PI MU EPSILON JOURNAL

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CONTENTS

PI MU EPSILON JOURNAL

THE OFFICIAL PUBLICATION OF THE HONORARY MATHEMATICAL FRATERNITY

David Ballew, Editor

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EDITORIAL

The Pi Mu Epsilon Journal is dedicated to undergraduate and beginning graduate students, and the Journal solicits articles of all kinds from them. We believe that many students solve problems, discover theorems, write papers, give talks and seminars, and complete projects that are suitable for submission to this publication. We encourage these students to prepare and submit these as articles for possible publication. Student papers will always be given first preference by this Editor. Many students have found it very advantageous to have had an article published in a refereed journal when considered for graduate school or employment.

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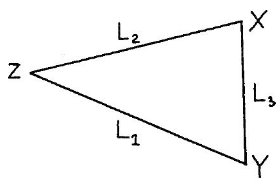
The Editor

THE AREA OF A TRIANGLE FORMED BY THREE LINES

by Michael L. Orrick, Jr.
Macalester College



One way to determine a triangle is to specify three noncollinear points $X(x_1, x_2)$, $Y(y_1, y_2)$ and $Z(z_1, z_2)$ to be used as vertices (Figure 1). It is well known [Noble, Daniel, 1977, p. 209] that the area, A, of the triangle is given by the formula:

$$(1) \qquad A = \pm\frac{1}{2}\begin{vmatrix} x_1 & x_2 & 1 \\ y_1 & y_2 & 1 \\ z_1 & z_2 & 1 \end{vmatrix}$$

where the sign is chosen to make A positive.

Another way to determine a triangle is to specify three non-current lines, no two parallel

$$(2) \qquad \begin{aligned} L_1&: a_1x + a_2y + a_3 = 0 \\ L_2&: b_1x + b_2y + b_3 = 0 \\ L_3&\ \ c_1x + c_2y + c_3 = 0 \end{aligned}$$

which enclose the triangle (Figure 1). Though it is an old result [Salmon, 1879, p. 32], it is not so well known that the area, A, of the triangle is also given by the formula

$$(3a) \qquad A = \pm\frac{1}{2}\left[\frac{1}{(b_1c_2-b_2c_1)(a_1c_2-a_2c_1)(a_1b_2-a_2b_1)}\right]\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2$$

The purpose of this note is to prove the formula (3a) using

notation and methods familiar to students taking a first course in linear algebra.

We begin by forming the coefficient matrix P of the system (2) and the matrix Q.

$$P = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \qquad Q = \begin{bmatrix} M_{a_1} & M_{a_2} & M_{a_3} \\ M_{b_1} & M_{b_2} & M_{b_3} \\ M_{c_1} & M_{c_2} & M_{c_3} \end{bmatrix}.$$

where M_{a_i}, M_{b_i} and M_{c_i} in Q are cofactors of elements a_i, b_i and c_i in P. For example,

$$(4) \qquad M_{a_3} = \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \qquad M_{b_3} = -\begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix} \qquad M_{c_3} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}.$$

We note that the condition that no two lines are parallel to each other implies that the cofactors M_{a_3}, $\setminus$, and M_{c_3} are all non-zero. Further-more, with this notation, formula (3a) becomes

$$(3b) \qquad A = \pm\frac{1}{2}\left[\frac{1}{(M_{a_3}\cdot M_{b_3}\cdot M_{c_3})}\right]\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2.$$

Is it possible for the determinant of matrix P, $\det P$, to equal zero? If it is, there will exist a non-trivialsolution (s_1, s_2, s_3) to the system

$$(5a) \qquad \begin{aligned} a_1s_1 + a_2s_2 + a_3s_3 &= 0 \\ b_1s_1 \text{ t } b_2s_2 + b_3s_3 &= 0 \\ c_1s_1 + c_2s_2 + c_3s_3 &= 0 \end{aligned}$$

If s_3 # 0, then $(s_1/s_3, s_2/s_3, 1)$ is also a solution to the system of equations in (5a). Thus all three lines of system (2) pass through the point $(s_1/s_3,\ s_2/s_3)$, violating the condition that these lines be non-concurrent. But if $s_3 = 0$, then (s_1, s_2) would be a non-trivial solution to the system

$$\text{(5b)}\qquad \begin{aligned} a_1s_1 + a_2s_2 &= 0 \\ b_1s_1 + b_2s_2 &= 0 \\ c_1s_1 + c_2s_2 &= 0\ . \end{aligned}$$

This is impossible, since all the determinants in (4) are non-zero.

Using Cramer's rule, we find that the coordinates of the vertices, $X(x_1,x_2)$, $Y(y_1,y_2)$ and $Z(z_1,z_2)$ are expressed as follows:

$$x_1 = \frac{\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}} = \frac{M_{c_1}}{M_{c_3}} \qquad x_2 = \frac{\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}}{-\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}} = -\frac{M_{c_2}}{M_{c_3}}$$

$$\text{(6)}\qquad y_1 = \frac{\begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix}} = \frac{M_{b_1}}{M_{b_3}} \qquad y_2 = \frac{\begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix}}{-\begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix}} = \frac{M_{b_2}}{M_{b_3}}$$

$$z_1 = \frac{\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}}{\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}} = \frac{M_{a_1}}{M_{a_3}} \qquad z_2 = \frac{\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}}{-\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}} = \frac{M_{a_2}}{M_{a_3}}$$

Using formula (1), we can obtain the area of the triangle

$$\text{(7)}\qquad A = \pm\frac{1}{2}\begin{vmatrix} \frac{M_{a_1}}{M_{a_1}} & \frac{M_{a_2}}{M_{a_2}} & 1 \\ \frac{M_{b_1}}{M_{b_3}} & \frac{M_{b_2}}{M_{b_3}} & 1 \\ \frac{M_{c_1}}{M_{c_3}} & \frac{M_{c_2}}{M_{c_3}} & 1 \end{vmatrix} = \pm\frac{1}{2}\left[\frac{1}{(M_{a_3}\cdot M_{b_3}\cdot M_{c_3})}\right]\begin{vmatrix} M_{a_1} & M_{a_2} & M_{a_3} \\ M_{b_1} & M_{b_2} & M_{b_3} \\ M_{c_1} & M_{c_2} & M_{c_3} \end{vmatrix}$$

$$= \pm\frac{1}{2}\left[\frac{1}{(M_{a_3}\cdot M_{b_3}\cdot M_{c_3})}\right]\det Q.$$

Evaluating $\det Q$, however, is a bit tedious. We therefore wish to simplify $\det Q$ to something more easily calculated. Consider the product $\det Q \det P$. Since a matrix and its transpose have the same determinant, $\det Q = \det Q^t$. Then:

$$\text{(8a)}\qquad \det P \det Q = \det PQ^t = \det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}\begin{bmatrix} M_{a_1} & M_{b_1} & M_{c_1} \\ M_{a_2} & M_{b_2} & M_{c_2} \\ M_{a_3} & M_{b_3} & M_{c_3} \end{bmatrix}.$$

Each entry on the main diagonal of the product, being the sum of the products of elements in a row of P multiplied by their respective co-factors, must equal det P. All other entries, being the sum of the elements in one row and the cofactors of a different row, must equal zero [Ficken, 1967, p. 263]. The product then simplifies to

$$\text{(8b)}\qquad \det P \det Q = \begin{vmatrix} \det P & 0 & 0 \\ 0 & \det P & 0 \\ 0 & 0 & \det P \end{vmatrix} = (\det P)^3\ .$$

Since $\det P \neq 0$, this implies that $\det Q = (\det P)^2$

We have then, for the area of the triangle,

$$(9) \qquad A = \pm \frac{1}{2}\left[\frac{1}{(M_{a_3} \cdot M_{b_3} \cdot M_{c_3})}\right](\det P)^2$$

which is the same result developed by Salmon in 1879.

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AN INFINITE NUMBER OF 4 × 4 MAGIC SQUARES

by *Stephen Ruberg*
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For many years magic squares have fascinated and intrigued mathematicians. The study of symmetry and unusual characteristics has 'been a favorite pasttime for those who like to dabble with numbers. A Normal magic square of order n is defined to be an arrangement of the first n^2 natural numbers in the cells of the square so that each row, column and diagonal sums to a magic constant. For this article, however, some variations -- using non-consecutive numbers, negative numbers and fractions -- will be employed. With this criterion, nonnormal magic squares for all integer sums and eventually any real number can be found for a square of order four.

The most familiar fourth order magic square is found by numbering the sixteen cells from right to left, top to bottom. Leaving the entries of the diagonals as they are, and exchanging the entries of the complementary cells (cells which are symmetric with respect to the center point of the square), a normal magic square is obtained (Fig. 1). The rows, columns, and diagonals all have the same magic constant, 34. Upon closer examination, however, the square has even more magic qualities. The four cells in the center, the four corners, the opposite pairs (the cells with numbers 5, 8, 9, 12 and 2, 3, 14, 15) and each quadrant also have the magic constant 34! These properties of doubly even magic squares have been known for quite some time. The question is what will happen if numbers other than the first sixteen integers are used.

With a few minor manipulations, magic squares with magic constants 35, 36, and 37 can be found. For the magic sum 35, the 34-square can be transformed by subtracting one from the cells containing 1 and 2 and adding one to the cells containing 11 through 16 (Fig. 2). For the sum of 36, add one to each cell containing the numbers 9 through 16 in the 34-square (Fig. 3). Once again, by increasing the cells containing 7 through 14 by one and the cells containing 15 and 16 by two, the 37-square is obtained (Fig. 4).

With these four squares as a basis, any 4 × 4 magic square can be obtained by adding an appropriate integer n to each of the sixteen cells of the square. In particular, subtracting eight from each cell of the 34-square and 35-square produces the 2-square and 3-square, respectively. Similarly, subtracting nine from each cell of the 36-square and 37-square produces the 0-square and the 1-square, respectively. Now adding an integer n to every cell of the 0, 1, 2, and 3-square, the general forms for any 4 × 4 magic square can be constructed (Fig. 5).

Because there have been fewer restrictions placed on the numbers which may fill a square, an infinite number of magic squares have been found. A magic square for a particular sum, however, need not be unique. If sixteen consecutive terms of any arithmetic sequence are placed in the cells in the same order as with the original 34-square, another magic square is created. Also, adding two magic squares or multiplying a magic square by a constant result in a magic square [3].

For a magic constant which is not an integer, several approaches can be used. To increase any integer sum by a decimal fraction (less than 1), add this fraction to any set of four cells which have one cell in each row, column, diagonal, the center, the corners and the opposite pair, forming a "complete set." An example is the cells containing the numbers 1, 3, 5 and 7 or 16, 14, 12 and 10 (the "complementary" numbers) in the 34-square. Also, adding one-half of the fraction to two complete sets or one-fourth of the fraction to all 16 squares will give the desired result. Another interesting technique is to consider the decimal part as an integer and find the magic square for it. Dividing by the appropriate negative power of 10 will reduce this magic square to its decimal form. Now add this decimal form to the integer magic square to obtain the desired magic square. For irrational sums, such as 4π or $17\sqrt{2}$, merely multiply the integer magic square by the appropriate irrational part. Thus, a magic square may be found for any real number magic constant.

The procedures here do not exhaust the possibilities. Many other combinations of numbers will produce magic squares possessing all of the magic properties mentioned here, perhaps even more. The symmetries and the order which are inherent in this size square and our number system are remarkable. There is a certain balance here, and the limitations seem to be only the limitations, if any, which are in the number system itself.

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1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16

FIG. 1
34-square

0	16	15	4
13	6	7	9
8	10	12	5
14	3	1	17

FIG. 2
35-square

1	16	15	4
13	6	7	10
8	11	12	5
14	3	2	17

FIG. 3
36-square

1	17	15	4
13	6	8	10
9	11	12	5
14	3	2	18

FIG. 4
37-square

n-8	n+7	n+6	n-5
n+4	n-3	n-2	n+1
n-1	n+2	n+3	n-4
n+5	n-6	n-7	n+8

(A)

n-8	n+8	n+6	n-5
n+4	n-3	n-1	n+1
n	n+2	n+3	n-4
n+5	n-6	n-7	n+9

(B)

n-7	n+7	n+6	n-4
n+4	n-2	n-1	n+1
n	n+2	n+3	n-3
n+5	n-5	n-6	n+8

(C)

n-8	n+8	n+7	n-4
n+5	n-2	n-1	n+1
n	n+2	n+4	n-3
n+6	n-5	n-7	n+9

(D)

FIG. 5

The general forms for magic squares having magic constants (A) 4n (B) 4n+1 (C) 4n+2 (D) 4n+3

GENERAL SOLUTION OF A GENERAL SECOND-ORDER LINEAR DIFFERENTIAL EQUATION

by R. S. Luthar
University of Wisconsin, Janesville

The following theorem was discovered while attempting to find a single method for solving a general second-order linear differential equation of the form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x). \tag{1}$$

The motivation for attack on such a method stemmed from the thought that when equation (1) is associated with a first-order linear differential equation of the form

$$\frac{dz}{dx} + B(x)z = R(x), \tag{2}$$

then z must necessarily be of the form

$$z = \frac{dy}{dx} + A(x)y\,. \tag{3}$$

Working backwards now with (3) and (2), we obtain

$$\frac{d}{dx}\left[\frac{dy}{dx} + A(x)y\right] + B(x)\left[\frac{dy}{dx} + A(x)y\right] = R(x),$$

which after taking derivative and rearranging terms takes the form

$$\frac{d^2y}{dx^2} + \left[A(x) + B(x)\right]\frac{dy}{dx} + \left[A(x)B(x) + A'(x)\right]y = R(x). \tag{4}$$

Comparing (1) and (4) we obtain

$$P(x) = A(x) + B(x)$$

and

$$Q(x) = A(x)B(x) + A'(x).$$

The above considerations lead us to state the following.

Theorem. A sufficient condition that the differential equation

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x) \qquad (1)$$

has a solution is that there exist two functions $A(x)$ and $B(x)$ such that

$$P(x) = A(x) + B(x)$$

and

$$Q(x) = A(x)B(x) + A'(x)$$

and in that case the solution is given by

$$y = e^{-\int A(x)dx}\left[\int e^{\int A(x)-B(x)\,dx}\left[\int R(x)e^{\int B(x)dx}dx + C_1\right]dx + C_2\right]$$

Proof. Under the given condition, (1) takes the form

$$\frac{d^2y}{dx^2} + [A(x) + B(x)]\frac{dy}{dx} + [A(x)B(x) + A'(x)]y = R(x)$$

$$\left[\frac{d^2y}{dx^2} + A(x)\frac{dy}{dx} + A'(x)y\right] + B(x)\frac{dy}{dx} + A(x)B(x)y = R(x)$$

$$\frac{d}{dx}\left[\frac{dy}{dx} + A(x)y\right] + B(x)\left[\frac{dy}{dx} + A(x)y\right] = R(x).$$

Letting

$$\frac{dy}{dx} + A(x)y = z, \qquad (5)$$

we have from the above equation

$$dz + B(x)z = R(x),$$

which is a linear differential equation of the first order whose solution by the usual methods is given by

$$z = e^{-\int B(x)dx}\left[\int R(x)e^{\int B(x)dx}dx + C_1\right].$$

Thus from (5) we have

$$\frac{dy}{dx} + A(x)y = e^{-\int B(x)dx}\left[\int R(x)e^{\int B(x)dx}dx + C_1\right],$$

which again is a linear differential equation of the first order whose solution by usual methods is given by

$$y = e^{-\int A(x)dx}\left[\int e^{-\int B(x)dx}\left[\int R(x)e^{\int B(x)dx}dx + C_1\right]e^{\int A(x)dx}dx + C_2\right]$$

$$= e^{-\int A(x)dx}\left[\int e^{\int (A(x)-B(x))dx}\left[\int R(x)e^{\int B(x)dx} + C_1\right]dx + C_2\right]. \qquad (6)$$

The function in (6) is the general solution of (1).

