = AX$.

Then

$$(x' + y' i_0 + z' j_0 + t' k_0) = (\alpha x - \beta y - \gamma z - \delta t)$$

$$+ (\beta x + \alpha y + \gamma z - \gamma t) i_0 + (\gamma x - \delta y + \alpha z + \beta t) j_0 + (\delta x + \gamma y - \beta z + \alpha t) k_0.$$

Using matrix notation, this may be rewritten as

$$\begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} = \begin{pmatrix} \alpha & -\beta & -\gamma & -\delta \\ \beta & \alpha & \delta & -\gamma \\ \gamma & -\delta & \alpha & \beta \\ \delta & \gamma & -\beta & \alpha \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} .$$

With this suggestive notation in mind, we proceed with the formal discussion. Our goal is to get a formula for the n th power of a quaternion, similar to that of Moivre ($[r(\cos 8 + i \sin 8)]^n = r^n[\cos(n\theta) + i \sin(n\theta)]$) for complex numbers.

Our first step is to show that the set of matrices

$$Q' = \left\{ \begin{pmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{pmatrix}, \quad a,b,c,d \in R \right\}$$

is isomorphic to Q, the division ring of quaternions over R.

So let $f: Q \to Q'$ be the map defined by

$$f(a + b i_0 + c j_0 + d k_0) = \begin{pmatrix} a & -b & -a & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & a & -b & a \end{pmatrix}$$

f being clearly surjective and injective. Also, it is obvious that if $w_1, w_2 \in Q$, then

$$f(w_1 + w_2) = f(w_1) + f(w_2).$$

Let's check that $f(w_1 w_2) = f(w_1) f(w_2)$; proceeding as follows:

Write

$$w_1 = a + b i_0 + c j_0 + d k_0, \quad w_2 = x + y i_0 + u j_0 + v k_0.$$

Then

$$w_1 w_2 = (ax - by - cu - dv) + (bx + ay + du - cv) i_0$$

$$+ (cx - dy + au + bv) j_0 + (dx + cy - bu + av) k_0,$$

so that

$$f(w_1)f(w_2) = \begin{pmatrix} a & -b & -c & d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{pmatrix} \begin{pmatrix} x & -y & -u & -v \\ y & x & v & \blacksquare \\ u & -v & x & \blacksquare \\ v & u & -y & x \end{pmatrix}$$

$$= \begin{pmatrix} ax-by-cu-dv & -ay-bx+cv-du & -au-bv-cx+dy & -av+bu-cy-dx \\ bx+ay+du-cv & -by+ax-dv-cu & -bu+av+dx+cy & -bv-au+dy-cx \\ cx-dy+au+bv & -cy-dx-av+bu & -cu-dv+ax-by & -cv+du+ay+bx \\ dx+cy-bu+av & -dy+cx+bv+au & -du+cv-bx-ay & -dv-cu-by+ax \end{pmatrix}$$

$$= f(w_1w_2).$$

This proves that Q' is indeed a division ring, isomorphic to Q.

Let's also note, for future use, that

$$\det\begin{pmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{pmatrix} = a\begin{vmatrix} a & d & -c \\ -d & a & b \\ c & -b & a \end{vmatrix} + b\begin{vmatrix} b & d & -c \\ c & a & b \\ d & -b & a \end{vmatrix} - c\begin{vmatrix} b & a & -c \\ c & -d & b \\ d & c & a \end{vmatrix} + d\begin{vmatrix} b & a & d \\ c & -d & a \\ d & c & -b \end{vmatrix}$$

$$= a(a^3 + a(b^2 + c^2 + d^2)) + b(b(a^2 + c^2 + d^2) + b^3)$$
$$- c(-c^3 - c(a^2 + b^2 + d^2)) + d(d(a^2 + b^2 + c^2) + d^3)$$
$$= (a^2 + b^2 + c^2 + d^2)^2.$$

Our original aim was to get a formula for

$$(a + bi_0 + cj0 + dk_0)^n .$$

With the above identification, we may hope to achieve this by **diagonalizing** the matrix $f(a + bi_0 + cj_0 + dk_0)$ and using its expression as the conjugate of a diagonal matrix to get an expression for its n^{th} power.

Let

$$w = \begin{pmatrix} \alpha & -\beta & -\gamma & -\delta \\ \beta & \alpha & \delta & -\gamma \\ \gamma & -\delta & \alpha & \beta \\ \delta & \gamma & -\beta & \alpha \end{pmatrix}.$$

By a previous computation, $\det(w - xId) = ((\alpha - x)^2 + \beta^2 + \gamma^2 + \delta^2)^2$, and hence the eigenvalues of w are $a \pm i\sqrt{\Delta}$, where $A = \beta^2 + \gamma^2 + \delta^2$.

If both γ and 6 are zero, then $a + \beta i_0 + \gamma j_0 + \delta k_0 = a + \beta i_0$, and the n^{th} power of $a + \beta i_0$ can be obtained using **Moivre's formula.**

So we assume that one of γ or 6 is non-zero. Writing $A = \gamma^2 + 6^2$, this is equivalent to assuming that $A \neq 0$ (and hence $A \neq 0$ since $\Delta = \Lambda + \beta^2$).

A being non-zero, w has two distinct eigenvalues, $a - i\sqrt{\Delta}$ and $a + i\sqrt{\Delta}$. The eigenspace of $a - i\sqrt{\Delta}$ is the **subspace** generated by

$$\left\{\begin{pmatrix} -i\sqrt{\Delta} \\ \beta \\ \gamma \\ \delta \end{pmatrix} \text{ and } \begin{pmatrix} \beta \\ i\sqrt{\Delta} \\ \delta \\ -\gamma \end{pmatrix}\right\}$$

and the **eigenspace of $\alpha + i\sqrt{\Delta}$** is the **subspace** generated by

$$\left\{\begin{pmatrix} -\gamma \\ \delta \\ i\sqrt{\Delta} \\ -\beta \end{pmatrix} \text{ and } \begin{pmatrix} \delta \\ \gamma \\ -\beta \\ i\sqrt{\Delta} \end{pmatrix}\right\}$$

Let

$$B = \begin{pmatrix} -i\sqrt{\Delta} & \beta & -\gamma & \delta \\ \beta & i\sqrt{\Delta} & \delta & \gamma \\ \gamma & \delta & i\sqrt{\Delta} & -\beta \\ \delta & -\gamma & -\beta & -i\sqrt{\Delta} \end{pmatrix}$$

Then $\det B = 4\Delta(\gamma^2 + 6^2) = 4AA$. Since we assumed $A \neq 0$, $A \neq 0$, we have $\det B \neq 0$, hence B is indeed invertible. We know that

$$w = B\begin{pmatrix} \alpha - i\sqrt{\Delta} & 0 & 0 & 0 \\ 0 & \alpha - i\sqrt{\Delta} & 0 & 0 \\ 0 & 0 & \alpha + i\sqrt{\Delta} & 0 \\ 0 & 0 & 0 & \alpha + i\sqrt{\Delta} \end{pmatrix}B^{-1}$$

and, therefore, that

$$w^n = B \begin{pmatrix} (\alpha-i\sqrt{\Delta})^n & 0 & 0 & 0 \\ 0 & (\alpha-i\sqrt{\Delta})^n & 0 & 0 \\ 0 & 0 & (\alpha+i\sqrt{\Delta})^n & 0 \\ 0 & 0 & 0 & (\alpha+i\sqrt{\Delta})^n \end{pmatrix} B^{-1}.(*)$$

The only remaining obstacle is B^{-1}. Lengthy computations lead to the following expression for B^{-1}:

$$B^{-1} = \frac{1}{\det B}\begin{bmatrix} 2i\sqrt{\Delta}(\gamma^2+\delta^2) & 0 & 2(\gamma\Delta-\beta\delta i\sqrt{\Delta}) & 2(\delta\Delta+\beta\gamma i\sqrt{\Delta}) \\ 0 & -2i\sqrt{\Delta}(\gamma^2+\delta^2) & 2(\delta\Delta+\beta\gamma i\sqrt{\Delta}) & -2(\gamma\Delta-\beta\delta i\sqrt{\Delta}) \\ -2(\gamma\Delta+\beta\delta i\sqrt{\Delta}) & 2(\delta\Delta-\beta\gamma i\sqrt{\Delta}) & -2i\sqrt{\Delta}(\gamma^2+\delta^2) & 0 \\ 2(\delta\Delta-\beta\gamma i\sqrt{\Delta}) & 2(\gamma\Delta+\beta\delta i\sqrt{\Delta}) & 0 & 2i\sqrt{\Delta}(\gamma^2+\delta^2) \end{bmatrix}.$$

We can now carry out the computations in equation (*). Using the isomorphism f^{-1}, we obtain the following formula:

$$(\alpha + \beta i_0 + \gamma j_0 + \delta k_0)^n = \alpha_n + \beta_n i_0 + \gamma_n j_0 + \delta_n i_0,$$

where

$$\alpha_n = \frac{1}{2}((\alpha - i\sqrt{\Delta})^n + (\alpha+i\sqrt{\Delta})^n),$$

$$\beta_n = \frac{\beta}{\sqrt{\Delta}}\frac{1}{2}((\alpha - i\sqrt{\Delta})^n - (\alpha+i\sqrt{\Delta})^n)i,$$

$$\gamma_n = \frac{\gamma}{\sqrt{\Delta}}\frac{1}{2}((\alpha - i\sqrt{\Delta})^n - (\alpha+i\sqrt{\Delta})^n)i,$$

$$\delta_n = \frac{\delta}{\sqrt{\Delta}}\frac{1}{2}((\alpha - i\sqrt{\Delta})^n - (\alpha+i\sqrt{\Delta})^n)i.$$

We check that a, β_n, γ_n, δ_n are real as follows:

Let

$$z = (\alpha-i\sqrt{\Delta})^n .$$

Then

$$z \in C,$$

and we have

$$a_n = \frac{1}{2}(z^n + z^{-n}) = Re(z^n) \in R,$$

$$\beta_n = \frac{\beta}{\sqrt{\Delta}}\frac{1}{2}(z^n - z^{-n})i = \frac{\beta}{\sqrt{\Delta}}(iIm(z^n))i + -\frac{\beta}{\sqrt{\Delta}}Im(z^n) \in R,$$

$$\gamma_n = -\frac{\gamma}{\sqrt{\Delta}}Im(z^n) \in R,$$

and

$$\delta_n = -\frac{\delta}{\sqrt{\Delta}}Im(z^n) \in R.$$

The formula itself can be easily verified by induction on n.

1979-80 STUDENT PAPER COMPETITION

The papers for the 1979-80 Student Paper Competition have been judged and the Winners are:

First Prize ($200) *Ziad Haddad, UCLA, "Two Remarks On The Quaternions",* This is the above article in this **Journal** starting on page 221.

Second Prize ($100) *Robert Smith, University of Arkansas, "Uniform Algebras and Scattered Spaces",* See the next article in this **Journal** starting on page 232.

Third Prize ($50) *Alma Posey, Hendrix College, "Rolling Cones",* This article appeared in the Fall 1980 Issue of this **Journal,** page 157.

This is an annual Student Paper Competition open to students who have not received their Master's degree at the time of submission. Papers may be submitted to the Editor at any time. Each Chapter which submits five or more papers creates a mini-contest among just those papers. The best will receive $20 and all such papers will be considered for the National Content. Two copies of submitted papers should be sent to the Editor at the address inside the front cover.

Second Prize Paper
National Student Paper Competition
1979-80

UNIFORM ALGEBRAS AND SCATTERED SPACES

by Robert C. Smith
University of Arkansas

Introduction.

In this paper, the relationship between scattered topological spaces and the spaces of continuous functions they support is explored. The first section contains the definition of a scattered space, and three equivalent formulations are developed there. The second section is devoted to the following result due to W. Rudin: If X is a compact scattered Hausdorff space, then X does not admit a proper uniform algebra. In the third section, a partial converse due to M. Rajagopalan is sketched, and there is a brief discussion of the question which yet remains to be resolved.

This article is based on a paper submitted to the University of Arkansas in partial fulfillment of the requirements for the degree of Bachelor of Science in Mathematics, with Honors, and the author would like to take this opportunity to thank Professor W.H. Summers for his advice and encouragement during the course of these investigations.

1. *Scattered topological spaces and equivalent formulations.*

A topological space X is called scattered (or dispersed) if every nonvoid subset contains an isolated point; i.e., for every $A \subset X$ so that $A \neq \phi$, there exists some $a \in A$ that is an isolated point in the relative topology on A.

Throughout this paper, we will assume that X is a compact Hausdorff space, and we further adopt the convention that the empty set is not perfect.

1.1. Proposition. If the space X is scattered, then X is totally disconnected and the set of isolated points in X is a dense subset of X.

Proof. Let C be a component of X. Then, because X is scattered, there exists an isolated point c in C. Since C is closed, $C \setminus \{c\}$ is closed, while $\{c\}$ is closed since X is a T_1-space. But, $(C \setminus \{c\}) \cap \{c\} = \phi$

and $(C \setminus \{c\}) \cup \{c\} = C$ whereby $C = \{c\}$ since C is connected. Now, put $A = \{a \in X : a \text{ is isolated in } X\}$ and suppose $X \setminus \bar{A} \neq \phi$. Then, since X is scattered, there exists an isolated point a, in $X \setminus \bar{A}$. But, $X \setminus \bar{A}$ is open and hence a is isolated in X, which is a contradiction. Thus, $X = \bar{A}$.

As shown by the following counterexample, the converse to Proposition 1.1 does not hold.

Example. Let C be the classical Cantor ternary set, let $A = \{x_i\}_{i \in \mathbb{N}}$ be an enumeration of the midpoints of the excluded intervals, and put $X = C \cup A$. Clearly, X is compact. For each $i \in \mathbb{N}$, let (a_i, b_i) be the excluded interval in which x_i sits. Then, $(a_i, b_i) \cap X = \{x_i\}$, $i \in \mathbb{N}$, so A consists of isolated points in X, and, since C is perfect, $A = \{x \in X : x \text{ is isolated (in } X)\}$. Let $x \in C$ and $\delta > 0$. Since C is perfect and totally disconnected, there exists some $i \in \mathbb{N}$ such that $(a_i, b_i) \subseteq (x - \delta, x + \delta)$. Hence $(x - \delta, x + \delta)$ contains some element of A; that is to say, $\bar{A} = X$.

Suppose D is a component of X. If D contains an element of A, say a, then $\{a\} = D$ (otherwise we would get a disconnection as in the proof of Proposition 1.1). On the other hand, if D is contained in C, then D is a singleton since C is totally disconnected. Thus, X is totally disconnected and the set of isolated points in X is dense in X. But the classical Cantor ternary set is a nonvoid subset of X which contains no isolated points, hence X is not scattered.

1.2. Proposition. The compact space X is scattered if, and only if, X contains no perfect subset.

Proof. Suppose $P \subset X$ is perfect. Since P is perfect, $P \neq \phi$ and P has no isolated points, so X is not scattered. In other words, if X is scattered, then X contains no perfect subset.

Suppose X does not contain any perfect subsets, and let $A \subset X$, $A \neq \phi$. Then $\bar{A}$ is not perfect, and so it follows that A must contain an isolated point; i.e., X is scattered.

Before stating our next result (1.4), we need the following definition and well known theorem (1.3).

Definition. A set E in a space X is said to be nowhere dense in X if $X \setminus \bar{E}$ is dense in X.

1.3. Baire Category Theorem (cf. [8]). A complete metric space X is not the union of a countable collection of nowhere dense sets; i.e., X is not of the first category.

1.4. Proposition. Assume that the compact space X is a subset of $\mathbb{R}^n$. Then X is scattered if, and only if, X is countable.

Proof. Let P be a perfect subset of X. Then P is a closed subset of the complete metric space $\mathbb{R}^n$ whence P is a complete metric space. Since P is perfect, $\{x\}$ is nowhere dense in P for each $x \in P$. Thus, by the Baire Category Theorem, P is uncountable. So, by Proposition 1.2, if X is countable, then X is scattered. Suppose, on the other hand, that X is scattered. Let a be an ordinal less than or equal to the cardinality of the power set of $\mathbb{R}^n$. Define X^α using transfinite induction by setting

$$X^\alpha = \begin{cases} X, & \alpha = 1 \\ (X^\beta)', & \beta + 1 = \alpha \text{ (a not a limit ordinal)} \\ \bigcap_{\beta < \alpha} X^\beta, & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

Put $A^\alpha = X^\alpha \setminus X^{\alpha+1}$. Then, $A^\alpha = \{x \in X^\alpha\colon x \text{ is isolated in } X^\alpha\}$. Fix an ordinal a. Since each $x \in A^\alpha$ is isolated in X^α, there exists a collection of basic open sets $\{U_x\}_{x \in A^\alpha}$ with the property that $U_x \cap X^\alpha = \{x\}$, $x \in A^\alpha$. By the second countability of $\mathbb{R}^n$, A^α must be countable. Another application of second countability shows that $A^\gamma = \phi$ for some ordinal y less than the first uncountable ordinal. So $X^\gamma = X^{\gamma+1}$ which implies that either $X^\gamma = \phi$ or X^γ is perfect. But X is scattered, and so, by Proposition 1.2, $X^\gamma = \phi$. Thus $X = \bigcup_{\alpha < \gamma} A^\alpha$ is countable.

