PI MU EPSILON JOURNAL

VOLUME 7 SPRING 1982 NUMBER 6

CONTENTS

Editorial 357

Quadric Surfaces Associated with Linear Differential Equations
Duane De Temple and Roland Starr 358

Relativity in Perspectivity
William Terkeurst 365

Singular Functions
Sandra L. Cousins 374

The Cubic Equation Revisited
F. Max Stein 382

A Day at the Races
William Tomcsanyi 396

Partial Differentiation of Functions of a Single Variable
Richard Katz and Stewart Venit 405

Chapter Report 406

Puzzle Section
David Ballew 407

Problem Department
Clayton W. Dodge 411

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PI MU EPSILON JOURNAL

THE OFFICIAL PUBLICATION OF THE HONORARY MATHEMATICAL FRATERNITY

David **Ballew**, Editor

Chapter reports, books for review, problems for solution and solutions to problems, **should be mailed directly** to the special editors found in this issue under the various sections. **Editorial correspondence**, including manuscripts (two copies) and news items should be mailed to **THE EDITOR** OF THE PI MU EPSILON JOURNAL. South Dakota School of Mines and **Technology**. Rapid City. **SD** 57701. For manuscripts, authors are requested to identify themselves as **to their class or** year if they are undergraduates or graduates, and the college or university they are **attending**, and as to position if they are faculty members or in a non-academic profession.

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EDITORIAL

The Pi Mu Epsilon Journal is dedicated to undergraduate and beginning graduate students, and the Journal solicits articles of all kinds from them. We believe that many students solve problems, discover theorems, write papers, give talks and seminars, and complete projects that are suitable for submission to this publication. We encourage these students to prepare and submit these as articles for possible publication. Student papers will always be given first preference by this Editor. Many students have found it very advantageous to have had an article published in a refereed journal when considered for graduate school or employment.

Pi Mu Epsilon encourages student research and the presentation/publication of that research through this Journal and many other means. The National Paper Competition awards prizes of $200, $100 and $50; all student paper submitted to the Journal are eligible for these awards. In addition, any one Chapter submitting five or more papers creates a mini-contest among just those papers with a top prize of $20 for the best. As most know, Pi Mu Epsilon sponsors many student paper conferences and student paper sessions in conjunction with other organizations such as the MAA.

This is a Call to all students who are writing those papers, those projects, giving those talks, proving theorems, etc., etc. Write up your results in the form of an article for this Journal and submit it to the Editor. This is a Call to faculty member to encourage your students and to help them with their paper. THIS IS YOUR JOURNAL--USE IT!

The Editor

QUADRIC SURFACES ASSOCIATED WITH LINEAR DIFFERENTIAL EQUATIONS

Duane De Temple (*Faculty*)
*Roland Starr** (*Student*)
Washington State University

1. *Introduction.*

We shall be concerned here with the real, first order linear autonomous system of differential equations

(1) $\dot{x} = Ax \qquad (\cdot = \frac{d}{dt})$

where $x \in R^n$ and A is a linear transformation of R^n. For the case $n = 2$ one of us has recently devised a purely geometrical method to construct the tangent vector $\dot{x}$ of the trajectory which passes through x, once the 2 x 2 matrix [A] representing A has been given [1]. The development there also revealed a remarkable geometric property of the planar phase portrait: there exists a family of homothetic ellipses (if det $A > 0$) or hyperbolas (if det $A < 0$) whose normals, when rotated through a constant angle, coincide with the direction of the tangents $\dot{x}$. The proof of this contained in [1] detours through complex analysis and is, as we show here, unnecessarily complicated. The linear algebra approach in the next section holds in any dimension. Of course, it is the geometrically visualizable cases $n = 2$ and 3 which are most interesting, and the concluding two sections illustrate these cases.

2. *Tangent Vectors as Rotated Normal Vectors of Quadric Surfaces.*

To avoid degenerate cases (which can be treated as limiting cases), we assume the linear transformation A is non-singular. Below we will show that A can always be decomposed into the form

(2) $A = RS$,

where R is proper rotation (i.e., an orthogonal transformation with det $R = +1$) and S is a symmetric non-singular transformation. For any such decomposition (it is not unique), the differential equation (1) takes the form

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(3) $\dot{x} = RSx.$

Next consider the one-parameter family of quadric surfaces

(4) $Q_\kappa = \{x : \frac{1}{2} \langle x, Sx \rangle = \kappa\}, \quad \kappa = \text{constant},$

where $\langle\, ,\rangle$ denotes the standard inner produce on R^n.