Example. 1.

Solve the differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 4y = e^x.$$

Solution.

Here $P(x) = -3$ and $Q(x) = -4$. We can thus take $A(x) = 1$ and $B(x) = -4$ and satisfy ourselves that

$$-3 = 1 + (-4)$$

and

$$-4 = (1)(-4) + 0.$$

Using (6) we have the solution

$$y = e^{-\int 1dx}\left[\int e^{\int (1-(-4))dx}\left[\int e^x . e^{\int -4dx}dx + C_1\right]dx + C_2\right]$$

$$= e^{-x}\left[\int e^{5x}\left[-\frac{1}{3}e^{-3x} + C_1\right]dx + C_2\right]$$

$$= -\frac{e^x}{6} + C_3e^{4x} + C_2e^{-x}$$

Example. 2.

Solve the differential equation

$$\frac{d^2y}{dx^2} - \frac{2}{x}\frac{dy}{dx} + \frac{2}{x^2}y = 4x.$$

Solution.

Here $P(x) = -\frac{2}{x}$ and $Q(x) = \frac{2}{x^2}$. We can thus take $A(x) = -\frac{1}{x}$ and $B(x) = -\frac{1}{x}$ and satisfy ourselves that

$$-\frac{2}{x} = \left(-\frac{1}{x}\right) + \left(-\frac{1}{x}\right) \text{ and } \frac{2}{x^2} = \left(-\frac{1}{x}\right)\left(-\frac{1}{x}\right) + \frac{d}{dx}\left(-\frac{1}{x}\right).$$

Hence by (6) we have the solution

$$y = e^{-\int -\frac{1}{x}\,dx}\left[\int e^{\int\left(-\frac{1}{x} - \left(-\frac{1}{x}\right)\right)dx}\left[\int 4xe^{-\int\frac{1}{x}\,dx}\,dx + C_1\right]dx + C_2\right]$$

$$= x\left[\int 1\left[\int 4x\cdot\frac{1}{x}\,dx + C_1\right]dx + C_2\right]$$

$$= 2x^3 + C_1x^2 + C_2x\ .$$

REGIONAL MEETINGS

Many regional meetings of the *Mathematical* Association of America *regularily* have sessions for undergraduate papers. If *two* or more colleges and at least one local chapter help sponsor or participate in such *undergraduate* sessions, *financial help* is *available* up to $50. Write to Dr. Richard Good, Department of Mathematics, University of Maryland, College Park, Maryland, 20742.

REFEREES

The following mathematicians have served as referees since the publication of the last issue; without their help, the Journal would not be possible, and the Editor is grateful for their cooperation: *Professor C. A. Grimm, Professor Roger Opp, Professor Ronald Weger, Professor Dale Rognlie, Professor Sailes Sengupta, Professor Harold Carda, all of The South Dakota School of Mines and Technology; Professor Barbara Reynolds, Professor Clayton Dodge of the University of Maine; Professor Steven Janke of Colorado College; Professor John Schumaker of Rockford College.*

120° and 60° PYTHAGOREAN TRIPLES

by Benedict Pollina and Stephen Snover
University of Hartford

Section 1. Introduction.

Finding all Pythagorean triples, that is, all integer solutions of the equation $x^2 + y^2 = z^2$ is a familiar problem. [1] In this article we consider the following generalization.

If triangle ABC has an angle of θ° and integer sides k,l,m, we call the triple (k,l,m) a θ° Pythagorean triple. Finding all 90° Pythagorean triples is just the familiar problem mentioned above. The problems of finding all 120° and 60° Pythagorean triples are not quite as well known, even though solutions were given by L. E. Dickson [2] in 1908. It is these latter two problems that we address in this article.

All such triples satisfy either $k^2 + kl + l^2 = m^2$ or $k^2 - kl + l^2 = m^2$. We derive a set of parametric equations which generate all integer solutions to these equations. Furthermore, we give conditions on the parameters so that each essentially different solution is generated exactly once. This latter consideration does not appear to have been addressed in the literature. We conclude by observing that the solutions to the 60°, 90°, and 120° problems provide a complete list of θ° Pythagorean triples with θ rational in degrees.

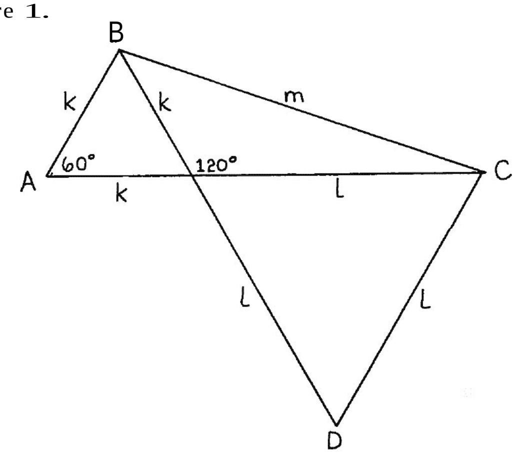
Section 2. Reduction of the Problems.

From the law of cosines it follows that if (k,l,m) is a 120° Pythagorean triple with side m opposite the 120° angle, then $m^2 = k^2 + l^2 - 2kl\cos(120^\circ) = k^2 + kl + l^2$. Similarly, if (k,l,m) is a 60° Pythagorean triple, then $m^2 = k^2 - kl + l^2$. We turn our attention, therefore, to the problem of finding all integer solutions to these two equations. Our Pythagorean triples are then just the solutions in which k, l, and m are all positive.

The following lemmas indicate that it is sufficient to solve the $k^2 + kl + l^2 = m^2$ problem. Both are easily established by straightforward computation.

Lemma 1. If the triple (k,l,m) satisfies $k^2+kl+l^2=m^2$, then the triples $(x,y,z) = (k,k+l,m)$ and $(x,y,z) = (l,k+l,m)$ satisfy $x^2-xy+y^2=z^2$.

Remark. If we assume that k,l,m are all positive so that (k,l,m) is a 120° Pythagorean triple, then a geometric proof of Lemma 1 is indicated by Figure 1.



ABC has a 60° angle at A and integer sides k, $k+l$, m.
DBC has a 60° angle at D and integer sides l, $k+l$, m.

Lemma 1 states that each solution to $k^2+kl+l^2=m^2$ generates two solutions to $k^2-kl+l^2=m^2$. The next lemma shows that any solution to $k^2-kl+l^2=m^2$ arises in this way.

Lemma 2. If (k,l,m) satisfies $k^2-kl+l^2=m^2$, then $(x,y,z) = (k, l-k,m)$ satisfies $k^2+kl+l^2=m^2$.

Section 3. The Equations $k^2+kl+l^2=m^2$ and k^2+kl+l^2

Precisely stated, we wish to find all integers k,l,m which satisfy $k^2+kl+l^2=m^2$. While this problem has been studied in the literature [2, 3, 5] we present a complete, self-contained solution because other solutions seem to be incomplete or too general to be readily understood.

First we define a solution triple to be a triple of integers (k,l,m) satisfying $k^2+kl+l^2=m^2$. Next define a primitive triple to be a solution triple in which the integers are pairwise relatively prime, that is $(k,l) = (l,m) = (k,m) = 1$.

In order to establish the fact that any solution triple is a scalar multiple of some primitive triple, we first show

Lemma 3. If (k,l,m) is a solution triple, then $(k,l) = (l,m) = (k,m)$.

Proof. Let $d=(k,l)$, $e=(l,m)$, and $f=(k,m)$. Since $k^2+kl+l^2=m^2$, we have $d^2|m^2$. Hence $d|m$, and therefore $d|e$ and $d|f$. Let $e=de'$, $f=df'$, $k=dk'$, $l=dl'$, and $m=dm'$. Then $1=(k',l')$, $e'=(l',m')$, $f'=(k',m')$ and $k'^2+k'l'+l'^2=m'^2$. If p is any prime divisor of e', $p|l'$ and $p|m'$. Thus $p|m'^2-l'^2=(k'+l')k'$ and so either $p|k'+l'$ or $p|k'$. But $p|l'$, so in either case $p|k'$. Hence $p|(k',l')=1$ and $p=1$. Thus $e'=1$. Similarly $f'=1$, proving Lemma 3.

The next lemma reduces our problem to one of finding all primitive solution triples. Its proof is straightforward and so details are omitted

Lemma 4. If (k,l,m) is a solution triple and $d=(k,l)=(l,m)=(k,m)$, then (k',l',m') is a primitive triple where $k=k'd, l=l'd$, and $m=m'd$.

Lemma 5. If (k,l,m) is a primitive solution triple, then either $l-k\equiv m \pmod 3$ or $k-l\equiv m \pmod 3$, but not both.

Proof.

$$k^2+kl+l^2 = m^2$$

$$(k-l)^2 = m^2-3kl$$

$$(m+(k-l))(m-(k-l)) = 3kl$$

Thus $3|m+(k-l)$ or $3|m-(k-l)$. However, if 3 divides both, then $3|k$ or $3|l$. If $3|k$, then $3|m-l$ and $3|m+l$, which implies $3|2l$ which in turn implies $3|l$. This contradicts primitivity. A similar contradiction arises if $3|l$.

Now we proceed towards finding a parametric representation of all primitive solution triples. To begin we assume that the primitive triple (k,l,m) is always written so that $k-l\equiv m \pmod 3$. This can be achieved by interchanging k and l if necessary.

Theorem 1. Let p and q be integers with $(p,q)=1$ and $p\not\equiv q \pmod 3$. Then $k=p^2-q^2$, $l=2pq+q^2$, and $m=p^2+pq+q^2$ form a primitive solution triple for which $k-l\equiv m \pmod 3$.

Proof. Straightforward algebra shows $k^2+kl+l^2=m^2$ and $k-l\equiv m \pmod 3$. We must show that k,l, and m are pairwise relatively prime. To do this we use Lemma 6 below, which will also be useful later.

Lemma 6. Let p and q be integers with $(p,q)=1$. Then $(p^2-q^2, 2pq+q^2)=1$ if and only if $p\not\equiv q \pmod 3$.

Proof. Suppose $p\equiv q \pmod 3$; then $3|p-q$ and $3|p^2-q^2$. But $p-q\equiv p+2q \pmod 3$, so $3|p+2q$. We also have $3|(p-q)+(p+2q)=2p+q$, and hence $3|2pq+q^2$. Therefore $(p^2-q^2,\ 2pq+q^2)\neq 1$.

Now suppose $(p^2-q^2,\ 2pq+q^2) = d \neq 1$. Let x be any prime divisor of d. Now x divides neither p nor q; this follows since $x|p^2-q^2$ and hence if x divides one it also divides the other, contradicting $(p,q)=1$. Since $x|2pq+q^2$ we must have $x|2p+q$. Also $x|p^2-q^2$ implies $x|p-q$ or $x|p+q$. If $x|2p+q$ and $x|p+q$, then $x|p$, a contradiction. Hence $x|2p+q$ and $x|p-q$ and these in turn imply $x|3p$. Thus $x=3$ and $3|p-q$ making $p\equiv q \pmod 3$.

To complete the proof of Theorem 1 note that by Lemma 3 it is enough to show that any two of k,l,m are relatively prime. Lemma 6 now gives us $(k,l)=1$.

Next we prove the converse to Theorem 1. Define (k,l,m) to be a trivial solution triple if it is of the form $(-k,k,k)$ or $(k,0,k)$.

<u>*Theorem 2*</u>. If (k,l,m) is a non-trivial primitive solution triple, then there exist unique integers p and q with $p>0$, $(p,q)=1$ and $p\not\equiv q \pmod 3$ such that $k=p^2-q^2$, $l=2pq+q^2$, and $m=p^2+pq+q^2$.

Proof. Since $k^2+kl+l^2=m^2$ we have $l(k+l)=m^2-k^2=(m+k)(m-k)$, and since (k,l,m) is non-trivial we can write

$$\frac{l}{m+k} = \frac{m-k}{k+l} = t$$

where t is a non-zero rational number. This gives us a system of equations:

$$l-kt=mt\ , \qquad lt+(t+1)k=m\ .$$

Solving this system for k and l we get

$$k = \frac{(1-t^2)m}{t^2+t+1}\ , \qquad l = \frac{(t^2+2t)m}{t^2+t+1}$$

Now since t is rational, we let $t = q/p$ where p and q are integers with $p>0$ and $(p,q)=1$. Clearly the p and q satisfying these conditions are unique. Substituting for t we get

$$k = \frac{(p^2-q^2)m}{p^2+pq+q^2}\ , \qquad l = \frac{(2pq+q^2)m}{p^2+pq+q^2}\ .$$

Recall that $l/(m+k) = t = q/p$, and by assumption $k-l\equiv m \pmod 3$.

Lemma 5 implies that $l-k\not\equiv m \pmod 3$ and thus $l\not\equiv m+k \pmod 3$. These facts together with $(p,q)=1$ imply $p\not\equiv q \pmod 3$. Lemmas 3 and 6 now tell us that $(p^2-q^2,\ p^2+pq+q^2)=1$ and $(2pq+q^2, 2p^2+pq+2q^2)=1$. Thus $p^2+pq+q^2|m$, and since $(k,l)=1$ we must in fact have $p\ +pq+q\ =m$. Thus we can write

$$k=p^2-q^2\ , \qquad l=2pq+q^2\ , \qquad m=p^2+pq+q^2\ .$$

This concludes the proof of Theorem 2.

Summarizing the previous results we have

<u>*Theorem 3*</u>. All integer triples (k,l,m) for which $k^2+kl+k^2=m^2$ and $k-l\equiv m \pmod 3$ are either:

(1) trivial, that is of the form $(-k,k,k)$ or $(k,0,k)$

(2) non-trivial, that is, there exist unique integers p,q,r with $p>0$, $q\neq 0$, $r\neq 0$, $(p,q)=1$, and $p\not\equiv q \pmod 3$ such that $k=(p^2-q^2)r$, $l=(2pq+q^2)$, and $m=(p^2+pq+q^2)r$.

To find all solutions to $k^2-kl+l^2=m^2$ we can use Theorem 3 and Lemmas 1 and 2. First note that primitivity is defined as before and that any solution triple is a multiple of a primitive one. The next lemma indicates that there are essentially two distinct families of primitive solutions.

<u>*Lemma 7*</u>. If the triple of integers is a primitive solution to $k^2-kl+l^2=m^2$, then $k+l\equiv m \pmod 3$ or $k+l\equiv -m \pmod 3$, but not both.

The proof is similar to the proof of Lemma 5 and is omitted.