Definition. If (A, B) is a pair of closed sets in a topological space, then we put $1 \cdot A = A$ and $-1 \cdot A = B$. A family $\{A_i, B_i\}_{i \in I}$ of pairs of closed sets is said to be interlocking if, for each finite set $J \subset I$ and each collection $\{\varepsilon_j\}_{j \in J}$ where each $\varepsilon_3 \in \{-1, 1\}$ we have that $\bigcap_{j \in J} \varepsilon_j A_J \neq \phi$.

1.5. Proposition ([2]). The compact space X contains a perfect subset if, and only if, there is an interlocking sequence of closed sets $\{A_n, B_n\}_{n \in \mathbb{N}}$ in X such that $A \cap B_n = \phi, n \in \mathbb{N}$.

Proof. Suppose $\{A_n, B_n\}_{n \in \mathbb{N}}$ is an interlocking sequence in X with $A \cap B_n = \phi$, and put $C = \bigcap_{n=1}^{\infty} (A \cup B_n)$. Then C is clearly compact. Next, define

$$f\colon C \to \{-1,1\}^{\mathbb{N}} \text{ by } f(x)(n) = \begin{cases} 1, & x \in A_n \\ -1, & x \in B_n. \end{cases}$$

Observe, furthermore, that $A_n \cap C$ and $B_n \cap C$ are both open in C for each $n \in \mathbb{N}$. Let $\psi_0 \in \{-1, 1\}^{\mathbb{N}}$, and let U be a basic compact-open neighborhood of $ in $\{-1, 1\}^{\mathbb{N}}$; i.e., there exists $m \in$ so that if $\psi(k) = \psi_0(k)$ for each $k \in F_m = \{n \in \mathbb{N}\colon n \le m\}$, then $\psi \in U$. Put $\epsilon_3 = \psi_0(j)$, $j \in F_m$, and put $V = \bigcap_{j \in F_m} ((\epsilon_j A_j) \cap C)$. Then V is a nonvoid open subset of C and $x \in V$ implies $f(x)(k) = \epsilon_k = \psi_0(k)$, $k \in F_m$, so that $f(V) \subset U$; that is to say, f is continuous. Now let $\psi \in \{-1, 1\}^{\mathbb{N}}$. Since $\{A_n, B_n\}_{n \in \mathbb{N}}$ is interlocking, $\bigcap_{j \in J} \psi(j) A_j \neq \phi$ for any finite set $J \subset \mathbb{N}$. Thus $\{\psi(j)A_j\}_{j \in \mathbb{N}}$ has the finite intersection property, and the compactness of X implies $\bigcap_{j=1}^{\infty} \psi(j) A_j \neq \phi$.

But $\bigcap_{j=1}^{\infty} \psi(j) A_j \subset C$ whence f is surjective. Put $S = \{A \subset C\colon A$ is closed and $f(A) = \{-1, 1\}^{\mathbb{N}}\}$. S is nonvoid since $C \in S$, and set inclusion provides a partial ordering on S. Let $\{A_\delta\}_{\delta \in \Delta}$ be a totally ordered subfamily of S, and let $\psi \in \{-1, 1\}^{\mathbb{N}}$. Put $E_\delta = \{x \in A_\delta\colon f(x) = \psi\}$, and let $\{A_{\delta_i}\}_{i=1}^{n}$ be a finite subcollection of $\{A_\delta\}_{\delta \in \Delta}$ with $A_{\delta_1} \supset A_{\delta_2} \supset \ldots \supset A_{\delta_n}$. Since $\bigcap_{i=1}^{n} E_{\delta_i} = E_{\delta_n} \neq \phi$, $\{E_\delta\}_{\delta \in \Delta}$ is a collection of closed sets with the finite intersection property.

Moreover, since C is compact, $\bigcap_{\varepsilon \Delta} E_\delta \neq \phi$. This implies $f(\bigcap_{\delta \in \Delta} A_\delta) = \{-1, 1\}^{\mathbb{N}}$, and so $\bigcap_{S \in A} A_5$ is a lower bound in S for $\{A_\delta\}_{\delta \in \Delta}$.

By Zorn's Lemma, there exists a minimal element, say P, of S; i.e., there exists some subset P of C such that $P \in S$ and f maps no proper closed subset of P onto $\{-1,1\}^{\mathbb{N}}$. For $\psi \in \{-1, 1\}^{\mathbb{N}}$, however, there exists a sequence of distinct points $\{x_n\} \subset P$ such that $f(x_n)$

converges to ψ. But since P is compact, this implies there is some $x \in P'$ for which $f(x) = \psi$. Since $f(P') = f(P)$, $P' = P$ whereby P is a perfect subset of C and therefore of X.

For the converse, let P be a perfect subset of X, and let x, y e P so that x iffy. Since P is a perfect subset of a compact Hausdorff space, P is normal and $P' = P$. Thus there exists open sets, U_1, V_1 in P so that $x \in U_1$, $y \in V_1$, and $\bar{U}_1 \cap \bar{V}_1 = \phi$. Let $n \in \mathbb{N}$ and suppose $\{U_j, V_j\}_{j=1}^{n}$ have been chosen so that $U_j \cap \bar{V}_j = \phi$ and $\bigcap_{j=1}^{n} \varepsilon_j \cdot U_j = \phi$, where $\varepsilon_j \in \{-1, 1\}$, $1 \cdot U_j = U_j$, and $-1 \cdot U_j = V_j$. Let $\{C_j\}_{j=1}^{2^n}$ be the family of all intersections obtained from all sequences $\{\epsilon_j\}_{j=1}^{n}$. This collection is pairwise disjoint since $U_j \cap V_j = \phi$ for all j. Since C_j is a finite, nonvoid intersection of open sets, there exist nonvoid open sets O_{j1}, $O_{j2} \subset C_j$, so that $\bar{O}_{j1} \cap \bar{O}_{j2} = \phi$. Put $U_{n+1} = \bigcup_{j=1}^{2^n} O_{j1}$ and $V_{n+1} = \bigcup_{j=1}^{2^n} O_{j2}$. Then $\bar{U}_{n+1} \cap \bar{V}_{n+1} = \phi$, and $\bigcap_{j=1}^{n+1} \epsilon_j U_j \neq \phi$ for any $\{\epsilon_j\}_{j=1}^{n+1}$. By finite induction, $\{\bar{U}_j, \bar{V}_j\}_{j \in \mathbb{N}}$ is defined and interlocking.

1.6. Corollary. The compact space X is scattered if, and only if, there is no interlocking sequence of closed sets $\{A_j, B_j\}_{j \in \mathbb{N}}$ in X, such that $A_j \cap B_j = \phi$, $j \in \mathbb{N}$.

Proof. This is immediate from Propositions 1.2 and 1.5.

2. *Uniform Algebras on Scattered Spaces*.

Let X be a compact Hausdorff space. Then $C(X)$, the set of complex valued continuous functions on X, is a complex algebra under the usual pointwise operations. For $f \in C(X)$, putting $||f|| = \sup\{|f(x)| : x \in X\}$ defines a norm on $C(X)$ under which $C(X)$ is a Banach space. Given $f, g \in C(X)$, $||fg|| \leq ||f||\,||g||$, and so $(C(X), ||\cdot||)$ is also a Banach algebra. Indeed, $C(X)$ is a commutative semisimple Banach algebra with identity (cf. [6]).

Definition. A subalgebra A of $C(X)$ which satisfies the following conditions is called a uniform algebra on X:

1) A is closed in $C(X)$;
2) A separates the points of X;
3) $1 \in A$.

If, in addition, $A \neq C(X)$ we will say that A is a proper uniform algebra on X; in this case X will be said to support a proper uniform algebra.

2.1. Proposition ([7]). If there exists a continuous mapping f of a compact scattered Hausdorff space X onto a compact Hausdorff space Y, then Y is scattered.

Proof. Suppose Y contains a perfect subset P. Put $D = \{E \subset X;$ E is closed and $f(E) = P\}$. Since $f^{-1}(P) \in D$, D is nonvoid, and D is partially ordered by set inclusion. Let $\{E_\delta\}_{\delta \in \Delta}$ be a totally ordered subfamily of D, and let $y \in P$. Put $C_\delta = \{e \in E : f(e) = y\}$. Let $\{E_{\delta_i}\}_{i=1}^{n}$ be a finite subcollection of $\{E_\delta\}_{\delta \in \Delta}$ with $E_{\delta_1} \supset E_{\delta_2} \supset \ldots \supset E_{\delta_n}$. So $\bigcap_{i=1}^{n} C_{\delta_i} = C_{\delta_n} \neq \phi$. Thus, $\{C_\delta\}_{\delta \in \Delta}$ is a collection of closed subsets of X with the finite intersection property whereby $\bigcap_{\delta \in \Delta} C_\delta \neq \phi$. This implies $f(\bigcap_{\delta \in \Delta} E_\delta) = P$ whence, by Zorn's Lemma, there exists a minimal element $M \in D$. M is compact, and, since X is scattered, there is some isolated point $m \in M$ so that $M \setminus \{m\}$ is compact. Since M is minimal, $f(M \setminus \{m\})$ is a proper compact subset of P. But P is perfect so that there exists $y \in P \setminus f(M \setminus \{m\})$, $y \neq f(m)$. Since $f^{-1}(y) \cap (M \setminus \{m\}) \neq \phi$, this is a contradiction.

We have need for the following two theorems. Since these results are well known and readily accessible, however, proofs have been omitted.

2.2. Mergelyan's Theorem (see [8]). If K is a compact set in the complex plane whose complement is connected, if f is a continuous complex function which is analytic in the interior of K, and if $\epsilon > 0$, then there exists a complex polynomial P such that $|f(z) - P(z)| < \epsilon$ for all $z \in K$.

2.3. Stone-Weierstrass Theorem (see [6]). Let X be a compact Hausdorff space, and let A be a closed subalgebra of $C(X)$ which separates the points of X and contains the constant functions. If A is self-adjoint, then $A = C(X)$.

2.4. *Theorem* (Rudin [7]). If X is a compact scattered Hausdorff space, then X supports no proper uniform algebra.

Proof. Let A be a uniform algebra on X, and take $f \in A$. Then $f(X)$ is scattered by Proposition 2.1. By Proposition 1.4, $f(X)$ is countable, and so the complement of $f(X)$ is readily seen to be connected. Moreover, since $f(X)$ is compact and countable, it is nowhere dense in $\mathbb{C}$. Fixing $n \in \mathbb{N}$, Mergeylan's theorem implies that there exists a complex polynomial P_n so that $|\bar{z} - P_n(z)| \leq \frac{1}{n}$, for all $z \in f(X)$; put $f_n(x) = P_n(f(x))$. Then $||\bar{f} - f_n|| \leq \frac{1}{n}$, and so it follows that $\bar{f} \in A$. Now, by the Stone–Weierstrass theorem, we have $A = C(X)$.

With Theorem 2.4 in hand, the problem of recognizing $C(X)$ among the other uniform algebras on X becomes an easy matter when X is a compact scattered Hausdorff space. It is now natural to ask if the scattered spaces are the only ones not admitting proper uniform algebras, and we take up this question in what follows.

3.. *A Partial Converse*

The converse of Theorem 2.4 remains an open question, but M. Rajagopalan ([5]) has shown that the converse is true for compact ordered spaces. As corollary, it can be shown that the converse is true for all metrizable compact spaces. First of all, however, we proceed to establish a reduction theorem (3.5) via several lemmas (cf.[5]).

3.1. *Lemma*. Let X be a compact Hausdorff space, and let Y be a closed subset of X which supports a proper uniform algebra A. Then X supports a proper uniform algebra.

Proof. Define $B = \{f \in C(X) : f|Y \in A\}$; B is clearly a subalgebra of $C(X)$. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in B which converges to $f \in C(X)$. Then $\{f_n \mid Y\}_{n=1}^{\infty}$ converges to $f \mid Y$, and, since A is a uniform algebra, $f \mid Y \in A$. Thus, B is closed. Since A is a uniform algebra, A contains the constants, and hence B also contains the constant functions. Let $x, y \in X$ with $x \neq y$.

Case 1: Suppose $x, y \in Y$. Then there is some $g \in A$ so that $g(x) \neq g(y)$, and the Tietze extension theorem yields $f \in B$ with $f(x) \neq f(y)$.

Case 2: Suppose $x \in X \setminus Y$, $y \in X$. By Urysohn's lemma, there is some continuous $f{:}X \to [0,1]$ so that $f(x) = 0$ and $f(z) = 1$, for $z \in Y \cup \{y\}$. But $f|y \in A$, and so $f \in B$. Thus, B separates the points of X. Since $A \neq C(Y)$, another application of Tietze's theorem shows that $B \neq C(X)$, and so B is a proper uniform algebra supported by X.

3.2. *Lemma*. Let X be a compact Hausdorff space, and suppose that X is not scattered. Then, there is a separable closed subspace Y of X and a continuous function h from Y onto the classical Cantor ternary set K. Moreover, Y and h can be chosen so that h maps no proper closed subset of Y onto K.

Proof. Since X is not scattered, there is a continuous surjection $f{:}X \to [0,1]$ from X onto $[0,1]$ (see [3]). Put $D = \{E: E \subset X, E$ closed in X, and $f(E) = K\}$; $f^{-1}(K) \in D$, so D is nonvoid. By a Zorn's lemma argument similar to that used in the proof of Lemma 2.1, there is some $Y \in D$ which is minimal under set inclusion. Put $h = f|Y$. By construction, h maps no proper closed subset of Y onto K, and it therefore follows that Y is separable.

3.3. *Theorem*. If every compact, separable, and nonscattered Hausdorff space supports a proper uniform algebra, then every compact non-scattered Hausdorff space supports a proper uniform algebra.

Proof. This is immediate from Lemmas 3.1 and 3.2.

3.4. *Lemma*. Let X be a compact, ordered, and nonscattered (Hausdorff) space. Then there is a closed subset Y of X and a continuous function $h{:}Y \to K$ from Y onto the Cantor ternary set K with the following properties:

1) $h(Y) = K$ and $h(G) \neq K$ for every proper closed subset $G \subset Y$;
2) Y is totally disconnected.
3) Y is separable;
4) Y is perfect;
5) Y is first countable.

Proof. There is a closed subset $Y \subset X$ and a continuous function h which satisfies 1) and 3) by Lemma 3.2. To prove 4), let $x_0 \in Y$ and $x_0 \in Y \setminus \{Y'\}$. Then $Y \setminus \{x_0\}$ is closed whereby $h(Y \setminus \{x_0\}) \subsetneq K$. But K is perfect, and this implies $K \setminus h(Y \setminus \{x_0\})$ contains more than one point which provides the desired contradiction; i.e., Y is perfect. Since Y is a (compact) ordered separable space, Y is first countable. Finally, let F be a component in Y, and suppose that F contains more than one point. Since X is an ordered space, there exist $a, b \in F$ so that

$(a,b) \neq \phi$ and F contains (a,b). Then $Y \setminus (a,b)$ is a proper closed subset of Y that h maps onto K. Since this contradicts 1), we have that 2) holds.

3.5. *Reduction* **Lemma**. Suppose all compact, ordered, totally disconnected, perfect, separable spaces support proper uniform algebras. Then every compact, ordered nonscattered space supports a proper uniform algebra.

Proof. Let X be a compact, ordered nonscattered space. Then, by Lemma 3.4, there is some $Y \subset X$ so that Y is compact, ordered, totally disconnected, perfect, and separable. The result is now immediate from Lemma 3.1.

Definition. Let $F \subset [0, 1]$ and let $F \ni D_2$ where $D_2 = \{\frac{m}{2^n} : 0 \leq \frac{m}{2^n} \leq 1;\ m,n \in \mathbb{N} \cup \{0\}\}$. Let X_F be the subset of the complex plane given by $X_F = (([0, 1] \times \{0\}) \cup (F \times \{1\})) \setminus \{(0, 0), (1, 1)\}$. Put the lexicographic order on X_F. So, if $(x,y), (u,v) \in X_F$ then $(x,y) \leq (u,v)$ if, and only if, $x < u$ or $x = u$ and $y \leq v$. X_F is a compact ordered space in the lexicographic order. This compact ordered space X_F is said to be "obtained from $[0, 1]$ by splitting the points of F."

The next result is due to S. Purisch.

3.6. *Theorem* (cf. [4]). Let X_F be as in the above definition. Then X_F is a compact, ordered, separable, perfect, totally disconnected space. Conversely, every compact, ordered, separable, perfect, totally disconnected space is homeomorphic to some X_F where X_F is as in the above definition.

3.7. *Theorem* (Rajagopalan [5]). All compact, ordered, nonscattered spaces support proper uniform algebras.

A detailed proof may be found in [5]; the main idea is to show that compact ordered spaces of the form X_F as above do support proper uniform algebras, and then use Lemma 3.5 and Theorem 3.6.

3.8. Corollary. Every uncountable compact metric space supports a proper uniform algebra.

Proof. Any uncountable compact metric space contains a Cantor set (see [1]). Since the Cantor set is ordered it admits a proper uniform algebra by the preceding theorem, and hence every compact metric uncountable space does support a proper uniform algebra.

Despite these contributions, the basic question still remains: Does every nonscattered compact Hausdorff space support a proper uniform algebra?

