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PI MU EPSILON JOURNAL
THE OFFICIAL PUBLICATION
OF THE HONORARY MATHEMATICAL FRATERNITY

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This is a Call to all students who are writing those papers, those projects, giving those talks, proving theorems, etc., etc. Write up your results in the form of an (Vitiate. for this Journal and submit it to the Editor. This is a Call to faculty members to encourage your students and to help them with their papers. THIS IS YOUR JOURNAL—USE IT!

The Editor
THE AREA OF A TRIANGLE FORMED BY THREE LINES

by Michael L. Orrick, Jr.
Macalester College

One way to determine a triangle is to specify three noncollinear points \(X(x_1, x_2), Y(y_1, y_2)\) and \(Z(z_1, z_2)\) to be used as vertices (Figure 1). It is well known [Noble, Daniel, 1977, p. 209] that the area, \(A\), of the triangle is given by the formula:

\[
A = \frac{1}{2} \begin{vmatrix} x_1 & x_2 & 1 \\ y_1 & y_2 & 1 \\ z_1 & z_2 & 1 \end{vmatrix}
\]

where the sign is chosen to make \(A\) positive.

Another way to determine a triangle is to specify three noncurrent lines, no two parallel

\[
L_1: a_1 x + a_2 y + a_3 = 0 \\
L_2: b_1 x + b_2 y + b_3 = 0 \\
L_3: c_1 x + c_2 y + c_3 = 0
\]

which enclose the triangle (Figure 1). Though it is an old result [Salmon, 1879, p. 32], it is not so well known that the area, \(A\), of the triangle is also given by the formula

\[
(3a) \quad A = \frac{1}{2} \left| \frac{1}{(b_1 c_2 - b_2 c_1)(a_1 c_2 - a_2 c_1)(a_1 b_2 - a_2 b_1)} \right|
\]

The purpose of this note is to prove the formula (3a) using notation and methods familiar to students taking a first course in linear algebra.

We begin by forming the coefficient matrix \(P\) of the system (2) and the matrix \(Q\).

\[
P = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}
Q = \begin{bmatrix} M_{a_1} & M_{a_2} & M_{a_3} \\ M_{b_1} & M_{b_2} & M_{b_3} \\ M_{c_1} & M_{c_2} & M_{c_3} \end{bmatrix}
\]

where \(M_{a_i}, M_{b_i}\) and \(M_{c_i}\) in \(Q\) are cofactors of elements \(a_i, b_i\) and \(c_i\) in \(P\). For example,

\[
M_{a_3} = \begin{bmatrix} b_1 & b_2 \\ c_1 & c_2 \end{bmatrix}, \quad M_{b_3} = \begin{bmatrix} a_1 & a_2 \\ c_1 & c_2 \end{bmatrix}, \quad M_{c_3} = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}
\]

We note that the condition that no two lines are parallel to each other implies that the cofactors \(M_{a_3}, M_{b_3}\) and \(M_{c_3}\) are all non-zero. Furthermore, with this notation, formula (3a) becomes

\[
(3b) \quad A = \frac{1}{2} \left| \frac{1}{(b_1 c_2 - b_2 c_1)(a_1 c_2 - a_2 c_1)(a_1 b_2 - a_2 b_1)} \right|
\]

Is it possible for the determinant of matrix \(P\), \(\text{det}P\), to equal zero? If it is, there will exist a non-trivial solution \((s_1, s_2, s_3)\) to the system

\[
(5a) \quad a_1 s_1 + a_2 s_2 + a_3 s_3 = 0 \\
b_1 s_1 + b_2 s_2 + b_3 s_3 = 0 \\
c_1 s_1 + c_2 s_2 + c_3 s_3 = 0
\]

If \(s_3 \neq 0\), then \((s_1/s_3, s_2/s_3, 1)\) is also a solution to the system of equations in (5a). Thus all three lines \((2)\) pass through the
point \((s_1/s_3, s_2/s_3)\), violating the condition that these lines be non-concurrent. But if \(s_3 = 0\), then \((s_1, s_2)\) would be a non-trivial solution to the system

\[
\begin{align*}
    a_1s_1 + a_2s_2 &= 0 \\
    b_1s_1 + b_2s_2 &= 0 \\
    c_1s_1 + c_2s_2 &= 0
\end{align*}
\]

(5b)

This is impossible, since all the determinants in (4) are non-zero.

Using Cramer's rule, we find that the coordinates of the vertices, \(X(x_1, x_2), Y(y_1, y_2)\) and \(Z(z_1, z_2)\) are expressed as follows:

\[
\begin{align*}
    x_1 &= \frac{a_2a_3}{a_1a_2} + b_3 - b_2b_3 \\
    x_2 &= \frac{a_1a_3}{a_1a_2} + b_1b_3 - b_1b_2 \\
    y_1 &= \frac{a_2a_3}{a_1a_2} + b_3 - b_2b_3 \\
    y_2 &= \frac{a_1a_3}{a_1a_2} + b_1b_3 - b_1b_2 \\
    z_1 &= \frac{b_2b_3}{a_1a_2} + a_3 - a_2a_3 \\
    z_2 &= \frac{b_1b_3}{a_1a_2} + a_2 - a_1a_2
\end{align*}
\]

(6)

Using formula (1), we can obtain the area of the triangle

\[
\frac{1}{2} \det P = \frac{1}{2} \begin{vmatrix}
    a_1 & a_2 & a_3 \\
    b_1 & b_2 & b_3 \\
    c_1 & c_2 & c_3
\end{vmatrix}
\]

(7a)

Evaluating \(\det Q\), however, is a bit tedious. We therefore wish to simplify \(\det Q\) to something more easily calculated. Consider the product \(\det Q \det P\). Since a matrix and its transpose have the same determinant, \(\det Q = \det Q^T\). Then:

\[
\det P \det Q = \det P^T \det Q = \det \begin{vmatrix}
    a_1 & a_2 & a_3 \\
    b_1 & b_2 & b_3 \\
    c_1 & c_2 & c_3
\end{vmatrix}
\]

Each entry on the main diagonal of the product, being the sum of the products of elements in a row of \(P\) multiplied by their respective cofactors, must equal \(\det P\). All other entries, being the sum of the elements in one row and the cofactors of a different row, must equal zero [Ficken, 1967, p. 263]. The product then simplifies to

\[
\begin{vmatrix}
    \det P & 0 & 0 \\
    0 & \det P & 0 \\
    0 & 0 & \det P
\end{vmatrix} = (\det P)^3
\]

(8b)

Since \(\det P \neq 0\), this implies that \(\det Q = (\det P)^2\).

We have then, for the area of the triangle,
which is the same result developed by Salmon in 1879.

References


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AN INFINITE NUMBER OF 4 × 4 MAGIC SQUARES

by Stephen Rubag
Miami University, Oxford, Ohio

For many years magic squares have fascinated and intrigued mathematicians. The study of symmetry and unusual characteristics has been a favorite pastime for those who like to dabble with numbers. A Normal magic square of order \( n \) is defined to be an arrangement of the first \( n^2 \) natural numbers in the cells of the square so that each row, column and diagonal sums to a magic constant. For this article, however, some variations -- using non-consecutive numbers, negative numbers and fractions -- will be employed. With this criterion, nonnormal magic squares for all integer sums and eventually any real number can be found for a square of order four.

The most familiar fourth order magic square is found by numbering the sixteen cells from right to left, top to bottom. Leaving the entries of the diagonals as they are, and exchanging the entries of the complementary cells (cells which are symmetric with respect to the center point of the square), a normal magic square is obtained (Fig. 1). The rows, columns, and diagonals all have the same magic constant, 34. Upon closer examination, however, the square has even more magic qualities. The four cells in the center, the four corners, the opposite pairs (the cells with numbers 5, 8, 9, 12 and 2, 3, 14, 15) and each quadrant also have the magic constant 34! These properties of doubly even magic squares have been known for quite some time. The question is what will happen if numbers other than the first sixteen integers are used.

With a few minor manipulations, magic squares with magic constants 35, 36, and 37 can be found. For the magic sum 35, the 34-square can be transformed by subtracting one from the cells containing 1 and 2 and adding one to the cells containing 11 through 16 (Fig. 2). For the sum of 36, add one to each cell containing the numbers 9 through 16 in the 34-square (Fig. 3). Once again, by increasing the cells containing 7 through 14 by one and the cells containing 15 and 16 by two, the 37-square is obtained (Fig. 4).
With these four squares as a basis, any $4 \times 4$ magic square can be obtained by adding an appropriate integer $n$ to each of the sixteen cells of the square. In particular, subtracting eight from each cell of the $34$-square and $35$-square produces the $2$-square and $3$-square, respectively. Similarly, subtracting nine from each cell of the $36$-square and $37$-square produces the $0$-square and the $1$-square, respectively. Now adding an integer $n$ to every cell of the $0$, $1$, $2$, and $3$-square, the general forms for any $4 \times 4$ magic square can be constructed (Fig. 5).

Because there have been fewer restrictions placed on the numbers which may fill a square, an infinite number of magic squares have been found. A magic square for a particular sum, however, need not be unique. If sixteen consecutive terms of any arithmetic sequence are placed in the cells in the same order as with the original $34$-square, another magic square is created. Also, adding two magic squares or multiplying a magic square by a constant result in a magic square [3].

For a magic constant which is not an integer, several approaches can be used. To increase any integer sum by a decimal fraction (less than $1$), add this fraction to any set of four cells which have one cell in each row, column, diagonal, the center, the corners and the opposite pair, forming a "complete set." An example is the cells containing the numbers $1, 3, 5$ and $7$ or $16, 14, 12$ and $10$ (the "complementary" numbers) in the $34$-square. Also, adding one-half of the fraction to two complete sets or one-fourth of the fraction to all $16$ squares will give the desired result. Another interesting technique is to consider the decimal part as an integer and find the magic square for it. Dividing by the appropriate negative power of $10$ will reduce this magic square to its decimal form. Now add this decimal form to the integer magic square to obtain the desired magic square. For irrational sums, such as $\frac{1}{4}$ or $11\sqrt{2}$, merely multiply the integer magic square by the appropriate irrational part. Thus, a magic square may be found for any real number magic constant.

The procedures here do not exhaust the possibilities. Many other combinations of numbers will produce magic squares possessing all of the magic properties mentioned here, perhaps even more. The symmetries and the order which are inherent in this size square and our number system are remarkable. There is a certain balance here, and the limitations seem to be only the limitations, if any, which are in the number system itself.