From advanced calculus, we know

(5) $n(x) = \nabla(\frac{1}{2}\langle x, Sx\rangle) = Sx$

is normal to Q at $x \in Q_\kappa$. Comparing (3) and (5), we see that

(6) $\dot{x} = Rn(x).$

That is, there is a constant proper rotation R which takes any gradient $n(x)$ of the quadric surface Q onto the tangent vectors $\dot{x}$ of the trajectory through $x \in Q$.

We now examine what possibilities exist to represent A in the form $A = RS$. From the polar decomposition theorem (e.g., see Halmos [2], p 170), we know A may be uniquely factored as $A = UP$, where U is orthogonal and P is the unique positive square root of A^*A. For some orthonormal basis the matrix of P will be diagonal, say $[P] = \text{diag}[\rho_1, \ldots, \rho_n]$. Then define the linear transformation E by $[E] = \text{diag}\,[\varepsilon_1, \ldots, \varepsilon_n]$, where the $\varepsilon_j \in \{-1, +1\}$ are chosen in any of the ways for which det E = det V. We observe that E is symmetric, orthogonal, $E^2 = I$ = identity, and EP = PE. Choosing S = EP and R = UE, it is clear S is symmetric and R is a proper rotation. Moreover, A = UP = $UEEP$ = RS.

Conversely, suppose A = RS. Then $A^*A = SR^*RS = S^2$, and so $S^2 = P^2$, where as before P is the unique positive square root of A^*A. As S is symmetric, there is some orthonormal basis in which $[S] = \text{diag}[\sigma_1, \ldots, \sigma_n]$. It then follows that $[P^2] = [S^2] = \text{diag}[\sigma_1^2, \ldots, \sigma_n^2]$, and so for some sequence $\sigma_j \in \{-1, +1\}$ we have $\sigma_j = \varepsilon_j\rho_j$, as before the ρ_j denote the eigenvalues of P. In summary, the construction of the preceding paragraph produces all decompositions of the form A = RS.

The number of distinct families of quadric surfaces can also be found. If the eigenvalues of P are distinct, the corresponding n linear eigenspaces are the principal axes of any of these families of quadric surfaces. In # if det $A > 0$, the number of ε_j which are taken as -1 must be even; this means there are $\binom{n}{0} + \binom{n}{2} + \ldots = 2^{n-1}$ families. These will be distinct when n is odd; however, when n is even, both $-S$