REFERENCES

1. Kuratowski, K., *Topologie I*, Warwaw, 1948.
2. Lacey, H. E. and Hardy, J., Notes on *Perfectness* and Total Disconnectedness, Amer. Math. Monthly 75, 1968, 602-606.
3. Mrowka, S., Rajagopalan, M. and Soundararajan, T., *A Characterization* of Compact Scattered Spaces *Through* Chain Limits, Topo-72 (General Topology and Its Applications, Proceedings of Second Pittsburgh International Conference), Lecture Notes in Mathematics 378, Springer-Verlag, Berlin, 1974, 288-297.
4. Purisch, S., *Orderability and Suborderability* Problems, Ph.D. Dissertation, Carnegie-Mellon University, 1973.
5. Rajagopalan, M., Uniform Algebras and Scattered Spaces (to appear).
6. Rickart, C.E., General Theory of Banach Algebras, Van Nostrand, Princeton, 1960.
7. Rudin, W., Continuous Functions on Compact Spaces Without Perfect Subsets, Proc. Amer. Math. Soc. 8, 1957, 39-42.
8. Rudin, W., *Real and* Complex *Analysis*, McGraw-Hill, New York, 1966.

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1981 NATIONAL PI MU EPSILON MEETING

IT IS TIME TO BE MAKING PLANS TO SEND AN UNDERGRADUATE DELEGATE OR SPEAKER FROM YOUR CHAPTER TO THE ANNUAL MEETING OF Pi Mu Epsilon AT THE UNIVERSITY OF PITTSBURGH IN AUGUST OF 1981. EACH SPEAKER WHO PRESENTS A PAPER WILL RECEIVE TRAVEL UP TO $500 AND EACH DELEGATE, UP TO $250 (ONLY ONE SPEAKER OR DELEGATE *CAN* BE FUNDED FROM A SINGLE CHAPTER, BUT OTHERS CAN ATTEND).

THE NUMBER OF BRIDGES AND OUTPOINTS IN A CUBIC GRAPH

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Abstract. If G is a cubic connected graph with b bridges and a ≥ 1 cutpoints, then b $+ 1 \leq c \leq 2b$ and these bounds are best possible. Furthermore, for any positive integers b, c with a even, satisfying these inequalities, there is a cubic connected graph with b bridges and c cutpoints which we will show how to construct. As a byproduct, we find that the number of cutpoints in any cubic graph is even.

1. Concepts about graphs.

The complete graph K_p has a set V of $p \geq 1$ points, and in K_P every pair of distinct points are joined by a line (are adjacent). The first graph of Figure 1 is the complete graph K_4. The set of lines of K_p is denoted by E_p. A graph G with p points has the same point set as K and its line set E is a subset of E_p. Following the terminology o f , a bridge of a connected graph G is a line whose removal disconnects G, and a **cutpoint** is such a point of G. In a cubic graph, each point has degree three. We show in Figure 1 the three smallest cubic graphs and also a cubic graph G with p = 10 points having a bridge e and a **cutpoint** u.

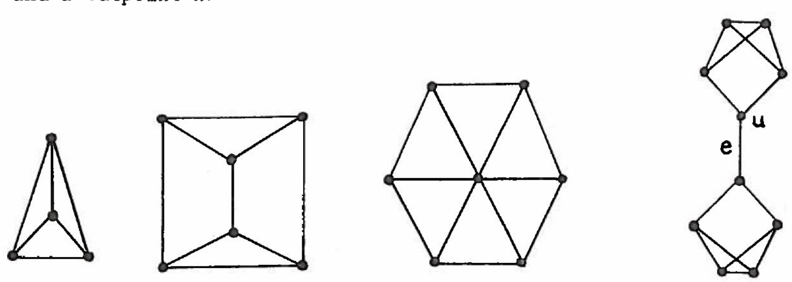


FIGURE 1
Some. cubic graphs.

In order that this note be self contained, we include here additional material from [1,p.32,42]. A tree is a connected graph with no cycles; it can also be defined as a graph in which every two distinct points are joined by a unique path. In a forest, each connected component is a tree. Thus a graph is a forest if and only if every line is a bridge.

For any graph H, we write $p(H)$ and $q(H)$ for the number of points and lines of H. For a tree T, $p(T) = q(T) + 1$ is well known. Hence for a forest F, $p(F) \geq q(F) + 1$, in fact, $p(F) = q(F) + n$ where n is the number of connected components.

1. Bounds on the number of bridges and cutpoints in a cubic graph .

The first observation is a structural lemma for cubic graphs, which is stated as an exercise in [1, p. 30].

Lemma 1. In a cubic graph, every **cutpoint** has a bridge incident with **it**.

Proof. Let v be a **cutpoint** in a cubic graph G. Then the points of G - v can be partitioned into two sets U and W such that every U-W path contains v. Since G is cubic, either U or W has exactly one line joining **it** with v. Say u is the only point of U adjacent to v. Then the line e = uv is a bridge, since the removal of e disconnects U and W.

let band a be the number of bridges and outpoints of the graph G, and let p_1 be the number of points of degree 1 (endpoints).

Lemma 2. If G is a connected graph with at least one cutpoint, then b $+ 1 \leq c + p_1$.

Proof. Let $G' \subset$ G be the **subgraph** of G consisting of all the bridges of G, as shown in Figure 2. Obviously G' is a forest, since **it** is impossible for a cycle of G to contain a bridge (and here every line of a cycle in G' **would** be a bridge of G), so $q(G') + 1 \leq p(G')$.

Let v be a point of the forest G'. Then v is either an endpoint or a **cutpoint** of G' and hence of G, thus $p(G') = c + p_1$. By definition of G', b = $g(G')$; therefore

$$b + 1 = q(G') + 1 \leq p(G') = a + p_1.$$

The following theorem gives best possible bounds on b and a **for** a cubic graph G.

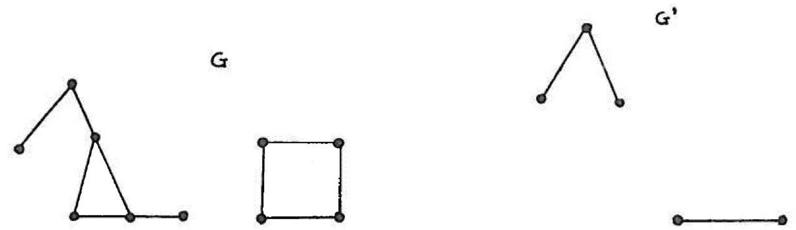


FIGURE 2
Subgraph consisting of the bridges

Theorem 1. If G is a connected cubic graph with at least one cutpoint, then $b + 1 \le c \le 2b$.

Proof. Since G is cubic, G has no endpoints. Thus by Lemma 2, $b + 1 \le c$. In a cubic graph every cutpoint has a bridge incident with it by Lemma 1, and each bridge joins at most two cutpoints, so $c \le 2b$. Thus $b + 1 \le c \le 2b$.

3. *Construction of cubic graphs with prescribed b and c.*

If b and c are any positive integers with $b + 1 \le c \le 2b$ and c even, there is a cubic graph with b bridges and c cutpoints, which we will show how to construct. We will also show that the restriction that c be even is necessary:

As in the proof of Lemma 2, again let G' be the subgraph of the cubic graph G consisting of its bridges. The next result shows that G' is a cubic forest, i.e., that each point has degree 1 or 3.

Lemma 3. If G is cubic, then G' is a cubic forest, i.e., has no points of degree 2.

Proof. Suppose G has a point v such that two of the lines e_1, e_2 incident with it are bridges and one, e_3, is not. As e_3 is not a bridge, it is on a cycle C in G which must contain either e_1 or e_2, say e_2. Then e_2 is on a cycle in G, contradicting the assumption that it is a bridge.

Thus in order to construct a connected cubic graph with b bridges and c cutpoints it is first necessary to find a cubic forest F with b lines and c points. Here we will see that it is necessary that c is even since the number of points in F of odd degree is even [1, p. 14]. We now prove this statement.

Theorem 2. The number c of cutpoints in a cubic graph is even.

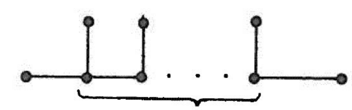
Proof. In Lemma 3, we showed that the subgraph G' of a cubic graph G consisting of the bridges of G and their points is a cubic forest with b lines and c points. It is shown in [1, p. 14] that in any graph, the number of points of odd degree is even. Since every point in the cubic forest G' has odd degree (1 or 3), the number c of points in G' must be even.

We will see that the fact that c is even is the only constraint besides $b + 1 \le c \le 2b$, and $b > 0$ for the existence of a cubic graph with b bridges and c cutpoints. For this purpose we require a preliminary result.

Lemma 4. For c even and $b > 0$ with $b + 1 \le c \le 2b$, there is a cubic forest having c points and b lines.

Proof. For any even integer $2n + 2 > 0$ define T_{2n+2} to be the tree obtained from the path P_n with n points by joining each of its points to new points so that the degree of every point in P_n becomes 3.

T_{2n+2}:



n points

FIGURE 3
A cubic tree

Clearly T_{2n+2} is a cubic tree with $c = 2n + 2$ points and $b = c - 1$ lines, for any even value of $c \ge 2$ (when $c = 2$, Figure 3 reduces to the 2-point tree K_2). Let $F = F_{b,c,k}$ be the union of T_{2k+2} and n-k copies of K_2. Then $F_{b,c,k}$ is defined for $0 \le k \le n$ and has $(2k+2) + 2(n-k) = 2n + 2$ points and $b = (2k+2-1) + (n-k) = n + k + 1$ lines. Thus there is a cubic forest with b lines and c points whenever $n + 1 \le b \le 2n+1$, that is, if

$c = 2n+2$ and $b + 1 \le c \le 2b$.

Now let H and $K_4 - e$ be as shown in Figure 4; they are useful as building blocks for cubic graphs with a prescribed number of cutpoints and bridges. Of course $K_4 - e$ is the complete graph on four points minus any edge e.

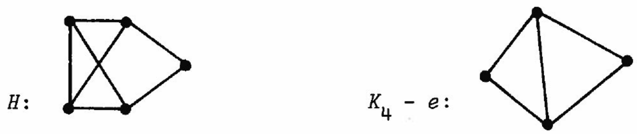


FIGURE 4
Two building blocks for cubic graphs.

Theorem 3. For any positive integers b, c with c even such that $b + 1 \le c \le 2b$, there is a connected cubic graph with b bridges and c cutpoints.

Proof. A cubic graph with b bridges and c cutpoints will be constructed by joining copies of H and $K_4 - e$ to the endpoints of the forest $F_{b,c,k}$ defined in Lemma 4 in such a way that the resulting graph is connected and cubic. Label the components of $F_{b,c,k}$ as trees $T_1, \ldots, T_m$ where $m = c - b = n - k + 1$. Since each of $T_1, \ldots, T_m$ has at least two points of degree 1, we can join endpoints of T_i and T_{i+1} with a copy of $K_4 - e$ for all $i = 1, 2, \ldots, m-1$, so that the points of degree 1 in T_i and T_{i+1} are identified with the points of degree 2 in $K_4 - e$. The resulting graph is connected and each point in it has degree 1 or 3. If we adjoin a copy of H to each point of degree 1 in this graph by identifying the endpoint with the unique point of degree 2 in H, the resulting graph G is cubic and connected with b lines and c outpoints, as illustrated in Figure 5 for $b = 7$, $c = 10$.

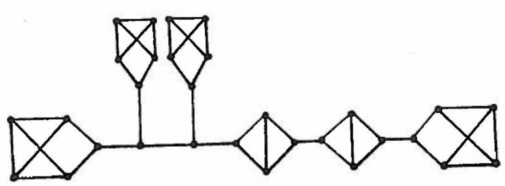


FIGURE 5
A cubic graph with 10 cutpoints and 7 bridges.

This completes the construction of a cubic connected graph with b bridges and c cutpoints for all even c and $b > 0$ with $b + 1 \le c \le 2b$. In fact it can be seen that in a cubic graph $c = b + n$, where n is the number of connected components of the forest G'.

REFERENCE

1. Harary, F., Graph Theory, Addison-Wesley, Reading, 1969.

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SOME DIVISIBILITY PROPERTIES OF BINOMIAL COEFFICIENTS

by Jean Ezell
University of Mississippi

It is well-known that if p is prime, then for all k, $0 < k < p^n$, the binomial coefficient $\binom{p^n}{k}$ is divisible by p. In the present work we investigate values of n for which the binomial coefficient $\binom{n}{k}$ are relatively prime to p for all k. In addition, for certain other values of n, we are able to say exactly how many of the coefficients $\binom{n}{k}$ are divisible by p.

First we consider the values of n for which all $\binom{n}{k}$ are prime to p. A reasonable conjecture emerges, at least for the case $p = 2$, when one examines the first few rows of Pascal's Triangle:

$n = 0$ 1
$n = 1$ 1 1
$n = 2$ 1 2 1
$n = 3$ 1 3 3 1
$n = 4$ 1 4 6 4 1
$n = 5$ 1 5 10 10 5 1
$n = 6$ 1 6 15 20 15 6 1
$n = 7$ 1 7 21 35 35 21 7 1
$n = 8$ 1 8 28 56 70 56 28 8 1

Examination of the above data leads one almost immediately to guess that the coefficients $\binom{n}{k}$ are all odd if n is one less than a power of 2. It is encouraging to note that the coefficients in rows 2 = 3 - 1 and 8 = 3 - 1 are all relatively prime to 3. But this is also true of the entries in row 5.

Theorem 1. If p is a prime, ℓ is a positive integer, $1 \le m < p$, and $0 \le k \le mp^\ell - 1$, then the binomial coefficient $\binom{mp^\ell - 1}{k}$ is relatively prime to p.

Proof.

$$\binom{mp^\ell - 1}{k} = \frac{(mp^\ell - 1)!}{(mp^\ell - 1 - k)!\,k!}$$
$$= \frac{(mp^\ell - 1)(mp^\ell - 2)\ \ldots\ (mp^\ell - k)}{1\ (2)\ \ldots\ (k)}$$
$$= \left(\frac{mp^\ell - 1}{1}\right)\left(\frac{mp^\ell - 2}{2}\right)\ \ldots\ \left(\frac{mp^\ell - k}{k}\right).$$

We now introduce the notation $h_p(n)$ for the largest integer r such that p^r divides n. We claim that for each i, $1 \le i \le k$, $h_p(mp^\ell - i) = h_p(i)$. To see this, let $h_p(i) = t$ so that $i = p^t s$ with s relatively prime to p. Then $mp^\ell - i = mp^\ell - sp^t$, and, since t is certainly less than or equal to ℓ, this is $p^t(mp^{\ell-t} - s)$. Now $mp^{\ell-t} - s$ is relatively prime to p. This is obvious if $t < \ell$, since $mp^{\ell-t}$ is divisible by p, but s is not. If $t = \ell$, $mp^{\ell-t} - s = m - s < m < p$. Hence, since $m - s \ge 1$, $mp^{\ell-t} - s$ is not divisible by p. It follows that $h_p(mp^\ell - i) = t = h_p(i)$. We now know that the highest power of p in the numerator of each factor $\left(\frac{mp^\ell - i}{i}\right)$ is the same as the highest power of p in the corresponding denominator. Hence p does not divide the binomial coefficient $\binom{mp^\ell - 1}{k}$.

Now, in case $p = 2$, we will show that the values specified in Theorem 1 are in fact the only ones. First some lemmas:

<u>**Lemma A.**</u> For a positive integer n,

$$h_2((2n + 1)!) = h_2((2n)!) = n + h_2(n!)\ .$$

Proof. Clearly $h_2((2n + 1)!) = h_2((2n)!)$ since $2n + 1$ is odd. Recall that $h_p(k!) = \sum_{j=1}^{\infty} \left[\frac{k}{p^j}\right]$, where square brackets denote the greatest integer function. Now $h_2((2n)!) = \sum_{j=1}^{\infty}\left[\frac{2n}{2^j}\right] =$

$$n + \sum_{j=2}^{\infty}\left[\frac{2n}{2^j}\right] = n + \sum_{j=1}^{\infty}\left[\frac{n}{2^j}\right] = n + h_2(n!)\ .$$

<u>**Lemma B.**</u> If n is a positive integer, then $h_2(n!) \le n - 1$, and $h_2(n!) = n - 1$ if and only if n is a power of 2.

Proof. *suppose* $2^k \le n < 2^{k+1}$. Then

$$h_2(n!) = [\tfrac{n}{2}] + [\tfrac{n}{4}] + \dots + [\tfrac{n}{2^k}]$$

$$\le \frac{n}{2} + \frac{n}{4} + \dots + \frac{n}{2^k}$$

$$= n(\frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^k})$$

$$= n(1 - \frac{1}{2^k}) < n.$$

Hence $h_2(n!) \le n - 1$. *If* $h_2(n!) = n - 1$, then *we have* $n - 1 \le n - \frac{n}{2^k}$, *since* $h_2(n!) \le n(1 - \frac{1}{2^k}) = n - \frac{n}{2^k}$. *Therefore,* $\frac{n}{2^k} \le 1$, or $n \le 2^k$. But $n \ge 2^k$, so $n = 2^k$.

Theorem 2. *If* n *is a positive integer with the property that* $\binom{n}{k}$ *is odd for each* k, $0 \le k \le n$, *then* $n = 2^{\ell} - 1$ *for some* I.