Observe that unlike the $k^2+kl+l^2=m^2$ problem we cannot choose which congruence we want to hold, since interchanging k and l leaves both unaffected. Combining Lemma 2 with Theorem 3 we have:

<u>*Theorem 4*</u>. All integer triples (k,l,m) for which $k^2-kl+l^2=m^2$ are either

(1) trivial, that is of the form $(-k,0,k)$, $(k,0,k)$, (k,k,k), $(0,k,k)$, $(-k,-k,k)$, or $(0,-k,k)$;

(2) non-trivial, that is, there exist unique integers p,q,r with $p>0$, $q\neq 0$, $r\neq 0$, $(p,q)=1$, and $p\not\equiv q \pmod 3$ such that

$$k=(p^2-q^2)r\ , \qquad l=(2pq+p^2)r\ , \qquad m=(p^2+pq+q^2)r\ ,$$

or

$$k=(2pq+q^2)r\ , \qquad l=(2pq+p^2)r\ , \qquad m=(p^2+pq+q^2)r\ .$$

It is an easy matter to see that the first non-trivial family corresponds to $k+l\equiv -m \pmod 3$ while the second corresponds to $k+l\equiv m \pmod 3$.

Section 4. Uniqueness of Solutions.

The solutions to the equations $k^2 \pm kl+l^2=m^2$ given in the previous section contain certain redundancies which arise because of the symmetry of the equations. For example, the integers $p=3$, $q=-1$, $r=1$ generate the solution triple $(k,l,m)=(8,-5,7)$ while $p=3$, $q=-2$, $r=1$ generate the triple $(k,l,m)=(5,-8,7)$. Both of these satisfy $k^2+kl+l^2=m^2$; however, it makes sense to regard these as essentially the same solution. In fact it is easy to see that for each equation, whenever (k,l,m) is a solution, several other triples are also solutions. Obvious examples of triples that are also solutions are $(k,l,-m)$, (l,k,m), $(-k,-l,m)$, and $(-l,-k,m)$. In total there are 24 related solutions for each of the equations. These are indicated in Table 1.

Table 1

$k^2+kl+l^2=m^2$	$k^2-kl+l^2=m^2$
(k,l,m)	(k,l,m)
$(k,l,-m)$	$(k,l,-m)$
$(l,k,\pm m)$	$(l,k,\pm m)$
$(-k,-l,\pm m)$	$(-k,-l,\pm m)$
$(-l,-k,\pm m)$	$(-l,-k,\pm m)$
$(-k,k+l,\pm m)$	$(-k,l-k,\pm m)$
$(k+l,-k,\pm m)$	$(l-k,-k,\pm m)$
$(k,-k-l,\pm m)$	$(k,k-l,\pm m)$
$(-k-l,k,\pm m)$	$(k-l,k,\pm m)$
$(-l,k+l,\pm m)$	$(-l,k-l,\pm m)$
$(k+l,-l,\pm m)$	$(k-l,-l,\pm m)$
$(l,-k-l,\pm m)$	$(l,l-k,\pm m)$
$(-k-l,l,\pm m)$	$(l-k,l,\pm m)$

Remark. If we take a solution triple (k,l,m) and consider all possible transformed triples of the form $(ak+bl,\ ck+dl,\ em)$ where a,b,c,d, and e are integers, then Table 1 contains precisely the transformed triples which are again solutions of the corresponding equation.

Clearly, for each of the two equations the related solutions form an equivalence class. By restricting the solutions given in Section 3 to solutions involving one and only one member of each equivalence class, we can eliminate redundant solutions. To do this we need to handle the two equations separately. The following lemma is easily established; we omit the proof.

Lemma 8. Among the equivalent primitive solutions to $k^2+kl+l^2=m^2$ given in Table 1, there is precisely one with k, l and m positive and $k-l \equiv m \pmod 3$.

We now restrict the values of p,q, and r in Theorem 3 so that we obtain only the representative given in Lemma 8. This gives us the following:

Theorem 5. The following is a complete non-redundant set of non-trivial solutions to $k^2+kl+l^2=m^2$:

$$k=(p^2-q^2)r \quad , \quad l=(2pq+q^2)r \quad , \quad m=(p^2+pq+q^2)r \quad ,$$

where p,q,r run through all integers satisfying $p>q>0$, $r>0$, $(p,q)=1$, and $p \not\equiv q \pmod 3$.

Proof. From Theorem 3 and Lemma 8 we have the following. The condition $m>0$ implies $r>0$, since $m=(p^2+pq+q^2)r$ and p^2+pq+q^2 is always positive for $p>0$. The condition $k-l \equiv m \pmod 3$ implies $p \not\equiv q \pmod 3$ as shown in the proof of Theorem 2. Next $k>0$ implies $p>q$, since $k=(p-q)(p+q)r$ and $p-q<0$ would require $q>p>0$ and $p+q>0$, contradicting $k>0$. Finally $l>0$ implies $q>0$ since

(i) $k>0$ and $l>0$ imply $k+l=(2pq+p^2)r>0$, which together with $p>0$ implies $q>-(p/2)$;

(ii) $l>0$ implies $(2pq+q^2)=q(2p+q)>0$, which in turn implies either $q>0$ or $q<0$ and $2p+q<0$. This latter choice implies $q<-2p<-(p/2)$, which contradicts (i).

Corollary 7. Once solutions to $k^2+kl+l^2=m^2$ are obtained with the restrictions given above, then all solutions are obtained by expanding the solution set 24-fold according to Table 1.

Eliminating redundancies in the solutions to $k^2-kl+l^2=m^2$ is achieved in a similar fashion. We begin with the following easily checked lemma.

Lemma 9. Among the equivalent primitive solutions to $k^2-kl+l^2=m^2$, there is precisely one with k,l, and m positive, $k<l$, and $k+l \equiv -m \pmod 3$.

Theorem 6. The following is a complete non-redundant set of non-trivial solutions to $k^2-kl+l^2=m^2$.

$$k=(p^2-q^2)r \quad , \quad l=(2pq+p^2)r \quad , \quad m=(p^2+pq+q^2)r \quad ,$$

where p,q,r run through all integers satisfying $p>q>0$, $r>0$, $(p,q)=1$, and $p\not\equiv q$ (mod 3).

Proof. Use Lemma 1 and Theorem 5 to observe that we no longer obtain two families of solutions as we did in Theorem 4, since Lemma 9 eliminates the second family.

Corollary. As in the previous case, all solutions to $k^2-kl+l^2=m^2$ may now be obtained by a 24-fold expansion of the above solution set.

Section 5. The 120° and 60° Pythagorean Triples.

As mentioned earlier, the 120° and 60° Pythagorean triples are simply the positive solutions to the equations $k^2 \pm kl+l^2=m^2$. We now show how to give a complete list of these.

Theorem 5 gives us solutions to $k^2+kl+l^2=m^2$ in which k,l, and m are positive, and so these form 120° Pythagorean triples. To get any others we take a triple generated by Theorem 5 and look among its 24 related solutions of Table 1 for other triples in which all three entries are positive. A quick glance at Table 1 shows that if k,l,m are all positive, then only one other entry has this property, namely (l,k,m). Clearly we should regard this Pythagorean triple as being the same as (k,l,m). Hence we may conclude that Theorem 5 also gives a complete non-redundant set of 120° Pythagorean triples. Table 2 below lists the first few 120° triples.

Table 2

Primitive 120° Pythagorean Triples ($p\leq 5$)

p	q	r	k	l	m
2	1	1	3	5	7
3	1	1	8	7	13
3	2	1	5	16	19
4	3	1	7	33	37
5	1	1	24	11	31
5	3	1	16	39	49
5	4	1	9	56	61

For the 60° Pythagorean triples we proceed similarly. We begin with the solutions generated by Theorem 6 in which k,l,m are positive and $k<l$. For each such solution we check among the 24 related solutions for other triples having all entries positive. This time we find several. If k,l,m are all positive, then so are the entries of (l,k,m), $(l-k,l,m)$, and $(l,l-k,m)$. The latter two triples are not obviously the same as (k,l,m) but should be regarded as equivalent to each other.' Hence we should include one of them, say $(l-k,l,m)$, in our list of 60° triples. This gives us the following:

Theorem 7. The following is a complete non-redundant set of 60° Pythagorean triples:

(1) $k=(p^2-q^2)r$, $l=(2pq+p^2)r$, $m=(p^2+pq+q^2)r$;

or (2) $k=(2pq+q^2)r$, $l=(2pq+p^2)r$, $m=(p^2+pq+q^2)r$,

where p,q,r run through all integers satisfying $p>q>0$, $r>0$, $(p,q)=1$, and $p\not\equiv q$ (mod 3).

Table 3

Primitive 60° Pythagorean Triples ($p\leq 5$)

p	q	r	k	l	m	k	l	m
2	1	1	3	8	7	5	8	7
3	1	1	8	15	13	7	15	13
3	2	1	5	21	19	16	21	19
4	3	1	7	40	37	33	40	37
5	1	1	24	35	31	11	35	31
5	3	1	16	55	49	39	55	49
5	4	1	9	65	61	56	65	61

Section 6. Conclusion.

Now that we have found all 120° and 60° Pythagorean triples, we can group them with the well-known 90° triples and observe that this collection is a list of all θ° Pythagorean triples with 9 rational in degrees. First, note that if (k,l,m) is a θ° Pythagorean triple, then cosθ is rational. According to Niven [4], if 0 is rational in degrees and cosθ is rational, then cosθ =0, 1/2, or ±1. Hence, if (k,l,m) is a θ° Pythagorean triple with θ rational in degrees, then 0 must be 120°, 90°, or 60°.

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THE EQUILIC QUADRILATERAL

by J. Garfunkel
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Every student of geometry knows that of all plane figures the triangle is one of the most prolific in producing theorems. In this article we show that the quadrilateral is also a rich source for investigation. Quite a number of special quadrilaterals have already been investigated. Examples are the cyclic quadrilateral whose vertices lie on the same circle, the circumscriptible or pericyclic quadrilateral whose sides are tangent to the same circle and the orthodiagonal quadrilateral whose diagonals are perpendicular. Furthermore, there are quadrilaterals that are both cyclic and pericyclic, cyclic and orthodiagonal and so on. To the quadrilaterals with interesting properties, we add a new quadrilateral which we will call equilic.

Definition. Quadrilateral $ABCD$ is said to be equilic if $AD = BC$ and if angle A + angle $B = 120^\circ$.

Note that the quadrilateral need not be convex. See Figures 1A and 1B.

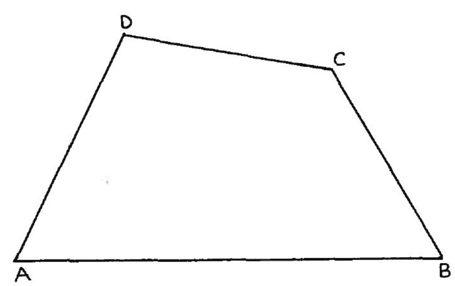


FIG. 1A

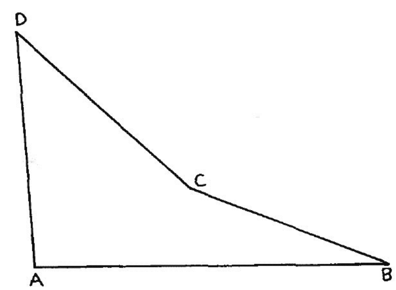


FIG. 1B

* We are deeply grateful to the referees for the meticulous care they took in greatly enchancing this article. Thanks.

We begin by stating a fairly obvious fact.

Theorem 1. A quadrilateral which is both cyclic and equilic is an isosceles trapezoid with $A = B = 60°$.

The proof of Theorem 1 is left to the reader.

Definition. If one angle of an equilic quadrilateral is equal to $90°$, the quadrilateral is called right equilic.

Theorem 2. In a right equilic quadrilateral, a diagonal is equal to an unequal side.

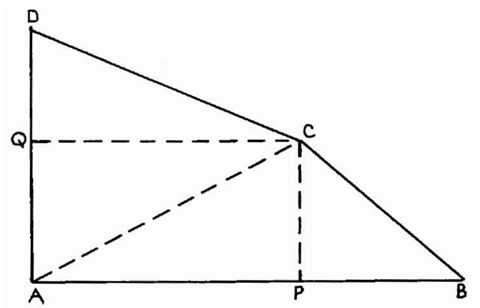


FIGURE 2

Proof. Quadrilateral ABCD is right equilic with the right angle at A and with AD = BC. From C drop perpendiculars CP and CQ to sides AB and AD respectively. Then $CP = 1/2CB = 1/2AD$, since angle B measures $30°$. Hence, C lies on the perpendicular bisector of AD, which makes triangle ACD isosceles and diagonal AC equal to side DC.

The proof is similar for the right angle at any other vertex.

Theorem 2A. In an equilic quadrilateral, if a diagonal is equal to an unequal side then the quadrilateral is right equilic.

Proof. In equilic quadrilateral ABCD with angle A + angle B = $120°$, AD = BC, and AC = CD, erect a perpendicular to AD at A to cut line BC at K. Drop perpendiculars CF and CG to AD and AK respectively. Let AD and BC extended meet at H, so angle H = $60°$. Because CDA is an isosceles triangle, F is the midpoint of AD. Then CG = FA = $1/2AD$ = 1/2 BC. Since CG is parallel to DA, then angle KCG = $60°$, so CG = 1/2 CK. Now

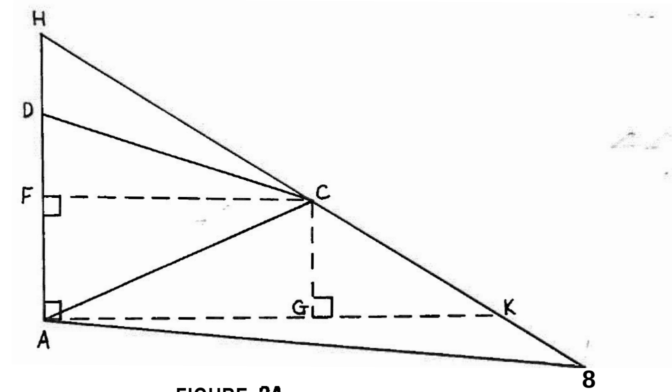


FIGURE 2A

$CK = CB$, so B and K coincide, making angle DAB = angle DAK = $90°$.

In the next two theorems we investigate some interesting relations between the equilic quadrilateral and equilateral triangles.

Theorem 3. If equilateral triangle ABP is constructed interiorly on side AB of equilic quadrilateral ABCD, then triangle PCD is also equilateral.

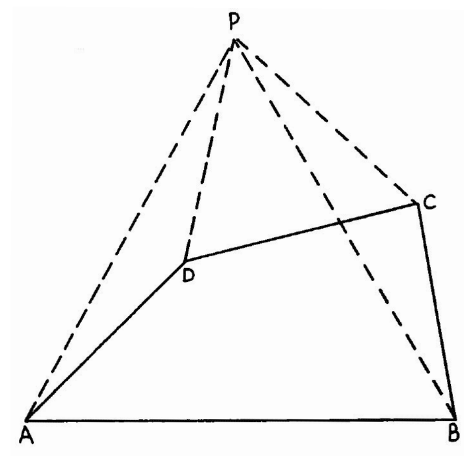


FIGURE 3

Proof. Refer to Figure 3. Let ABCD be an equilic quadrilateral. Assume, without loss of generality, that angle $ABC >$ angle BAD. Construct equilateral triangle ABP interiorly and draw PC and PD. Because AD and BC meet at $60°$ and AP and BP also meet at $60°$, then angle $PAD =$ angle PBC.