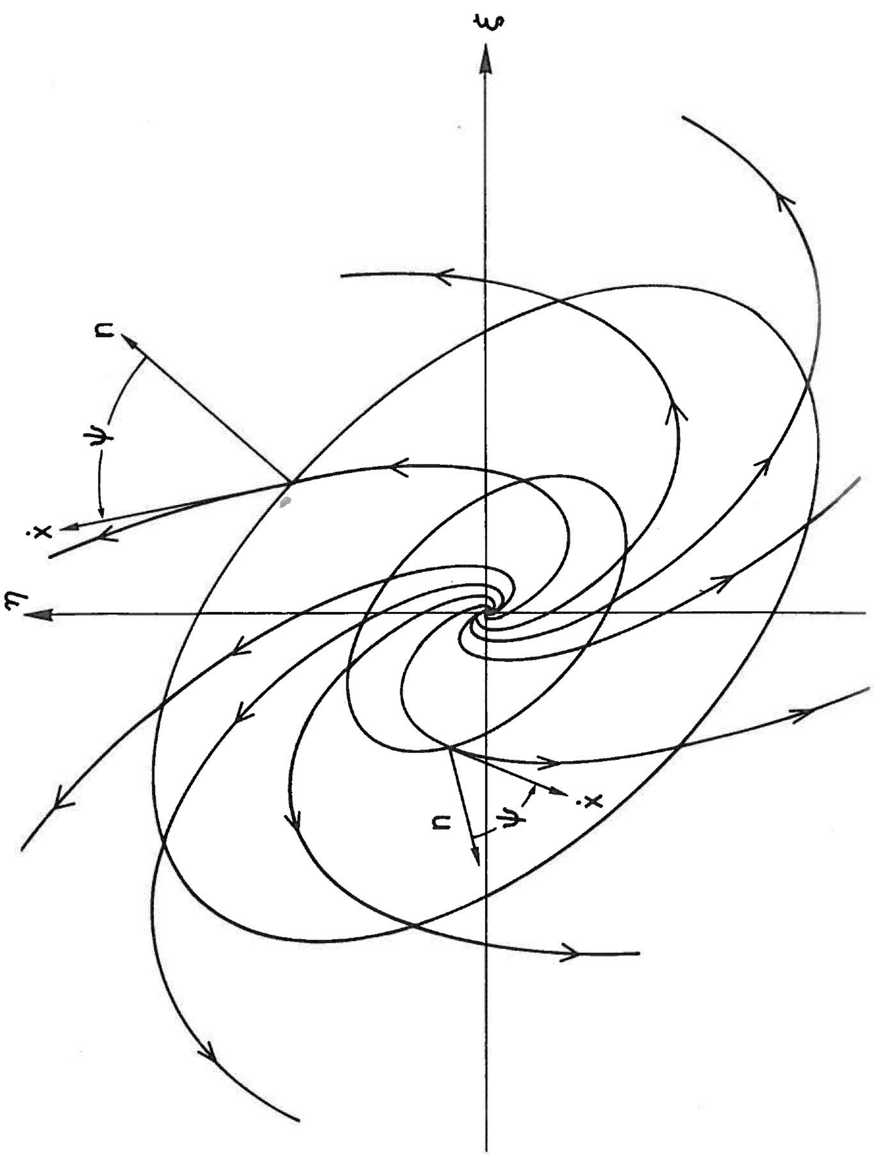

FIGURE 1

and $+S$ give the same family (and $-R$ is still a proper rotation), meaning there are 2^{n-2} distinct families. An analogous argument shows that the number of families of quadric surfaces remains the same in the case $\det A < 0$.

If P has any multiple eigenvalues, the axes of the quadrics within the corresponding eigenspaces can be rotated arbitrarily within these subspaces.

3. The. *Two-dimensional Case.*

Here $n = 2$ is even, so the number of families of conics is $2^{2-2} = 1$. This will be a family of homothetic central ellipses (when $\det A > 0$) or conjugate hyperbolas (when $\det A < 0$). As an illustrative case consider the differential equation $\dot{x} = Ax$, where

$$[A] = \begin{bmatrix} 2 & -5 \\ 11 & 10 \end{bmatrix}$$

in the standard basis of R^2. Then

$$[A] = [R]\ [S] = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 10 & 5 \\ 5 & 10 \end{bmatrix},$$

where R is a rotation by $\psi \doteq 53^\circ$ and S is a symmetric linear transformation with eigenvalues 15 and 5 and corresponding eigenvectors (1,1) and (1,-1). The phase plane is shown in Figure 1; two ellipses of the family

$$\frac{1}{2}\langle x, Sx\rangle = 5\xi^2 + 5\xi\eta + 5\eta^2 = \text{constant}$$

are also shown, where $x = (\xi,\eta)$. The RS decomposition of A can be obtained by purely geometric means, including the rotation angle ψ and the eigenvectors and eigenvalues of S[1].

4. The *Three-dimensional Case.*

Consider the differential equation $\dot{x} = Ax$ in R^3, where

$$[A] = \frac{1}{3}\begin{bmatrix} 24 & 26 & -45 \\ -6 & -4 & -15 \\ -9 & -1 & 15 \end{bmatrix}$$