- **Fig. 1 (34-square)**
  
<table>
<thead>
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<th>14</th>
<th>4</th>
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<td>7</td>
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<td>11</td>
<td>5</td>
</tr>
<tr>
<td>13</td>
<td>3</td>
<td>2</td>
<td>16</td>
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- **Fig. 2 (35-square)**
  
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<th>15</th>
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<tr>
<td>14</td>
<td>3</td>
<td>1</td>
<td>17</td>
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- **Fig. 3 (36-square)**
  
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- **Fig. 4 (37-square)**
  
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<tr>
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<td>3</td>
<td>2</td>
<td>18</td>
</tr>
</tbody>
</table>

**REFERENCES**

The general forms for magic squares having magic constants
(A) \( 4n \)  (B) \( 4n+1 \)  (C) \( 4n+2 \)  (D) \( 4n+3 \)

FIG. 5

The general solution of a general second-order linear differential equation

The following theorem was discovered while attempting to find a single method for solving a general second-order linear differential equation of the form

\[
\frac{d^2y}{dx^2} + P(x) \frac{dy}{dx} + Q(x)y = R(x).
\]  

(1)

The motivation for attack on such a method stemmed from the thought that when equation (1) is associated with a first-order linear differential equation of the form

\[
\frac{dz}{dx} + B(x)z = R(x),
\]

(2)

then \( z \) must necessarily be of the form

\[
z = \frac{dy}{dx} + A(x)y.
\]

(3)

Working backwards now with (3) and (2), we obtain

\[
\frac{d}{dx} \left[ \frac{dy}{dx} + A(x)y \right] + B(x) \left[ \frac{dy}{dx} + A(x)y \right] = R(x),
\]

which after taking derivative and rearranging terms takes the form

\[
\frac{d^2y}{dx^2} + \left[ A(x) + B(x) \right] \frac{dy}{dx} + \left[ A(x)B(x) + A'(x) \right] y = R(x).
\]

(4)

Comparing (1) and (4) we obtain

\[
P(x) = A(x) + B(x)
\]

and

\[
Q(x) = A(x)B(x) + A'(x).
\]

The above considerations lead us to state the following.
Theorem. A sufficient condition that the differential equation
\[ \frac{d^2 y}{dx^2} + P(x) \frac{dy}{dx} + Q(x)y = R(x) \]  
has a solution is that there exist two functions \( A(x) \) and \( B(x) \) such that
\[ P(x) = A(x) + B(x) \]
and
\[ Q(x) = A(x)B(x) + A'(x) \]
and in that case the solution is given by
\[ y = e^{-\int A(x) dx} \left( \int e^{\int B(x) dx} dx + C_1 \right) dx + C_2 \]

Proof. Under the given condition, (1) takes the form
\[ \frac{d^2 y}{dx^2} + \left( A(x) + B(x) \right) \frac{dy}{dx} + \left( A(x)B(x) + A'(x) \right)y = R(x) \]
\[ \left( \frac{d^2 y}{dx^2} + A(x) \frac{dy}{dx} + A'(x)y \right) + B(x) \frac{dy}{dx} + A(x)B(x)y = R(x) \]
\[ \frac{d}{dx} \left( \frac{dy}{dx} + A(x)y \right) + B(x) \frac{dy}{dx} + A(x)y = R(x). \]
Letting
\[ \frac{dy}{dx} + A(x)y = z, \]
we have from the above equation
\[ \frac{dz}{dx} + B(x)z = R(x), \]
which is a linear differential equation of the first order whose solution by the usual methods is given by
\[ z = e^{-\int B(x) dx} \left( \int R(x) e^{\int B(x) dx} dx + C_1 \right). \]
Thus from (5) we have
\[ \frac{dy}{dx} + A(x)y = e^{-\int B(x) dx} \left( \int R(x) e^{\int B(x) dx} dx + C_1 \right), \]
which again is a linear differential equation of the first order whose solution by usual methods is given by
\[ y = e^{-\int A(x) dx} \left( \int e^{\int B(x) dx} dx + C_1 \right) e^{\int A(x) dx} dx + C_2 \]
\[ = e^{-\int A(x) dx} \left[ \int e^{\int B(x) dx} dx + C_1 \right] dx + C_2 \]
The function in (6) is the general solution of (1).

Example 1.
Solve the differential equation
\[ \frac{d^2 y}{dx^2} - x y = 3. \]
Solution.
Here \( P(x) = -x \) and \( Q(x) = -4 \). We can thus take \( A(x) = 1 \) and \( B(x) = -4x \) and satisfy ourselves that
\[ -x = 1 + (-4) \]
and
\[ -4 = (1)(-4) + 0. \]
Using (6) we have the solution
\[ y = e^{-\int A(x) dx} \left[ \int e^{\int B(x) dx} dx + C_1 \right] dx + C_2 \]
\[ = e^{-x} \left[ \int e^{\int B(x) dx} dx + C_1 \right] dx + C_2 \]
\[ = - e^{-x} + C_3 e^{-x} + C_2 e^{-x}. \]

Example 2.
Solve the differential equation
\[ \frac{d^2 y}{dx^2} - \frac{2}{x} \frac{dy}{dx} + \frac{2}{x^2} y = 0. \]
Solution.
Here \( P(x) = -\frac{2}{x} \) and \( Q(x) = \frac{2}{x^2} \). We can thus take \( A(x) = -\frac{1}{x} \) and
\[ B(x) = -\frac{1}{x} \]
and satisfy ourselves that
\[ -\frac{2}{x} = \left( -\frac{1}{x} \right) + \left( -\frac{1}{x} \right) \]
and
\[ \frac{2}{x^2} = \left( -\frac{1}{x} \right) \left( -\frac{1}{x} \right) + \frac{d}{dx} \left( -\frac{1}{x} \right). \]
Hence by (6) we have the solution
Many regional meetings of the Mathematical Association of America regularly have sessions for undergraduate papers. If two or more colleges and at least one local chapter help sponsor or participate in such undergraduate sessions, financial help is available up to $50. Write to Dr. Richard Good, Department of Mathematics, University of Maryland, College Park, Maryland, 20742.

The following mathematicians have served as referees since the publication of the last issue; without their help, the Journal would not be possible, and the Editor is grateful for their cooperation: Professor C. A. Grimm, Professor Roger Opp, Professor Ronald Wenger, Professor Dale Rognlie, Professor Saises Sengupta, Professor Harold Carda, all of The South Dakota School of Mines and Technology; Professor Barbara Reynolds, Professor Clayton Dodge of the University of Maine; Professor Steven Janke of Colorado College; Professor John Schumaker of Rockford College.

Section 1. Introduction.

Finding all Pythagorean triples, that is, all integer solutions of the equation \( x^2 + y^2 = z^2 \) is a familiar problem. [1] In this article we consider the following generalization.

If triangle \( ABC \) has an angle of \( 60^\circ \) and integer sides \( k, l, m \), we call the triple \( (k, l, m) \) a \( 60^\circ \) Pythagorean triple. Finding all \( 90^\circ \) Pythagorean triples is just the familiar problem mentioned above. The problems of finding all \( 120^\circ \) and \( 60^\circ \) Pythagorean triples are not quite as well known, even though solutions were given by L. E. Dickson [2] in 1908. It is these latter two problems that we address in this article.

All such triples satisfy either \( k^2 + kl + l^2 = m^2 \) or \( k^2 - kl + l^2 = m^2 \). We derive a set of parametric equations which generate all integer solutions to these equations. Furthermore, we give conditions on the parameters so that each essentially different solution is generated exactly once. This latter consideration does not appear to have been addressed in the literature. We conclude by observing that the solutions to the \( 60^\circ, 90^\circ \), and \( 120^\circ \) problems provide a complete list of \( \theta^\circ \) Pythagorean triples with \( \theta \) rational in degrees.

Section 2. Reduction of the Problems.

From the law of cosines it follows that if \( (k, l, m) \) is a \( 120^\circ \) Pythagorean triple with side \( m \) opposite the \( 120^\circ \) angle, then \( m = k + l - 2k\cos(120^\circ) = k + kl \). Similarly, if \( (k, l, m) \) is a \( 60^\circ \) Pythagorean triple, then \( m^2 = k^2 - kl + l^2 \). We turn our attention, therefore, to the problem of finding all integer solutions to these two equations. Our Pythagorean triples are then just the solutions in which \( k, l, \) and \( m \) are all positive.

The following lemmas indicate that it is sufficient to solve the \( k^2 + kl + l^2 = m^2 \) problem. Both are easily established by straightforward computation.
Lemma 1. If the triple \((k, l, m)\) satisfies \(k^2 + kl + l^2 = m^2\), then the triples \((x, y, z) = (k, k+l, m)\) and \((z, y, x) = (l, k+l, m)\) satisfy \(x^2 - xy + y^2 = z^2\).

Remark. If we assume that \(k, l, m\) are all positive so that \((k, l, m)\) is a \(120^\circ\) Pythagorean triple, then a geometric proof of Lemma 1 is indicated by Figure 1.

Figure 1 shows a \(120^\circ\) Pythagorean triple. ABC has a \(60^\circ\) angle at A and integer sides \(k, k+l, m\). DBC has a \(60^\circ\) angle at D and integer sides \(1, k, m\).

Lemma 1 states that each solution to \(k^2 + kl + l^2 = m^2\) generates two solutions to \(k^2 - kl + l^2 = m^2\). The next lemma shows that any solution to \(k^2 - kl + l^2 = m^2\) arises in this way.

Lemma 2. If \((k, l, m)\) satisfies \(k^2 - kl + l^2 = m^2\), then \((x, y, z) = (k, k - l, m)\) satisfies \(x^2 + xy + y^2 = z^2\).

Section 3. The Equations \(k^2 + kl + l^2 = m^2\) and \(k^2 + kl + l^2 = m^2\)

Precisely stated, we wish to find all integers \(k, l, m\) which satisfy \(k^2 + kl + l^2 = m^2\). While this problem has been studied in the literature \([2, 3, 5]\) we present a complete, self-contained solution because other solutions seem to be incomplete or too general to be readily understood.

First we define a solution triple to be a triple of integers \((k, l, m)\) satisfying \(k^2 + kl + l^2 = m^2\). Next we define a primitive triple to be a solution triple in which the integers are pairwise relatively prime, that is \((k, l) = (l, m) = (k, m) = 1\).

In order to establish the fact that any solution triple is a scalar multiple of some primitive triple, we first show

Lemma 3. If \((k, l, m)\) is a solution triple, then \((k, l) = (l, m) = (k, m) = 1\).

Proof. Let \(d = (k, l)\) and \(f = (l, m)\). Since \(k^2 + kl + l^2 = m^2\), we have \(d^2 | m^2\). Hence \(d | m\) and therefore \(d | e\) and \(d \not| f\). Let \(e = d^2\), \(f^2 = d^2\), \(k = dk\), \(l = dl\), and \(m = dm\). Then \((k, l) = (l, m) = (k, m) = 1\) and \(k^2 + kl + l^2 = m^2\). If \(p\) is any prime divisor of \(e\) or \(f\), then \(p\) is either \(k + l\) or \(k\). But \(p\) is odd, so in either case \(p\) is a prime. Hence \(k^2 + l^2 = m^2\). Similarly \(f^2 = 1\), proving Lemma 3.

The next lemma reduces our problem to one of finding all primitive solution triples. Its proof is straightforward and so details are omitted.

Lemma 4. If \((k, l, m)\) is a solution triple with \(d = (k, l) = (l, m) = (k, m)\), then \((k', l', m')\) is a primitive triple where \(k = k'\), \(l = l'\), and \(m = m'\).

Lemma 5. If \((k, l, m)\) is a primitive solution triple, then either \(k - l \equiv m \pmod{3}\) or \(l - k \equiv m \pmod{3}\), but not both.

Proof. \(k^2 + kl + l^2 = m^2\) \(\iff (m + (k - l))(m - (k - l)) = 3k^2\).

Thus \(3 | m + (k - l)\) or \(3 | m - (k - l)\). If \(3 | k\), then \(3 | m - 2l\) and \(3 | m + l\), which implies \(3 | 2l\) which in turn implies \(3 | l\). This contradicts primitivity. A similar contradiction arises if \(3 | l\).

Next we proceed towards finding a parametric representation of all primitive solution triples. To begin we assume that the primitive triple \((k, l, m)\) is always written so that \(k - l \equiv m \pmod{3}\). This can be achieved by interchanging \(k\) and \(l\) if necessary.