Proof. We need only consider odd n, *since if* n *is even* $\binom{n}{1}$ *is even. Suppose* n *is odd, say* $n = 2k + 1$. *Now consider* $\binom{2k+1}{k} = \frac{(2k+1)!}{k!\ (k+1)!}$. *Since* $\binom{2k+1}{k}$ *is odd,* $h_2(\binom{2k+1}{k}) = h_2((2k+1)!) - h_2(k!) - h_2((k+1)!) = k + h_2(k!) - h_2(k!) - h_2((k+1)!) =$

$k - h_2((k+1)!) = 0$. So $h_2((k+1)!) = k$, and by *Lemma B* this *means* $k + 1$ *is a power of 2, making* $k = 2 \quad - 1$ *for some* ℓ.

In the introductory remarks, it was mentioned that the coefficients $\binom{p^n}{k}$ *are all divisible by* p, *for* $0 < k < p^n$. *This result is a corollary of the following theorem.*

Theorem 3. *If* $n = mp^{\ell} + j$, *where* $1 \le m < p$, *and* $0 \le j < p$, *then* p *is relatively prime to exactly* $(m + 1)(j + 1)$ *of the coefficients* $\binom{n}{k}$.

Proof.

$$\binom{n}{k} = \binom{mp^I + j}{k} = (\frac{IT^I + j}{k})(\frac{mp^{\ell} + j - 1}{k - 1}) \cdots (\frac{IT + j - k + 1}{1})..$$

If $k = 0$, *certainly* $\binom{n}{k}$ *is relatively prime to* p. *If* $0 < k \le j$, *then none of the numerators of these fractions is divisible by* p, *and so, for*

$k \le j$, $\binom{n}{k}$ is relatively prime to p. If $j < k \le mp^{\ell} + j$, then $0 < k - j \le mp^{\ell}$. In this case

$$\binom{n}{k} = \binom{mp^{\ell} + j}{k} = (\frac{mp^{\ell} + j}{k})(\frac{mp^{\ell} + j - 1}{k - 1}) \cdots (\frac{mp^{\ell}}{k - j})(\frac{mp^{\ell} - 1}{k - j - 1}).$$

The last factor is relatively prime to p *by Theorem 1. Hence*

$$h_p(\binom{n}{k}) = \sum_{i=0}^{j} (h_p(mp^{\ell} + j - i) - h_p(k - i)).$$

Since $i \le j < p$, $mp^{\ell} + j - i$ *is divisible by* p *only* in case $i = j$, *in which case* $h_p(mp^{\ell}) = \ell$. So $h_p(\binom{n}{k}) = \ell - \sum_{i=0}^{j} h_p(k - i)$, *and it follows that* $\binom{n}{k}$ *is relatively prime to* p *if and only if* $\sum_{i=0} h_p(k - i) = \ell$.

Since $j < p$, *at most one* $h_p(k - i)$ *may be nonzero, for* $0 \le i \le j$, *and if* $h_p(\binom{n}{k})$ *is to be zero, then* $h_p(k - i)$ *must be* ℓ *for some* i, $0 \le i \le j$. *That is,* $k - i = rp^{\ell}$ *for some* r, *with* r *relatively prime to* p. *Hence* $k = rp^I + i \le mp^I + j$. *For any of the* $j + 1$ *possible values of* i, $0 \le i \le j$, *there are* m *values of* r, $1 \le r \le m$, *which yield values of* k *for which* $h_p(\binom{n}{k}) = 0$. *This yields* $m(j + 1)$ *values of* k. *Recalling the* $j + 1$ *values* $0 \le k \le j$ *determined earlier, we have a total of* $(m + 1)(j + 1)$.

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THE KERNEL OF THE LAPLACE TRANSFORM

by *David C. Sutherland*
Hendrix College

Introduction.

Is there any correlation between the use of the term kernel as the kernel of an integral transform and the kernel of a homomorphism between algebraic structures? The Laplace transform is given by $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = F(s)$ for $s > S$, the abscissa of convergence of F. In this paper the kernel, $K(s,t) = e^{-st}$, is exhibited as the kernel of a homomorphism between two multiplicative semigroups of functions.

Setting. Let S be the set of all functions $f:[0,\infty)\to R$ such that f is piecewise continuous on each interval $[0,T]$, $T > 0$, and is of exponential order (i.e., $|f(t)| \le Me^{\alpha t}$, $t > T$, for some M, a, T ? 0). When necessary assume $f(t) = 0$ for $t < 0$. Define convolution multiplication on S by

$$f * g(t) = \int_0^t f(t-u)g(u)du.$$

Theorem 1. $(S,*)$ is a commutative semigroup.

Proof. We first note that if f, $g \in s$, then $f*g$ is continuous. The Proof follows Churchill [1], problem 9, p. 49.

Suppose $f, g \in s$. Since f and g are of exponential order, we may choose M_1, a, $T > 0$ such that $|f(t)| \le M_1 e^{\alpha t}$ and $|g(t)| \le M_1 e^{\alpha t}$ for $t \ge T$. Further require that M_1 is sufficiently large so that $|f(t)| \le M_1$ and $|g(t)| \le M_1$ for $0 \le t \le T$. Let $\varepsilon > 0$ and denote by M_2 the maximum of $(\frac{2}{\alpha}+t)e^{-\varepsilon t}$ on $[0,\infty)$. If $t > 2T$, then

$$\begin{aligned}|f*g(t)| &\le \left|\int_0^T f(t-u)g(u)du\right| + \left|\int_T^{t-T} f(t-u)\,g(u)du\right| \\ &\quad + \left|\int_{t-T}^t f(t-u)\,g(u)du\right| \\ &\le M_1^2\int_0^T e^{\alpha(t-u)}du + M_1^2\int_T^{t-T} e^{\alpha t}du + M_1^2\int_{t-T}^t e^{\alpha u}du \\ &= M_1^2 e^{\alpha t}\left[\tfrac{2}{\alpha}(1-e^{-\alpha T}) + t - 2T\right] \le M_1^2\left(\tfrac{2}{\alpha}+t\right)e^{\alpha t} \\ &= M_1^2\left(\tfrac{2}{\alpha}+t\right)e^{-\varepsilon t}\,e^{(\alpha+\varepsilon)t} \le Me^{(\alpha+\varepsilon)t},\end{aligned}$$

where $M = M_1^2 M_2$. Therefore, $f*g$ is of exponential order.

We have shown that convolution is closed in S. To show associativity suppose f, g, h $\in$ S and let $t \in [0,\infty)$.

$$(f*g)*h(t) = \int_0^t\int_0^{t-u} f(t-u-v)g(v)h(u)dvdu. \qquad (1)$$

Substituting $w = u+v$, K $= u$, A? $= dv$, and changing the order of integration in (2) we get

$$\begin{aligned}(f*g)*h(t) &= \int_0^t\int_u^t f(t-w)\,g(w-u)h(u)dwdu \\ &= \int_0^t\int_0^w f(t-w)g(w-u)h(u)dudw \\ &= \int_0^t f(t-w)g*h(w)\,dw = f*(g*h)\,(t).\end{aligned}$$

Therefore, $(S,*)$ is a semigroup. Commutativity involves a straightforward substitution and is left to the reader.

Let T be the set of all continuous functions F: $(s_F,\infty)\to R$, $-\infty \le s_F < \infty$. Define pointwise multiplication on T by $F\cdot G(s) = F(s)G(s)$, $s > \max\{s_F, s_G\}$.

Theorem I. $(T,\cdot)$ is a semigroup.

The proof to Theorem 2 is straightforward and left to the reader.

Theorem 3. The Laplace transform, $\mathcal{L}$, acts as a mapping from S into T.

The existence of $\mathcal{L}\{f(t)\} = F(s)$ on an interval (s_F,∞) for each $f \in$ S is shown in most standard differential equations texts (see [2], for example). The continuity of F is shown in Churchill [1], pp. 41–43, and Widder [6], pp. 373–375. A proof of the following theorem and other properties of the Laplace transform also can be found in these references.

Theorem 4. (Convolution Theorem) If f, g $\in S$, then

$$\mathcal{L}\{f*g(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} = F\cdot G(s),\ s > \max\left\{s_F, s_G\right\}.$$

As a consequence of the preceding theorem we have that $\mathcal{L}: S\to T$ is a homomorphism.

Dirac delta function. At this point S does not have an identity. For that purpose we introduce the Dirac delta function. Let H denote the unit step function (or Heaviside function) defined by

$$H(t) = \begin{cases} 0, & t < 0 \\ 1, & t \ge 0.\end{cases}$$

Let $h > 0$ and $a \geq 0$. Define $d_h : R \to R$ by

$$d_h(t) = (1/h)\{H(t) - H(t-h)\}.$$

There are several ways to define the Dirac-delta function. One way is to consider it as a generalized derivative of H (for example, see [3]). Another conventional approach is to consider the Dirac-delta function as the generalized limit of d_h as $h \to 0+$ (for example, see [5]). In [4], this function is introduced through convolution quotient rings. Thus, the Dirac-delta function is not actually a function in the usual sense but rather belongs to the class of generalized functions or distributions.

The following theorem summarizes properties of the Dirac-delta function which will be used later.

Theorem 5. Let $a,b \geq 0$ and $f \in s$. If $a \geq 0$ $\delta(t-a)$ is $\delta_a(t)$.

(i) $f * \delta_a(t) = \delta_a * f(t) = H(t-a-)f(t-a-)$.

(ii) $\delta_a * \delta_b(t) = \delta_{a+b}(t)$.

(iii) $\mathcal{L}\{\delta_a(t)\} = e^{-as}$, $s > 0$.

Proofs for this theorem can be found in [3], pp. 58-69.

Kernel. Let $S_1 = S \cup \{\delta_a : a \geq 0\}$. The fact that S_1 is a semigroup under convolution follows from Theorem 5, (i) and (ii). Furthermore, the Laplace transform is defined on S_1, and it is easily verified that the Convolution Theorem still holds. For example, if $f \in s$ and $a \geq 0$.

$$\mathcal{L}\{f * \delta_a(t)\} = \mathcal{L}\{H(t-a-)f(t-a-)\} = e^{-as}\mathcal{L}\{f(t-)\}$$
$$= \mathcal{L}\{f(t)\} \quad \mathcal{L}\{\delta_a(t)\}.$$

Therefore, $\mathcal{L}$ is a homomorphism from S_1 into T. Let $T_1 = \mathcal{L}(S_1)$.

Before proceeding to our main result we need the following

Definition. A congruence on a semigroup S is an equivalence relation ρ on S such that if $s \rho t$ and $u \rho v$, then $su \rho tv$.

Now define the relation ρ on S_1 such that $f \rho g$ if, and only if, $F(s) = \mathcal{L}\{f(t)\} = e^{-as}\mathcal{L}\{g(t)\} = e^{-as}G(s)$ for some $a \in R$ and all $s > s_0 \geq \max\{s_F, s_G\}$.

Theorem 6. The relation ρ is a congruence on S_1.

Proof. We first show that ρ is an equivalence relation. For the reflexive property simply choose $a = 0$. Suppose $f, g \in S_1$ such that $F(s) = e^{-as}G(s)$ for some $a \in R$ and all $s > s_1$. Then $G(s) = e^{as}F(s)$ which gives symmetry. In addition assume that $h \in S_1$ and $G(s) = e^{-bs}H(s)$ for some $b \in R$ and $s > s_2$. Then $F(s) = e^{-as}G(s) = e^{-as}e^{-bs}H(s) = e^{-(a+b)s}H(s)$, $s > \max\{s_1, s_2\}$, demonstrating transitivity.

Suppose $f \rho g$ and $h \rho k$ in S_1. Choose $a, b \in R$ such that $F(s) = e^{-as}G(s)$ and $H(s) = e^{-bs}K(s)$, $s > s_0$. By the Convolution Theorem extended to S_1,

$$\mathcal{L}\{f * h(t)\} = F(s)H(s) = e^{-as}G(s)e^{-bs}K(s)$$
$$= e^{-(a+b)s}G(s)K(s)$$
$$= e^{-(a+b)s}\mathcal{L}\{g * k(t)\}, \ s > s_0.$$

Therefore $f * h \rho g * k$ and the theorem is proved.

Let S_1/ρ be the quotient semigroup of all ρ-classes in S_1 with multiplication defined by $[f][g] = [f * g]$. From Theorem 5, $\mathcal{L}\{f * \delta(t)\} = \mathcal{L}\{f(t)\}$ for all $f \in S_1$. Therefore, $[\delta] = [\delta_a]$, $a \geq 0$, is the identity for S_1/ρ.

Finally, define the mapping $\alpha : T_1 \to S_1/\rho$ so that $\alpha(F) \supset \mathcal{L}^{-1}(F)$ for all $F \in T_1$.

Theorem 7. α is a homomorphism, and the kernel of α is $\{F \in T_1 : F(s) = e^{-sa}, a \geq 0, s > 0\}$.

Proof. Let $F, G \in T_1$. Choose $f, g \in S_1$ such that $\mathcal{L}\{f(t)\} = F(s)$ and $\mathcal{L}\{g(t)\} = G(s)$. Then $f \in \alpha(F)$ and $g \in \alpha(G)$. Furthermore, $\mathcal{L}\{f * g(t)\} = F \cdot G(s)$ from the Convolution Theorem. Therefore,

$$\mathcal{L}^{-1}(F \cdot G) \subset [f * g] = [f][g] = \alpha(F)\alpha(G),$$

from which it follows that $\alpha(F)\alpha(G) = \alpha(F \cdot G)$. Theorem 5, (iii) implies that $\{F \in T_1 : F(s) = e^{-sa}, a \geq 0, s > 0\}$ is the kernel of α.

In conclusion, let us regard the kernel of the Laplace transform, $K(s,t) = e^{-st}$, as a one-parameter subsemigroup of T_1 for $s > 0$ and $t \geq 0$. That is, for each $t \geq 0$ let $K(\cdot,t) \in T_1$ such that $K(s,t) = e^{-st}$, $s > 0$. Then the subsemigroup of T_1 associated with K by the mapping $t \to K(\cdot,t)$ is precisely the kernel of the homomorphism α.

Referee's Note - Some of the results discussed in this paper are also covered in Louis Brand's Differential *and Difference Equations*, *Wiley*, 1966, using the *Mikusinski* Operational Calculus.

REFERENCES

1. Churchill, R.V., *Operational Mathematics,* 3rd ed., McGraw-Hill Book Co., New York, 1972.
2. Derrick, W.R., and Grossman, S.I., *Elementary Differential Equations with Applications,* Addison-Wesley Publishing Co., Reading, Mass., 1976.

Doetsch, G., *Introduction to the Theory and Application of the Laplace Transformation,* Springer-Verlag, New York, 1974.

Erdélyi, A., *Operational Calculus and Generalized Functions,* Holt, Rinehart and Winston, New York, 1962.

Lighthill, M.J., *Introduction to Fourier Analysis and Generalized Functions,* Cambridge University Press, New York, 1958.

Widder, D.V., *Advanced Calculus,* Prentice-Hall, New York, 1947.

This paper was written under the direction of Dr. Robert Eslinger while the author was a sophomore.

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EMPLOYMENT OPPORTUNITIES IN INDUSTRY FOR NON-PH.D. MATHEMATICIANS

by *V. Bushaw*
Washington State University

Introduction. Although much information about nonacademic employment opportunities for Ph.D. mathematicians can now be found (see especially the *Notices of the American Mathematical Society),* much less has been written about such opportunities for mathematicians who do not have the doctorate. This is unfortunate, because it is probably for this latter group that far more opportunities exist.

In the late seventies, my university received a grant from the National Science Foundation to consider ways of modifying the graduate program in mathematics to provide better preparation for nonacademic careers*. In connection with this project, dozens of present or potential employers of mathematicians were interviewed to see what they had to say about desirable qualifications for mathematicians. This interviewing was accompanied by a review of the pertinent literature and by participation in several conferences at Washington State University and at Clemson University where representatives of government and industry were able to express their views on the same subject. The following remarks constitute mainly a summary of conclusions drawn from these experiences:

Opportunities for holders of Bachelor's or Master's Mathematics Degrees. There is a considerable demand for suitably trained holders of a bachelor's or master's degree in mathematics. Rightly or wrongly, many employers regard Ph.D. mathematicians as an expensive luxury, while those with master's degrees and no Ph.D. are considered to have almost as much to offer, and to "fit in" better in the typical nonacademic working environment. (In fact, recent surveys indicate that the ratio of master's degrees to Ph.D.'s in nonacademic employment for mathematicians is about four to one.) Furthermore, the potential opportunities in industry for

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*NSF Grant No. SED75-17322. A report on this project and on the parallel project at Clemson University, *New Opportunities in Applied Mathematics* (October, 1979) is available from the author of this note.

mathematicians have probably been hardly scratched. Many possible employers do not yet realize, but are often easy to convince, that they should consider hiring mathematicians in this category. Even so, there are plenty of opportunities now, The two universities mentioned above, and others with similar programs, have experienced no difficulty in recent years in placing their master's degree graduates in a great variety of appropriate jobs. Recent M.S. graduates from my university, for example, have found positions with the Navy, in the lumber industry, in the aerospace industry, with manufacturers of calculators, in diverse consulting firms, and so on. Of course, some of them have chosen to go into teaching or to continue graduate work.