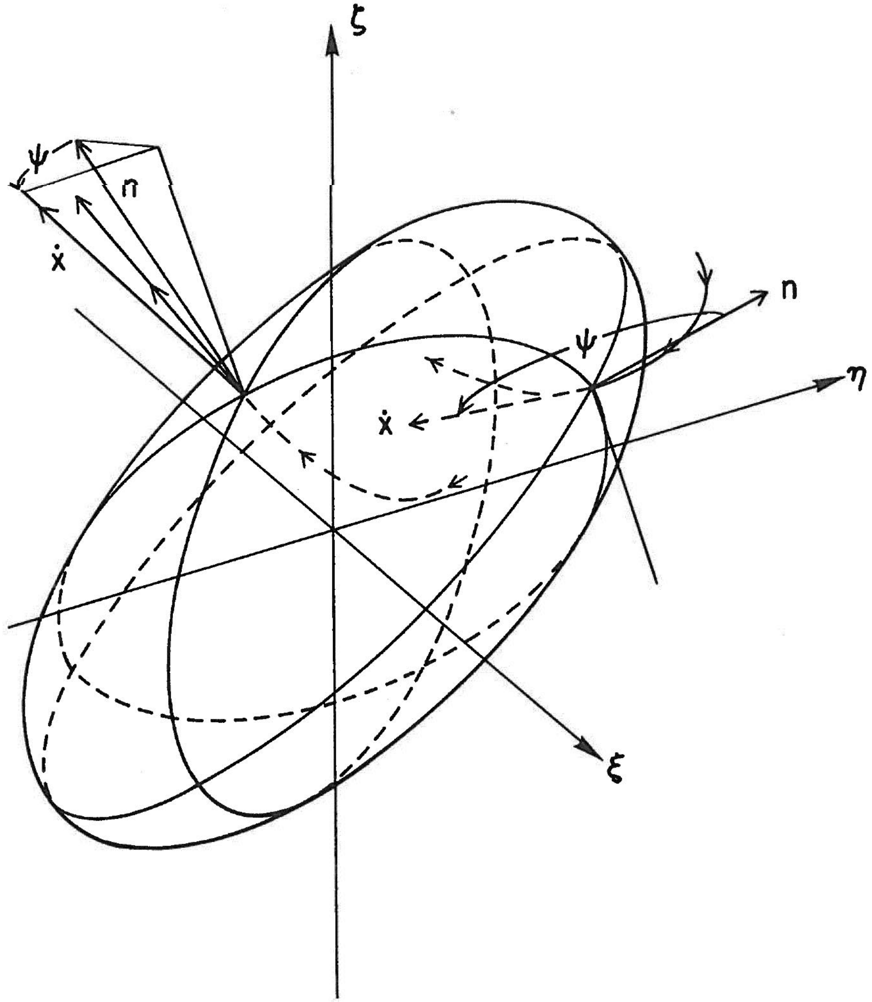

FIGURE 2

in the standard basis. The polar decomposition of A is found to be

$$[A] = [U][P] = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 6 & 4 & -5 \\ 4 & 6 & -5 \\ -5 & -5 & 15 \end{bmatrix} .$$

The eigenvalues and corresponding eigenvectors of P are

$$\rho_1 = 20, [e_1] = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}; \ \rho_2 = 5, [e_2] = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}; \ \rho_3 = 2, [e_3] = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} .$$

The directions of e_1, e_2, e_3 define the principal axes of the families of quadric surfaces.

Since det $U = +1$, U is a rotation and UP is one of the four RS decompositions of A. A unit eigenvector corresponding to the eigenvalue 1 is $b_1 = (1/\sqrt{2}, 0, -1/\sqrt{2})$. Completing this to an ordered orthonormal basis $\mathcal{B} = (b_1, b_2, b_3)$, we have the form

$$[U; \mathcal{B}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{bmatrix};$$

and so the cosine of the rotation angle is'

$$\cos\psi = \tfrac{1}{2}(tr\ U - 1) = \tfrac{1}{2}(\tfrac{1}{3} - 1) = -\tfrac{1}{3} .$$

Since $U(0,1,0) = (\frac{2}{3}, -\frac{1}{3}, \frac{2}{3})$, we see $\psi = \arccos(-\frac{1}{3}) \in (90^\circ, 180^\circ)$.

That is, U is a right-hand rotation about $(1,0,-1)$ by $\psi = 109.5^\circ$.

Figure 2 shows how a trajectory cuts through an ellipsoid from the family

$$\tfrac{1}{2}\langle x, Px\rangle = 3\xi^2 + 6\eta^2 + 15\zeta^2 + 4\xi\eta - 5\xi\zeta - 5\eta\zeta = \text{constant},$$

where $x = (\xi, \eta, \zeta)$.

The remaining three families of quadric surfaces are hyperboloids of the sort shown in Figure 3.

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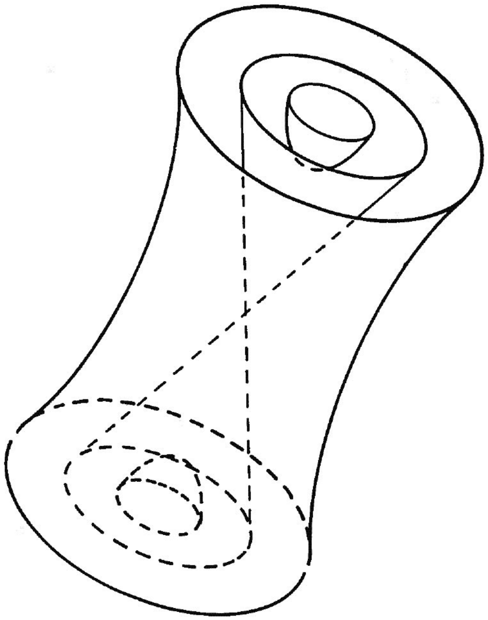


FIGURE 3

REGIONAL MEETINGS

Many regional meetings of the Mathematical Association of America regularly have sessions for undergraduate papers. If two or more colleges and at least one local chapter help sponsor or participate in such undergraduate sessions, financial help is available up to $50. Write to Dr. Richard Good, Department of Mathematics, University of Maryland, College Park, Maryland, 20742.