Theorem 1. Let \(p\) and \(q\) be integers with \((p, q) = 1\) and \(p \not\equiv q \pmod{3}\). Then \(k = p - q^2\), \(l = 2pqq^2\), and \(m = p^2 + q^2\) form a primitive solution triple for which \(k - l \equiv m \pmod{3}\).

Proof. Straightforward algebra shows \(k^2 + kl + l^2 = m^2\) and \(k - l \equiv m \pmod{3}\). We must show that \(k\), \(l\), and \(m\) are pairwise relatively prime. To do this we use Lemma 6 below, which will also be useful later.

Lemma 6. Let \(p\) and \(q\) be integers with \((p, q) = 1\). Then \((p^2 - q^2, 2pqq^2) = 1\) if and only if \(p \not\equiv q \pmod{3}\).
Proof. Suppose \( p \equiv q \pmod{3} \); then \( 3 \mid p-q \) and \( 3 \mid p^2-q^2 \). But
\[ p-q \equiv p+2q \pmod{3}, \]
so \( 3 \mid p+2q \). We also have \( 3 \mid (p-q)+(p+2q) = 2p+q \), and hence \( 3 \mid 2pq+q^2 \). Therefore \( (p^2-q^2, 2pq+q^2) \neq 1 \).

Now suppose \( (p^2-q^2, 2pq+q^2) = d \neq 1 \). Let \( x \) be any prime divisor of \( d \). Now \( x \) divides neither \( p \) nor \( q \); this follows since \( x \mid p^2-q^2 \) and hence if \( x \) divides one it also divides the other, contradicting \( (p,q) = 1 \).

Since \( x \mid 2pq+q^2 \) we must have \( x \mid 2p+q \). Also \( x \mid p^2-q^2 \) implies \( x \mid p-q \) or \( x \mid p+q \). If \( x \mid 2pq \) and \( x \mid p+q \), then \( x \mid p \), a contradiction. Hence \( x \mid 2pq \) and \( x \mid p-q \) and these in turn imply \( x \mid 3p \). Thus \( x=3 \) and \( 3 \mid p-q \) making \( p \equiv q \pmod{3} \).

To complete the proof of Theorem 1 note that by Lemma 3 it is enough to show that any two of \( k, l, m \) are relatively prime. Lemma 6 now gives us \( (k,l)=1 \).

Next we prove the converse to Theorem 1. Define \((k,l,m)\) to be a trivial solution triple if it is of the form \((-k,k,k)\) or \((k,0,k)\).

**Theorem 2.** If \((k,l,m)\) is a non-trivial primitive solution triple, then there exist unique integers \( p \) and \( q \) with \( p>0 \), \((p,q)=1\) and \( p \not\equiv q \pmod{3} \) such that \( k=\frac{p^2-q^2}{2} \), \( l=2pq+q^2 \), and \( m=p^2+pq+q^2 \).

**Proof.** Since \( k^2+kl+l^2=m^2 \) we have \( l(k+1)=m^2-k^2=(m+k)(m-k) \), and since \((k,l,m)\) is non-trivial we can write
\[ \frac{m+k}{m-k} = \frac{t}{k+l} \]
where \( t \) is a non-zero rational number. This gives us a system of equations:
\[ \begin{align*}
l-k &= mt, \\
l(t+t+1) &= km.
\end{align*} \]
Solving this system for \( k \) and \( l \) we get
\[ \begin{align*}
k &= \frac{(l-t^2)m}{t^2+t+1}, \\
l &= \frac{(2p+q)m}{t^2+t+1}.
\end{align*} \]
Now since \( t \) is rational, we let \( t = \frac{q}{p} \) where \( p \) and \( q \) are integers with \( p>0 \) and \((p,q)=1\). Clearly the \( p \) and \( q \) satisfying these conditions are unique. Substituting for \( t \) we get
\[ \begin{align*}
k &= \frac{(p^2-q^2)m}{p^2+pq+q^2}, \\
l &= \frac{2pq+q^2}{p^2+pq+q^2}.
\end{align*} \]
Recall that \( \frac{l}{m+k} = t = \frac{q}{p} \), and by assumption \( k-l \equiv m \pmod{3} \).

Lemma 5 implies that \( k \not\equiv m \pmod{3} \) and thus \( l \equiv m+k \pmod{3} \). These facts together with \((p,q)=1\) imply \( p \not\equiv q \pmod{3} \). Lemmas 3 and 6 now tell us that \((p^2-q^2, 2pq+q^2)=1 \) and \((2pq+q^2, p^2+pq+q^2)=1 \). Thus \( p^2+pq+q^2 \not\equiv 0 \pmod{3} \), and since \((k,l)=1\) we must in fact have \( p+q \equiv 0 \). Thus we can write
\[ \begin{align*}
k &= p^2-q^2, \\
l &= 2pq+q^2, \\
m &= p^2+pq+q^2.
\end{align*} \]
This concludes the proof of Theorem 2.

Summarizing the previous results we have

**Theorem 3.** All integer triples \((k,l,m)\) for which \( k^2+kl+l^2=m^2 \) and \( k-l \equiv m \pmod{3} \) are either:

1. trivial, that is of the form \((-k,k,k)\) or \((k,0,k)\)
2. non-trivial, that is, there exist unique integers \( p,q,r \) with \( p>0 \), \( q>0 \), \((p,q)=1\) and \( p \not\equiv q \pmod{3} \) such that \( k=\frac{p^2-q^2}{2} \), \( l=2pq+q^2 \), and \( m=p^2+pq+q^2 \).

To find all solutions to \( k^2+kl+l^2=m^2 \) we can use Theorem 3 and Lemmas 1 and 2. First note that primitivity is defined as before and that any solution triple is a multiple of a primitive one. The next lemma indicates that there are essentially two distinct families of primitive solutions.

**Lemma 7.** If the triple of integers is a primitive solution to \( k^2+kl+l^2=m^2 \), then \( k+l \equiv m \pmod{3} \) but not both.

The proof is similar to the proof of Lemma 5 and is omitted.

Observe that unlike the \( k^2+kl+l^2=m^2 \) problem we cannot choose which congruence we want to hold, since interchanging \( k \) and \( l \) leaves both unaffected. Combining Lemma 2 with Theorem 3 we have:

**Theorem 4.** All integer triples \((k,l,m)\) for which \( k^2+kl+l^2=m^2 \) are either:

1. trivial, that is of the form \((-k,0,k)\), \((k,0,k)\), \((0,k,k)\), \((-k,-k,k)\), or \((0,-k,-k)\);
2. non-trivial, that is, there exist unique integers \( p,q,r \) with \( p>0 \), \( q>0 \), \((p,q)=1\) and \( p \not\equiv q \pmod{3} \) such that \( k=\frac{p^2-q^2}{2} \), \( l=2pq+q^2 \), and \( m=p^2+pq+q^2 \).

It is an easy matter to see that the first non-trivial family corresponds to \( k+l \equiv m \pmod{3} \) while the second corresponds to \( k \equiv l \equiv m \pmod{3} \).
Section 4. Uniqueness of Solutions.

The solutions to the equations \( k^2 + k + 2 = m^2 \) given in the previous section contain certain redundancies which arise because of the symmetry of the equations. For example, the integers \( p=3, q=-1, r=1 \) generate the solution triple \((k, l, m) = (8, -5, 7)\) while \( p=3, q=-2, r=1 \) generate the triple \((k, l, m) = (5, -8, 7)\). Both of these satisfy \( k^2 + k + 2 = m^2 \); however, it makes sense to regard these as essentially the same solution. In fact it is easy to see that for each equation, whenever \((k, l, m)\) is a solution, several other triples are also solutions. Obvious examples of triples that are also solutions are \((k, l, -m)\), \((-k, k, m)\), \((-k, -l, m)\), and \((-l, -k, m)\). In total there are 24 related solutions for each of the equations. These are indicated in Table 1.

<table>
<thead>
<tr>
<th>( k^2 + k + 2 = m^2 )</th>
<th>( k^2 - k + 2 = m^2 )</th>
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<tbody>
<tr>
<td>((k, l, m))</td>
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<td>((k, l, -m))</td>
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</table>

Remark. If we take a solution triple \((k, l, m)\) and consider all possible transformed triples of the form \((ak+bl, ck+dl, em)\) where \(a, b, c, d,\) and \(e\) are integers, then Table 1 contains precisely the transformed triples which are again solutions of the corresponding equation.

Clearly, for each of the two equations the related solutions form an equivalence class. By restricting the solutions given in Section 3 to solutions involving one and only one member of each equivalence class, we can eliminate redundant solutions. To do this we need to handle the two equations separately. The following lemma is easily established; we omit the proof.

**Lemma 8.** Among the equivalent primitive solutions to \( k^2 + k + 2 = m^2 \) given in Table 1, there is precisely one with \( k, l, \) and \( m \) positive and \( k - 2 \equiv m \pmod{3} \).

We now restrict the values of \( p, q, \) and \( r \) in Theorem 3 so that we obtain only the representative given in Lemma 8. This gives us the following:

**Theorem 5.** The following is a complete non-redundant set of non-trivial solutions to \( k^2 + k + 2 = m^2 \):

\[ k = (p^2 - q^2)r, \quad l = (2pq + q^2)r, \quad m = (p^2 + pq + q^2)r, \]

where \( p, q, r \) run through all integers satisfying \( p > q > 0, \) \( r > 0, \) \( (p, q) = 1, \) and \( p \not\equiv q \pmod{3} \).

**Proof.** From Theorem 3 and Lemma 8 we have the following. The condition \( m > 0 \) implies \( r > 0, \) since \( m = (p^2 + pq + q^2)r \) and \( p^2 + pq + q^2 \) is always positive for \( p > 0. \) The condition \( k - 2 \equiv m \pmod{3} \) implies \( p \not\equiv q \pmod{3} \) as shown in the proof of Theorem 2. Next \( k > 0 \) implies \( p > q, \) since \( k = (p - q)(p + q)r \) and \( p - q \) would require \( q > p > 0 \) and \( p + q > 0, \) contradicting \( k > 0. \) Finally \( l > 0 \) implies \( q > 0 \) since

(i) \( k > 0 \) and \( l > 0 \) imply \( k + l = (2pq + q^2)p > 0, \) which together with \( p > 0 \) implies \( q > (p/2); \)

(ii) \( l > 0 \) implies \( (2pq + q^2) = q(2p + q) > 0, \) which in turn implies either \( q > 0 \) or \( q < 0 \) and \( 2p + q < 0. \) This latter choice implies \( q < -2p = -(p/2), \) which contradicts (i).

**Corollary 7.** Once solutions to \( k^2 + k + 2 = m^2 \) are obtained with the restrictions given above, then all solutions are obtained by expanding the solution set 24-fold according to Table 1.

Eliminating redundancies in the solutions to \( k^2 - k + 2 = m^2 \) is achieved in a similar fashion. We begin with the following easily checked lemma.

**Lemma 9.** Among the equivalent primitive solutions to \( k^2 - k + 2 = m^2 \), there is precisely one with \( k, l, \) and \( m \) positive, \( k < l, \) and \( k + l = m \pmod{3} \).

**Theorem 6.** The following is a complete non-redundant set of non-trivial solutions to \( k^2 - k + 2 = m^2 \):

\[ k = (p^2 - q^2)r, \quad l = (2pq + p^2)r, \quad m = (p^2 + pq + p^2)r, \]

\[ (k, l, m) \]
where \( p, q, r \) run through all integers satisfying \( p > q > 0, r > 0, (p, q) = 1, \) and \( p \neq q \) (mod 3).