Thus, triangles *AD7* and *BCP* are congruent *by SAS*. Hence a 60° rotation about *P* carries triangle *ADP* into *BCP*, so angle *DPC* = 60° and triangle *PCD* is equilateral.

Theorem 4. The midpoints of the diagonals and the midpoint of an unequal side of an equilic quadrilateral are vertices of an equilateral triangle.

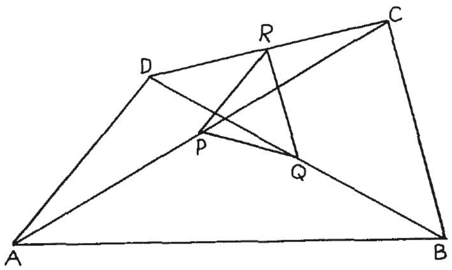


FIGURE 4

Proof. In Figure *4 P* , Q and *R* are the midpoints of *AC*, *BD* and *CD* , respectively. Clearly, PR is parallel to *AD* and equal to 1/2 *4D* , and *RQ* is parallel to *BC* and equal to 1/2 *BC*. Since *AD* = *BC*, triangle *PQR* is isosceles with PR = *QR*. Since the angle between *AD* and *BC* is 60°, then so also is angle *PRQ* = 60° and triangle *PQR* is equilateral. Similarly, triangle *PQS* is equilateral where *S* is the midpoint of side *AB*. Moreover, RQSP is a rhombus.

The reader is encouraged to carry out the proof of Theorem 4 in the case of a non- convex equilic quadrilateral.

Definition. An equilic quadrilateral *ABCD* is called isosceles equilic if *AD* = *DC* = *CB*.

Theorem 5. The point *P* of the intersection of the diagonals of an isosceles equilic quadrilateral is the circumcenter of triangle *ABQ*, where *Q* is the point of intersection of sides *BC* and *AD*.

'Proof. In Figure 5, *ABCD* is an isosceles equilic quadrilateral. Through *A* and *C* draw lines parallel to *DC* and *DA*, respectively, to intersect in *E*. Then, *AECD* is a rhombus. Denote the equal angles DAP, *PAE*, *DCP*, and *PCE by* a, angle *EAB* by β and angle *DQP* by γ. Angle *ECB* = 60°, since *CE* is parallel to *AD*. Hence, triangle *BCE* is equilateral

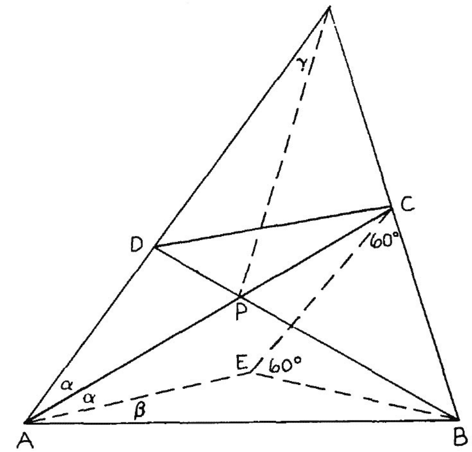


FIGURE 5

and triangle *ARE* is isosceles with *AE* = *BE*. Since angle *BAQ* + angle *ABQ* = 120°, (2 a + β) + β + 60° = 120° and a + β = 30°. In isosceles triangle *BCD*, 2 a + 60° + 2(angle *CDB*) = 180°, so angle *CDB* = 60° - a. Thus, angle *DBE* = a and triangle *ABP* is isosceles with *AP* = *BP* and angle *APB* = 120°. It remains to prove that *AP* = *PQ*. Since angle *APB* = 120° - angle *DPC*, quadrilateral *DPCQ* is cyclic and angle a = angle *DCP* = angle *DQP* = angle γ. Hence, triangle *APQ* is isosceles and AP = *PQ*. This proves that *P* is the circumcenter of triangle *ABQ*.

It should be noted that Theorem 5 holds if point E falls outside the equilic quadrilateral. The reader is encouraged to carry out the proof in the case of a non-convex isosceles equilic quadrilateral.

The proof of the following corollary is left as an exercise for the reader.

Corollary. The opposite angles of an isosceles equilic quadrilateral are in the ratio of 1:2.

The next few theorems are of a more sophisticated nature.

Theorem 6. If equilateral triangles PAD, QDC, and RBC are erected on consecutive sides AD, DC, and *CB* of equilic quadrilateral ABCD, exteriorly on side *CD* and on AD and interiorly on side *BC*, then triangle *PQR* is equilateral. (By symmetry, the result holds if the roles of AD and *BC* are interchanged).

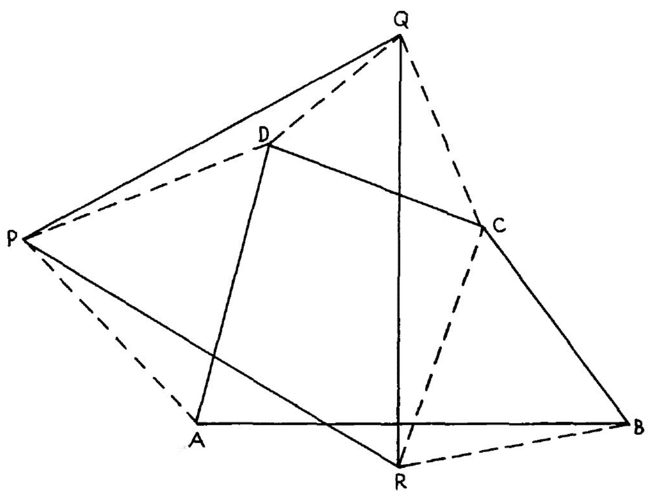


FIGURE 6

Proof. Angle *QCR* = angle *DCB* since both are equal 60° + angle DCR.
Angle PDQ = $360^\circ - 120^\circ$ - angle ADC
= 240° - (360° - angle A - angle *B* - angle *DCB*)
= 240° - (240° - angle *DCB*) = angle *DCB* = angle QCR.

Also, *QC* = QD and DP = *CR*, so that triangles PQD and RQC are congruent. Since a 60° rotation about point Q carries triangle PQD into triangle RQC, it follows that angle PQR = 60° and that triangle *PQR* is equilateral.

**Theorem 6a.* If equilateral triangles are erected exteriorly on sides DA, *AB*, and *BC* of equilic quadrilateral ABCD,, then their third vertices are vertices of an equilateral triangle.

Proof. The proof is similar to that of Theorem 6. Let the appended equilateral triangles be RAD, PBA, and QCB, then RA = *QB*, AP = PB and angle RAP = angle QBP = $120^\circ + B$. Therefore, triangles PRA and PQB are congruent and a 60° rotation about P carries one into the other. Since PR = PQ and angle RPQ = 60°, triangle PQR is equilateral.

Theorem 6b. If equilateral triangles *QBC*, *PCD*, and RDA are erected interiorly on sides *BC*, CD, and DA, respectively, then triangle PQR is equilateral.

Proof. We have PD = PC, DR = CQ and angle PCQ = angle *PDR* = 120° - angle *ADC*, so triangles PDR and PCQ are congruent. Again, a 60° rotation about P carries one into the other and we argue as before.

Theorem 7. If *ABCD* is an equilic quadrilateral and if equilateral triangles are erected as follows: PCA on the same side of CA as *B*, *QBD* on the same side of BD as A, and *RBA* exteriorly, then triangle *PQR* is equilateral.

Proof. A 60° rotation about *B* carries AD into QR and a 60° rotation about *A* carries *RP* into *BC*. It follows that QR = PR and the angle between them is 60°.

Theorem 7a. If *PCA*, *QCD*, and RBD are equilateral triangles erected away from side BA of equilic quadrilateral ABCD, then *P*, *Q*, and *R* are collinear.

Proof. By Theorem 3, a 60° rotation about A carries *PQ* into *CB* and a 60° rotation about *B* carries AD into QR. Since AD and *BC* intersect

* Again I must thank the referees for Theorems 6, 6a, and 6b.

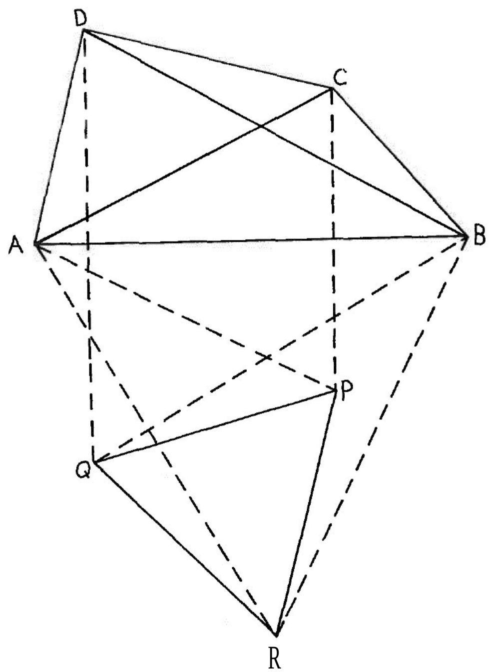


FIGURE 7

at 60°, the result follows.

We offer a second proof of Theorem 7 since this method of proof is useful in proving a later theorem and is interesting in itself. We will employ vectors in the complex plane and make use of the following facts:

1. If v is a vector, then ωv represents the same vector rotated 120° in the counterclockwise direction, where ω represents a cube root of unity.
2. $\omega^3 = 1$, $1 + \omega + \omega^2 = 0$.

Proof. Let $\vec{AB} = p$ and $\vec{BC} = q$, then $\vec{DA} = \omega q$, $\vec{BR} = \omega^2 p$ and $\vec{RA} = \omega p$.
Since $\vec{AC} = p + q$, then $\vec{PA} = \omega p + \omega q$ and $\vec{CP} = \omega^2 p + \omega^2 q$.
Since $\vec{DB} = p + \omega q$, then $\vec{QD} = \omega p + \omega^2 q$ and $\vec{BQ} = \omega^2 p + q$.
It follows that $\vec{RP} = \vec{RA} - \vec{PA} = -\omega q$, and that

$$\vec{PQ} = \vec{PA} + \vec{AB} + \vec{BQ} = \omega p + \omega q + p + q + \omega^2 p = (\omega^2 p + \omega p + p) + (\omega q + q) = -\omega^2 q.$$

Thus, $\vec{PQ} = \omega(\vec{RP})$, which proves the theorem.

Definition. The side AB of the equilic quadrilateral $ABCD$ is called the base.

A most interesting hexagon results when an equilic quadrilateral is reflected in the line containing its base.

Theorem 8. An equilic quadrilateral reflected in the line of its base forms a hexagon with the property that if equilateral triangles are erected exteriorly on any three alternate sides, then their third vertices form an equilateral triangle.

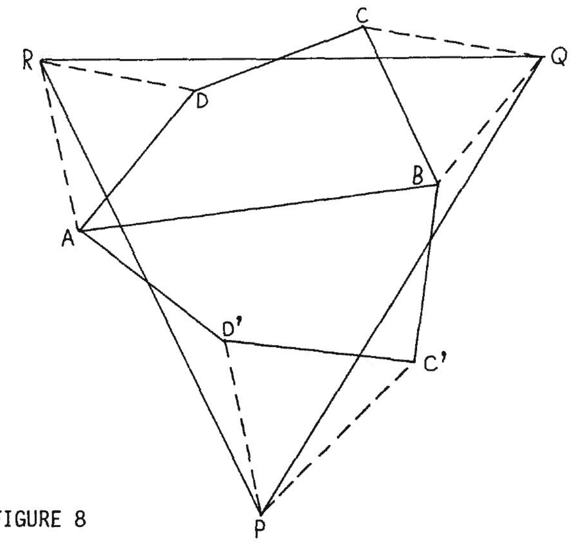


FIGURE 8

Proof. Let C' and D' be the reflections of C and D, respectively, in line AB. A 60° rotation about P takes $PC'BQ$ into $PD'AR$ and, therefore, Q into R since angle $PC'B$ = angle $C + 60^\circ$ = angle $PD'A$ and angle $C'BQ = 120^\circ +$ angle A = angle $D'AR$.

The reader is encouraged to investigate whether Theorem 8 holds if the word interiorly replaces exteriorly.

NOTE: It has been pointed out by one of the referees that "Problem 3524 [1932, page 559] of the *American Mathematical Monthly* states: To the vertices of an equilateral triangle ABC let there be hinged three equilateral triangles AKM, BNR, and CPQ of any sizes and positions, all four sensed counterclockwise. Then the midpoints of the segments in the trio (RP, KQ, MN) form a counterclockwise equilateral triangle. Counterclockwise equilateral triangles are also formed by the midpoints of segments in each of the trios (BQ, CN, RP), (KB, MN, RA) and (AP, KQ, CM).

In Figure 8, triangle PQR has three equilateral triangles RAD, $PC'D'$ and QCB hinged to its vertices. Problem 3524 applies to this figure, so the midpoints of CD, AQ and $C'B$ form an equilateral triangle. By symmetry, so also do the midpoints of $C'D'$, AD, and CB."

Theorem 9. The equilateral triangle formed by joining the midpoints of the diagonals and the midpoint of side AB of equilic quadrilateral ABCD and the equilateral triangle PQR of Theorem 7 are perspective.

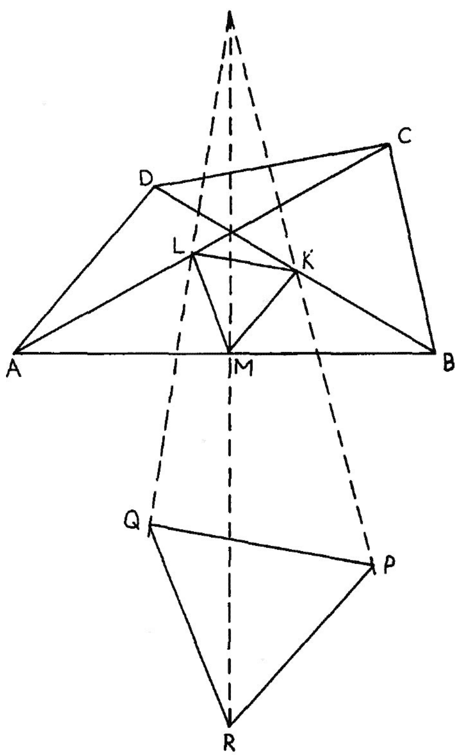


FIGURE 9

Proof. In the second proof of Theorem 7, we have shown that $\vec{RP} = -\omega q$, $\vec{PQ} = -\omega^2 q$, and $\vec{QR} = -q$. Now LM is parallel to and equal to one-half of BC; therefore $\vec{LM} = -1/2q$ and LM is thus parallel to QR.

Similarly, KM is parallel to and equal to one-half of DA. Therefore, $\vec{KM} = -1/2\,\omega q$ and KM is thus parallel to RP. It follows that QP is parallel to LK, completing the proof of the perspectivity of triangles PQR and KLM.

Surely, the reader can find additional properties of this prolific equilic quadrilateral. But we turn to a matter of perhaps greater significance. A most interesting procedure that leads to the discovery of new results and to simple proofs of known results is to allow a figure to degenerate to a familiar figure and to observe the properties as they transform after the degeneration. This will become clear, and the reader will be in a better position to appreciate the advantages of this method of discovery, after a few illustrations.

Example 1. Let equilic quadrilateral ABCD degenerate into a 30°, 60°, 90° right triangle where angle BCD = 180° with C the midpoint of the hypotenuse, then the equilateral triangle erected interiorly on AB and that erected exteriorly on DC have the same vertex.