RELATIVITY IN PERSPECTIVITY

by *William Terkeurst*
Hope College, Holland, Michigan

"Those Who Hear, Forget.
Those Who See, Remember.
Those Who Do, Understand." - Anonymous

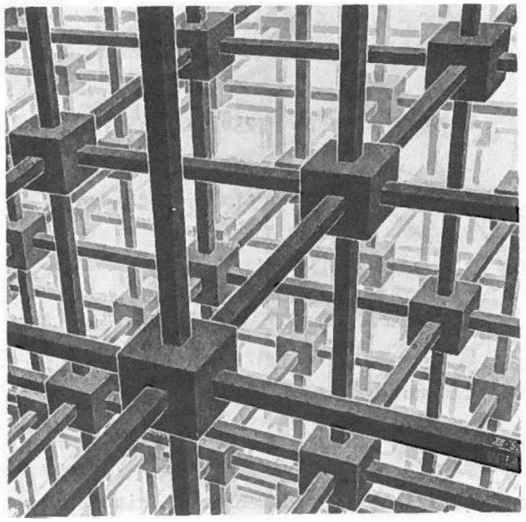
Under the influence of Dr. Elliot Tanis (Hope College) and my own fascination with the art work of M.C. Escher, I was compelled to delve into researching Escher's work myself, especially since Escher was an artist, not a mathematician. He just created his works from his imagination and some outside ideas, yet there is the mathematics in his work. With my chief interests and background stemming from architectural drafting and rendering, I became intrigued with the way in which Escher created some of his perspective drawings. I noticed, however, that his use of perspective was different from my own experience, which was mostly orthogonal perspective. So I decided to find out why.

Starting with the question, "What is a perspective drawing and where does it come from?", I discovered it comes from projective geometry. There are five basic axioms for plane projective geometry[1]. They are:

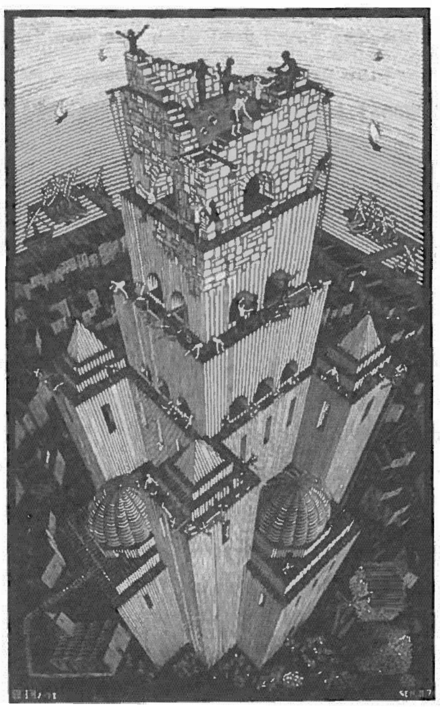
1. Any two distinct points are incident with exactly one line.
2. Any two lines are incident with at least one point.
3. There exist four points of which not three are collinear.
4. The three diagonal points of a completequadrangle are never collinear.
5. If a projectivity leaves invariant each of three distinct points on a line, it leaves invariant every point on the line.

The basic undefined terms are point, line, and incidence, the same as in other geometries.

The first two axioms are very direct. Axiom 1 says that two points determine a line (same as Euclidean), and Axiom 2 says that any two lines meet somewhere, or in other words, no parallelism (not like Euclidean). A drawing will help to clarify the others.



CUBIC SPACE DIVISION



TOWER OF BABEL

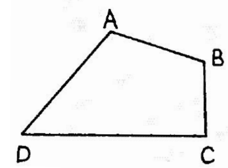


Figure 1a

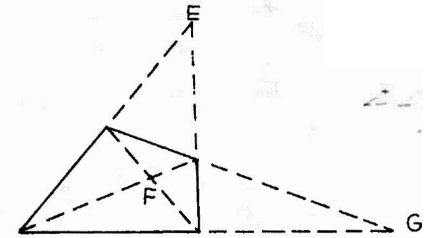


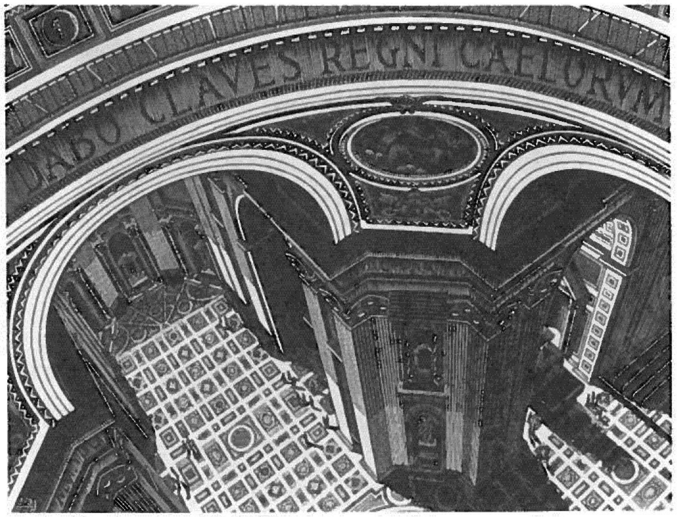
Figure 1b

Figure 1a shows a complete quadrangle A,B,C,D (Axiom 3), and Figure 1b illustrates the three diagonal points E,F, and G (Axiom 4) formed by extending opposite pairs of sides of the quadrangle. Axiom 5 states that when you project a line, project all of it.