These placements, however, are not always automatic. Opportunities are not always obvious, and finding them sometimes requires a certain amount of imagination and aggressiveness on the part of the graduate and his or her faculty advisors. For instance, it is a mistake to look only at advertisements announcing positions specifically for mathematicians. Engineers, physical scientists, computer scientists, business administration graduates, and others have often been hired to do what should be a mathematician's work. This is sometimes a result of an often misguided but not always unshakable belief that such people are more "practical". but probably more often of an inadequate appreciation of what well-trained mathematicians can contribute. Mathematics students in search of employment are well advised to invade this territory, which should rightly be theirs. Recruiters from government agencies and private firms seldom resent this tactic. and often count it in favor of the student as a sign of initiative.

It may happen that energetic students. with the help of their instructors and often of a college placement office, can create positions for themselves by persuading employers' representatives that they Would he valuable additions to their staffs. The sound advice in the literature of job-seeking applies *fortissimo* in this situation.

A recipe for an absolutely employable degree in mathematics. The M.A. - or M.S. -holding mathematician who has been the subject of the preceding paragraphs has been assumed to he suitably trained. While the traditional degree in pure mathematics hardly disqualifies a person for nonacademic employment, it has become clear that certain features. some novel. can greatly enhance the graduate's opportunities. The inquiries

described in the Introduction have led to a loose consensus about what those features might be. While the claim in the heading of this section may be a bit extravagant, there is ample evidence that the student who completes a program with most or all of the following characteristics may confidently look forward to a rosy professional future.

1. *Core Mathematics.* Mathematics departments considering a move in the direction of preparation for nonacademic careers are sometimes tempted to throw overboard all of the traditional pure mathematics. This is almost surely a mistake, as many industrial mathematicians would be among the first to say. Besides legitimizing the use of the word "mathematics" in the name of the degree, a significant graduate-level experience with some of the most important ideas of traditional mainstream mathematics is invaluable background for further learning and for doing most kinds of applied mathematics. There is naturally some disagreement about how much is "significant", which ideas are "most important", and how firmly core requirements should be prescribed.

2. *Computing and numerical analysis.* There is wide agreement that nowadays anyone who receives a degree in mathematics should understand computing to a reasonable extent. This is certainly true of those contemplating careers as mathematicians in industry or government. Again, opinions about what "a reasonable extent" is vary greatly. Some qualified people would claim that a broad background in computing is sufficient, and that detailed knowledge of languages, systems, and software should be developed on the job; others would expect the equivalent of an undergraduate major in computer science. A tolerable middle position would seem to be that a nonacademic mathematician should know enough about computing to avoid being responsible for unduly expensive computer runs.

3. *Applied statistics.*

4. *Operations research or mathematical programming.* While extensive sequence- of course? in these areas are hardly necessary in all cases, some acquaintance with the basic concepts and techniques is invaluable.

5. *Modeling.* The student should be given some experience in modeling real world problems, preferably authentic problems with

real data. The experience may be obtained in separate courses or seminars, or along the way in otherwise ordinary courses.

6. *Field experience*. Something by way of an off-campus internship, besides adding several dimensions to the student's education, offers the student and at least one possible employer an opportunity to look each other over. Making such arrangements and supervising them is usually a matter of personal diplomacy, and therefore requires a good deal of faculty time; but, by all indications, it is worth it. Plans such as that of the Mathematics Clinic at the Claremont Colleges provide interesting alternatives. In general, when off-campus arrangements are not practical, on-campus internships in other departments or in nonacademic research units may serve many of the same purposes.

7. *Broad interests*. Employers tend to expect their scientific personnel to have a good "world view", including an appreciation of both the technical and nontechnical(political, economic, etc.) milieus by which their work might be affected and on which it might impinge. To some extent this attribute may be strengthened by encouraging or even requiring students to take appropriate courses in other fields.

Well-developed avocational interests also come under this heading. A healthy interest in games, civic causes, and so on is much more likely to count for than against an employee.

8. *Communication skills*. The importance of being able to communicate effectively, orally and in writing, with colleagues of many kinds can hardly be exaggerated.

9. *Attitudes*. The mathematician in government or industry should be not only competent, but eager to help others with their problems and willing to work with them on their own terms. In particular, he or she should not be obsessed with mathematical rigor. One of the most frequent complaints against traditionally trained mathematicians is that they often do not know when enough time has been spent on a problem and something else should be taken up.

10. *Leadership ability*. Even employees recruited for their scientific knowledge are often expected, after some years, to be able to move into the ranks of management. Failure to do so is

frequently seen as failure indeed.

It may seem a hopelessly tall order to compress all of these elements into the two years or less of the usual master's degree program. But by ingenious management of the curriculum, and a sincere effort to impart the less technical elements throughout the program, something very like it can be done, and is being done at a number of universities. Students at universities where no such program has been formalized can usually, with a little help from the faculty, custom-build something of the kind for themselves.

The bachelor's degree. The standard bachelor's degree with a mathematics major is not generally regarded in industry as a terminal professional degree. Nevertheless, almost all holders of the bachelor's degree, especially if they know some computing, are now finding suitable positions. For example, it has recently been reported (e.g., in *The Wall Stree Journal* of September 12, 1980) that many firms are now hiring people with B.A.'s and B.S.'s in mathematics for positions that would formerly have been filled by holders of the Master of Business Administration degree.

With the rapid mathematization of so many aspects of modern life, this trend will surely continue. In fact, it can be expected to accelerate if more mathematics departments modify their undergraduate programs along the lines suggested above for the master's degree. This has already happened in many colleges, and the move is under study at many others. Specific recommendations in this area, based on extensive consultations and discussions, will appear in the forthcoming report of the CUPM Panel on a General Mathematical Sciences Program.

Conclusion. Students who love mathematics, but have been discouraged from considering a career in mathematics because of persistent rumors about the unemployability of mathematicians, should stop worrying! With a bit of thoughtful planning and guidance, they may be assured of gratifying professional lives with mathematics itself -- especially if they obtain suitable mathematics degrees.

PROBLEM DEPARTMENT

Edited by Leon Bankoff
Los Angeles, California
and
Clayton W. Dodge
University of Maine

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also acceptable. The choice of proposals for publication will be based on the editor's evaluation of their anticipated reader response and also on their intrinsic interest. Proposals should be accompanied by solutions if available and by any information that will assist the editor. Challenging conjectures and problem proposals not accompanied by solutions will be designated by an asterisk (*).

Problem proposals offered for publication should be sent to Dr. Leon Bankoff, 6360 Wilshire Boulevard, Los Angeles, California 90048.

To facilitate consideration of solutions for publication, solvers should submit each solution on separate sheets (one side only) properly identified with name and address and mailed before December 1, 1981 to Clayton W. Dodge, Mathematics Department, University of Maine, Orono, Maine 04469.

Contributors desiring acknowledgment of their proposals and solutions are requested to enclose a stamped and self-addressed postcard or, for those outside the U.S.A., an unstamped card or mailing label.

Problems for Solution

***456.** [Fall 1979, Fall 1980] (Restated) Proposed by Paul Erdös, Spacership Earth.

Let me restate problem 456. I want a path on visible lattice points (with relatively prime coordinates) which does not pass through a point (p,q) where both coordinates are primes and where both coordinates tend to infinity. Explanation: (u,v) has four neighbors, $(u+1,v)$ $(u-1,v)$, $(u,v+1)$, $(u,v-1)$, and a point can be joined only to one of its neighbors.

I offer 50 dollars for a path which goes through visible lattice points and avoids (p,q) and moves monotonically away from the origin, i.e., (u,v) can be joined only to $(u+1,v)$ or $(u,v+1)$. The start of the path can be any $(u,v) = 1$. I pay also for a non-existence proof. I do not know the solution and I apologize for the unclearly and incorrectly stated problem 456. My old age and stupidity is, I believe, adequate explanation and excuse.

476. [Fall 1980] (Corrected) Proposed by Jack Garfunkel, Queens College, Flushing, New York.

If A, B, C, D are the internal angles of a convex quadrilateral, that is, if $A + B + C + D = 360^\circ$, then $\sqrt{2}\,[\cos(A/2) + \cos(B/2) + \cos(C/2) + \cos(D/2)] \le [\cot(A/2) + \cot(B/2) + \cot(C/2) + \cot(D/2)]$, with equality when $A = B = C = D = 90^\circ$.

486. Proposed by Chuck Allison and Peter Chu, San Pedro, California.

Swimmers A and B start from opposite sides of a river and swim to their corresponding opposite sides and then back again, each swimming at his own constant rate. If on the first pass they meet each other x feet from A's starting side, and on the second pass they meet at a point y feet from B's starting side, how wide is the river in terms of x and y?

487. Proposed by Solomon W. Golomb, University of Southern California.

We know that $1/7 = .142857\ldots$ repeating with period 6. With $A = 142$ and $B = 857$, the first and second halves of the period, respectively, we observe that $A + B = 999$, and $B = 6A + 5$. Prove this generalization:

> If p is prime, and the decimal expansion of $1/p$ has period $2t$, where A and B are the first and second halves of the period, then $A + B$ consists of "all 9's", and when B is divided by A, there is a quotient of $p - 1$ with a remainder of $p - 2$.

Can you also generalize from the relation $14 + 28 + 57 = 99$? Finally, what happens if the expansions are in base b and p is merely relatively prime to b? (Note: In base $b > 1$, b is always equal to 10, but not necessarily equal to ten.)

488. *Proposed by Herb Taylor, South Pasadena, California.*

Take the numbers from 1 to 24 and put them into 8 disjoint 3-sets $[a,b,c]$ such that in each 3-set, $a + b = a$.

489. *Proposed by Michael W. Ecker, Pennsylvania State University, Worthington Scranton Campus.*

Let k and n be positive integers with $k<n$. Two players take turns choosing, on each turn, a positive integer 5 k. A running total is kept, and the player to achieve n as the sum is the winner.

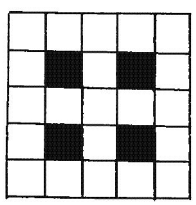
State and prove winning strategy results for this game. (The game with $n = 50$ and $k = 6$ has been used as a teaching tool, with modest popularity, at the elementary and secondary school levels.)

490. *Proposed by Joyce W. Williams, North University of Lowell.*

The function $f(n)$ is to be constructed to give the number of days in a year through the nth month for $n = 0, 1, \ldots, 12$. That is, $f(0) = 0$, $f(1) = 31, \ldots, f(12) = 365$. Leap year is to be ignored. What is the simplest solution?

491. *Proposed by Charles W. Trigg, San Diego, California.*

From a square grid of side $2n + 1$ alternate squares are removed to form a sieve. (a) What is the smallest sieve that can be dissected and the parts assembled into two squares with integer sides? (b) What is the smallest number of pieces into which the sieve must be cut to accomplish this assembly?

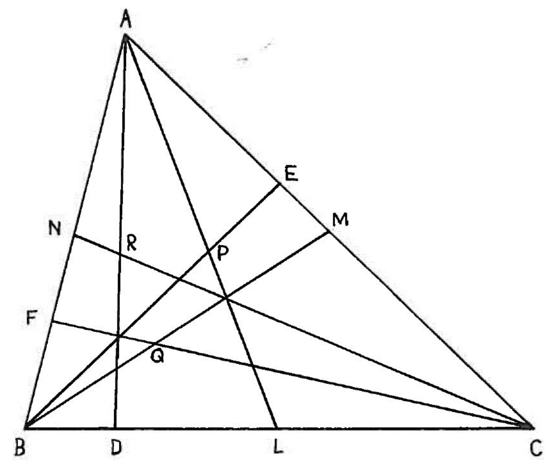


492. *Proposed by Jack Garfunkel, Queens College, Flushing, N.Y.*

Given an acute triangle ABC with altitudes denoted by h_a, h_b, h_c and medians by m_a, m_b, m_c to sides a, b, c respectively. The points P, Q, R are determined by the intersections $m_a \cap h_b$, $m_b \cap h_c$, and $m_c \cap h_a$, respectively. Prove:

$$\frac{AP}{PL} + \frac{BQ}{QM} + \frac{CR}{RN} \geq 6,$$

where L, M, N are the feet of the medians.

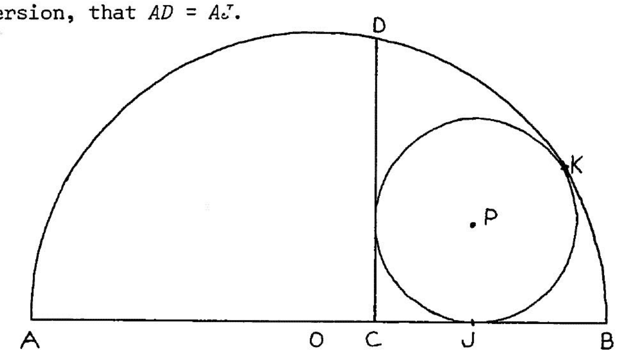


493. *Proposed by Kenneth M. Wilke, Topeka., Kansas.*

Determine the greatest power which divides n!. Prove that for $n>21$ it is a square. (This is a restatement of problem 467. [Spring 1980]).

494. *Proposed by Zelda Katz, Beverly Hills, California.*

In the annexed figure CD is a half-chord perpendicular to the diameter AB of the semicircle (O), and the inscribed circle (P) touches AB in J and the arc DB in K. Show by elementarv plane geometry, without using inversion, that $AD = AJ$.



495. *Proposed by Richard Hess, Palos Verdes, California.*

A regular pentagon is drawn on ordinary graph paper. Prove that no more than two of its vertices lie on grid points.

496. *Proposed by Donald Canard, Anaheim, California.*

P is any point within a triangle ABC, whose sides are a, b, c, whose semiperimeter is s and whose orthocenter is H. Let x denote the distance from P to BC and let R denote the circumradius of triangle ABC. Show that

$$PA^2 = PH^2 + b^2+c^2 - 4R^2 - \frac{ax}{2s}(b^2+c^2 - a^2).$$

497. *Proposed by Scott Kim, Artificial Intelligence Laboratory, Stanford University.*

Three drummers are positioned at the corners of a large equilateral triangle, say 1 mile on a side. Each drummer beats his drum at a constant rate r, with the time between beats being equal to the time it takes for the sound to travel the length of one side of the triangle. The drums are synchronized so that a listener standing in the center of the triangle would hear all three beats simultaneously. This means that it seems to each drummer that the other two drums are in synch with his own drum (actually they are delayed by one beat).

Problem: Where else can a listener stand (besides the center and corners) and hear all three drums in synchronization?

Unsolved (untried): What if the drummers beat at a rate of nr, for $n = 2,3,4,\ldots$?

Solutions

423 [Spring 1978; Spring 1979; Spring 1980] *Proposed by Richard S. Field, Santa. Monica, California.*

Find all solutions in positive integers of the equation

$A - B^D = C^C$ where D is a prime number.

Comment by the proposer.

The published solution does not generate all solutions. Counter-examples include (A,B,C,D) equal to (14, 13, 3, 2), (65, 63, 4, 2), and a general form

$$\left(\frac{(2n+1)^{2n+1}+1}{2}, \frac{(2n+1)^{2n+1}-1}{2}, 2n+1, 2\right).$$

Editorial *Comment.* Since we do not have a complete solution, this problem remains open and solutions are solicited.

438. [Spring 1979; Spring 1980; Fall 1980] *Proposed by Ernst Straus, University of California at Los Angeles.*

Prove that the sum of the lengths of alternate sides of a hexagon with concurrent major diagonals inscribed in the unit circle is less than 4.

II. *Solution by Paul Kelly, University of California at Santa. Barbara.*

Let the hexagon be ABCDEF with major diagonals meeting at P. If we keep A, B, C, D fixed and let P move on AD then we maximize EF by maximizing $\measuredangle BPC$; that is, by choosing P so that the circle $K(BPC)$ is tangent to AD.

We now want to prove that the only nondegenerate critical case is obtained for the regular hexagon (which does not give a maximum).

So, assume the hexagon is critical with

$K(BCP)$ tangent to AD
$K(DEP)$ tangent to FC
$K(AFP)$ tangent to BE

and invert on a circle centered at P. The original circle goes to some new circle containing the image hexagon $A'B'C'D'E'F'$ whose major diagonals still meet at P. The circles $K(BCP)$, $K(DEP)$, and $K(AFP)$ map to $B'C' \parallel A'D'$, $D'E' \parallel F'C'$, and $A'F' \parallel B'E'$ repectively. Such a configuration leads to $\overline{B'C'} = \overline{A'F'} = \overline{D'E'}$ and $\overline{A'B'} = \overline{C'D'} = \overline{E'F'}$, creating several isosceles trapezoids. But this is possible only if P is the center of the new circle, and hence was the center of the old circle.

So maximum is attained only for P on the boundary in which case clearly the sum is ≤ 4.

462. [Spring 1980] *Proposed by the. late R. Robinson Rowe..*

A pilot down at Aville asked a native how far it was to Btown and was told, "It's south 1500 miles, then east 1000 miles, or east 500 miles and south 1500 miles." How far was it directly?