Proof. Use Lemma 1 and Theorem 5 to observe that we no longer obtain two families of solutions as we did in Theorem 4, since Lemma 9 eliminates the second family.

Corollary. As in the previous case, all solutions to \( k^2 - kl + l^2 = m^2 \) may now be obtained by a 24-fold expansion of the above solution set.

Section 5. The 120° and 60° Pythagorean Triples.

As mentioned earlier, the 120° and 60° Pythagorean triples are simply the positive solutions to the equations \( k^2 + kl + l^2 = m^2 \). We now show how to give a complete list of these.

Theorem 5 gives us solutions to \( k^2 + kl + l^2 = m^2 \) in which \( k, l, \) and \( m \) are positive, and so these form 120° Pythagorean triples. To get any others we take a triple generated by Theorem 5 and look among its 24 related solutions of Table 1 for other triples in which all three entries are positive. A quick glance at Table 1 shows that if \( k, l, m \) are all positive, then only one other entry has this property, namely \( (l, k, m) \).

Table 2 below lists the first few 120° Pythagorean triples.

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( r )</th>
<th>( k )</th>
<th>( l )</th>
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For the 60° Pythagorean triples we proceed similarly. We begin with the solutions generated by Theorem 6 in which \( k, l, m \) are positive and \( k \neq l \). For each such solution we check among the 24 related solutions for other triples having all entries positive. This time we find several. If \( k, l, m \) are all positive, then so are the entries of \( (l, k, m) \), \( (l-k, l, m) \), and \( (l, l-k, m) \). The latter two triples are not obviously the same as \( (k, l, m) \) but should be regarded as equivalent to each other. Hence we should include one of them, say \( (l-k, l, m) \), in our list of 60° triples. This gives us the following:

Theorem 7. The following is a complete non-redundant set of 60° Pythagorean triples:

1. \( k = (p^2 - q^2)r \), \( l = (2pq + p^2)r \), \( m = (p^2 + pq^2)r \);
2. \( k = (2pq + p^2)r \), \( l = (2pq + p^2)r \), \( m = (p^2 + pq^2)r \),

where \( p, q, r \) run through all integers satisfying \( p > q > 0, r > 0, (p, q) = 1, \) and \( p \neq q \) (mod 3).

Table 3

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
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<th>( k )</th>
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Section 6. Conclusion.

Now that we have found all 120° and 60° Pythagorean triples, we can group them with the well-known 90° triples and observe that this collection is a list of all 6° Pythagorean triples with 9 rational in degrees. First, note that if \( (k, l, m) \) is a 6° Pythagorean triple, then \( \cos \theta \) is rational. According to Niven [4], if \( \theta \) is rational in degrees and \( \cos \theta \) is rational, then \( \cos \theta = 0, 1/2, \) or \( \pm 1 \). Hence, if \( (k, l, m) \) is a 6° Pythagorean triple with \( \theta \) rational in degrees, then \( \theta \) must be 120°, 90°, or 60°.
REFERENCES


THE EQUILIC QUADRILATERAL

by J. Garfunkel
Queens College, N.Y.

Every student of geometry knows that of all plane figures the triangle is one of the most prolific in producing theorems. In this article we show that the quadrilateral is also a rich source for investigation. Quite a number of special quadrilaterals have already been investigated. Examples are the cyclic quadrilateral whose vertices lie on the same circle, the circumscribable or pericyclic quadrilateral whose sides are tangent to the same circle and the orthodiagonal quadrilateral whose diagonals are perpendicular. Furthermore, there are quadrilaterals that are both cyclic and pericyclic, cyclic and orthodiagonal and so on. To the quadrilaterals with interesting properties, we add a new quadrilateral which we will call equilic.

**Definition.** Quadrilateral $ABCD$ is said to be equilic if $AD = BC$ and if angle $A + angle B = 120^\circ$.

Note that the quadrilateral need not be convex. See Figures 1A and 1B.

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# References


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# The Equilic Quadrilateral

**Definition.** Quadrilateral $ABCD$ is said to be **equilic** if $AD = BC$ and if angle $A + angle B = 120^\circ$.

Note that the quadrilateral need not be convex. See Figures 1A and 1B.

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We are "deeply grateful to the referees for the meticulous care they took in greatly enhancing this article. Thanks."
We begin by stating a fairly obvious fact.

**Theorem 1.** A quadrilateral which is both cyclic and equilic is an isosceles trapezoid with \( A = B = 60° \).

The proof of Theorem 1 is left to the reader.

**Definition.** If one angle of an equilic quadrilateral is equal to \( 90° \), the quadrilateral is called right equilic.

**Theorem 2.** In a right equilic quadrilateral, a diagonal is equal to an unequal side.

![Figure 2](image)

**FIGURE 2**

Proof. Quadrilateral \( ABCD \) is right equilic with the right angle at \( A \) and with \( AD = BC \). From \( C \) drop perpendiculars \( CP \) and \( CQ \) to sides \( AB \) and \( AD \) respectively. Then \( CP = 1/2CB : 1/2AD \), since angle \( B \) measures \( 30° \). Hence, \( C \) lies on the perpendicular bisector of \( AD \), which makes triangle \( ACD \) isosceles and diagonal \( AC \) equal to side \( DC \).

The proof is similar for the right angle at any other vertex.

**Theorem 2A.** In an equilic quadrilateral, if a diagonal is equal to an unequal side then the quadrilateral is right equilic.

Proof. In equilic quadrilateral \( ABCD \) with angle \( A + B = 120° \), \( AD = BC \), and \( AC = CD \), erect a perpendicular to \( AD \) at \( A \) to cut line \( BC \) at \( K \). Drop perpendiculars \( CF \) and \( CG \) to \( AD \) and \( AX \) respectively. Let \( AD \) and \( BC \) extended meet at \( H \), so angle \( H = 60° \). Because \( CD \) is an isosceles triangle, \( F \) is the midpoint of \( AD \). Then \( CG : FA : 1/2AD : 1/2BC \). Since \( CG \) is parallel to \( DA \), then angle \( KCG : 60° \), so \( CG : 1/2 CK \). Now \( CK = CB \), so \( B \) and \( K \) coincide, making angle \( DAB \) : angle \( DAK \) : \( 90° \).

In the next two theorems we investigate some interesting relations between the equilic quadrilateral and equilateral triangles.

**Theorem 3.** If equilateral triangle \( ABP \) is constructed interiorly on side \( AB \) of equilic quadrilateral \( ABCD \), then triangle \( PCD \) is also equilateral.

![Figure 3](image)

**FIGURE 3**

Proof. Refer to Figure 3. Let \( ABCD \) be an equilic quadrilateral. Assume, without loss of generality, that \( \text{angle} ABC > \text{angle} BAD \). Construct equilateral triangle \( ABP \) interiorly and draw \( PC \) and \( PD \). Because \( AD \) and \( BC \) meet at \( 60° \) and \( AP \) and \( BP \) also meet at \( 60° \), then angle \( \text{PAD} = \text{angle} FBC \).
Thus, triangles $\triangle ADP$ and $\triangle BCP$ are congruent by SAS. Hence a $60^\circ$ rotation about $P$ carries triangle $\triangle ADP$ into $\triangle BCP$, so angle $DPC = 60^\circ$ and triangle $\triangle PCD$ is equilateral.

**Theorem 4.** The midpoints of the diagonals and the midpoint of an unequal side of an equilic quadrilateral are vertices of an equilateral triangle.

![Figure 4](image)

**Proof.** In Figure 4, $Q$ and $R$ are the midpoints of $AC$, $BD$, and $CD$, respectively. Clearly, $PR$ is parallel to $AD$ and equal to $1/2 AD$, and $RQ$ is parallel to $BC$ and equal to $1/2 BC$. Since $AD = BC$, triangle $\triangle PQR$ is isosceles with $PR = QR$. Since the angle between $AD$ and $BC$ is $60^\circ$, then so also is angle $\angle PQR = 60^\circ$ and triangle $\triangle PQR$ is equilateral. Similarly, triangle $\triangle PQR$ is equilateral where $\triangle PQR$ is the midpoint of side $AB$. Moreover, $\triangle RQP$ is a rhombus.

The reader is encouraged to carry out the proof of Theorem 4 in the case of a non-convex equilic quadrilateral.

**Definition.** An equilic quadrilateral $ABCD$ is called isosceles equilic if $AD = DC = CB$.

**Theorem 5.** The point $P$ of the intersection of the diagonals of an isosceles equilic quadrilateral is the circumcenter of triangle $\triangle ABQ$, where $Q$ is the point of intersection of sides $BC$ and $AD$.

**Proof.** In Figure 5, $ABCD$ is an isosceles equilic quadrilateral. Through $A$ and $C$, draw lines parallel to $DC$ and $DA$, respectively, to intersect in $E$. Then, $\triangle AEC$ is a rhombus. Denote the equal angles $\angle DAP$, $\angle PAE$, $\angle DCP$, and $\angle PCE$ by $\alpha$, angle $\angle EAB$ by $\beta$ and angle $\angle DQP$ by $\gamma$. Angle $\angle BCB = 60^\circ$, since $CE$ is parallel to $AD$. Hence, triangle $\triangle BCE$ is equilateral and triangle $\triangle ABE$ is isosceles with $AE = BE$. Since angle $\angle BMQ + \angle ABQ = 120^\circ$, $(2\alpha + \beta) + \beta + 60^\circ = 120^\circ$ and $\alpha + \beta = 30^\circ$. In isosceles triangle $\triangle BCD$, $2\alpha + 60^\circ + 2(\angle CDB) = 180^\circ$, so angle $\angle CDB = 60^\circ - \alpha$. Thus, angle $\angle DBE = \alpha$ and triangle $\triangle ABP$ is isosceles with $AP = BP$ and angle $\angle APB = 120^\circ$. It remains to prove that $\angle AP = \angle PQ$. Since angle $\angle APB = 120^\circ - \angle DPC$, quadrilateral $\triangle DPCQ$ is cyclic and angle $\angle DPC = \angle DQP = \angle y$. Hence, triangle $\triangle AQP$ is isosceles and $AP = PQ$. This proves that $P$ is the circumcenter of triangle $\triangle ABQ$.

It should be noted that Theorem 5 holds if point $E$ falls outside the equilic quadrilateral. The reader is encouraged to carry out the proof in the case of a non-convex isosceles equilic quadrilateral.
The proof of the following corollary is left as an exercise for the reader.

**Corollary.** The opposite angles of an isosceles equilic quadrilateral are in the ratio of $1:2$.

The next few theorems are of a more sophisticated nature.

**Theorem 6.** If equilateral triangles $PAD$, $QDC$, and $RBC$ are erected on consecutive sides $AD$, $DC$, and $CB$ of equilic quadrilateral $ABCD$, exteriorly on side $CD$ and on $AD$ and interiorly on side $BC$, then triangle $PQR$ is equilateral. (By symmetry, the result holds if the roles of $AD$ and $BC$ are interchanged).

**Proof.** Angle $QCR = \angle DCB$ since both are equal $60^\circ$ + angle DCR.
Angle $PDQ = 360^\circ - 120^\circ - \angle ADC = 240^\circ - (360^\circ - \angle A - \angle B - \angle DCB) = 240^\circ - (240^\circ - \angle DCB) = \angle DCB$ = angle $QCR$.