In order to follow along, we must get the terminology straight. Diagonal points, as referred to in Axiom 4, are commonly called vanishing points. Vanishing points can be further classified as either distance, zenith, or nadir points. A distance point is any vanishing point off to either side, a zenith point is straight up or overhead, and a nadir point is directly below or down.

How does all of this fit in with Escher's work? Let us take a look at one of Escher's prints, which is a classic example of a projective geometry, CUBIC SPACE DIVISION, July of 1952. If we follow the direction of the "girders" (or the sides of the cubes) we can extend them off the page and see that they would meet at a distance point in the upper right, another distance point at left center, and a nadir point at the bottom center. (Remember, none of those lines are parallel, Axiom 2). There are three vanishing points just like our Axioms say (keep in mind we are now considering 3-dimensional objects, the cube, and not restricting ourselves to a plane surface, a side of the cube). A very interesting aspect of Escher's work is in the way in which he uses relativity. It is amazing how often it lies hidden in a print. For example, CUBIC SPACE DIVISION looks rather uncomplicated. However, simply rotating the page in multiples of 90 degrees changes the orientation of your viewpoint, from upper right to lower right to lower left to upper left moving clockwise.

One of Escher's earliest perspective drawings is TOWER OF BABEL,



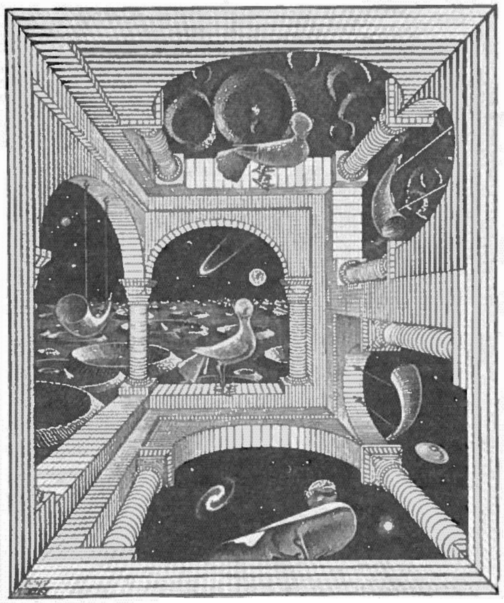
SAINT PETER'S OF ROME

OTHER WORLD I

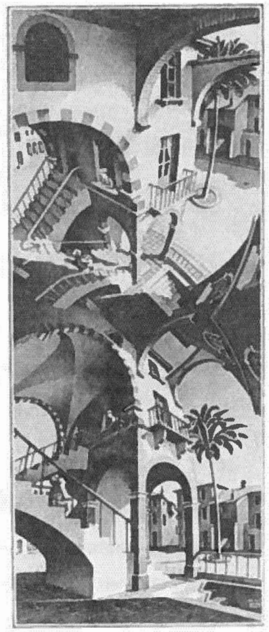
February of 1928. Here Escher has a definite interest in the nadir point, with distance points off to both sides. This is also true in SAINT PETER'S OF ROME done in March of 1935. His early obsession with what is sometimes called a "birds eye view" is simple: he is an artist. Artists try to convey a message or feeling in their work. In both of these prints, Escher is trying to express the feeling your stomach gets when you look down from a high place.

A few years later, in the wake of another world war, Escher again was influenced by the world around him. The advancing technology, interest in space, and Einstein's idea about how things are measured with respect to other things, helps Escher think along the same lines. In the final month of 1946, he introduced a strange concept in OTHER WORLD. In this case, all of the lines vanish toward the center of the print. Concerning the question of whether this vanishing point is a distance, nadir, or zenith point, the answer is yes. It depends on which side of the print you view, as to which role the vanishing point takes on. All of the sides are of the same scene, looking along a tunnel with arched openings in which is standing a "simurgh" (a note for the trivialist: the simurgh is a Persian man-bird given to Escher by his father-in-law who bought it in Baku, Russia, or so I am told) under a fish lamp. In other words, if we look at the top of the print we are looking down the tunnel (nadir point), if we look at the bottom of the print we are looking up (zenith point), while looking to either side of the print we are looking along the tunnel (distance point). As an artist, Escher was not overly pleased with this print. It has a long dark shady tunnel and it took four sides to convey three worlds.