Solution *by Morris Katz, Mocwahoc, Maine.*

Let N denote the north pole, A = Aville, B = Btown, C be 1500 miles south of A, D be 1500 miles north of B, and O the center of the earth, assumed to be a sphere of radius 3950 miles. Let planes perpendicular to ON through A and D and through B and C cut ON at Q and R respectively. Let α denote the angle between great circles NDB and NAC, so $\alpha = \angle DQA = \angle BRC$. Let β and γ be the angles of inclination from the plane of the equator of OB and OA respectively. We measure all angles in radians. We have

$$\alpha \cdot QD = 500 \text{ and } \alpha \cdot RB = 1000, \text{ so } QD = \tfrac{1}{2}RB.$$

Also

$$1500 = 3950(\gamma-\beta), \quad \text{whence } \gamma-\beta = 0.3797468.$$

Since

$$\tfrac{1}{2}\cos\beta = \cos\gamma = \cos(\beta+(\gamma-\beta)) = \cos\beta\cos(\gamma-\beta) - \sin\beta\sin(\gamma-\beta),$$

$$\tan\beta = \frac{\cos(\gamma-\beta)-\frac{1}{2}}{\sin(\gamma-\beta)} = 1.1566643,$$

so

$$\beta = 0.8579127 \quad\text{and}\quad \gamma = 1.2376596,$$

$$QD = 3950\cos\gamma = 1291.6853 \text{ mi},$$

and

$$RB = 3950\cos\beta = 2583.3706 \text{ mi}.$$

Finally

$$\alpha = \frac{500}{QD} = \frac{1000}{RB} = 0.3870912.$$

Let $\delta = \angle BOC$ and $\varepsilon = \angle AOD$. Then

$$\cos\delta = 1 - \cos^2\beta(1-\cos\alpha) = 0.9683520,$$

$$\cos\varepsilon = 1 - \cos^2\gamma(1-\cos\alpha) = 0.9920880,$$

so

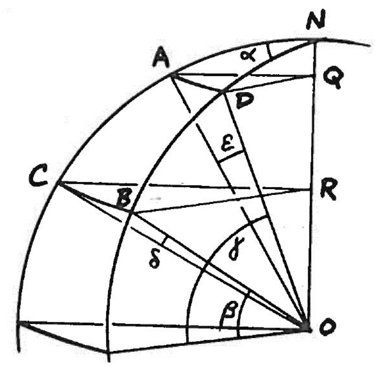
$$\delta = 0.2522552 \quad\text{and}\quad \varepsilon = 0.1258766.$$

If E is the midpoint of BC, then NEC is a right angle, whence in triangle NEC, $\cos NC = \cot\angle N \cot\angle C$; that is,

$$\cot\angle C = \cos NC \tan\angle N = \cos(\tfrac{\pi}{2}-\beta)\tan\tfrac{\alpha}{2} = 0.1482692$$

and

$$\angle C = \angle ACB = 1.4235995.$$



Now apply the law of cosines to triangle ABC to get

$$\begin{aligned}\cos c &= \cos a\cos b + \sin a\sin b\cos\angle C\\ &= \cos\delta\cos(\gamma-\beta) + \sin\delta\sin(\gamma-\beta)\cos\angle C\\ &= 0.9129345\end{aligned}$$

so $c = \angle AOB = 0.4203785$. The length of side AB is therefore given by

$$AB = 3950c = 1660.4952 \text{ mi}.$$

Also solved by MARK EVANS, IRWIN JUNGREIS, *and the Proposer.*

EVANS *and the Proposer* independently pointed out that a very accurate approximate solution is obtained by assuming $ABCD$ to be a plane trapezoid. Then

$$AB = ((1500^2 - 250^2) + 750^2)^{1/2} = 1658.3 \text{ mi},$$

an error of only 2 miles.

463. [Spring 1980] Proposed by C.S. Venkataraman, Sree Kerala Varma College, Trichur, South India.

Let $f(n)$ be a function defined over positive integers and

$$\sum_{d|n} f(d) = n.$$

Then, prove that $f(n) = \phi(n)$, the Euler's function denoting the number of integers prime to and not greater than n.

I. Solution by Michael W. Ecker, Pennsylvania State University, Scranton.

The problem really has two parts, an implicit existence portion with an explicit uniqueness assertion.

Existence -- First, ϕ is such a function; i.e. $\sum_{d|n} \phi(d) = n.$

This is fairly well-known (e.g. found on page 97 of the 2nd edition of Niven & Zuckerman's *An Introduction to the Theory of Numbers).*

Uniqueness -- This is trivial from induction or well-ordering. Let n = smallest integer for which $f(n) \neq \phi(n)$. Clearly, $n > 1$. From $f(d) = \phi(d)$ for all $d < n$ we have now

$$n = f(n) + \sum_{\substack{d|n \\ d<n}} f(d) = \phi(n) + \sum_{\substack{d|n \\ d<n}} \phi(d)$$

implying $f(n) = \phi(n)$.

II. Solution by Ferrell Wheeler, Texas A & M University.

It is well known that $\sum_{d|n} \phi(d) = n$, and we are given $\sum_{d|n} f(d) = n$ for some number-theoretic function f. Using the Möbius inversion formula for both of these equations gives us

$$\phi(n) = n \sum_{d|n} \frac{\mu(d)}{d}$$

and

$$f(n) = n \sum_{d|n} \frac{\mu(d)}{d}$$

therefore $f(n) = \phi(n)$.

Also solved by MIKE CALL, (2 solutions), MARCO A. ETTRICK, IRWIN JUNGREIS, MARK F. KRUELLE, HENRY S. LIEBERMAN, PETER A LINDSTROM, BOB PRIELIPP, GALI SALVATORI, DWIGHT SAWYER, I. PHILIP SCALISI, KENNETH M. WILKE, and *the Proposer.*

SALVATORI and WILKE offered references for the Möbius inversion formula:

1. Adams and Goldstein, *Introduction to Number Theory*, Prentice-Hall, 1976, Ex. 14, p. 152.
2. Carmichael, *Theory of Numbers*, Dover, p. 32.
3. Niven and Zuckerman, *Introduction to the Theory of Numbers*, 3rd ed., John Wiley and Sons, pp. 86, 88.

464. [Spring 1980] *Proposed by Solomon W. Golomb, University of, Southern California, Lob Angeles.*

For all positive integers a and b with $1 < a < b$, show that $(a!)^{b-1} < (b!)^{a-1}$.

Solution by Robert A. Stump, Hopewell, Virginia.

Clearly $a! \leq a^{a-1}$. Let $b = a + k$ (k a positive integer), then

$$\begin{aligned}(a!)^{b-1} &= (a!)^{a+k-1} \\ &= (a!)^{a-1}(a!)^k \\ &\leq (a!)^{a-1}(a^{a-1})^k \\ &< (a!)^{a-1}[(a+1)^{a-1}(a+2)^{a-1}\ldots(a+k)^{a-1}] \\ &= [(a+k)!]^{a-1} \\ &= (b!)^{a-1}\end{aligned}$$

Also solved by MIKE CALL, MARK EVANS, MARTIE FIELDS, SAMUEL GUT, IRWIN JUNGREIS, MORRIS KATZ, ZELDA KATZ, JAMES A. PARSLY, BOB PRIELIPP. DWIGHT SAWYER, JEFF SHALLIT, FERRELL WHEELER, *and the. Proposer.*

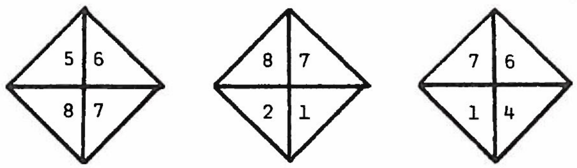
465. [Spring 1980] *Proposed by Charles W. Trigg, San Diego, California.*

What is the shortest strip of equilateral triangles of side k that, while remaining intact. can be folded along the sides of the triangles so as to completely cover the surface of an octahedron with edges k?

Solution by Was. Call, Rose-Hulman Institute of Technology, Terre Haute, Indiana.

In analysis of this problem, a strip of twelve triangles, numbered 1 through 12, may be useful. Also label an octahedron as shown in Figure 1

If the strip is not folded back upon itself. at most 6 sides of the octahedron may be covered; e.g., sides 1,2,3,5,6,7,1,2,3,... . The analysis may be broken down casewise according to when in the 6-cycle the strip is folded upon itself. Folding back after the 1st triangle is useless. Foldingbackafter the 3rd triangle results in 1,2,3,3,4,... with the possible choices of repeating side 1, or folding back on side 4. Hence, at least 10 triangles are necessary. Folding back after the 4th results in 1,2,3,5,5,8,... with the choices of repeating side 2 or folding back upon side 8. Again, at least 10 choices are necessary. Folding back after the 5th triangle results in 1,2,3,5,6,6,4,... with the choices of repeating 3 or folding back on side 4. Again, at least 10 are necessary. Folding back after the 6th triangle results in 1,2,3,5,6,7,7,8,... with the choices of repeating 5, or folding back upon 8. Folding back after the second triangle, however, results in 1,2,2,3,4,7,8,5,5,6 which uses exactly 10 triangles. Having covered all possible cases, it may be said that the minimal strip length is 10.



Top View Front View Right Side

FIGURE 1

Comment by the Proposer.

The 8-triangle strip of Figure 2 will cover the surface of a concave octahedron which polyhedron can be formed from three regular tetrahedrons with a common edge. Fold over the strip so that side a coincides with b and f with g, then bring i into contact with d. This will cause c and j and e and h to coincide, thus completing the surface of a concave octahedron with an 8-triangle strip.

Also solved by the Proposer. Solution of length 11 by RALPH KING.

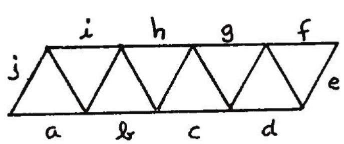
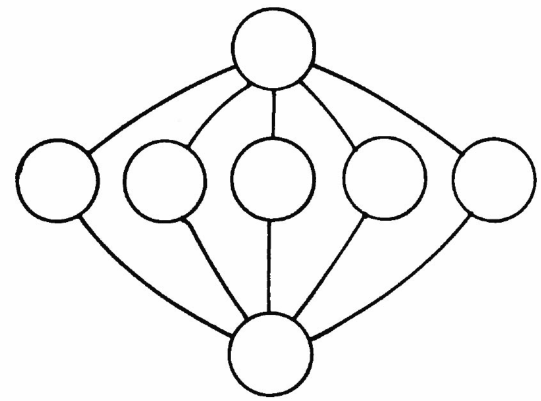


FIGURE 2

466. [Spring 19801 *Proposed by Herbert Taylor, South Pasadena, California.*

Let the adversary put four distinct symbols in each box (node) of this graph. Prove or disprove: No matter what pattern of symbols he puts, we can choose two symbols from each box in such a way that adjacent boxes have disjoint chosen 2-sets.



Only one (incorrect) solution has been received from our readers, so we extend the deadline and encourage you to submit solutions to this problem.

467. [Spring 19801 *Paopobed by Paul Erdös, Spaceship Earth, and John L. Selfridge, University of Michigan.*

Determine the greatest power which divides $n!$ Prove that for $n \geq 6$ it is a square.

1. *Solution by Jean E. Ezell, University, Mississippi, and Irwin Jungreis, No. Woodmere, New York, independently.*

As stated, the problem is incorrect. Consider

$$21! = 2^{18} \cdot 3^{9} \cdot 5^{4} \cdot 7^{3} \cdot 11 \cdot 13 \cdot 17 \cdot 19.$$

The greatest power which divides $21!$ is $h = (2^6 \cdot 3^3 \cdot 5 \cdot 7)^3$ and the greatest square which divides $21!$ is $(2^9 \cdot 3^4 \cdot 5^2 \cdot 7)^2 = \frac{5}{21} h$.

II. *Solution bq Kenneth M. Wilke, Topeka, Kansas.*

Counterexample: For $n = 21$, $21! = 51090942171709440000 = 2^{18} \cdot 3^9 \cdot 5^4 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 19 = (7257600)^2 \cdot 21 \cdot 46189 = (60480)^3 \cdot 5 \cdot 46189$, where $46189 = 11 \cdot 13 \cdot 17 \cdot 19$ and $(60480)^3 =$

221225582592000 > 52672757760000 = $(7257600)^2$. Here $60480 = 2^6 \cdot 3^3 \cdot 5 \cdot 7$ which is the root of the largest cube which divides 21!. Likewise $7257600 = 2^9 \cdot 3^4 \cdot 5^2 \cdot 7$ which is the root of the largest square which divides 21!.

Editor's Comment. See problem **493**, proposed in this issue of the *Pi Mu Epsilon Journal*.

468. [Spring 1980] *Proposed by Michael W. Ecker, Pennsylvania State University, Worthington Scranton Campus.*

A priori, the expression a^{b^c} is ambiguous in that it would mean either $(a^b)^c$ or $a^{(b^c)}$. Assuming a, b, and c are positive integers, find all triples (a, b, c) for which the two expressions are equal.

Solution by Ferrell Wheeler, Texas A & M University, Beaumont.

The first obvious set of solutions is $(1, b, c)$. Now for $a > 1$ $a^{bc} = a^{(b^c)}$ implies $bc = b^c$ which implies $c = b^{c-1}$ since $b > 0$. When $c = 1$ this is always true, therefore another set of solutions is $(a, b, 1)$. If $c = 2$, then $b = 2$, therefore the third set of solutions is $(a, 2, 2)$. For $b, c > 2$, $b^{c-1} > c$, therefore all of the solutions are given by

$$(1, b, c),\ (a, b, 1),\ \text{and}\ (a, 2, 2).$$

Also solved by MIKE CALL, MARK EVANS, ROBERT C. GEBHARDT, SAMUEL GUT, IRWIN JUNGREIS, BOB PRIELIPP, JEFF SHALLIT, CHRIS THOMAS, CHARLES W. TRIGG, KENNETH M. WILKE, *an unsigned solver, and the Proposer.*

469 [Spring 1980] *Proposed by Richard I. Hub, Palos Verdes, California.*

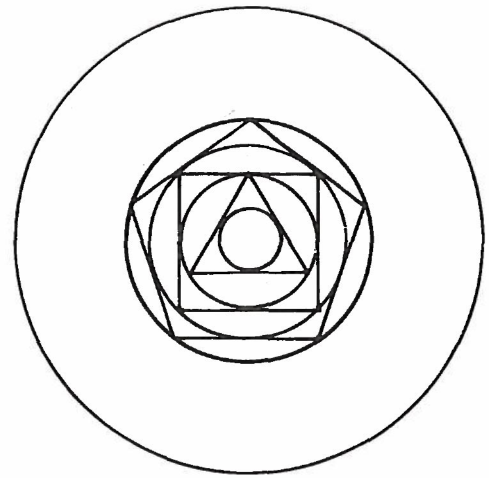
Start with a unit circle and circumscribe an equilateral triangle about it. Then circumscribe a circle about the triangle and a square about the circle. Continue indefinitely circumscribing circle, regular pentagon, circle, regular hexagon, etc.

a) Prove that there is a circle which contains the entire structure.

* b) Find the radius of the smallest such circle.

1. *Solution to part (a) by Michael W. Eckm, Scranton, Pennsylvania.*

Let r_n = radius of incircle of the n-gon (for $n = 3,4,5,\ldots$). then r_{n+1} = radius of circumcircle of the n-gon. We also have



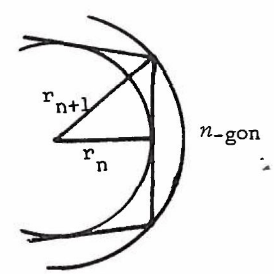
$$\cos\frac{\pi}{n} = \frac{r_n}{r_{n+1}}.\quad \text{Thus,}\quad r_{n+1} = \sec\frac{\pi}{n}\cdot r_n = \sec\frac{\pi}{n}\cdot\sec\frac{\pi}{n-1}\cdot r_{n-1} = \ldots$$

$$= \left(\prod_{k=4}^{n}\sec\frac{\pi}{k}\right)\cdot r_4.$$ It suffices to show that $\lim_{n\to\infty}\prod_{k=4}^{n}\sec\frac{\pi}{k}$ exists.

For this, we need only show that $\sum_{k=4}^{\infty}\left(\sec\frac{\pi}{k} - 1\right)$ converges. This is so by comparison: $0 < \sec\frac{\pi}{k} - 1 < \frac{\pi^2}{k^2}$. This last inequality may be verified as follows:

$$\frac{1}{\cos\frac{\pi}{k}} = \frac{1}{1 - \frac{\pi^2/k^2}{2!} + \varepsilon} < \frac{1}{1 - \frac{\pi^2/k^2}{2!}} < 1 + \frac{\pi^2}{k^2} \quad \text{where } \varepsilon > 0$$

because $1 < \left(1 - \frac{\pi^2/k^2}{2}\right)\left(1 + \frac{\pi^2}{k^2}\right)$.



II. *Solution to part (b) by Harry L. Nelson, Journal of Recreational Mathematics, Livermore, California.*

By computer, we find that r_∞ is 8.70005 ± .00005.

Also solved by MIKE CALL, ($r_\infty \approx 8.69999$ ...), IRWIN JUNGREIS, *part [a] only*, HARRY L. NELSON, JOYCE W. WILLIAMS (8.70001 < r_∞ < 8.70004), *and the Proposer* ($r_\infty \approx 8.700036$ ...).

470. [Spring 1980] *Proposed by Tom Apostol, California lmLiAf.de of Technology.*

Given integers $m > n > 0$. Let

$$\alpha = a\sqrt{m} + b\sqrt{n}$$

$$\beta = c\sqrt{m} + d\sqrt{n}$$

where a, b, c, d are rational numbers.

(a) If $ad + bc = 0$ or if mn is a square, prove that both α and β are rational or both are irrational.