Also, $QC = QD$ and $DP = QR$, so that triangles $PDQ$ and $QRC$ are congruent. Since a $60^\circ$ rotation about point $Q$ carries triangle $PDQ$ into triangle $QRC$, it follows that angle $PQR = 60^\circ$ and that triangle $PQR$ is equilateral.

**Theorem 6a.** If equilateral triangles are erected exteriorly on sides $DA$, $AB$, and $BC$ of equilic quadrilateral $ABCD$, then their third vertices are vertices of an equilateral triangle.

**Proof.** The proof is similar to that of Theorem 6. Let the appended equilateral triangles be $RAD$, $PBA$, and $QCB$, then $RA = QB$, $AP = PB$ and angle $PAP = \angle QBP = 120^\circ + B$. Therefore, triangles $PBA$ and $QCB$ are congruent and a $60^\circ$ rotation about $P$ carries one into the other. Since $PR = PQ$ and angle $RPQ = 60^\circ$, triangle $PQR$ is equilateral.

**Theorem 6b.** If equilateral triangles $QBC$, $PCD$, and $RDA$ are erected interiorly on sides $BC$, $CD$, and $DA$, respectively, then triangle $PQR$ is equilateral.

**Proof.** We have $PD = PC$, $DR = CQ$ and angle $PCQ = \angle PDR = 120^\circ$ + angle $ADC$, so triangles $PDR$ and $PCQ$ are congruent. Again, a $60^\circ$ rotation about $P$ carries one into the other and we argue as before.

**Theorem 7.** If $ABCD$ is an equilic quadrilateral and if equilateral triangles are erected as follows: $PCA$ on the same side of $CA$ as $B$, $QBD$ on the same side of $BD$ as $A$, and $RBA$ exteriorly, then triangle $PQR$ is equilateral.

**Proof.** A $60^\circ$ rotation about $B$ carries $AD$ into $QR$ and a $60^\circ$ rotation about $A$ carries $BP$ into $BC$. It follows that $QR = PR$ and the angle between them is $60^\circ$.

**Theorem 7a.** If $PCA$, $QCD$, and $RBD$ are equilateral triangles erected away from side $BA$ of equilic quadrilateral $ABCD$, then $P$, $Q$, and $R$ are collinear.

**Proof.** By Theorem 3, a $60^\circ$ rotation about $A$ carries $PQ$ into $CB$ and a $60^\circ$ rotation about $B$ carries $AD$ into $QR$. Since $AD$ and $BC$ intersect again I must thank the referees for Theorems 6, 6a, and 6b.
We offer a second proof of Theorem 7 since this method of proof is useful in proving a later theorem and is interesting in itself. We will employ vectors in the complex plane and make use of the following facts:

1. If \( v \) is a vector, then \( wv \) represents the same vector rotated 120° in the counterclockwise direction, where \( w \) represents a cube root of unity.
2. \( u^3 = 1, 1 + w + u^2 = 0. \)

**Proof.** Let \( AB = p \) and \( BC = q \), then \( \overrightarrow{BA} = wq, \overrightarrow{BC} = w^2p \), and \( \overrightarrow{BQ} = wp \).

Since \( \overrightarrow{AC} = p + q \), then \( \overrightarrow{CA} = wp + uq \) and \( \overrightarrow{CA} = w^2p + uq \).

Since \( \overrightarrow{BQ} = p + uq \), then \( \overrightarrow{CA} = wp + u^2q \) and \( \overrightarrow{CA} = w^2p + q \).

It follows that \( \overrightarrow{CA} - \overrightarrow{CA} = -uq \), and that \( \overrightarrow{CQ} = \overrightarrow{CA} + \overrightarrow{CA} \)

\[ = w^2p + uq + p + q + w^2p \]

\[ = (w^2p + wp + p) + (uq + q) = -u^2q. \]

Thus, \( \overrightarrow{CQ} = w(\overrightarrow{CA}) \), which proves the theorem.

**Definition.** The side \( AB \) of the equilic quadrilateral \( ABCD \) is called the base.

A most interesting hexagon results when an equilic quadrilateral is reflected in the line containing its base.

**Theorem 8.** An equilic quadrilateral reflected in the line of its base forms a hexagon with the property that if equilateral triangles are erected exteriorly on any three alternate sides, then their third vertices form an equilateral triangle.

**Proof.** Let \( C' \) and \( D' \) be the reflections of \( C \) and \( D \), respectively, in line \( AB \). A 60° rotation about \( P \) takes \( PC'BQ \) into \( PD'AR \) and, therefore, \( Q \) into \( R \) since angle \( PC'B = \angle C + 60^\circ = \angle PDA \) and angle \( C'BQ = 120^\circ = \angle PDA \).

The reader is encouraged to investigate whether Theorem 8 holds if the word interiorly replaces exteriorly.

**Note:** It has been pointed out by one of the referees that "Problem 3524 [1932, page 559] of the American Mathematical Monthly states: To the vertices of an equilateral triangle \( ABC \) let there be hinged three equilateral triangles \( AKM, BNR, \) and \( CPQ \) of any sizes and positions, all four sensed counterclockwise. Then the midpoints of the segments in the trio \( (RP, QK, MN) \) form a counterclockwise equilateral triangle. Counterclockwise equilateral triangles are also formed by the"
midpoints of segments in each of the trios \((BQ, CN, RP), (KB, MN, RA)\) and \((AP, KQ, CM)\).

In Figure 8, triangle PQR has three equilateral triangles RAD, PC'D' and QCB hinged to its vertices. Problem 3524 applies to this figure, so the midpoints of CD, E and C'B form an equilateral triangle. By symmetry, so also do the midpoints of C'D', AD, and CB.

**Theorem 9.** The equilateral triangle formed by joining the midpoints of the diagonals and the midpoint of side AB of equilic quadrilateral ABCD and the equilateral triangle PQR of Theorem 7 are perspective.

![Figure 9](image)

**FIGURE 9**

Proof. In the second proof of Theorem 7, we have shown that \(\overrightarrow{RP} = -q\), \(\overrightarrow{PQ} = -q\), and \(\overrightarrow{QR} = -q\). Now LM is parallel to and equal to one-half of BC; therefore \(\overrightarrow{LM} = -\frac{1}{2}q\) and LM is thus parallel to QR.

Similarly, KM is parallel to and equal to one-half of DA. Therefore, \(\overrightarrow{KM} = -\frac{1}{2}q\) and KM is thus parallel to RP. It follows that QR is parallel to LK, completing the proof of the perspectivity of triangles PQR and KLM.

Surely, the reader can find additional properties of this prolific equilic quadrilateral. But we turn to a matter of perhaps greater significance. A most interesting procedure that leads to the discovery of new results and to simple proofs of known results is to allow a figure to degenerate to a familiar figure and to observe the properties as they transform after the degeneration. This will become clear, and the reader will be in a better position to appreciate the advantages of this method of discovery, after a few illustrations.

**Example 1.** Let equilic quadrilateral ABCD degenerate into a \(30^\circ, 60^\circ, 90^\circ\) right triangle where angle BCD = \(180^\circ\) with C the midpoint of the hypotenuse, then the equilateral triangle erected interiorly on AB and that erected exteriorly on DC have the same vertex.

Proof. The proof follows immediately from Theorem 3. See Figure 10.

![Figure 10](image)

**Example 2.** Using the same degeneration and applying Theorem 2, we get diagonal AC = CD = \(\frac{1}{2}BD\).
Example 3. Using the same degeneration and applying Theorem 7, we get an interesting theorem about a $30^\circ$, $60^\circ$, $90^\circ$ right triangle. If equilateral triangles are erected on $AC$, $BD$, and $AB$ in the directions as in Theorem 7, then their vertices are the vertices of an equilateral triangle.

Example 4. Again with the same degeneration, we obtain another interesting result about a $30^\circ$, $60^\circ$, $90^\circ$ right triangle by applying Theorem 4. The midpoint of the hypotenuse, the midpoint of the median to the hypotenuse and the midpoint of the side opposite the $90^\circ$ angle are the vertices of an equilateral triangle.

Other degenerations can be made with interesting consequences. Thus, if we allow equilic quadrilateral $ABCD$ to degenerate so that angle $ABC = 60^\circ$, as in Figure 11, some novel theorems emerge.

Furthermore, as in Figure 12, we can degenerate our figure into a negative equilic quadrilateral and consider the properties of this figure.

Example 5. The lines joining the midpoints of $AC$ and $BD$ with the midpoint of either $AB$ or $CD$ form an equilateral triangle. The proof follows from Theorem 4. See Figure 11A.

It should be noted that the angle at $S$ is negative in this case. Finally, we can consider the equilateral triangle itself as a degenerate equilic quadrilateral, with $DC = 0$. Again, applying the properties of the equilic quadrilateral, some well-known facts about the equilateral triangle pop out without effort.

We hope that we have convinced the reader that the equilic quadrilateral deserves a place alongside the well-known quadrilaterals.

REFERENCES
1. N. A. Court, College Geometry, Barnes and Noble, N.Y., 1960.

POSTERS AVAILABLE FOR LOCAL ANNOUNCEMENTS
We have a supply of 10 x 14-inch Fraternity Crests available. One in each color will be sent free to each local chapter on request. Additional posters may be ordered at the following rates:

(1) Purple on Goldenrod stock ---------------$1.50/dozen
(2) Purple on Lavendar on Goldenrod--------$2.00/dozen.
ADVENTURES IN (COMPUTERIZED) TOPOLOGY

by Gary Ricard
South Dakota School of Mines and Technology
SUMMARIES OF CHAPTER REPORTS

ARKANSAS BETA (HENDRIX COLLEGE) The Chapter had its usual very active year with the following presentations:

- Sandra Cousins, Hendrix College; "A Report on the Summer Meetings in Am. Arbor"
- Ben Schumacher, Hendrix College; "Polarimetry and the Clouds of Venus"
- Dr. Richard Rolleigh, Hendrix College; "The Calculus of Variations and the Brachistochrone Problem"
- Jerry Cober, Hendrix College; "Mathematical Games"
- Sandra Cousins, David Sutherland, Carol Smith, Hendrix College; "The Development of the Mathematical Method"
- Dr. Boris Schein, Univ. of Arkansas; "Mathematics Education in the Soviet Union"

David Sutherland, Hendrix College; "Non-Linear Derived Functions"
Sandra Cousins, Hendrix College; "Infinite Compositions"
Ben Schumacher, Hendrix College; "Exponential Calculus"
Carol Smith, Hendrix College; "Infinite Sums of Derivatives"
Dr. Jay McDaniel, Hendrix College; "Religion and Mathematics"

The Chapter hosted the Sixth Annual Conference on Undergraduate Mathematics on April 10-11, 1981 in which there were thirty-six student speakers and talks by six prominent mathematicians. Awards were won in the following categories:

- McKenry-Lane Freshman Math Award: Karen Cornell, David McCallum
- Hogan Senior Math Award: Sandra Cousins, Mike Pinter
- Phillip Parker Undergraduate Research Award: David Sutherland

GEORGIA BETA (GEORGIA INSTITUTE OF TECHNOLOGY) The Chapter presented book awards to The Outstanding Graduates in Mathematics:

- John J. Crittenden
- Brenda Jean Knowles
- Lynn Marie Ramsey

GEORGIA GAMMA (ARMSTRONG STATE COLLEGE) The heard the following papers at regular meetings:

- Dr. Natherton, Armstrong State; "An Algorithm for Computer Science Problem Solving"
- Dr. Netherton, Armstrong State; "In Pursuit of a Prime Number Generator"
- Dr. Charles Shipley, Armstrong State; "Paradoxes In and Around Mathematics"
- Andrew Zeigler, Armstrong State; "Contract Programming"
- Stephan Suchower, Armstrong State; "Theory of Superconducting Magnets"
- Dr. Richard Summerville, Christopher Newport College; "The Mathematical Context"

The award for The Outstanding Senior in Mathematics was given to Stephan Suchower.