Proof. The proof follows immediately from Theorem 3. See Figure 10.

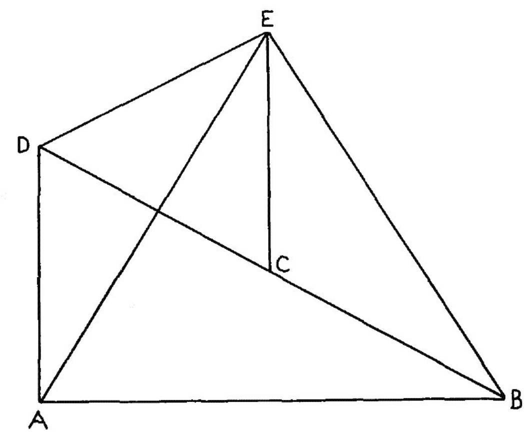


FIGURE 10

Example 2. Using the same degeneration and applying Theorem 2, we get diagonal AC = CD = $1/2BD$.

Example **3.** Using the same degeneration and applying Theorem 7, we get an interesting theorem about a 30°, 60°, 90° right triangle. If equilateral triangles are erected on *AC*, *BD*, and *AB* in the directions as in Theorem 7, then their vertices are the vertices of an equilateral triangle.

Example **4.** Again with the same degeneration, we obtain another interesting result about a 30°, 60°, 90° right triangle by applying Theorem 4. The midpoint of the hypotenuse, the midpoint of the median to the hypotenuse and the midpoint of the side opposite the 60° angle are the vertices of an equilateral triangle.

Other degenerations can be made with interesting consequences. Thus, if we allow equilic quadrilateral *ABCD* to degenerate so that angle $ABC = 0^\circ$, as in Figure 11, some novel theorems emerge.

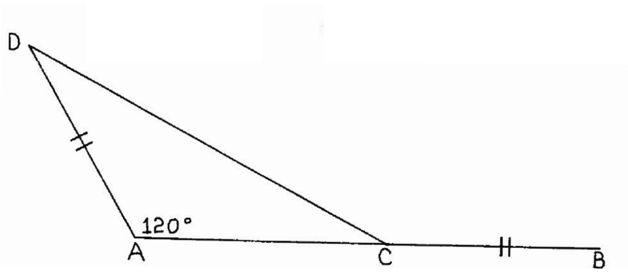


FIGURE 11

Example **5.** The lines joining the midpoints of *AC* and *DB* with the midpoint of either *AB* or *CD* form an equilateral triangle. The proof follows from Theorem 4. See Figure 11A.

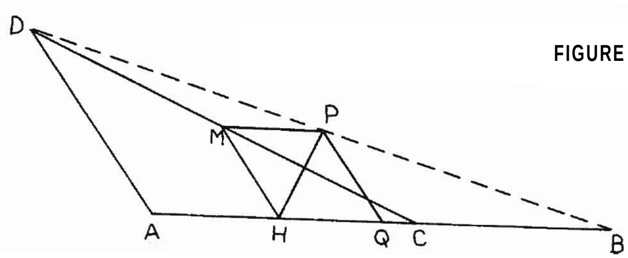


FIGURE 11A

Furthermore, as in Figure 12, we can degenerate our figure into a negative equilic quadrilateral and consider the properties of this figure.

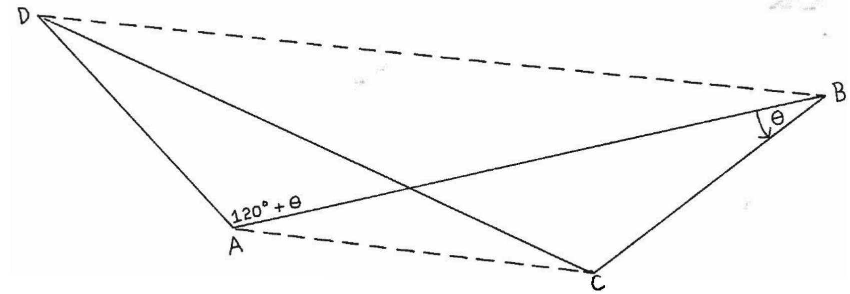


FIGURE 12

It should be noted that the angle at *B* is negative in this case Finally, we can consider the equilateral triangle itself as a degenerate equilic quadrilateral, with $DC = 0$. Again, applying the properties of the equilic quadrilateral, some well-known facts about the equilateral triangle pop out without effort.

We hope that we have convinced the reader that the equilic quadrilateral deserves a place alongside the well-known quadrilaterals.

REFERENCES

1. N. A. Court, *College Geometry*, Barnes and Noble, N.Y., 1960.
2. I. M. Yaglom, *Geometric Transformations*, Random House, N.Y., 1962.

POSTERS AVAILABLE FOR LOCAL ANNOUNCEMENTS

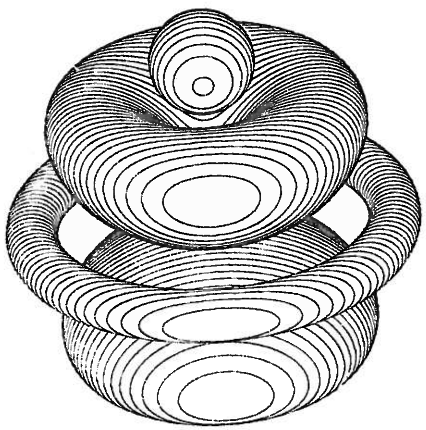
We have a supply of 10 x 14-inch Fraternity Crests available. One in each color will be sent free to each local Chapter on request. Additional posters may be ordered at the following rates:

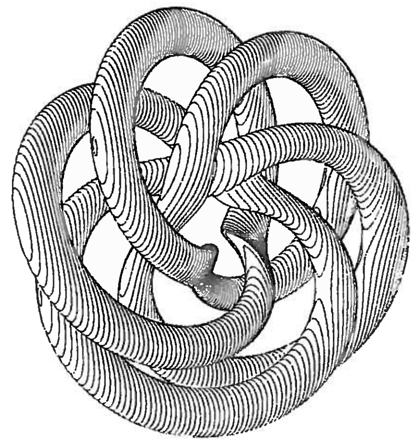
(1) Purple on Goldenrod stock--------------$1.50/dozen

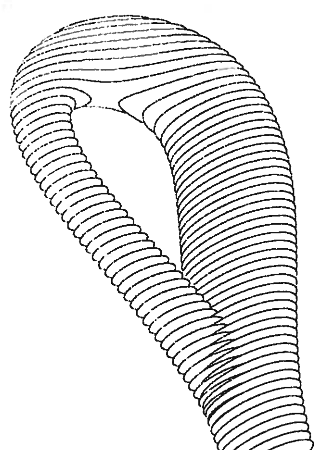
(2) Purple on Lavendar on Goldenrod--------$2.00/dozen.

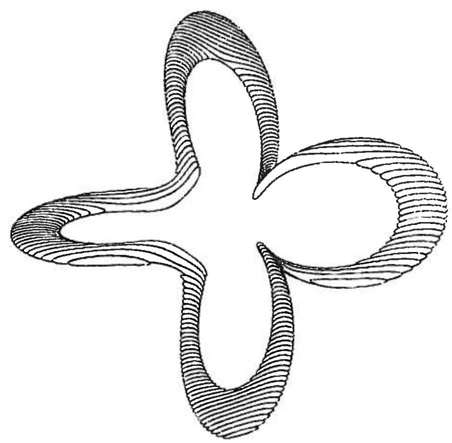
ADVENTURES IN (COMPUTERIZED) TOPOLOGY

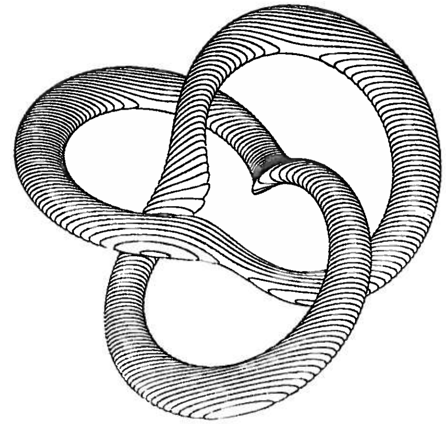
by Gary Ricard
South Dakota School of Mines and Technology

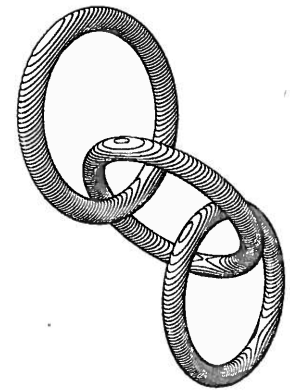


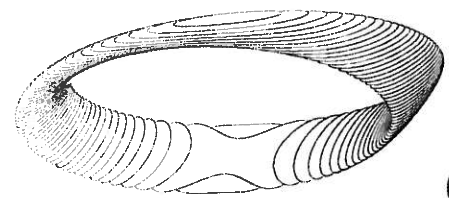


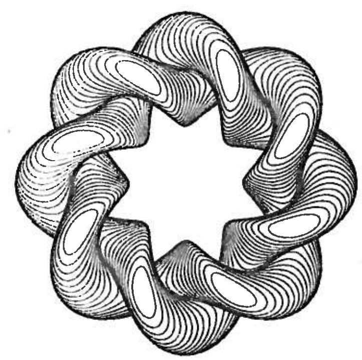


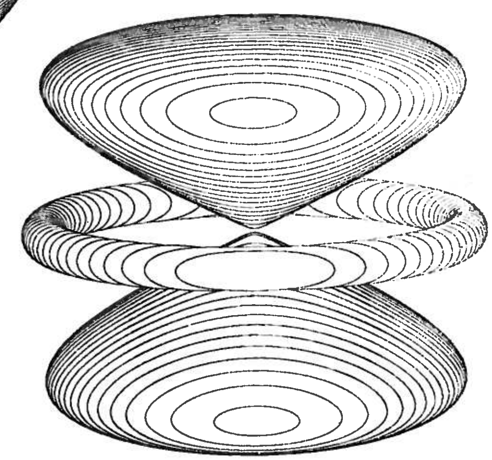












SUMMARIES OF CHAPTER REPORTS

ARKANSAS BETA (HENDRIX COLLEGE) The Chapter had its usual very active year with the following presentations:

Sandra Cousins, Hendrix College; "A Report on the Summer Meetings in Ann Arbor"

Gene Weber, Southern Methodist; "Operations Research: Graduate Study, Applications, and Employment Opportunities"

Ben Schumacher, Hendrix College; "Polarimetry and the Clouds of Venus"

Dr. Richard Rolleigh, Hendrix College; "The Calculus of Variations and the Brachistrochrone Problem"

Jerry Coker, Hendrix College; "Mathematical Games"

Sandra Cousins, David Sutherland, Carol Smith, Hendrix College; "The Development of the Mathematical Method"

Dr. Boris Schein, Univ. of Arkansas; "Mathematics Education in the Soviet Union"

David Sutherland, Hendrix College; "Non-Linear Derived Functions"

Sandra Cousins, Hendrix College; "Infinite Compositions"

Ben Schumacher, Hendrix College; "Exponential Calculus"

Carol Smith, Hendrix College; "Infinite Sums of Derivatives"

Dr. Jag McDaniel, Hendrix College; "Religion and Mathematics"

The Chapter hosted the Sixth Annual Conference on Undergraduate Mathematics on April 10-11, 1981 in which there were thirty-six student speakers and talks by six prominent mathematicians. Awards were won in the following categories:

McKenry-Lane Freshman Math Award: Karen Cornell, David McCallum

Hogan Senior Math Award: Sandra Cousins, Mike Pinter

Phillip Parker Undergraduate Research Award: David Sutherland

GEORGIA BETA (GEORGIA INSTITUTE OF TECHNOLOGY) The Chapter presented book awards to **The Outstanding Graduates in Mathematics:**

John J. Crittenden | Holly Beth Shulman
Brenda Jean Knowles | Hon Wah Tom
Lynn Marie Ramsey

GEORGIA GAMMA (ARMSTRONG STATE COLLEGE) The heard the following papers at regular meetings:

Dr. Netherton, Armstrong State.; "An Algorithm for Computer Science Problem Solving"

Stephan Suchower, Armstrong State.; "In Pursuit of a Prime Number Generator"

Dr. Charles Shipley, Armstrong State.; "Paradoxes In and Around Mathematics"

Andrew Zeigler, Armstrong State; "Contract Programming"

Stephan Suchower, Armstrong State; "Theory of Superconducting Magnets"

Dr. Richard Summerville, Christopher Newport College; "The Mathematical Context"

The award for **The Outstanding Senior in Mathematics** was given to Stephan Suchower.

ILLINOIS ZETA (SOUTHERN ILLINOIS UNIVERSITY-EDWARDSVILLE) The Chapter sponsored the Regional Illinois Council of Teachers of Mathematics High School Competition. They also organized a used mathematics textbook sale for the university with many of the books donated to area high schools.

IOWA ALPHA (IOWA STATE UNIVERSITY) Activities included the following talks:

Prof. Jerold Mathews, Iowa State.; "Ancient Mathematical Models"

Prof. James Cornette, Iowa State; "Polynomial Approximation"

Prof. Richard Sprague, Iowa State.; "Construction of Regular Polygons"

Joyce Schneider, Honeywell; "The Uses of Mathematics in Industry"

Prof. James Carlson, Univ. of Utah; "Prime Numbers and Codes"

Departmental awards were presented as follows:

Outstanding Achievement on the Putnam Exam: William Somiky

Pi Mu Epsilon Scholarships: John Klem, Sudirman Maulim

Dio Lewis Holl Award: Lee Roberts

Gertrude Herr Adamson Awards for demonstrated ingenuity in Mathematics:

Rebecca Potter | Barbara Rus
William Somsky | Phillip McKinley
Steven Seda | Gary McGraw
Gregory Anderson | Wade Johnson

LOUISANNA DELTA (SOUTHEASTERN LOUISANNA UNIVERSITY) During the past year the Chapter heard the following presentations:

Dr. Billy Joe Holmes, Nicholls State; "PERT"

Dale Nasser, Southeastern Louisanna; "The Effects of Projecting Two Mutually Perpendicular Simple Periodic Motions on a Screen"

In addition, the following awards were presented:

Thomas K. Maddox Pi Mu Epsilon Award: Nancy Gautier, Frederick Day
Margo David Award: Jari A. McGee

MINNESOTA DELTA (ST. JOHN'S COLLEGE) The Chapter sponsored the **Mathematics and Humanities Conference** on April 30 and May 1. The Conference had guest lecturers: *Doris Schattschneider, Leonard Gillman and Donald Koehler* plus papers by undergraduate students.