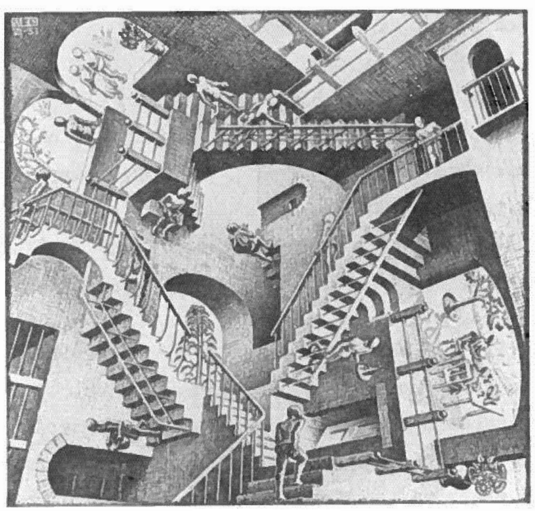
A month later, January of 1947, Escher created OTHER WORLD again. This time, however, we find ourselves in a strange five-sided room instead of a tunnel. As far as the perspective goes, the emphasis is still the center and it still can be a distance, nadir, or zenith point. What is so different about the room? Well, depending on which "window" you look in or out of, concepts like right, left, above, below, in front, and behind are totally interchangable. There is another geometry called Affine Geometry, which explains the parallel lines in both of the OTHER WORLD prints. Affine geometry is a proper subgroup of projective geometry, and is merely an extension of the Euclidean plane in which parallelism is preserved. Affine Geometry also explains orthogonal perspective for the



OTHER WORLD II



RELATIVITY



HIGH AND LOW

architectural draftsman.

In July of 1957 Escher gave the world **HIGH** AND LOW which is a couple of towers with curved lines. This is a classic example of artistic license. What we really have here is two drawings in one, the upper half and the lower half. In the upper half is a tower whose lines are vanishing toward a point in the center of the print, a nadir point. There is also a distance point off to each side near the top of the print. Likewise, in the lower half is a tower whose lines are curving toward a point in the center of the print, a zenith point, along with a distance point off to each side near the bottom of the print. The nadir of the upper half and the zenith of the lower half coincide in the center of the print. So we still have six vanishing points for two drawings, everything is legal except for those curved lines.

The concept of drawing "straight lines" in a curved manner is one arrived at by both artists and draftsmen alike and has a simple derivation. The projection of straight parallel lines X,Y (Figure 2a) onto a cylinder produces a couple of semi-elipses, X' and Y'. Slicing the cylinder in half, ABCD, and laying it out flat, produces a nice sinusoidal curve, Figure 2b.

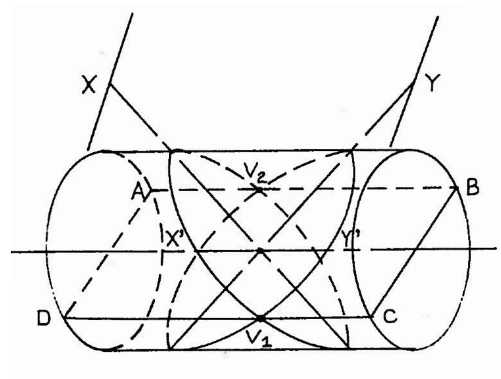


Figure 2a

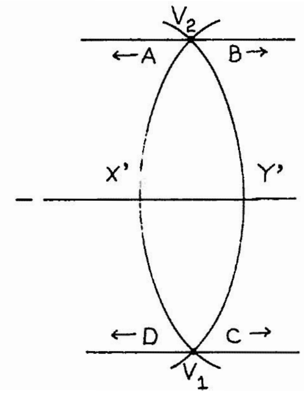


Figure 2b

Other analogous situations are the multi-photograph effect, or the telegraph-wire effect [2]. The amazing thing is that Escher said it was simply the way he saw it, and careful measurements of some of his drawings show them to be almost sine curves. Please note that the upper and lower halves are really the same scene (a boy sitting on some steps talking to a girl in the tower). Also notice that the doors, windows, stairs, and even the palm trees are in the same positions in both halves. This print allows the observation of the same scene from two vantage points at the same time. Something to contemplate: What does the boy in the lower half see, if he tilts his head back a little further and looks up?