ILLINOIS ZETA (SOUTHERN ILLINOIS UNIVERSITY-EDWARDSVILLE) The Chapter sponsored the Regional Illinois Council of Teachers of Mathematics High School Competition. They also organized a used mathematics textbook sale for the university with many of the books donated to area high schools.

IOWA ALPHA (IOWA STATE UNIVERSITY) Activities included the following talks:

- Prof. Jerold Mathews, Iowa State; "Ancient Mathematical Models"
- Prof. James Corrette, Iowa State; "Polynomial Approximation"
- Prof. Richard Sprague, Iowa State; "Construction of Regular Polygons"
- Joyce Schneider, Honeywell; "The Uses of Mathematics in Industry"
- Prof. James Carlson, Univ. of Utah; "Prime Numbers and Codes"

Departmental awards were presented as follows:

Outstanding Achievement on the Putnam Exam: William Somiky
Pi Mu Epsilon Scholarships: John Klem, Sudirman Maulin
Dio Lewis Holl Award: Lee Roberts
Gertrude Herr Adamson Awards for demonstrated ingenuity in Mathematics:
  - Rebecca Potter
  - William Somiky
  - Steven Seda
  - Gregory Anderson

LOUISIANA DELTA (SOUTHEASTERN LOUISIANA UNIVERSITY) During the past year the Chapter heard the following presentations:

- Dr. Billy Joe Holmes, Nicholls State; "PERT"
- Dale Nasser, Southeastern Louisiana; "The Effects of Projecting Two Mutually Perpendicular Simple Periodic Motions on a Screen"

In addition, the following awards were presented:

- Thomas K. Maddox Pi Mu Epsilon Award: Nancy Gautier, Frederick Day
- Margo David Award: Jari A. McGee
MINNESOTA DELTA (ST. JOHN'S COLLEGE) The Chapter sponsored the Mathematics and Humanities Conference on April 30 and May 1. The Conference had guest lecturers: Doris Schattschneider, Leonard Gillman, and Donald Koehler plus papers by undergraduate students.

MISSOURI GAMMA (ST. LOUIS UNIVERSITY, FONTEBONNE COLLEGE, AND MARYVILLE COLLEGE) The Chapter had an active year with papers and presentations as follows:

- Sister Harriet Ann Padberg, Maryville College; "A Mathematical Model: An Historic Note"; The James E. Case S.J. Memorial Lecture
- Prof. Robert Hogg, Univ. of Iowa; "Size of Loss Distribution" and "Statistics, Actuarial Science and the Future"
- Susan Burns, Culver-Stockton College; "A Mathematical Look at Tonality"
- Robert Gregory, SIU-Carbondale; "A Look at Solving Differential Equations Using a Separation of Variables Technique"
- Michael May, S.J., St. Louis Univ; "Some Sums of Sums and the Calculus of Finite Differences"
- Steven Lazorchak, SIU-Carbondale; "Sinusoidal Steady-State Analysis of Electrical Circuits Using the Phasor Concept"
- Freedy Desai, SIU-Carbondale; "Program Verification"
- Barney Smith, St. Louis University; "Magic Cards, Squares and Cubes"
- Prasanna Balabatala, SIU-Carbondale; "Microprogramming"
- Michael May, S.J., St. Louis University; "Notions of Infinity"
- Jagdish Singh, SIU-Carbondale; "Bit Slices in Microprogramming"

The Chapter's award presentation list is quite extensive and includes:

- James W. Garneau Mathematics Award: Thomas Blackwell
- Francis Regan Scholarship: Michael May, S.J.
- Missouri Gamma Undergraduate Award: Jeanne Dulle
- Missouri Gamma Graduate Award: Mark Hopfinger
- The Pi Mu Epsilon Contests: Senior Winner: Daniel Kinner; Junior Winner: James Shamess
- John J. Andrews Graduate Service Award: Kara Ryan
- Berardinio Family Fraternityship Award: Michael May, S.J.

NEW JERSEY DELTA (SETON HALL UNIVERSITY) The Chapter held two meetings which were problem solving sessions conducted by John Saccoman.

NEW YORK EPSILON (ST. LAWRENCE UNIVERSITY) The Chapter sponsored the 37th annual Pi Mu Epsilon Interscholastic Mathematics Contest. The Chapter made the following award:

- The O. Kenneth Bates Award: Denise Martinez

NEW YORK ALPHA ALPHA (QUEENS COLLEGE OF CUNY) The following talks were presented:

- Vh. Joel Stemple, Queens College; "The Four Color Problem"
- Steven Kahan, Queens College; "Alphabets: Letters Where the Numbers Ought to Be"

The Pi Mu Epsilon Prize for Excellence in Mathematics and Service was won by Joel Kreitzer and Wendy Cannel.

OHIO NU (UNIVERSITY OF AKRON) The Chapter awarded a prize to Kendall Cmey for Excellence in Mathematics.

OKLAHOMA GAMMA (CAMERON UNIVERSITY) Among their many activities the Chapter heard the following paper:

- Dr. William Ray, University of, Oklahoma; "Discrete Predator-Prey Problems"

PENNSYLVANIA NU. Among the various talks presented were the following:

- James Watson; "Iteration Techniques"
- Dr. John Lane; "Number Density"

SOUTH CAROLINA GAMMA (COLLEGE OF CHARLESTON) Chapter members are very involved on the college campus and in the surrounding communities. Some of the members are currently involved in a Junior high school project where they teach sixth, seventh and eighth graders how to use computers. One of the members is helping to prepare packets for Computer Assisted Instruction in Mathematics. Several members do volunteer tutoring for the Charleston County PATHE program. The Chapter sponsored the 4th Annual Math Meet with over 600 high school students participating.

SOUTH DAKOTA BETA (SOUTH DAKOTA SCHOOL OF MINES AND TECHNOLOGY) The Chapter constructed a What Math Do I Take brochure for distribution to state high schools. The following papers were presented:

- Prof. Roger Opp, SSVSMST; "A Classification of Projectile Paths"
- Prof. David Balles, SSVSMST; "Employment Opportunities"
- Prof. Al Grimm, SSVSMST; "The Cubic Equation"
- Dr. Francis Florey, Univ. of Wisconsin-Superior; "Generalized Inner Products With Applications of Fourier Series"
- Janet Potts, SSVSMST; "Robotics"
- Dean Vogch, SSVSMST; "An Application of Game Theory in Taking Tests"
- Leon Nelson, SSVSMST; "On Trisecting the Angle"
- Gary Ricard, SSVSMST; "Parametric Methods in Computer Graphics"
Brian Bunsness, SDSMST; "Analyticity and Taylor's Series"  
Colleen Quatter, SDSMST; "The Evolution of Computer Languages"  
The Chapter sponsored the Annual West River Mathematics Contest for High School Students, and initiated the South Dakota Collegiate Mathematics Contest won by Northern State College.

THE SUMMER MEETING OF PI MU EPSILON, 1981

The following papers were presented at the Summer Meeting in Pittsburgh:

Beth Snyder, Miami University: "Introduction to Box-Jenkins Time Series"

Dean Mrog, S. D. School of Mines and Technology: "Stokes' Theorem For Quaternion Integral Operators"

Dean Shea, St. John's University; "A Look at Formal Theory"

Edward D. Lowy, Western Washington University; "An Approximation to the Normal Distribution"

Bro. Longinus Anganu, Morgan State University; "The Gamma Function and Extensions"

Ravi Salgia, Loyola University; "Dirichlet Integrals and Their Applications"

Brian Summer, University of Denver; "Transformation of Computer Programs into Functions"

Robert Kear, East Carolina University; "A Complex Parabola in Four Dimensions"

Margaret R. Wallace, Miami University; "Using the Method of Maximum Likelihood Estimation in Genetics"

Kevin Saylors, Pomona College; "Rubio's Magic Cube"

James F. Goche, S.J., St. Louis University; "Descartes: Philosopher or Mathematician?"

Dean Follmann, Northern Illinois University; "A Non-Parametric Multiple Comparison Test for Differences in Variances"

Brian Bunsness, S. D. School of Mines and Technology; "The Relations of Differentiable Functions and the Power Series"

Donna I. Ford, Miami University; "Mazes and Their Passage"

Elias Kosmas, Oklahoma State University; "Mathematical Analysis of Inflation"

The J. Sutherland Frame Lecture

Professor E. P. Miles, Jr., Florida State University; "The Beauties of Mathematics Revealed in Color Block Graphs"

PUZZLE SECTION
Edited by
David Ballow

This Department is for the enjoyment of those readers who are addicted to working crossword puzzles or who find an occasional mathematical puzzle attractive. We consider mathematical puzzles to be problems who's solutions consist of answers immediately recognizable as correct by simple observation and requiring little formal proof. Material submitted and not used here will be sent to the Problems Editor if deemed appropriate for that Department.

Address all proposed puzzles and puzzle solutions to David Ballow, Editor of the Pi Mu Epsilon Journal, Department of Mathematical Sciences, South Dakota School of Mines and Technology, Rapid City, South Dakota, 57701. Deadlines for puzzles appearing in the Fall issue will be the next February 15, and puzzles appearing in the Spring issue will be due the next September 15.

Mathacrostic No. 13
submitted by Joseph D. E. Konhauser
Macalester College, St. Paul, Minnesota

Like the proceeding puzzles, this puzzle (on the next page) is a keyed anagram. The 248 letters to be entered in the diagram in the numbered spaces will be identical with those in the 29 keyed words at matching numbers, and the key letters have been entered in the diagram to assist in constructing your solution. When completed, the initial letters will give a famous author and the title of his book; the diagram will be a quotation from that book. (See an example solution in the solutions section of this Department.)