MISSOURI GAMMA (ST. LOUIS UNIVERSITY, FONTBONNE COLLEGE, AND MARYVILLE COLLEGE) The Chapter had an active year with papers and presentations as follows:

Sister Harriet Ann Padberg, Maryville College; "A Mathematical Model: An Historic Note"; **The James E. Case S.J. Memorial Lecture**

Prof. Robert Hogg, Univ. of Iowa; "Size of Loss Distribution" and *"Statistics, Acturial Science and the Future"*

Susan Burns, Culver-Stockton College; "A Mathematical Look at Tonality"

Robert Gregory, SIU-Carbondale; "A Look at Solving Differential Equations Using a Seperation of Variables Technique"

Michael May S.J., St. Louis Univ; "Some Sums of Sums and the Calculus of Finite Differences"

Steven Lazorchak, SIU-Carbondale; "Sinusodial Steady-State Analysis of Electrical Circuits Using the Phasor Concept"

Freny Desai, SIU-Carbondale; "Program Verification"

Barney Smith., St. Louis University; "Magic Cards, Squares **and** *Cubes"*

Prasanna Balakantala, SIU-Carbondale; "Microprogramming"

Michael May, S.J., St. Louis University; "Notions of Infinity"

Jagdish Singh, SIU-Carbondale; "Bit Slices in Microprogramming"

The Chapter's award presentation list is quite extensive and includes:

James W. Garneau Mathematics Award: *Thomas Blackwell*

Francis Regan Scholarship: *Michael May, S.J.*

Missouri Gamma Undergraduate Award: *Jeanne Dulle*

Missouri Gamma Graduate Award: *Mark Hopfinger*

The Pi Mu Epsilon Contests: **Senior Winner:** *Daniel Kirner*
Junior Winner: *James Shamess*

John J. Andrews Graduate Service Award: *Kara Ryan*

Beradino Family Fraternityship Award: *Michael May, S.J.*

NEW JERSEY DELTA (SETON HALL UNIVERSITY) The Chapter held two meetings which were problem solving sessions conducted by *John Saccoman.*

NEW YORK EPSILON (ST. LAWRENCE UNIVERSITY) The Chapter sponsored the 37th annual **Pi Mu Epsilon** Interscholastic Mathematics Contest. The Chapter made the following award:

The O. Kenneth Bates Award: *Denise Martinez*

NEW YORK ALPHA ALPHA (QUEENS COLLEGE OF CUNY) The following talks were presented:

Vh. Joel Stemple, Queens College; "The Four Color Problem"

Steven Kahan, Queens College; "Alphametics: Letters Where the Numbers Ought to Be".

The Pi Mu Epsilon Prize for Excellence in Mathematics and Service was won by *Joel Kreitzer* and *Wendy Carnel.*

OHIO NU (UNIVERSITY OF AKRON) The Chapter awarded a prize to *Kendall Cmey* for **Excellence in Mathematics.**

OKLAHOMA GAMMA (CAMERON UNIVERSITY) Amoung their many activities the Chapter heard the following paper:

Dr. William Ray, University of, Oklahoma; "Discrete Predator-Prey Problems".

PENNSYLVANIA NU. Amoung the various talks presented were the following:

James Watson; "Iteration Techniques"

Dr. John Lane; "Number Density"

SOUTH CAROLINA GAMMA (COLLEGE OF CHARLESTON) Chapter members are very involved on the college campus and in the surrounding communities. Some of the members are currently involved in a Junior high school project where they teach sixth, seventh and eighth graders how to use computers. One of the members is helping to prepare packedts for Computer Assisted Instruction in Mathematics. Several members do volunteer tutoring for the Charleston County PATHE program. The Chapter sponsored the 4th Annual Math Meet with over 600 high school students participating.

SOUTH DAKOTA BETA (SOUTH DAKOTA SCHOOL OF MINES AND TECHNOLOGY) The Chapter constructed a *What Math Do I Take* brouchure for distribution to state high schools. The following papers were presented:

Prof. Roger Opp, SVSMST; "A Classification of Projectile Paths"

Prof. David Ballew, SDSM&T; "Employment Opportunities"

Prof. Al Grimm, SVSMST; "The Cubic Equation"

Dr. Francis Florey, Univ. of Wisconsin-Superior; "Generalized Inner Products With Applications ot Fourier Series"

Janet Potts, SVSMST; "Robotics"

Dean Mogck, SVSMST; "An Application of Game Theory in Taking Tests"

Leon Nelson, SVSMST; "On Trisecting the Angle"

Gary Ricard, SVSMST; "Parametric Methods in Computer Graphics"

Brian Bunsness, SDSM&T; "Analyticity and Taylor's Series"

Colleen Quatier, SDSMST; "The Evolution of Computer Languages"

The Chapter sponsored the Annual West River Mathematics Contest for High School Students, and initiated the South Dakota Collegiate Mathematics Contest won by Northern State College.

THE SUMMER MEETING OF PI MU EPSILON, 1981

The following papers were presented at the Summer Meeting in Pittsburgh:

Beth Snyder, Miami University; "Introduction to Box-Jenkins Time Series"

Dean Mogck, S. D. School of Mines and Technology; "Stokes' Theorem For Quaternion Integral Operators"

Dean Shea, St. John's University; "A Look at Formal Theory"

Edward D. Lowry, Western Washington University; "An Approximation to the Normal Distribution"

Bro. Longinus Anyanwu, Morgan State. University; "The Gamma Function and Extensions"

Ravi Salgia, Loyola University; "Dirichlet Integrals and Their Applications"

Brian Sumner, University of Denver; "Transformation of Computer Programs into Functions"

Robert Kear, East Carolina University; "A Complex Parabola in Four Dimensions"

Margaret R. Wallace, Miami University; "Using the Method of Maximum Likelihood Estimation in Genetics"

Kevin Saylors, Pomona College; "Rubic's Magic Cube"

James F. Goeke, S.J., St. Louis University; "Descartes: Philosopher or Mathematician?"

Dean Follmann, Northern Illinois University; "A Non-Parametric Multiple Comparison Test for Differences in Variances"

Brian Bunsness, S. D. School of Mines and Technology; "The Relations of Differentiable Functions and the Power Series"

Donna 1. Ford, Miami University; "Mazes and Their Passage"

Elias Kosmas, Oklahoma State University; "Mathematical Analysis of Inflation"

The J. Sutherland Frame Lecture

Professor E. P. Miles, Jr., Florida State University; "The Beauties of Mathematics Revealed in Color Block Graphs"

PUZZLE SECTION

Edited by

David Ballew

This Department is for the enjoyment of those readers who are addicted to working crossword puzzles or who find an occasional mathematical puzzle attractive. We consider mathematical puzzles to be problems whos solutions consist of answers immediately recognizable as correct by simple observation and requiring little formal proof. Material submitted and not used here will be sent to the Problems Editor if deemed appropriate for that Department.

Address all proposed puzzles and puzzle solutions to David Ballew, Editor of the Pi Mu Epsilon Journal, Department of Mathematical Sciences, South Dakota School of Mines and Technology, Rapid City, South Dakota, 57701. Deadlines for puzzles appearing in the Fall issue will be the next February 15, and pussies appearing in the Spring issue will be due the next September IS.

Mathacrostic No. 13

submitted bq Joseph D. E. Konhouser

Macalester College, St. Paul, Minnesota

Like the preceeding puzzles, this puzzle (on the next page) is a keyed anagram. The 248 letters to be entered in the diagram in the numbered spaces will be identical with those in the 29 keyed words at matching numbers, and the key letters have been entered in the diagram to assist in constructing your solution. When completed, the initial letters will give a famous author and the title of his book; the diagram will be a quotation from that book. (See an example solution in the solutions section of this Department.)

Cross Word Puzzle

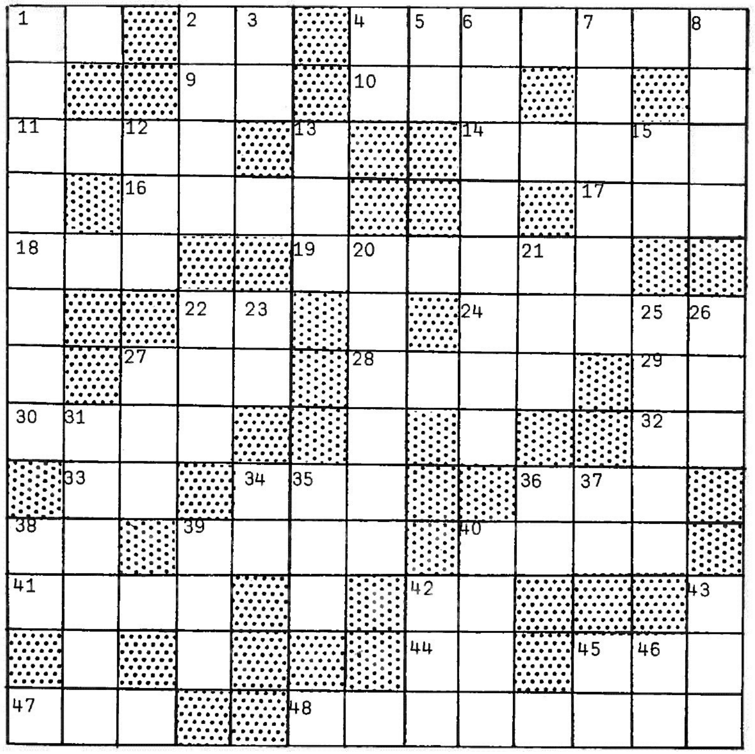
submitted by Alexander Mehaffey Jh. and Curt Olson

The University of South Dakota.

This crossword puzzle (three pages forward) is a standard crossword puzzle with a mathematical flavor.

1 R	2 G	3 J	4 b	5 I	6 Y	7 V		8 T	9 B		10 U	11 c	12 M
13 O	14 P	15 F	16 L	17 H	18 A		19 E	20 C		21 I	22 c	23 B	
24 W	25 V	26 Q		27 R	28 X	29 P	30 N		31 U	32 J	33 Z	34 X	35 b
36 A		37 E	38 D	39 T		40 Y	41 B	42 G	43 D	44 N		45 O	46 C
	47 P	48 F	49 c	50 b	51 A	52 U		53 Q	54 F	55 L	56 R		57 b
58 X	59 H	60 a	61 T	62 J	63 P	64 V	65 L		66 N	67 O		68 Y	69 R
70 N	71 F	72 A	73 L	74 G		75 X	76 I	77 S	78 b	79 H	80 O		81 U
82 R	83 N		84 T	85 H		86 C	87 A	88 R		89 Z	90 D	91 N	92 P
93 c		94 J	95 C	96 R	97 U		98 G	99 D		100 B	101 b	102 E	
103 M	104 A	105 C	106 U	107 Q	108 F	109 Z	110 H	111 b		112 Y	113 O		114 I
115 c	116 C		117 L	118 F	119 O	120 a	121 S	122 U		123 L	124 T		125 N
126 X	127 G	128 R		129 A	130 E	131 W	132 X	133 S	134 J	135 N	136 U	137 R	138 b
139 H	140 L	141 Z		142 P	143 K	144 J	145 I	146 S	147 F	148 a	149 C	150 E	
151 G	152 Z		153 D	154 b	155 A	156 F		157 O	158 E		159 U	160 I	161 Q
162 T	163 L	164 F	165 P	166 S	167 H	168 c		169 X	170 a	171 b	172 L	173 J	174 F
	175 P	176 O	177 C		178 N	179 B	180 Q	181 S		182 K	183 G	184 E	185 b
	186 D	187 S	188 W	189 J		190 A	191 M	192 R	193 H	194 K	195 c	196 F	197 U
198 Y	199 Z		200 Q	201 P	202 K	203 J	204 Z		205 B	206 C	207 U		208 G
209 X	210 J	211 A	212 H	213 b	214 I		215 P	216 W	217 C	218 L		219 c	220 E
221 D		222 A	223 N	224 T	225 I	226 H	227 C	228 O	229 S	230 P	231 G		232 V
233 F	234 S	235 T	236 M		237 D	238 Z	239 Q	240 U	241 Y	242 S	243 R		244 H
245 V	246 F	247 b	248 N										

Definitions	Words
A. a ball-and-socket joint	104 18 51 211 155 87 36 72 129 222 190
B. a redundant account	41 23 179 205 9 100
C. analysis which admits infinitely small quantities (comp.)	206 46 20 227 86 95 149 177 217 105 116
D. S. European climbing plant bearing fragrant flowers (2 wds.)	186 153 43 99 90 237 221 38
E. "The laws of nature are but the mathematical ______ of God." Kepler	184 220 19 130 150 37 158 102
F. spiral with polar coordinate equation $a = r\cos^3(\theta/3)$	15 174 108 156 71 147 196 54 233 118 164 48 246
G. providing aid or direction in the solution of a problem	98 231 151 2 42 74 127 183 208
H. Cauchy's single-limit analysis	110 193 226 139 167 17 85 79 212 59 244
I. a number greater than half of a total	160 114 145 225 76 5 21 214
J. creative power (2 wds.)	210 94 3 134 144 32 203 173 189 62
K. heating element (plug) in a Diesel engine	202 194 143 182
L. "All human knowledge thus begins with ______," Kant, Critique of Pure Reason (followed by Z and b)	55 172 117 73 16 218 123 140 65 163
M. one of Thom's seven elementary catastrophes	12 191 236 103
N. Cantor discard (2 wds.)	223 125 44 83 135 248 178 91 66 70 30
O. specialist in diagnosis and treatment of non-surgical diseases	228 67 45 13 119 176 157 80 113
P. systematic or random repetition	47 92 14 230 165 142 175 215 63 201 29
Q. polygon divisible into congruent ones similar to it (comp.)	239 26 161 53 180 200 107
R. perfectly simple (comp.)	128 192 88 96 69 82 1 56 27 137 243
S. manipulative puzzle rage of the 1980's (2 wds.)	133 187 229 77 234 181 242 146 166 121
T. contrary; antithetical	84 224 61 162 39 8 124 235
U. nonsense; something trivial (coB~.)	106 159 122 97 52 31 240 81 10 207 136 197
V. "father" of descriptive geometry (1746-1818)	232 64 25 7 245
W. truncate	188 216 24 131
X. its ears are its radiators	132 209 58 75 34 28 126 169
Y. in De Thiende (1585), he introduced decimal fractions for general purposes (1548-1620)	198 40 241 68 112 6
Z. "proceeds thence to ______," (follows L, followed by b)	204 89 141 33 238 152 109 199
a. Danish poet, designer, inventor of Hex and Soma Cube, b. 1905	170 60 148 120
b. "and ______." (follows Z, 3 wds.)	50 78 111 138 4 154 213 185 171 57 35 101 247
c. carrying back	93 195 49 11 22 168 115 219



Across

1. *c/d*
2. Symbol for the population mean
4. Used in *calculus* proofs
9. *Killed* in a duel *(initials)*
10. XXIV *hours*
11. $y_2 - y_1$
14. Unit of length
16. One element of a set
17. Possible *nickname* for a popular *Calculus* book.
18. Above x indicates a simple mean
19. $x^2 + y^2 + z^2 = a^2$
22. Greek Letter
24. Doughnut
27. Vertex
28. Irrational number (archaic)
29. Initials of 12 down in phone book.
30. *Norwegian* mathematician
32. Sometimes cast out
33. Eleven across goes over this (without vowel)
34. Professional Organization *(ab)*
36. Cantor's concept
38. Has *minimal* area *amoung* the 50. *(ab)*
39. Gauss' first name
40. Three in cards
41. *Moebius* strip has but one
42. Connective
44. $x\,dy = y\,dx$ *(ab)*
45. Perfect score
47. *Student's distribution*
48. The "Slasher"

Down

1. Conic Section
2. Operation in Lattice Theory
3. Calculus is taken by the *(ab)*
4. Every journal needs one *(ab)*
5. The 1981 Meeting was in this state *(ab)*
6. This puzzle has little of this
7. A mathematician wrote this to communicate results to others before *journals*
8. Product of a complex number and its conjugate
12. Isaac Newton was called this
13. Mathematician's Organization
15. Transposition of breakfast cook's order *(ab)*
20. Man of triangle fame
21. Unit of length
22. *Published* first non-Euclidean geometry (initials)
23. Positive direction of y-axis
25. Multiplicative Identity
26. First perfect number
27. *Bo's* number
31. A critical edge of a graph'
34. Ph. D. -1
35. *Projective* geometry received impetus from this area
36. Fourth year of *college (ab)*
37. Type of engineer *(ab)*
38. A step then a *(ab)*
39. Elliptical orbits (possible nickname for proponent)
40. Connected graph with no cycles
42. Not even
43. Number to Pythagorean
45. First "female" number
46. One to a *circuit*

SOLUTIONS

Mathacrostic No. 12. (See Spring 1981 issue) ***(Proposed by J.D.E. Konhauser)***

Definitions and Key:

A. Left bower	H. Student	O. Raw data	V. Entwine
B. Algorithm	I. Paving	P. One two three	W. Arbelos
C. Not in your eyes	J. At the bath	Q. Ululant	X. Grapevine
D. Catastrophes	K. Clifford	R. Gnomon	Y. Ephemeris
E. Zermelo	L. Ether	S. Half hitch	Z. Swineshead
F. Octahedron	M. Two handles	T. Troytown	
G. Stone	N. Hyades	U. Human mind	

First Letters: Lanczos Space Through The Ages

Quotation: *The observations are the primary thing. Then comes the theory. Little would he have guessed that a few years later he himself would be Plato's astronomer who gazed down rather than up and by contemptation found the inner meaning of Newton's Law and its correction.*

Solved by: Jeanette Bickley, Webster Groves High School, Missouri; Louis H. Cairoli, Kansas State University; Victor G. Feser, Mary College, Bismark; Robert Forsberg, Lexington, Mass.; Robert C. Gebhardt, Hapatcong, NJ; Roger E. Kuehl, Kansas City, MO; Henry S. Leibennan, John Hancock Mutual Life Ins. Co.; Robert Prielipp, Univ. of Wisc. at Oshkosh; Sister Stephanie Sloyan, Georgian Court College; the Proposer and the Editor.