(b) If $m = r^2$ and $n = s^2$ for some pair of integers $r > s > 0$ then α and β are both rational. Prove that the converse is also true if $ad \neq bc$.

I. *Disproof by Mike Call, Rose-Hulman 1n4LiAf.de of Technology, Terre tfaf.de, Indiana, and Bob Prielipp, University of Wisconsin-Oshkosh, independently.*

Let $m = 8$, $n = 2$, $a = c = 1$, $b = -2$, $d = 2$. Then

$$0 = \alpha = 1\sqrt{8} - 2\sqrt{2} \quad \text{and} \quad 4\sqrt{2} = \beta = 1\sqrt{8} + 2\sqrt{2},$$

even though $mn = 16$ and $ad + bc = 1(2) + 1(-2) = 0$. This shows that neither of the aforementioned conditions is sufficient for the conclusion.

II. *Solution by Ferrell Wheeler*, Texas A & M University, College Station.*

(a) Multiplying the two equations together we have

$$\alpha\beta = acm + bdn + (ad + bc)\sqrt{mn}. \qquad (1)$$

If either $(ad + bc) = 0$ or mn is a square then the right side of this equation is rational. This immediately implies that if $\alpha\beta \neq 0$ then α and β are both rational or both are irrational.

(b) We shall assume α and β are nonzero rational numbers. Since $ad \neq bc$, either ad or bc is nonzero. Without loss of generality, assume ad is nonzero which implies a and d are both nonzero. Now

$$(\alpha - a\sqrt{m})^2 = (b\sqrt{n})^2$$

or

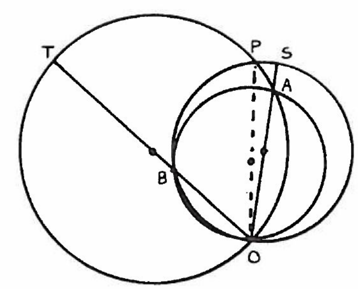
$$\alpha^2 - 2a\alpha\sqrt{m} + a^2m = b^2n$$

which implies $\sqrt{m} = (a^2m - b^2n + \alpha^2)/2a\alpha$ is a rational number. Therefore m must be a square number, say r^2. Now from equation (1) mn is a square which implies that n is a square, say s^2. Since $m > n > 0$, then we may take $r > s > 0$.

The following additional solvers who also discovered the disproof are indicated by an asterisk: MIKE CALL*, DAVID DEL SESTO, MARCO A. ETTRICK, ROBERT A. FULLER, *(part (a))*, HENRY S. LIEBERMAN, BOB PRIELIPP*, *and the Proposer.*

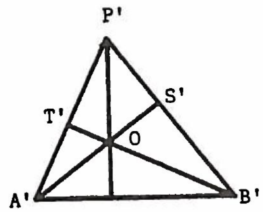
471. [Spring 1980] *Proposed by Clayton W. Dodge, University of Maine at Orono.*

Let two circles meet at O and P, and let the diameters OS and OT of the two circles cut the other circle at A and B. Prove that chord OP passes through the center of circle OAB.



Solution by Sister Stephanie Sloyan, Georgian Court College, Lakewood, New Jersey.

This problem is solved completely in the text (p. 88) of Shively, *Modern Geometry* (Wiley, 1939) by means of inversion. With appropriate changes of letters the solution follows: Invert the figure with O as center of inversion, and let P', A', and B' be the inverses of P, A and B respectively. Each of the lines PO, TO, and SO inverts into itself while the circles through APT, BPS, and AOB invert into the lines $P'A'$, $P'B'$, and $A'B'$ respectively. Moreover, since a diameter intersects its circle orthogonally, $A'O$ and $B'O$ are, by the conformal property of inversion, altitudes of triangles $P'A'B'$; hence $P'O$ is perpendicular to $A'B'$. Therefore PO is orthogonal to the circle OAB, from which it follows that it passes through the center of this circle.



Also solved by MIKE CALL, MANGHO AHUJA, *and the Proposer*.

CALL's solution was by analytic geometry, placing O at the origin and OAS along the x-axis. He showed the point of intersection of the perpendicular bisectors of OA and OB satisfied the equation of line OP.

472. [Spring 1980] *Proposed by R. S. Luthar, University of Wisconsin Center, Janesville.*

Evaluate $\displaystyle\int \frac{5}{16 + 9\cos^2 x}\,dx$

Solution by Yuan-Whay Chu, Janesville, Wisconsin

We have

$$\int \frac{5\,dx}{16 + 9\cos^2 x} = 5\int \frac{dx}{25\cos^2 x + 16\sin^2 x}$$

$$= 5\int \frac{\sec^2 x\,dx}{25 + 16\tan^2 x}$$

$$= \frac{1}{4}\int \frac{d(\frac{4}{5}\tan x)}{1 + (\frac{4}{5}\tan x)^2}$$

$$= \frac{1}{4}\arctan(\tfrac{4}{5}\tan x) + C.$$

Also solved by MANGHO AHUJA, MIKE CALL, MARCO A. ETTRICK, VICTOR G. FESER, ROBERT C. GEBHARDT, IRWIN JUNGREIS, RALPH KING, GUS MAVRIGIAN, BOB PRIELIPP, [illegible]. PHILIP SCALASI, FERRELL WHEELER, BARTON L. WILLIS, *and the Proposer*.

In addition to CHU, GEBHARDT used the substitution $u = \frac{4}{5}\tan x$, CALL and PRIELIPP used $u = \tan x$, SCALISI used $u = \frac{5}{4}\cot x$, FESSER, WILLIS, and the *Proposer* used $u = \tan(x/2)$, JUNGREIS used the substitution $\frac{4}{5}\tan x = \tan u$, and the other solvers used tables or undisclosed procedures.

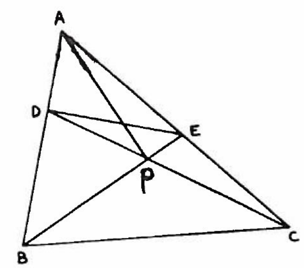
473. [Spring 1980] *Proposed by Jack Garfunkel, Forest Hills High School, Flushing, New York.*

In an acute triangle ABC with angle $A = 60^\circ$, P is a point within the triangle. D and E are the feet of the Cevians through P from C and B respectively.

a) If $BD = DE = EC$, prove that $AP = BP = CP$.

b) Conversely, if $AP = BP = CP$, prove that $BD = DE = EC$.

c) If angle PBC = angle $PCB = 30^\circ$, show that $BD = DE = EC$.



I. Solution to part (a) by Zelda Katz, Beverly Hills, California.

If $BD = DE = EC$, then

$$\measuredangle PCE + \measuredangle PBD = \measuredangle PDE + \measuredangle PED = \measuredangle PBC + \measuredangle PCB$$

since triangles DBE and EDC are isosceles and triangles PDE and PBC share vertical angles at P. Therefore

$$\measuredangle PBC + \measuredangle PCB = \tfrac{1}{2}(\measuredangle ABC + \measuredangle ACB) = 60^\circ$$

and $\measuredangle BPC = 120^\circ$. Thus A, D, P, and E are concyclic, whence

$$\measuredangle PAE = \measuredangle PDE = \measuredangle ECD.$$

Now triangle PAC is isosceles, so $AP = CP$. Similarly $AP = BP$ and the theorem follows.

II. Solution to parts (b) and (c) by Irwin Jungreis, North Woodmere, New York.

(b) Since, by part (a), if D and E exist with $BD = DE = EC$, then they are the feet of the cevians through the circumcenter and part (b) is proved, all we need show is that in every acute ΔABC with $\measuredangle A = 60^\circ$, there exist D and E on AB and AC respectively, with $BD = DE = EC$.

First, since $\measuredangle C < 90^\circ$ and $\measuredangle B > 30^\circ$ (because $\measuredangle A = 60^\circ$), we know $AB < 2AC$ and similarly $AC < 2AB$. Say $AB \le AC$. Let D_x be the point between A and B a distance x from B and E_x be the point between A and C a distance x from C. Let $f(x) = BD_x - D_xE_x$. Then $f(0) = BC < 0$ and $f(AB) = AB - (AC - AB) = 2AB - AC > 0$, so by continuity there is a point D between A and B with $f(BD) = 0$. That is the desired point.

(c) If $\measuredangle PBC = \measuredangle PCB = 30^\circ$ then $\measuredangle BDC = 120^\circ$ and $OB = OC$. Draw the circle centered at 0 and through B and C. Since arc $BC = 120^\circ$ and $\measuredangle BAC = 60^\circ$, then A is on the circle, and $AP = BP = CP$. Now apply part (b).

Complete solutions were submitted by both solvers above and solutions to parts (a) and (b) by the *Proposer.*

PROBLEMATICAL POSTSCRIPT

The following note was received from ROBERT C. BROWN *and* PAUL M. RIGGS *of the Southeastern Louisiana University::*

Please refer to the comment on page 206 of Volume 7, Fall 1980, by Miss Amanda B. Reckundwith.

On page 298 of the Historical and Biographical Notes of N. Altshiller Court's *College Geometry,* Second Edition 1952, is the following quote concerning the proposition 175.

The orthocenter -*"The three altitudes of a triangle are concurrent."* 175. The proposition is not included in the Elements of Euclid. It is found in the writings of Archimedes (287-212 B.C.) in an indirect form, and explicitly in Proclus (410-485), a commentator of Euclid.

PUZZLE SECTION

David Ballew

This department is for the enjoyment of those readers who are addicted to working crossword puzzles or who find an occasional mathematical puzzle attractive. We consider mathematical puzzles to be problems whose solutions consist of answers immediately recognizable as correct by simple observation and requiring little formal proof. Material submitted and not used here will be sent to the Problem Editor if deemed appropriate for that department.

Address all proposed puzzles and puzzle solutions to David Ballew, Editor of the Pi Mu Epsilon Journal, Department of Mathematical Sciences, South Dakota School of Mines and Technology, Rapid City, South Dakota, 57701. Deadlines for puzzles appearing in the Fall issue will be the next February 15, and puzzles appearing in the Spring issue will be due on the next September 15.

Mathacrostic No. 12

submitted by Joseph D. E. Konhauser

Macalester College, St. Paul, Minnesota

Like the preceeding puzzles, this puzzle (on the next page) is a keyed anagram. The 215 letters to be entered in the diagram in the numbered spaces will be identical with those in the 26 keyed words at matching numbers, and the key letters have been entered in the diagram to assist in constructing your solution. When completed, the initial letters will give a famous author and the title of his book; the diagram will be a quotation from that book.

Cross-Number Puzzles

submitted by Mark Isaak

Student, University of California, Berkeley

In the cross-number puzzles (starting two pages hence), each of the letters stands for a positive, nonzero integer. The algebraic expressions evaluate out to two to five digit numbers which fit in the squares as in a normal crossword puzzle. None of the numbers in the squares have any leading zeros; i. e., if there is room for a four digit number, that number will be at least 1000, never, for example, 0999.

1 G	2 Z	3 P		4 T	5 J	6 D	7 A	8 B	9 X	10 Q	11 H	12 C	13 M
14 U	15 Z		16 O	17 W	18 N		19 F	20 D	21 Y		22 X	23 A	24 Z
25 U	26 J	27 K	28 C		29 S	30 F	31 I	32 V	33 B		34 C	35 S	36 M
37 P		38 D	39 W	40 R	41 V	42 N		43 H	44 L	45 E		46 Q	47 U
48 X	49 A	50 Y	51 C		52 K	53 V	54 D	55 J	56 S	57 F		58 M	59 T
60 C	61 A	62 O		63 P	64 V		65 Y	66 O	67 I	68 Z		69 R	70 Q
71 W	72 D	73 M	74 C	75 Z		76 A	77 B	78 D	79 J		80 O		81 K
82 D	83 P		84 N	85 G	86 B	87 F	88 C		89 W	90 S	91 J	92 X	93 P
	94 M	95 E		96 S	97 B	98 Y	99 W	100 H	101 M	102 K		103 Z	104 R
105 U	106 E	107 H		108 W	109 P		110 D	111 Q	112 N	113 T	114 C	115 G	
116 I	117 H	118 P	119 D	120 B	121 V	122 R	123 U	124 L	125 O		126 V	127 S	128 F
	129 X	130 J	131 E	132 Z	133 K		134 F	135 D	136 A	137 I		138 T	139 M
140 B	141 J	142 L	143 E		144 O	145 N	146 Z	147 C		148 Q	149 Y		150 F
151 H	152 N		153 A	154 T		155 S	156 P	157 Z	158 D	159 J	160 E	161 I	162 B
163 U	164 L	165 X	166 F	167 R		168 A	169 C	170 H	171 T	172 M		173 P	174 J
175 A		176 U	177 F	178 X	179 C	180 L		181 B	182 Y	183 D	184 Q	185 K	186 G
187 I		188 E	189 S		190 U	191 P	192 O	193 M	194 K	195 C	196 Y		197 Q
198 X	199 T		200 W	201 R	202 U		203 Y	204 T	205 Z		206 K	207 G	208 X
209 C	210 Y	211 F	212 V	213 S	214 P	215 M							

A. in Euchre, the other jack of the same color as the trump suit jack (2 **wds.**) — 61 175 168 76 153 49 136 7 23

B. ordered formal calculation — 86 162 33 120 8 97 140 77 181

C. "The truth of a theory is in your mind, _____ . A. Einstein (4 **wds.**) — 195 169 34 12 147 51 114 60 209 74 28 179 88

D. fold, cusp, swallowtail, butterfly, wigwam — 38 183 158 78 6 54 119 135 110 20 82 72

E. first to prove the well-ordering theorem **(1871-1953)** — 131 95 143 160 45 106 188

F. dual of a cube — 166 211 19 150 30 57 134 87 128 177

G. a piece in backgammon — 115 1 207 186 85

H. pseudonym of Guiness brewery chemist, W. S. Gossett **(1876-1937)**, known for his distributions of t, not beer — 117 11 170 107 100 151 43

I. a filling of the plane without gaps or overlaps — 161 116 67 31 137 187

J. where Archimedes formulated his law of hydrostatics **(3 wds.)** — 26 79 55 174 159 5 130 91 141

K. British geometer **(1845-1879)**, who, in 1876, suggested there is a relationship between matter and curvature — 206 52 185 81 102 194 27 133

L. medium postulated to carry electromagnetic waves — 124 164 44 142 180

M. what a sphere needs to become homeomorphic to a double torus (2 **wds.**) — 193 58 13 94 139 215 172 101 36 73

N. star cluster in the face of the Bull — 145 84 112 152 18 42

O. a statistician's bread and butter (2 **wds.**) — 125 66 192 62 80 144 16

P. " _____ . Infinity"; title of **Gamow's popular** exploration of science (3 **wds.**) — 214 37 191 118 83 156 173 63 93 3 109

Q. howling; wailing — 148 111 70 197 10 184 46

R. ancient shadow clock; carpenter's square — 69 167 122 40 104 201

S. easily unfastened knot (2 **wds.**) — 127 90 56 189 35 213 29 155 96

T. Julian's Bower; labyrinth **(comp.)** — 113 138 4 154 204 59 199 171

U. "Mathematics is the glory of the _____ . G. Leibniz (2 **wds.**) — 47 105 123 163 190 25 176 14 202

V. twist together — 41 32 212 126 53 121 64

W. curve known as the shoemaker's knife — 200 17 108 71 89 39 99

X. a communication channel of sorts — 129 208 198 22 48 9 165 178 92

Y. table of assigned places of a celestial body for regular intervals — 182 149 65 21 98 210 50 203 196

Z. Cistercian monk **(c.** 1345) who was called "the calculator" — 15 103 24 157 132 205 2 68 146 75

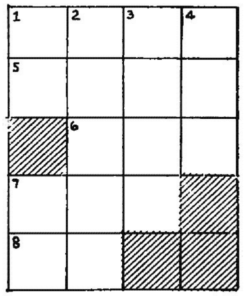
#1

ACROSS

1. AB
5. A^2
6. C
7. 3A - 18
8. $(1/9)A^2 - D$

DOWN

1. A/E
2. B^3
3. $E^6 + 1000$
4. BC/89
7. E^2



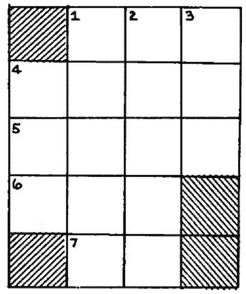
#2

ACROSS

1. 5AB/91
4. 66C + 82
5. $(\sqrt{11B} + 22)^2$
6. D
7. (7D + 208)/20

DOWN

1. 5AB
2. 619(B + C + E) - 30
3. C + 7E
4. 6C + 3



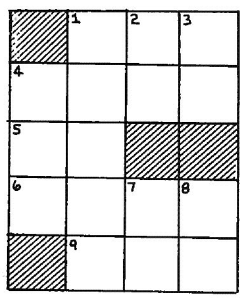
#3

ACROSS

1. A^2
4. BC
5. 63C/5
6. $3D\sqrt{5E}$
9. B

DOWN

1. 10(B + F) - 2
2. E
3. G
4. F
7. $\sqrt{E} - \sqrt{G} + H$
8. D



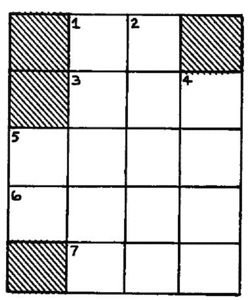
#4

ACROSS

1. $A^4 + BC + C$
3. D
5. $(E + 1)^4F$
6. $((C/B) + A)^G$
7. H(A + G)(C - 2A)