Cross Word Puzzle

submitted by Alexander Neffegey Jr. and Curt Olson
The University of South Dakota

This crossword puzzle (three pages forward) is a standard crossword puzzle with a mathematical flavor.
Definitions

A. a ball-and-socket joint

B. a redundant account

C. analysis which admits infinitely small quantities (comp.)

D. S., European climbing plant bearing fragrant flowers (2 wds.)

E. "The laws of nature are but the mathematical ______ of God." Kepler

F. spiral with polar coordinate equation \( r = \cos^3(8 \pi) \)

G. providing aid or direction in the solution of a problem

H. Cauchy's single-limit analysis

I. a number greater than half of a total

J. creative power (2 wds.)

K. heating element (plug) in a --- Diesels engine

L. "All human knowledge thus begins with ___.," Kant, Critique of Pure Reason (followed by Z and b)

M. one of Thorn's seven elementary catastrophes

N. Cantor discard (2 wds.)

O. specialist in diagnosis and treatment of non-surgical diseases

P. systematic or random repetition

Q. polygon divisible into congruent ones similar to it (comp.)

R. perfectly simple (cap.)

S. manipulative puzzle rage of the 1980's (2 wds.)

T. contrary; antithetical

U. nonsense; something trivial (COB-)

V. "father" of descriptive geometry (1746-1818)

W. truncate

X. its ears are its radiators

Y. in De Thiende (1585), he introduced decimal fractions for general purposes (1548-1620)

Z. "proceeds thence to ___." (follows L, followed by b)
a. Danish poet, designer, inventor of Hex and Soma Cube, b. 1905
b. "and ______." (follows E, 3 wds.)
c. carrying back

table with rows and columns, each containing alphanumeric characters.
**Across**

<table>
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<th>1.</th>
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<td>Symbol for the population mean</td>
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<td>Symbol for the population mean</td>
<td>Every journal needs one</td>
<td>Transposition of breakfast cook's order</td>
<td>Isaac Newton was called</td>
<td>First woman number</td>
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<td>Killed in a duel (initials)</td>
<td>XIV hours</td>
<td>x^2 + y^2 + z^2 = a^2</td>
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<td>Above x indicates a simple mean</td>
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<td>Eleven across goes over this (without vowel)</td>
<td>Professional Organization (ab)</td>
<td>Cantor's concept</td>
<td>Minimal area among the 50. (ab)</td>
<td>Gauss' first name</td>
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<td>Connective</td>
<td>Connective</td>
<td>x dy = y dx (ab)</td>
<td>Perfect score</td>
<td>Student's distribution</td>
<td>The &quot;Slasher&quot;</td>
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</tbody>
</table>

**Down**

- 1. Conic Section
- 2. Operation in Lattice Theory
- 3. Calculus is taken by the (ab)
- 4. Every journal needs one (ab)
- 5. The 1981 Meeting was in this state (ab)
- 6. This puzzle has little of this
- 7. A mathematician wrote this to communicate results to others before journals
- 8. Product of a complex number and its conjugate nickname for proponent
- 12. Isaac Newton was called this
- 13. Mathematician's Organization
- 15. Transposition of breakfast cook's order (ab)
- 20. Min of triangle fame
- 21. Unit of length
- 22. Published first non-Euclidean geometry (initials)
- 23. Positive direction of y-axis

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**Definitions and Key:**

| A. Left bower | H. Student | 0. Raw data | V. Entwine |
| B. Algorithm | I. Paving | P. One two three | W. Arbelos |
| C. Not in your eyes | J. At the bath | Q. Ululant | X. Grapevine |
| D. Catastrophes | K. Clifford | R. Gnomon | Y. Ephebmeris |
| E. Zermelo | L. Ether | S. Half hitch | Z. Swineshead |
| F. Octahedron | M. Two handles | T. Troytown |
| G. Stone | N. Hyades | U. Human mind |

**Solutions**

Mathacrostic No. 12. (See Spring 1981 issue) [Proposed by J.D.E. Konhauser]

**Quotation:** The observations are the primary thing. Then comes the theory. Little would he have guessed that a few years later he himself would be Plato's astronomer who gazed down rather than up and by contemplation found the inner meaning of Newton's Law and its correction.

*Solved by:* Jeanette Bickley, Webster Groves High School, Missouri; Louis H. Cairoli, Kansas State University; Victor G. Feser, Mary College, Bismark; Robert Forsberg, Lexington, Mass.; Robert C. Gebhardt, Hatapon, NJ; Roger E. Kuehl, Kansas City, MO; Henry S. Leibennan, John Hancock Mutual Life Ins. Co.; Robert Prielipp, Univ. of Wisc. at Oshkosh; Sister Stephanie Sloyan, Georgian Court College; the Proposer and the Editor.
Cross Number Puzzles: (See Spring 1981 Issue) [Proposed by Mark Isaak]

Solved by: Dan Essig, Houston, Texas; Victor G. Feser, Mary College, Bismarck; Martha Hasting, St. Louis University; Murray Katz, Penn State University; Roger Kuehl, Kansas City, MO; The Proposer and the Editor.

Problem Department

Edited by Clayton W. Dodge
University of Maine
and
Leon Bankoff
Los Angeles, California

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also acceptable. The choice of proposals for publication will be based on the editor's evaluation of their anticipated reader response and also on their intrinsic interests. Proposals should be accompanied by solutions if available and by any information that will assist the editor. Challenging conjectures and problem proposals not accompanied by solutions will be designated by an asterisk (*).

Problem proposals offered for publication should be sent to Dr. Leon Bankoff, 6360 Wilshire Boulevard, Los Angeles, California 90048.

To facilitate consideration of solutions for publication, solvers should submit each solution on separate sheets (one side only) properly identified with name and address and mailed before July 1, 1982 to Clayton W. Dodge, Mathematics Department, University of Maine, Orono, Maine 04469.

Contributors desiring acknowledgment of their proposals and solutions are requested to enclose a stamped and self-addressed postcard or, for those outside the U.S.A., an unstamped card or mailing label.

Problems For Solution

498. Proposed by R. S. Luthan, University of Wisconsin, Janesville.
Find the general solution of
\[ x^3 + y^3 + 3xy = 1. \]

The array below is defined by the following properties:

i) The entries are distinct positive integers.

ii) In each column, the entries are consecutive integers, top to bottom.

iii) In each row, each integer (except the first one, of course) is a multiple of the integer at its left.

1 7 511
2 8 512
3 9 513

a) Find a fourth column for this array

b) Find the minimal fourth column for this array, and show it is minimal.

c) Construct an array of 4 rows and 4 columns with the same properties. Is it minimal?

500. Proposed by Chuck Allison and Peter Chu, San Pedro, California.

A condemned prisoner is given a chance to escape execution. He is given two boxes capable of holding sixteen bottles each, and is required to place eight bottles of water and eight bottles of clear poison in those boxes leaving no box empty. He will then summon the guard who will then pick one box at random and then select a bottle from that box which the prisoner must drink. How should the prisoner arrange the bottles in the two boxes to maximize his probability of survival, and what is that probability?


A rectangle is inscribed inside a circle. The area of the circle is twice the area of the rectangle. What are the proportions of the rectangle?


Consider

\[ 2^k + 2^k = 1^k + 3^k \]

for \( k = 1 \),

and

\[ 2^k + 2^k + 2^k = 1^k + 1^k + 1^k + 3^k \]

for \( k = 1, 2 \).

Complete the equations

\[ 2^k + 2^k + 2^k + 2^k = ? \]

for \( k = 1, 2, 3 \),

and

\[ ? = ? \]

for \( k = 1, 2, 3, 4 \),

where the left side is a function of \( 2^k \) only, and the right side is a function of \( 1^k \) and \( 3^k \) only.

503. Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, Pennsylvania.

Find the equation of the ellipsoid \( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \) with minimum volume which shall pass through the point \( p(r, s, t) \), \( 0 < r < a \), \( 0 < s < b \), \( 0 < t < c \).

504. Proposed by Charles W. Trigg, San Diego, California.

In the square array of the nine non-zero digits

\[
\begin{array}{ccc}
9 & 2 & 6 \\
4 & 1 & 7 \\
8 & 3 & 5 \\
\end{array}
\]

the sum of the digits in each 2-by-2 corner array is 16. Find another arrangement of the nine digits in which the sum of the digits in each corner array is five times the central digit.

505. Proposed by John W. Howell, Littlerock, California.

A baseball team has all .300 hitters. They never steal a base, get picked off base or hit into a double play. And men on base advance only one base when there is a hit.

a) What is the probability of this team getting one or more runs in an inning?

b) What is the expected number of runs scored by this team per inning?


"The addition cryptarithm \( IN + THE = MOOD \) is not difficult, but the solution cannot be unique because \( N \) and \( E \) can be interchanged, and so can \( I \) and \( H \)."

"Even taking account of those interchanges," his friend replied, "there are still many different solutions."

"That is so," agreed the first, "but let me tell you the value of one of those four letters."

I could not hear the letter and the value he whispered to his colleague, but the reply was quite clear. "Ah, now the solution is unique except, of course, for the interchange of the two letters of the other pair, and it uses every digit that is an odd prime, too."
507. Proposed by Herbert R. Bailey, Robe. Polytechnic Institute, Terre Haute, Indiana.

A unit square is to be covered by three circles of equal radius. Find the minimum necessary radius.

508. Proposed by Bruce W. King, Burnt Hills, New York.

When Professor Umbigo asked his calculus class to find the derivative of \( y^2 \) with respect to \( x^2 \) for the function \( y = x^2 - x \), his nephew Socrates Umbigo found \( \frac{dy}{dx} \cdot \frac{y}{x} \) and obtained the correct answer. Help the professor to enlighten his nephew about taking derivatives.

509. Proposed by Jack Garfunkel, Queens College, New York.

Given a triangle \( ABC \) with its incircle \( I \), touching the sides of the triangle at points \( L, M, N \). Let \( P, Q, R \) be the midpoints of arcs \( NL, LM, MN \) respectively. Form triangle \( DEF \) by drawing tangents to the circle at \( P, Q, \) and \( R \). Prove that the perimeter of triangle \( DEF \) is \( \perimeter \) the triangle \( AN \).

Solutions


A pilot down at Aville asked a native how far it was to Btown and was told, "It's south 1500 miles, then east 1000 miles, or east 500 miles and south 1500 miles." How far was it directly?

Comment by Jimmy Griffith, Charlotte, North Carolina.

Once we know \( a, \beta, \) and \( \gamma \), we may find \( a \) at once by

\[
\cos \alpha = \cos \left( \frac{\beta}{2} - \beta \right) \cos \left( \frac{\gamma}{2} - \gamma \right) + \sin \left( \frac{\beta}{2} - \beta \right) \sin \left( \frac{\gamma}{2} - \gamma \right) \cos \alpha
\]

so the rest of the featured solution on pages 268-269 seems unnecessary.

Late solutions were received from Mike Call and George W. Rainey, Jr.

474. [Fall 1980] Proposed by Scott Kim, Artificial Intelligence Laboratory, Stanford University.

Knotted path: Consider a 2 by 3 by 7 block of unit cubical cells. Your task is to find a path moving from cell to adjacent cell, returning to the original cell so that the path traced is a 3-dimensional knot.

Each cell must be visited exactly once; two cells are adjacent only if they share a face.

Solution by the PROPOSER.

Number the elements on the lower level first row 1 to 7, second row 8 to 14, and third row 15 to 21. To each of these element numbers add 21 to get the corresponding element in the upper level. A path then is


See Figure 1.

Comment by Morris Katz, Namahoe, Maine.

It is easy to show that, if \( DE/EC \) is rational, then triangle ICG has sides proportional to a Pythagorean triangle, and conversely, if triangle
Since 2520 is the smallest number divisible by each of 2 through 10, the 2519th row is the first such row. Any row numbered 2520k - 1, k a positive integer, has this property. For k = 1, these elements are

1, 2519, 2519, 2518, 2519, 2518, 2517, ...


PIGS = ROOT + ROOT + ROOT.

but can only dig up a single solution when each different letter represents a distinct digit, and PIGS contains three consecutive odd digits.

What is the unique representation of the addition?

Solution by Kenneth H. Wilke, Topeka, Kansas.

Clearly neither T nor S can be 5 or 0, and R = 1, 2, or 3. Taking the statement that PIGS contains three consecutive odd integers true in both senses, they must be (1, 3, 5), (3, 5, 7), (5, 7, 9) or their reversals, and the remaining digit of PIGS must be 6 or 9. Of the 16 possibilities, only PIGS = 6357 yields a solution: ROOTS = 2119. If all 120 permutations of the 5 possible sets of numbers are tested, again only the stated solution survives.


479. [Fall 1980] Proposed by Herbert Taylor, South Pasadena, California.