Cross Number Puzzles: (See Spring 1981 Issue) (*Proposed by Mark Isaak*)

Solved by: Dan Essig, Houstion, Texas; Victor G. Feser, Mary College, Bismark; Martha Hasting, St. Louis University; Murray Katz, Penn State University; Roger Kuehl, Kansas City, MO; The Proposer and the Editor.

#1

ACROSS
1. AB
5. A^2
6. C
7. 3A - 18
8. $(1/9)A^2 - D$

DOWN
1. A/E
2. B^3
3. $E^6 + 1000$
4. BC/89
7. E^2

1	1	5	2
2	3	0	4
	8	9	0
1	2	6	
6	4		

#2

ACROSS
1. 5AB/91
4. 66C + 82
5. $(\sqrt{11B} + 22)^2$
6. D
7. (7D + 208)/20

DOWN
1. 5AB
2. 619(B + C + E) - 30
3. C + 7E
4. 6C + 3

	3	8	5
4	5	7	0
1	0	8	9
1	3	6	
	5	8	

#3

ACROSS
1. A^2
4. BC
5. 63C/5
6. $3D\sqrt{5E}$
9. B

DOWN
1. 10(B + F) - 2
2. E
3. G
4. F
7. $\sqrt{E} - \sqrt{G} + H$
8. D

	1	4	4
4	3	5	5
6	3		
2	2	9	5
	8	7	1

#4

ACROSS
1. $A^4 + BC + C$
3. D
5. $(E + 1)^4F$
6. $((C/B) + A)^G$
7. H(A + G)(C - 2A)

DOWN
1. J-DK
2. 2J + 20
4. GHK
5. B + C

	3	8	
	1	4	1
1	8	7	5
2	1	9	7
	2	4	5

#5

ACROSS
1. A
5. A + B
6. CD +(E/D)
7. F/2 - 1
9. (A + 7)/D

DOWN
1. 3C
2. E
3. DFG
4. B
7. D + (D/10)
8. G

1	6	9	3
2	0	8	9
		4	6
1	3	2	
1	7	0	

#6

ACROSS
1. A
3. B
4. $C^2 + BD - 43$
5. DE/2
6. F

DOWN
1. G^{H-F}
2. H + 4
3. J
4. E

		5	1
	3	9	0
6	2	0	1
4	5	4	3
9	9	9	

#7

ACROSS
1. 21A
4. $(2B - 1)^2$
5. $(C-17453)^2/20$
6. $(2B-1)^4-100D+4$
7. A

DOWN
1. 59E
2. $5C - B^3 + 17B^2 - 66B + 5$
3. 11F
5. (E + 5)/G

	9	8	7
	2	8	9
1	6	2	0
8	3	2	5
		4	7

#8

ACROSS
1. $(A - 32)^2$
4. (158/131)B
5. C
6. A^2
8. 67(3D - 66)

DOWN
1. 10EF + 3
2. 1331G + 1
3. 4BCH
6. $F^4 + 10F^2J + 13F$
7. G

4	8	4	
3	1	6	
	1	1	
2	9	1	6
4	2	2	1

PROBLEM DEPARTMENT

Edited by Clayton W. Dodge
University of Maine
and
Leon Bankoff
Los Angeles, California

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also acceptable. The choice of proposals for publication will be based on the editor's evaluation of their anticipated reader response and also on their intrinsic interests. Proposals should be accompanied by solutions if available and by any information that will assist the editor. Challenging conjectures and problem proposals not accompanied by solutions will be designated by an asterisk (*).

Problem proposals offered for publication should be sent to Dr. Lean Bankoff, 6360 Wilshire Boulevard, Los Angeles, California 90048.

To facilitate consideration of solutions for publication, solvers should submit each solution on separate sheets (one side only) properly identified with name and address and mailed before July 1, 1982 to Clayton W. Dodge, Mathematics Department, University of Maine, Orono, Maine 04469.

Contributors desiring acknowledgment of their proposals and solutions are requested to enclose a stamped and self-addressed postcard or, for those outside the U.S.A., an unstamped card or mailing label.

Problems For Solution

498. *Phopobed by R. S. Luthar, University of Wisconsin, Janesville.*
Find the general solution of

$$x^3 + y^3 + 3xy = 1.$$

499. *Proposed by Vision. G. Feser, Mary Cortege, Bismarck, North Dakota..*

The array below is defined by the following properties:

i) The entries are distinct positive integers.

ii) In each column, the entries are consecutive integers, top to bottom.

iii) In each row, each integer (except the first one, of course) is a multiple of the integer at its left.

1	7	511
2	8	512
3	9	513

a) Find a fourth column for this array

b) Find the minimal fourth column for this array, and show it is minimal.

c) Construct an array of 4 rows and 4 columns with the same properties. Is it minimal?

500. *Proposed by Chuck Allison and Peter Chu, San Pedro, California.*

A condemned prisoner is given a chance to escape execution. He is given two boxes capable of holding sixteen bottles each, and is required to place eight bottles of water and eight bottles of clear poison in those boxes leaving no box empty. He will then summon the guard who will then pick one box at random and then select a bottle from that box which the prisoner must drink. How should the prisoner arrange the bottles in the two boxes to maximize his probability of survival, and what is that probability?

501. *Proposed by Robert C. Gebhardt, Parsippany, New Jersey.*

A rectangle is inscribed inside a circle. The area of the circle is twice the area of the rectangle. What are the proportions of the rectangle?

502. *Proposed by Robert C. Gebhardt, Parsippany, New Jersey.*

Consider $\quad 2^k + 2^k = 1^k + 3^k \quad$ for $k = 1$,

and $\quad 2^k + 2^k + 2^k = 1^k + 1^k + 1^k + 3^k \quad$ for $k = 1,2$.

Complete the equations

$$2^k + 2^k + 2^k + 2^k = ? \qquad \text{for } k = 1,2,3,$$

and

$$? = ? \qquad \text{for } k = 1,2,3,4,$$

where the left side is a function of 2^k only, and the right side is a function of 1^k and 3^k only.

503. *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, Pennsylvania.*

Find the equation of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ with minimum volume which shall pass through the point $p(r,s,t)$, $0 < r < a$, $0 < s < b$, $0 < t < c$.

504. *Proposed by Charles W. Trigg, San Diego, California.*

In the square array of the nine non-zero digits

9	2	6
4	1	7
8	3	5

the sum of the digits in each 2-by-2 corner array is 16. Find another arrangement of the nine digits in which the sum of the digits in each corner array is five times the central digit.

505. *Proposed by John M. Howell, Littlerock, California.*

A baseball team has all .300 hitters. They never steal a base, get picked off base or hit into a double play. And men on base advance only one base when there is a hit.

a) What is the probability of this team getting one or more runs in an inning?

b) What is the expected number of runs scored by this team per inning?

506. *Proposed by Morris Katz, Macwahoc, Maine.*

"The addition cryptarithm IN + THE = MOOD is not difficult, but the solution cannot be unique because N and E can be interchanged, and so can I and H."

"Even taking account of those interchanges," his friend replied, "there are still many different solutions."

"That is so," agreed the first, "but let me tell you the value of one of those four letters."

I could not hear the letter and the value he whispered to his colleague, but the reply was quite clear. "Ah, now the solution is unique except, of course, for the interchange of the two letters of the other pair, and it uses every digit that is an odd prime, too."

507. *Proposed by Herbert R. Bailey, Rose Polytechnic Institute, Terre Haute, Indiana.*

A unit square is to be covered by three circles of equal radius. Find the minimum necessary radius.

508. *Proposed by Bruce W. King, Burnt Hills, New York.*

When Professor Umbugio asked his calculus class to find the derivative of y^2 with respect to x^2 for the function $y = x^2 - x$, his nephew Socrates Umbugio found $\frac{dy}{dx} \cdot \frac{y}{x}$ and obtained the correct answer. Help the professor to enlighten his nephew about taking derivatives.

509. *Proposed by Jack Garfunkel, Queens College, New York.*

Given a triangle ABC with its incircle I, touching the sides of the triangle at points L, M, N. Let P, Q, R be the midpoints of arcs NL, LM, and MN respectively. Form triangle DEF by drawing tangents to the circle at P, Q, and R. Prove that the perimeter of triangle $DEF \leq$ perimeter of triangle ABC.

Solutions

462. [Spring 1980; Spring 1981] *Proposed by the late R. Robinson Rowe.*

A pilot down at Aville asked a native how far it was to Btown and was told, "It's south 1500 miles, then east 1000 miles, or east 500 miles and south 1500 miles." How far was it directly?

Comment by *Jimmy Griffith, Charlotte, North Carolina.*

Once we know a, β, and γ, we may find a at once by

$$\cos c = \cos(\tfrac{\pi}{2} - \beta)\cos(\tfrac{\pi}{2} - \gamma) + \sin(\tfrac{\pi}{2} - \beta)\sin(\tfrac{\pi}{2} - \gamma)\cos a$$

$$= \sin\beta\sin\gamma + \cos\beta\cos\gamma\cos a,$$

so the rest of the featured solution on pages 268-269 seems unnecessary.

Late solutions were received from MIKE CALL and GEORGE W. RAINEY, JR.

474. [Fall 1980] *Proposed by Scott Kim, Artificial Intelligence Laboratory, Stanford University.*

Knotted path: Consider a 2 by 3 by 7 block of unit cubical cells. Your task is to find a path moving from cell to adjacent cell, returning to the original cell so that the path traced is a 3-dimensional knot. Each cell must be visited exactly once; two cells are adjacent only if they share a face.

Solution by the PROPOSER.

Number the elements on the lower level first row 1 to 7, second row 8 to 14, and third row 15 to 21. To each of these element numbers add 21 to get the corresponding element in the upper level. A path then is

1- 8-29-36-15-16-17-10-31-32-33-40-
19-20-21-14- 7- 6- 5- 4- 3- 2- 9-37-38-
39-18-11-12-13-34-41-42-35-28-27-26-
25-24-23-22- 1.

See Figure 1.

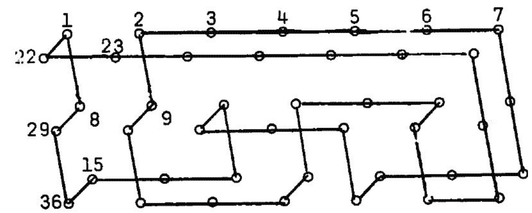
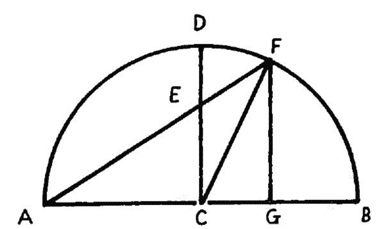


Figure 1.

475. [Fall 1980] *Proposed by Zelda Katz, Beverly Hills, California.*

In the accompanying diagram DC is the radius perpendicular to the diameter AB of the semicircle ADB; FG is a half-chord parallel to DC; AF cuts DC in E. Show that the sides of the triangle FCG are integers if and only if DE/EC or its reciprocal is an integer.



Comment by *Morris Katz, Macwahoc, Maine.*

It is easy to show that, if DE/EC is rational, then triangle FCG has sides proportional to a Pythagorean triangle, and conversely, if triangle

FGC is Pythagorean, then DE/EC is rational.

Let $CG = a$, $GF = b$, $CF = CA = CD = r$, $CE = x$, $ED = y$, $AE = u$, and $EF = v$. Then we have

$$\frac{r}{r+a} = \frac{x}{b}, \quad \text{so} \quad x = \frac{br}{r+a}$$

from similar triangles AEC and AFG. Then

$$\frac{DE}{EC} = \frac{y}{x} = \frac{r-x}{x} = \frac{r - br/(r+a)}{br/(r+a)} = \frac{r+a-b}{b},$$

so DE/EC is rational when a, b, and r are integral, establishing the converse. If $DE/EC = p/q$ is rational, with p even, take $m = q + p/2$ and $n = p/2$ and let $a' = 2mn$, $b' = m^2 - n^2$, and $r' = m^2 + n^2$. Then we have

$$\frac{r' + a' - b'}{b'} = \frac{m^2 + n^2 + 2mn - m^2 + n^2}{m^2 - n^2} = \frac{2n}{m-n} = \frac{p}{q},$$

but this Pythagorean triangle with sides a', b', and r' is similar to triangle CFG.

It is easy to find counterexamples showing that the stated theorem is quite false.

Counterexamples were submitted by RICHARD A. GIBBS, BOB PRIELIPP, and ROBERT A. STUMP.

Editor's *comment*. When confronted with her mistake, the proposer *purred* that "after *all*, rational and integral are almost the same thing." Such pussy-footing evades the problem, but Zelda is a regular contributor of high quality, so this is no catastrophe.

477. [Fall 1980] *Proposed by Solomon W. Golomb, University of Southern California.*

In the eleventh row of Pascal's Triangle, the first five terms (1, 11, 55, 165, 330) have the property that each is an integral multiple of its predecessor. Is there a row of Pascal's triangle where there are eleven consecutive terms with this property?