DOWN

1. J - DK
2. 2J + 20
4. GHK
5. B + C



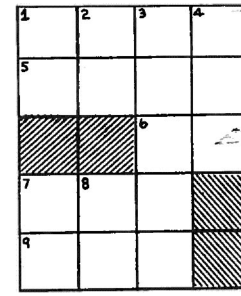
#5

ACROSS

1. A
5. A + B
6. CD + (E/D)
7. F/2 - 1
9. (A + 7)/D

DOWN

1. 3C
2. E
3. DFG
4. B
7. D + (D/10)
8. G



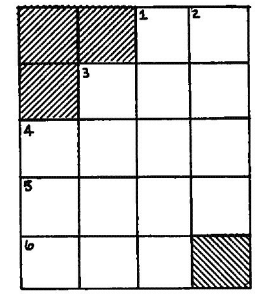
#6

ACROSS

1. A
3. B
4. $C^2 + BD - 43$
5. DE/2
6. F

DOWN

1. G^{H-F}
2. H + 4
3. J
4. E



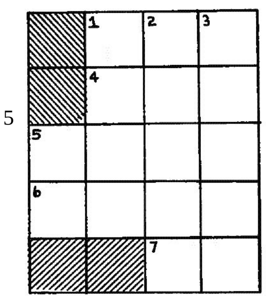
#7

ACROSS

1. 21A
4. $(2B - 1)^2$
5. $(C-17453)^2/20$
6. $(2B-1)^4-100D+4$
7. A

DOWN

1. 59E
2. $5C - B^3 + 17B^2 - 66B + 5$
3. 11F
5. (E + 5)/G



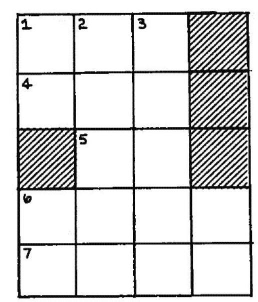
#8

ACROSS

1. $(A - 32)^2$
4. (158/131)B
5. C
6. A^2
8. 67(3D - 66)

DOWN

1. 10EF + 3
2. 1331G + 1
3. 4BCH
6. $F^4 + 10F^2J + 13F$
7. G



ERRORS AND MISTAKES

In the Cross-Number Puzzles published in the Fall 1980 Issue, there were several minor but crucial mistakes. These puzzles are republished as #1 through #4 on the preceding page., and the errors are marked with the arrows. Because of, the mistakes, solutions to these puzzles will be accepted for the Fall 1981 Issue of, this Journal. Sorry!

SOLUTIONS

Mathacrostic No. 11 (See Fall 1980 issue) (**Proposed by J. D. E. Konhauser**)

Definitions and Key:

A. Dissonant	H. Asterism	O. Dichotomy	V. Events
B. Risque	I. Dipole	P. Erlanger	W. Rhenish
C. Hebetate	J. Twisted cubic	Q. Litotes	X. Berkeley
D. Occultation	K. Elation	R. Empirical	Y. Atom smasher
E. Fixed point	L. Ramiform	S. Shift	Z. Catenate
F. Systole	M. Granny	T. Chladni	a. Halfway
G. The Quiz Kids	N. Ocarina	U. Hints	

First Letters: Dr Hofstadter Godel Escher Bach

Quotation: Besides, the drive to eliminate paradoxes at any cost, especially when it requires the creation of, highly artificial formalisms, puts too much stress on bland consistency, and too little on the quirky and bizarre, which make life and mathematics interesting.

Solved by: Jeanette Bickley, Webster Groves High School, Missouri; Louis H. Cairoli, Kansas State University; Victor Feser, Mary College; Robert Gebhardt, Hopatcong, N.J.; Roger E. Kuehl, Kansas City; Henry S. Lieberman, John Hancock Mutual Life Insurance Co.; D. C. Pfaff, Univ. of Nevada-Reno; Robert Prielipp, Univ. of Wisconsin-Oshkosh; John Oman, Univ. of Wisconsin-Oshkosh; Chris Thomas, Rose-Hulman Institute of Technology; The Editor and The Proposer. Victor Feser also included a solution to Mathacrostic No. 10.

Who Stole the Candy? (See Fall 1980 issue), **Proposed by Wayne M. Delia and Bernadette D. Barnes.**

Solution: If Ivan was the thief, he would have had to tell two lies (#1 and #3). Similarly, Ernie, Dennis, and Linda could not have been the thief. Hence, Sylvia is the thief, and the lies are #3, #4, #9, #11 and #15. This solution was presented by John Wesley Emert of Knoxville, Tenn.

Also Solved by: Kathy Ames; Jeanette Bickley, Webster Groves High School, Missouri; Victor Feser, Mary College, Bismark, N.D.; Robert C. Gebhardt, Hopatcong, N.J.; Samuel Gut, Brooklyn, N.Y.; John Kahila, Univ. of Washington; Roger E. Kuehl, Kansas City (Who noted that Sylvia's three statements and Dennis' three statements are superfluous); Sarah Lieberman, 8th grade, Meadowbrook Jr. High, Newton, Mass.; D. C. Pfaff, University of Nevada-Reno; The Editor and The Proposers.

PI MU EPSILON CONFERENCE
MIAMI UNIVERSITY
OXFORD, OHIO

Sept. 25, 26, 1981

The Ninth Annual Pi Mu Epsilon Conference of Miami Univ. will be September 25 and 26 at the Miami Campus in Oxford, Ohio. Papers are welcome on items hanging from expository to research, interesting applications, problems, etc. Presentation time should be from 15 to 30 minutes. Send abstracts (by Sept. 12) to:

Professor Milton Cox
Department of Mathematics and Statistics
Miami University
Oxford, OH 45056

HIGH SCHOOL MATHEMATICS CONTESTS

Many Pi Mu Epsilon Chapters either sponsor or contribute services to contests, competitions and "Math Days" among high school students. The Editor's office can act as an information source and a clearing house to swap examinations and ideas. If your Chapter is involved, please send the Editor an outline of how your examinations are conducted and send copies of your materials, about 25 copies of advertisements, brochures, and if possible, your examinations for swapping purposes. The Editor will send you copies of the materials on hand. If you are starting such and event or thinking about it, let the Editor know and materials can be sent to you. This could be a great help to us all!!

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POSTERS AVAILABLE FOR LOCAL ANNOUNCEMENTS

We have a supply of 10 x 14-inch Fraternity Crests available. One in each color will be sent free to each local chapter on request. Additional posters may be ordered at the following rates:

(1) Purple on goldenrod stock------------------$1.50/dozen,

(2) Purple on Lavendar on goldenrod------------$2.00/dozen.

CHAPTER REPORTS

FLORIDA EPSILON (UNIVERSITY OF SOUTH FLORIDA) The Chapter had an active program which included the following speakers and their talks: ***Christina Patterson***, *"Frames, A Knowledge Representation and Organization System for a Computer"*; ***Dr. Allan Wayne*** (Pasco-Hernandez Community College), *"A Census of Natural Triangles?* ***Prof. Athanasois Kartsatos***, *"Monotonicity in R^n"*; ***Dr. Joseph Carr***, *"Parade of Planets"*; ***Craig Hubbard***, *"Just What is the Gross National Product?"*; ***Dr. James Bell***, *"Mathematics As An Empirical Science"*; ***Dr. Richard Stark***, *"Exotic Constructions Using Baire's Theorem"*; ***Dr. Kenneth Pothoven***, *"A Little Bit About Pi"*; ***Paul Artola***, *"Musical Harmony and Mathematics"*; ***Dr. James Feller***, *"Mathematical Applications in Finance"*; ***Dr. Nicholas Passell***, *"An Arc Length Problem"*; ***John R. Kenyon***, *"Dynamic Programming"*, ***Alan Craig***, *"Logical Puzzles and Paradoxes in Mathematics"*; ***Dr. Willam Clark***, *"Using Number Theory to Construct Secret Codes"*. In addition the Chapter sponsored a Film Festival and hosted the Mu Alpha Theta Math Bowl, a competition amoung local high schools.

MISSOURI GAMMA (ST. LOUIS UNIVERSITY) The Chapter hosted (in conjunction with MARYVILLE COLLEGE OF ST. LOUIS) a **Pi Mu** Epsilon Conference with invited speaker ***Professor Robert V. Hogg*** of the University of Iowa. Professor **Hogg's** talk was *"Statistics, Actuarial Science and the Future"*. Information on Student papers will be in the Fall Issue.

MONTANA ALPHA (UNIVERSITY OF MONTANA) The Chapter heard the following talks: ***Stephen Eberhart***, *"Seeing the Imaginary"*; ***Peggy Krull-Bordewick***, *"Computers Applied to Satellite Photography"*, ***Roman Cakon***, *"Mathematics Graduate Training in Germany: A Personal View"*; and ***Robert Hollister***, *"African Mathematics-Counting and Measurements"*. In addition there was a film program and a special talk by ***Professor Kenneth Yale*** on *"Unorthodox Programming Features of the TI-58"*. The John Peterson Book Award for the outstanding graduating senior in mathematics education was presented to ***Brad Simshaw***.

NEW YORK ALPHA ALPHA (QUEENS COLLEGE OF CUNY) The Chapter sponsored several film programs, parties and other activities. ***Robbin Bura*** was given the first annual Pi Mu **Epsilon Prize** for excellence in mathematics and service to the Chapter.

OHIO DELTA (MIAMI UNIVERSITY) chapter activities included paper presentations at the Pi **Mu** Epsilon Conference of Sept. 28. Eight of the thirteen papers were given by Miami students. Papers contributed included: ***Patty Brueneman***, *"The Penrose Tiling in Bachelor Courtyard"*; ***Steve Ruberg***, *"The Tree Planting Problem"*; ***Kathy Reynolds***, *"Testing the Associativity of a Binary Operation: Reducing the Tedium"*; ***Tom Pastuszak***, *"Finite Projective Planes"*; ***Neil Gandal***, *"The Use of Statistics in Welfare Reform"*, ***Van Fremion***, *"Pentominoes"*; ***Chris Hawkins and Lee Josvanger***, *"So You Want to be a Systems Analyst?"*; and ***Jeff King***, *"A First Step Toward a Mathematical Theory of Ecology"*. In addition two Chapter Members gave papers at the **MAA** Section Meeting. They were: ***Neil Gandal***, *"The Use of Statistics in Welfare Reform"*: ***Bud Kostic***, *"Frucht's Theorem--An Example of the Interplay of Graph Theory* ***and*** *Algebra"*. Regular Chapter presentations included: ***Steve Ruberg***, *"Magic Squares"*, (See this Issue of the Journal); ***Dr. Schaefer***, *"Biostatistics"*; ***Dr. Clyde Hardin***, *"Random Walk* ***and*** *Gambler's Ruin"*; ***Elizabeth Roberts***, (Systems Engineer from Armco Steel), *"Mathematics, Statistics,* and *Systems Analysis at Armco"*; ***Mr. Mike Miller***, Actuary from State Farm Mutual, *"Actuarial Science* and *Job Opportunities"*: and ***Dr. Dunn***, *"Statistical Discrimination"*. The winners of the Pi Mu Epsilon Examination were ***Scott Buckman*** and ***Donna Ford***. There were ongoing discussions at the College Inn.

Talks at the Seventh Annual Pi **Mu** Epsilon Student Conference held at Miami University on Sept. 26, 27, 1980 were:

Karen Zielke (Miami Univ.)	*Data Collection Techniques at the Federal Reserve Board*
Gheg Taylor (Miami Univ.)	*Focus Forecasting*
Julie Griswold (Miami Univ.)	*Geometrical* ***Ideas*** *in Kaleidoscopes*
Cathryn Hallett (Oakland Univ.)	*Linear Generative Grammars or How to Make a One Stringed Harp*
Bruce Bullis (Miami Univ.)	*25 Points*
Cecil Ellard (Miami Univ.)	*Sequences of Integers With a Limited Number of* ***Prime Divisors***
Van Fremion (Miami Univ.)	*Designs With Hexagons*
Brenda Rood (Miami Univ.)	*Curve Stitching for Junior High*
Beverly Skeans (Marshall Univ.)	*Correctness Proofs for Flowchart Programs*
Daren Wilson (Oakland Univ.)	*Fractional Arithmetic--How to Make Two and Two Equal 4.520360*
Mark Bates (Rose-Hulman)	*Individual* ***Game*** *Probabilities Concerning the World Series*

Michael Call (Rose-Hulman)	*What is the Calculus of Finite Differences?*
Barbara Vano (Oakland Univ.)	*The Growth of Mathematics in Russia*
Henry Deist (Miami Univ.)	*Artificial Intelligence: Can Computers Think?*
Pun Pollak (Miami Univ.)	*Applications of the Programmable Calculator in Mathematics and Statistics*

OKLAHOMA GAMMA (CAMERON UNIVERSITY) The Chapter field trips, socials and the following three presentations: Dr. Morris Marx (University of Oklahoma), *"Shift Registers"*; Dr. Dwight Olson; *"The Buffon Needle Problem"*; Dr. Tommy Wright (Statistician at Union Carbide in Oak Ridge), *"Undercount Problems Encountered During the United States Census"*.

Is your Chapter Report appearing here? If not, send it detailing speakers, awards, and programs to the Editor. This is the best possible way to let other Chapters know what you are doing and to share ideas for programs.

AT PRESS TIME--

The Program for the Sixth Annual Conference on Undergraduate Mathematics arrived. This Conference was hosted by Arkansas Beta at Hendrix College. The Speakers and their topics were:

Professor John W. Neuberger, North Texas State University	*Differential Equations in Science and Mathematics*
Conrad Plaut, Guilford College	*More on Derived Sets of Well-ordered Sets*
John Steeley, Guilford College	*Topologies on Collections of Subsets Generated by Families of Selection Maps*
Julie Brinkworth, Oklahoma State University	*Bleed Air Contamination Analysis*
Sheri Thompbon, Oklahoma State University	*Differential Equations and Flight Paths*
Kathy Alexander, Guilford College	*Leximorphic Sets*

Scotty Hofer, Pan American University	*A Model For the Numbers in 2^N*
Efton Park, University of Oklahoma.	*Solutions of Directional Differentional Equations*
Rebecca Thomas, University of Arkansas	*A Numerical Technique for the Solution of Integro-Differential Equations*
Professor M. Z. Nashed, University of Delaware	*Glimpses Into Optimization Theory*
Ben Schumacheh, Hendrix College	*Exponential Calculus*
David Sutherland, Hendrix College	*Nonlinear Derived Functions*
Elias Cosmas, Oklahoma State. University	*Mathematical Analysis of Inflation*
Lisa Townsley, University of Santa Clara	*Applications of Set Theory and Topology To Economics*
Ruth Moore and Laura Restess, Salem College	*Right-Hand Derivatives Suffice*
Hassan Azima, Texas Tech University	*A Pseudonorm and Hyperbolic Functions*
Jeff Bowles, University of Oklahoma.	*Some Considerations a Computer Frequently Forgets*
Paul Kraght, Harvey Mudd College	*Computer Arithmetic Algorithms*
Carol Smith, Hendrix College	*Infinite Sums of Derivatives*
Sandra Cousins, Hendrix College	*Infinite Composition*
Miriam Ann Reilman, University of New Orleans	*Cluster Analysis For Univariate Data*
Mark Heuser, University of Central Florida	*Least Squares Fitting of Distributions Using Non-Linear Regression*
Professor R. H. Bing, University of Texas	*Examples and Counterexamples*
Professor Paul R. Halmos, Indiana	*Some Problems I Couldn't Solve*
Kevin Keating, Washington University	*The Conjuction of Cayley Diagrams*
John Campion, St. Olaf College	*What Kind of Basis Might a Module Have?*
Tim Cornelson, University of Oklahoma.	*Pythagoras In a Box*
Laurie Chism and Susan Lucas, Oklahoma. State University	*Trigonometry and Sound Waves*

William Butterworth University of Santa Clara	*Applications of Topology to Logic*
Hike Meyer University of Oklahoma.	*Automata Theory For Mathematicians*
Dale DeLaPorte University of Arkansas	*The Dynamics of Traffic Flaw*
Steven Lazorchak Southern Illinois University	*Sinusoidal Steady-State Analysis of Electric Circuits Using the Phasor Transform*
Professor Burton Jones University of Colorado	*Multiplicative Functions in Number Theory*
Edward Shpiz Washington University	*Lower Bounds For van der Waerden Numbers*
Ravi Salgia Layola University (Chicago)	*Dirichlet Integrals and Their Applications*
Stephen Semmes Washington University	*Symmetric Groups on Ordinals*
Professor ALL Amir-Moez Texas Tech University	*Haw One Makes a Simple Idea Impressive*
Margaret R. Devlin Cardinal Stritch College	*Mp Coloring: 'Planar and Not so Planar Results*
Kevin Fox University of the. South	*Significant Figures Via Interval Arithmetic*
Jean Ezell University of Mississippi	*Some Divisibility Properties of Bionomial Coefficients*
Morteza Samiepour Texas Tech University	*Analytic and Synthetic Treatment Of Envelopes*
Annette T. Herz Kearney State College	*The Fibonacci, Numbers*
Daren Wilson Oakland University	*Fractional Arithmetic*

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