Prove that the following statement is true whenever 0 < r ≤ n,

or else find a counterexample:

Given a 2n x n matrix of 0's and 1's, with each column sum equal to 2r and each row sum equal to r, it is always possible to mark 2n of the 1's in such a way that one 1 is marked in each row and two 1's are marked in each column.

Solution by Robert Henderson, South Pasadena, California.

Let B = \([A|A]\). Then B is 2n x 2n with all row and column sums = 2r. The theorem of Philip Hall shows that we can select 2n ones no two of which lie in the same row or column of B (e.g., H. J. Kyser, Combinatorial Mathematics, p. 57). These 2n ones, as ones of A, satisfy the conditions of the claim.

a) Cut the large piece at right into two pieces which can be reassembled with piece a into an 8 x 8 square.

b) Do the same, using piece b.

I. Solution by Roger E. Kuehl, Kansas City, Missouri.

11. Solution by Peter Szabaga, New York City.

a) Cut the figure along the dashed lines, turn the piece in the lower right counterclockwise 90°, and insert it with piece a as shown below.

b) Cut the figure along the dashed lines, turn the piece in the lower right clockwise 90°, and insert it with piece b as shown below.

Also solved by DANIEL ESSIG, STEPHEN HODGE, STEPHEN L. SNOVER, and the proposer.

481. [Fall 1980] Proposed by Clayton W. Dodge, University of Maine at Orono.

Find all roots of the polynomial equation

\[ ax^6 - x^5 + 4x^4 + 5x^3 + 4x^2 + 36x - 36 = 0, \]
given that it has two roots whose sum is zero.

Solution by Leo Sauvé, Algonquin College, Ottawa, Canada.

Let the two roots in question (which must be nonzero) be \( \pm t \). If the given equation is denoted by \( f(x) = 0 \), then \( 2t \) are also zeros of

\[ f(x) - f(-x) = -2ax(x+3)(x-3)(x^2+4). \]

Synthetic division soon yields \( f(\pm 3) = f(2i) = 0 \) and

\[ f(x) = (x+3)(x-3)(x^2+4)(x^2-2i). \]

The roots are \( \pm 3, \pm 2i, \) and \( -\frac{1}{2} \pm \frac{1}{2}i \sqrt{2} \).
Also solved by J. ANNULIS, JEANETTE BICKLEY, THOMAS D. DELSERTO,
DANIEL ESSIG, MARK EVANS, JACOB GARFUNKEL, ROBERT C. GEBHARDT, W. C. IGIPS. RALPH KING, JEAN LANE, HENRY S. LIEBERMAN, JAMES A. PARSLEY, BOB PRIELIPP, TAGHI REZAY-GARACANI, SAHIB SINGH, ROBERT A. STUMP, PETER SZABABA, KENNETH M. WILKE, and the PROPOSER.

482. [Fall 1980] Proposed by Ronald E. Shiffner, Georgia State University.

Let X be a continuous random variable having a uniform distribution with domain \([a, b]\) and mean and standard deviation represented by \(\mu\) and \(\sigma\), respectively. Verify that

\[
P(\mu - 2\sigma < X < \mu + 2\sigma) = 1
\]

Solution by Bob Priellp, University of Wisconsin-Oshkosh, Wisconsin.

It is known that \(\mu = (a+b)/2\) and \(\sigma^2 = (b-a)^2/12\) [see Table 4.1 on p. 83 of LINDGREN AND MCELRATH, Introduction to Probability and Statistics, The Macmillan Co., New York, 1959]. It follows that

\[
2\sigma = \frac{(b-a)}{\sqrt{3}}
\]

Because \(\mu\) is the midpoint of \([a, b]\), \(|X - \mu| \leq (b-a)/2\). But \(\sqrt{3} < 2\) so \(1/2 < 1/\sqrt{3}\), and hence \((b-a)/2 < (b-a)/\sqrt{3}\). Therefore \(|X - \mu| < \frac{(b-a)}{\sqrt{3}}\) = \(2\sigma\) so \(P(\mu - 2\sigma < X < \mu + 2\sigma) = 1\).

Also solved by DANIEL ESSIG, MARK EVANS, ROBERT C. GEBHARDT, JOHN M. HOWELL, W. C. IGIPS, HENRY S. LIEBERMAN, SAHIB SINGH, and the PROPOSER.

483. [Fall 1980] Proposed by Paul Erdos, Spaceship Earth.

Let \(n\) be the smallest integer for which \(\mu_n(u_n + 1) \equiv 0\) (mod \(n\)).

Prove \(\sum \frac{1}{\mu_n(u_n + 1)} < \infty\).

Solution by Irwin Jungreis, No. Woodmere, Nj York.

We have

\[
\sum_{n=1}^{\infty} \frac{1}{\mu_n(u_n + 1)} = \sum_{n=1}^{\infty} \frac{\lambda_n}{\mu_n(u_n + 1)}
\]

where \(\lambda_n\) is the number of values of \(n\) for which \(\lambda_n = \mu_n\). From the definition of \(\mu_n, \mu_n = \lambda_n \leq \tau(\lambda_n + 1)\) so \(\lambda_n \leq \tau(\lambda_n + 1)\). We know, however, that \(\tau(x) = o(x^2)\) for any \(\epsilon > 0\). Taking \(\epsilon = 1/4\), there is \(N\) such that

\[
\tau(n) < n^{1/4}.
\]

Then

\[
\sum_{\tau=1}^{\infty} \frac{\lambda_n}{\mu_n(u_n + 1)} < \sum_{\tau=1}^{\infty} \frac{\tau(\tau + 1)}{\tau(\tau + 1)} < \sum_{\tau=1}^{N} \frac{\tau(\tau + 1)}{\tau(\tau + 1)} + \sum_{\tau=N+1}^{\infty} \frac{1}{3/2} < \infty.
\]

Also solved by the PROPOSER.

484. [Fall 1980] Proposed by the late R. Robinson Rowe.

In a triangle with base \(AB\) and vertex \(C\), secants from \(A\) and \(B\) to points \(D\) and \(E\) on \(BC\) and \(AC\) divide the area into four subareas \(S, T, U, V\). In some order of \(S, T, U, V\), the points \(D\) and \(E\) can be located so that the subareas are in increasing arithmetical progression, or so that they are in decreasing arithmetical progression. Find that order and evaluate the subareas.

Solution by the PROPOSER and Morris Katz, Macauhac, Maine.

Let triangle \(ABC\) have base \(AB\) of length 2 and altitude to vertex \(C\) equal to 1, without loss of generality. Let \(AD\) and \(BE\) meet at \(F\), and let \(D, E, F\) have altitudes \(x, y\), and \(h\) from base \(AB\). Designate by \(S, T, U, V\) the areas of triangles \(FAB, AEF, BDF,\) and quadrilateral \(CDFE\). See Figure 1. From similar triangles obtain

\[
h = \frac{xy}{x + y - xy}.
\]

We also have

\[
S = h, T = y - h, U = x - h,\text{ and } V = 1 - x - y + h.
\]

Now \(S < U\) implies \(T < S\), so \(S\) lies between \(T\) and \(U\). Hence, by symmetry, we need consider just two orders: \(VTSU\) and \(TSUV\).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Figure 1}
\end{figure}

For the order \(VTSU\), either increasing or decreasing, we must have

\[
U - S = S - T = T - V,
\]

\[
x - 2h = 2h - y = 2y - 2h + x - 1,
\]

\[
\tau(x) = o(x^2)\text{ for any } \epsilon > 0\.
\]

Taking \(\epsilon = 1/4\), there is \(N\) such that

\[
\tau(n) < n^{1/4}.
\]

Then

\[
\sum_{\tau=1}^{\infty} \frac{\lambda_n}{\mu_n(u_n + 1)} < \sum_{\tau=1}^{\infty} \frac{\tau(\tau + 1)}{\tau(\tau + 1)} < \sum_{\tau=1}^{N} \frac{\tau(\tau + 1)}{\tau(\tau + 1)} + \sum_{\tau=N+1}^{\infty} \frac{1}{3/2} < \infty.
\]
whence \( y = \frac{1}{2} \). Now substituting into (1) and (2), obtain

\[ 2x^2 - 5x + 1 = 0, \]

so

\[ x = \frac{5 \pm \sqrt{17}}{4} \]

Only the negative sign permits \( x < 1 \), so

\[ x = \frac{5 - \sqrt{17}}{4}, \ y = \frac{1}{2}, \] and \( h = \frac{7 - \sqrt{17}}{16} \)

We find that

\[ V = \frac{3\sqrt{17} - 5}{16}, \ T = \frac{1 + \sqrt{17}}{16}, \ S = \frac{7 - \sqrt{17}}{16}, \] and \( U = \frac{13 - 3\sqrt{17}}{16} \)

For TVSU we use

\[ U - S = V - V = V - T, \]

(3) \hspace{1cm} \[ x - 2h = x + y - 1 = 1 - 2y = x + 2h. \]

The solution is not as simple here, but using (1) and (3) we eliminate \( x \) and \( h \) to get

\[ 4y^3 - 17y^2 + 14y - 3 = 0, \]

which has no rational roots. Its decimal roots are

\[ y_1 = .353116, \ y_2 = .655199, \] and \( y_3 = 3.2416 \).

We cannot use \( y_3 \) because \( y < 1 \). We have two solutions

\[ x_1 = \frac{3}{2} - 2y = .793678, \ y_1 = .353116, \ h_1 = \frac{1 - y}{2} = .323442, \]

\[ T_1 = .029674, \ V_1 = .176558, \ S_1 = .353442, \ U_1 = .470326 \]

and

\[ x_2 = .189803, \ y_2 = .655199, \ h_2 = .172451, \]

\[ T_2 = .482648, \ V_2 = .327549, \ S_2 = .172451, \ U_2 = .017351. \]

485. [Fall 1980] Proposed by R. S. Luther, University of Wisconsin, Madison.

A line \( l \) cuts two parallel rays emanating from \( L \) and \( M \) at \( A \) and \( B \) respectively. A point \( C \) is taken anywhere on \( l \). Lines through \( A \) and \( B \) respectively parallel to \( MC \) and \( LC \) intersect in \( P \). Find the locus of \( P \).

Solution by the Proposer.

Let \( AP \) intersect line \( LM \) at \( T \). Then, according to Problem 409 (this JOURNAL, Fall 1978, page 556) by ZELDA KATZ, \( BT \) must be parallel to \( CL \). Thus \( BP \) and \( BT \) coincide because they are both parallel to \( CL \). Hence \( T \) and \( P \) coincide; that is, \( P \) lies on line \( LM \).

Also solved by ROBERT C. GEBHARDT (by analytic geometry), RALPH KING (graphical solution), ROGER E. KUEHL, HENRY S. LIEBERMAN (two solutions, one by a theorem of Pappus, the other by Grassmann's geometric algebra), and SISTER STEPHANIE SLOYAN (by the converse to Pascal's theorem).
1980–81 STUDENT PAPER COMPETITION

The papers for the 1980–81 Student Papa Competition have been judged and the winners are:

First Prize ($200)
Gary Ricard, South Dakota School of Mines and Technology; "A Comparison of Computer Algorithms To Solve the Knight's Tour Problem"

SECOND PRIZE ($100)
Sandra Cousins, Hendrix College; "Singular Functions", To Appear in the Spring 1982 Issue.

THIRD PRIZE ($50)
Michael Orrick, Macalester College; "The Area of a Triangle Formed by Three Lines", Appearing in this Issue.

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