Essentially similar solutions were received from JEANETTE BICKLEY, *St. Louis, Missouri,* MARK EVANS, *Louisville, Kentucky,* ROBERT C. GEBHARDT, *Hopatcong, New Jersey,* W. C. IGIPS, *Danbury, Connecticut,* KRISHNAMOORTHY, *Troy, New York,* ROGER E. KUEHL, *Kansas City, Missouri,* HENRY S. LIEBERMAN, *Boston, Massachusetts,* BOB PRIELIPP, *University of Wisconsin-Oshkosh,* SAHIB SINGH, *Clarion State College, Pennsylvania,* ROBERT A. STUMP, *Hopewell, Virginia,* KENNETH M. WILKE, *Topeka, Kansas, and the* PROPOSER.

Since 2520 is the smallest number divisible by each of 2 through 10, the 2519th row is the first such row. Any row numbered $2520k - 1$, k a positive integer. has this property. For k = 1, these elements are

$$1,\ 2519,\ 2519 \cdot \frac{2518}{2},\ 2519 \cdot \frac{2518}{2} \cdot \frac{2517}{3},\ \ldots$$

478. [Fall 1980] *Proposed by Charles W. Trigg, San Diego, California.*

PIGS = ROOT + ROOT + ROOT,

but can only dig up a single solution when each different letter represents a distinct digit, and PIGS contains three consective odd digits. What is the unique representation of the addition?

Solution by Kenneth M. Wilke, Topeka, Kansas.

Clearly neither T nor S can be 5 or 0, and R = 1, 2, or 3. Taking the statement that PIGS contains three consecutive odd integers true in both senses, they must be (1, 3, 5), (3, 5, 7), (5, 7, 9) or their reversals, and the remaining digit of PIGS must be 6 or 9. Of the 16 possibilities, only PIGS = 6357 yields a solution: ROOTS = 2119. If all 120 permutations of the 5 possible sets of numbers are tested, again only the stated solution survives.

Also solved by JEANETTE BICKLEY. DANIEL ESSIG, MARK EVANS, JACKIE E. FRITTS, ROBERT C. GEBHARDT, W.C. IGIPS, ROGER E. KUEHL, KRISHNAMOORTHY. BOB PRIELIPP, TAGHI REZAY-GARACANI, SAHIB SINGH, DALE E. WATTS, *and the* PROPOSER.

479. [Fall 1980] *Proposed by Herbert Taylor, South Pasadena, California.*

Prove that the following statement is true whenever $0 < r \le n$, or else find a counterexample:

Given a $2n \times n$ matrix of 0's and 1's, with each column sum equal to $2r$ and each row sum equal to r, it is always possible to mark 2n of the 1's in such a way that one 1 is marked in each row and two 1's are marked in each column.

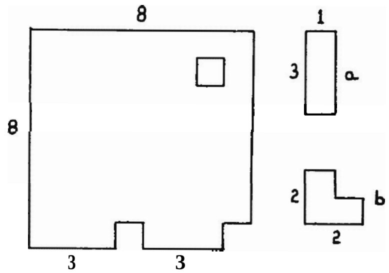
Solution by Robert Henderson, South Pasadena, California.

Let $B = [A|A]$. Then B is $2n \times 2n$ with all row and column sums $= 2r$. The theorem of Philip Hall shows that we can select $2n$ ones no two of which lie in the same row or column of B (e.g., H. J. KYSER, *Combinatorial Mathematics*, p. 57). These 2n ones, as ones of A, satisfy the conditions of the claim.

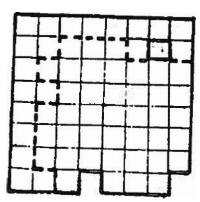
480. [Fall 1980] *Proposed by Richard I. Hess, Palos Verdes, California.*

a) Cut the large piece at right into two pieces which can be reassembled with piece a into an 8 x 8 square.

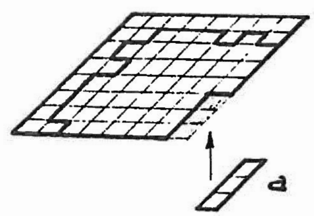
b) Do the same, using piece b.



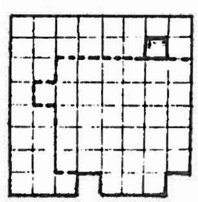
I. Solution by Roger E. Kuehl, Kansas City, Missouri.



CUT ---

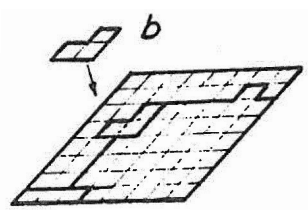


PART A



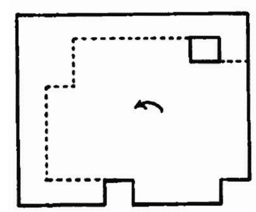
CUT ---

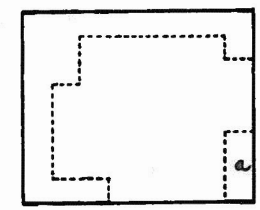
PART B



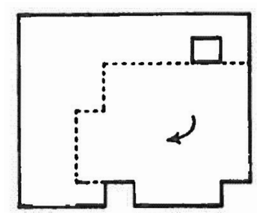
II. Solution by Peter Szabaga, New York City.

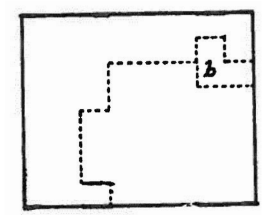
a) Cut the figure along the dashed lines, turn the piece in the lower right counterclockwise 90°, and insert it with piece a as shown below.





b) Cut the figure along the dashed lines, turn the piece in the lower right clockwise 90°, and insert it with piece b as shown below.





Also solved by DANIEL ESSIG, STEPHEN HODGE, STEPHEN L. SNOVER, and the PROPOSER.

481. [Fall 1980] *Proposed by Clayton W. Dodge., University of Maine at Orono.*

Find all roots of the polynomial equation

$$x^6 - x^5 - 4x^4 + 5x^3 - 41x^2 + 36x - 36 = 0,$$

given that it has two roots whose sum is zero.

Solution by Léo Sauvé, Algonquin College, Ottawa, Canada.

Let the two roots in question (which must be nonzero) be $\pm a$. If the given equation is denoted by $f(x) = 0$, then $\pm a$ are also zeros of

$$f(x) - f(-x) = -2x(x+3)(x-3)(x^2+4).$$

Synthetic division soon yields $f(\pm 3) = f(\pm 2i) = 0$ and

$$f(x) = (x+3)(x-2)(x^2+4)(x^2-x+1).$$

The roots are ± 3, $\pm 2i$, and $\frac{1 \pm i\sqrt{3}}{2}$.

Also solved by J. ANNULIS, JEANETTE BICKLEY, DAVID DELSESTO, DANIEL ESSIG, MARK EVANS, JACK GARFUNKEL, ROBERT C. GEBHARDT, W. C. IGIPS, RALPH KING, JEAN LANE, HENRY S. LIEBERMAN, JAMES A. PARSLY, BOB PRIELIPP, TAGHI REZAY-GARACANI, SAHIB SINGH, ROBERT A. STUMP. PETER SZABAGA, KENNETH M. WILKE, and the PROPOSER.

482. [Fall 1980] *Proposed by Ronald E. Shiffler, Georgia State University.*

Let X be a continuous random variable having a uniform distribution with domain $[a,b]$ and mean and standard deviation represented by μ and σ, respectively. Verify that $P(\mu - 2\sigma \le X \le \mu + 2\sigma) = 1$.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, Wisconsin.

It is known that $\mu = (a+b)/2$ and $\sigma^2 = (b-a)^2/12$ [see Table 4.1 on p. 83 of LINDGREN AND MCELRATH, Introduction to Probability and Statistics, The Macmillan Co., New York, 1959]. It follows that $2\sigma = (b-a)/\sqrt{3}$. Because μ is the midpoint of $[a,b]$, $|X - \mu| \le (b-a)/2$. But $\sqrt{3} < 2$ so $1/2 < 1/\sqrt{3}$ and hence $(b-a)/2 < (b-a)/\sqrt{3}$. Therefore $|X - \mu| < (b-a)/\sqrt{3} = 2\sigma$ so $P(\mu - 2\sigma \le X \le \mu + 2\sigma) = 1$.

Also solved by DANIEL ESSIG, MARK EVANS, ROBERT C. GEBHARDT, JOHN M. HOWELL, W. C. IGIPS, HENRY S. LIEBERMAN, SAHIB SINGH, and the PROPOSER.

483. [Fall 1980] *Proposed by Paul Erdos, Spaceship Earth.*

Let μ_n be the smallest integer for which $\mu_n(\mu_n + 1) \equiv 0 \pmod{n}$. Prove $\sum \frac{1}{\mu_n(\mu_n + 1)} < \infty$.

Solution by Irwin Jungreis, No. Woodmere, New York.

We have
$$\sum_{n=1}^{\infty} \frac{1}{\mu_n(\mu_n+1)} = \sum_{i=1}^{\infty} \frac{\lambda_i}{i(i+1)}$$
where λ_i is the number of values of n for which $i = \mu_n$. From the definition of μ_n, $\mu_n = i \Rightarrow n | i(i+1)$ so $\lambda_i \le \tau(i(i+1))$. We know, however, that $\tau(x) = o(x^{\varepsilon})$ for any $\varepsilon > 0$. Taking $\varepsilon = \frac{1}{4}$, there is N such that $n > N$ implies $\tau(n) < n^{1/4}$. Then
$$\sum_{i=1}^{\infty} \frac{\lambda_i}{i(i+1)} \le \sum_{i=1}^{\infty} \frac{\tau(i(i+1))}{i(i+1)} < \sum_{i=1}^{N} \frac{\tau(i(i+1))}{i(i+1)} + \sum_{i=N+1}^{\infty} \frac{1}{(i)^{3/2}} < \infty.$$

Also solved by the PROPOSER.

484. [Fall 1980] *Proposed by the late R. Robinson Rowe.*

In a triangle with base AB and vertex C, secants from A and B to points D and E on BC and AC divide the area into four subareas S, T, U and V. In some order of S, T, U, V, the points D and E can be located so that the subareas are in increasing arithmetical progression, or so that they are in decreasing arithmetical progression. Find that order and evaluate the subareas.

Solution by the Proposer and Morris Katz, Macwahoc, Maine.

Let triangle ABC have base AB of length 2 and altitude to vertex C equal to 1, without loss of generality. Let AD and BE meet at F, and let D, E, and F have altitudes x, y, and h from base AB. Designate by S, T, U, and V the areas of triangles FAB, AEF, BDF, and quadrilateral $CDFE$. See Figure 1. From similar triangles obtain

$$h = \frac{xy}{x + y - xy}. \tag{1}$$

We also have

$$S = h,\ T = y - h,\ U = x - h, \text{ and } V = 1 - x - y + h.$$

Now $S < U$ implies $T < S$, so S lies between T and U. Hence, by symmetry, we need consider just two orders: $VTSU$ and $TSVU$.

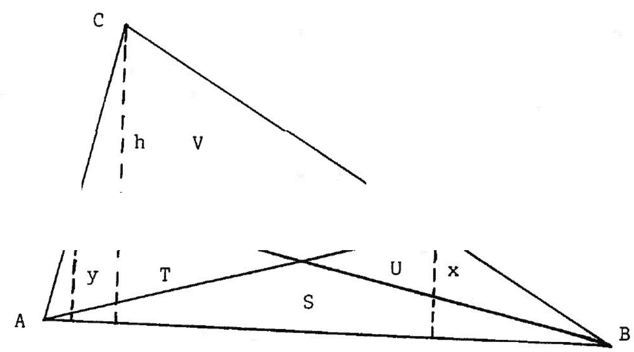


Figure 1

For the order $VTSU$, either increasing or decreasing, we must have

$$U - S = S - T = T - V,$$

$$x - 2h = 2h - y = 2y - 2h + x - 1, \tag{2}$$

whence $y = 1/2$. Now substituting into (1) and (2), obtain

$$2x^2 - 5x + 1 = 0,$$

$$\text{so } x = \frac{5 \pm \sqrt{17}}{4}$$

Only the negative sign permits $x < 1$, so

$$x = \frac{5 - \sqrt{17}}{4}, \quad y = \frac{1}{2}, \text{ and } h = \frac{7 - \sqrt{17}}{16}$$

We find that

$$V = \frac{3\sqrt{17} - 5}{16}, \quad T = \frac{1 + \sqrt{17}}{16}, \quad S = \frac{7 - \sqrt{17}}{16}, \text{ and } U = \frac{13 - 3\sqrt{17}}{16}$$

For TVSU we use

$$U - S = S - V = V - T,$$

$$(3) \qquad x - 2h = x + y - 1 = 1 - 2y - x + 2h .$$

The solution is not as simple here, but using (1) and (3) we eliminate x and h to get

$$(4) \qquad 4y^3 - 17y^2 + 14y - 3 = 0,$$

which has no rational roots. Its decimal roots are

$$y_1 = .353116, \quad y_2 = .655199, \text{ and } y_3 = 3.2416 .$$

We cannot use y_3 because $y < 1$. We have two solutions

$$x_1 = \frac{3}{2} - 2y = .793678, \quad y_1 = .353116, \quad h_1 = \frac{1 - y}{2} = .323442,$$

$$T_1 = .029674, \quad V_1 = .176558, \quad S_1 = .353442, \quad U_1 = .470326$$

and

$$x_2 = .189803, \quad y_2 = .655199, \quad h_2 = .172451,$$

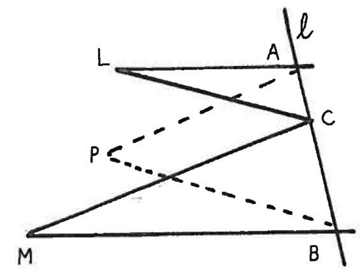
$$T_2 = .482648, \quad V_2 = .327549, \quad S_2 = .172451, \quad U_2 = .017352 .$$

485. [Fall 1980] *Proposed by* R. S. *Luthar, University of, Wisconsin, Janesville.*

A line l cuts two parallel rays emanating from L and M in A and B respectively. A point C is taken anywhere on l. Lines through A and B respectively parallel to MC and LC intersect in P. Find the locus of P.

Solution by the. Proposer.

Let AP intersect line LM at T. Then, according to Problem 409 (this *JOURNAL*, Fall 1978, page 556) by **ZELDA KATZ**, BT must be parallel



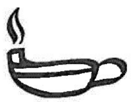
to CL. Thus BP and BT coincide because they are both parallel to CL. Hence T and P coincide; that is, P lies on line LM.

Also solved by ROBERT C. GEBHARDT *(by analytic geometry)*, RALPH KING *(graphical solution)*, ROGER E. KUEHL, HENRY S. LIEBERMAN *(two solutions, one by a theorem of, Pappus, the. other by Grassmann's geometric algebra), and* SISTER STEPHANIE SLOYAN *(by the converse to Pascal's theorem)*.

1980-81 STUDENT PAPER COMPETITION

The papers for the. 1980-81 Student Papa Competition have been judged and the winners are:

First Prize ($200)	*Gary Ricard, South Dakota School of, Mines and Technology;* "A Comparison of Computer Algorithms To Solve the Knight's Tour Problem''
SECOND PRIZE ($100)	*Sandra Cousins, Hendrix College;* "Singular Functions", To Appear in the Spring 1982 Issue.
THIRD PRIZE ($50)	*Michael Orrick, Macalester College;* 'The Area of a Triangle Formed by Three Lines", Appearing in this Issue.



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