Our journey leads us finally to a print most appropriately entitled RELATIVITY, completed in July of 1953. The three vanishing points are now all off the page forming an equilateral triangle over two meters a side (the print is 28 x 29 cm). Already in our brief exploration of the bizarre world of M.C. Escher, we have come to accept the possibility of one vanishing point being three different vanishing points in order to stay within the axioms of projective geometry. However our boundaries are stretched even further as we notice that each vanishing point now serves as a zenith point, a right distance point, and a left distance point. With each point being three-fold in function, we have, relativistically, about nine vanishing points. This unique situation is actually three worlds combined into one. Those worlds can be identified as the uprighters - the figures whose bodies point upward (the figure at the bottom center coming upstairs), the right-leaders, whose bodies point to the right (figure coming downstairs with a tray in hand), and the left-leaners, whose bodies lean leftward (figure with basket). And though these figures share the same environment, they each have different ideas about what to call things. For example, one group calls a surface a floor, another group calls it a wall or a ceiling. Likewise, a door to one might be a trap-door to another. As a result of this strange cohabitation, we run across some rather baffling situations, such as stairways which allow figures to walk on opposite sides of the staircase, or stairways that allow two figures to walk on the same side of the staircase, walk in the same direction (left to right across the page), yet one is going upstairs while the other is going downstairs. Also, each world has its own patio, but can every figure reach his/her respective patio?

It is possible to spend hours studying this print, rotating it, trying to look around corners, and so on. But if you really want to experience and understand it, you must build one. It is possible, and I am sure you will find, as I have, that it will aid in answering many questions. At the same time, however, while answering questions, it also

poses new ones (i.e., for example, like three more worlds in opposite directions of the present ones). Try it.

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This paper *won* Second *Prize* in the 1980-81 *National* Student Paper Competition.

SINGULAR FUNCTIONS

by Sandra L. Cousins
Hendrix College, Conway, Arkansas

Introduction.

In calculus, a student is taught to think of integration and differentiation as inverse operators. This notion is set forth in the Fundamental Theorem of Calculus, which is stated here in two parts.

FTC I. If f is continuous on $[a,b]$ and $c \in [a,b]$, then $f(x) = \frac{d}{dx}\int_c^x f(t)dt$ for every $x \in (a,b)$.

If the condition of continuity of f is relaxed to just integrability, then f is bounded and continuous almost everywhere on $[a,b]$. Therefore, $f(x) = \frac{d}{dx}\int_c^x f(t)dt$ for every x at which f is continuous, that is, almost everywhere on $[a,b]$. Thus, we have the following theorem:

Theorem. If f is integrable on $[a,b]$ and $c \in [a,b]$, then $f(x) = \frac{d}{dx}\int_c^x f(t)dt$ almost everywhere on $[a,b]$.

Next we look at the second part of the Fundamental Theorem.

If f' is continuous on $[a,b]$ and $c \in [a,b]$, then $\int_c^x f'(t)dt = f(x)-f(c)$ for all $x \in [a,b]$.

Does an analogous theorem result when continuity is relaxed here: The answer to this question is negative and the examples of so-called "singlular" functions which follow exhibit this pathology.

A singular function is a monotonic continuous function $f: [a,b] \to \mathbb{R}$ such that $f'(x) = 0$ almost everywhere (i.e., on the complement of a set of measure zero). A set of measure zero is a set which can be covered by a countable collection of segments, the sum of whose lengths is arbitrarily small.

Cantor-Lebesgue Function. The first example we will consider is the Cantor-Lebesgue singular function 1, pp. 135-137 , which is based upon the Cantor set S. Recall that this set is formed as follows: $S_0 = [0,1]$. Remove the open middle third $(1/3, 2/3)$, thus leaving $S_1 = [0, 1/3] \cup [2/3, 1]$. Continue to remove the open middle third from each of the remaining intervals at each step in the process. Then the Cantor set $S = \bigcap_{n=0}^{\infty} S_n$. Note that since the sum of the lengths of the intervals constituting S_n is $(2/3)^n$ which approaches zero as $n \to \infty$, then S has measure zero. Each point in S can be represented uniquely by its ternary expansion of the form $\sum_{n=1}^{\infty} a_n 3^{-n}$, where $a_n \in \{0,2\}$, $n = 1,2,\ldots$.

Turning now to the construction of the Cantor-Lebesgue function C, let $C(x) = 0$ for $x < 0$ and let $C(x) = 1$ for $x > 1$. If $x \in S$, write $x = \sum_{n=1}^{\infty} a_n 3^{-n}$, $a_n \in \{0,2\}$, $n = 1, 2, \ldots$. Define $C(x) = \sum_{n=1}^{\infty} a_n 2^{-(n+1)}$. If $x \in [0,1]$ and $x \notin S$, then there exists a "middle third" or complementary interval (y,z) such that $y < x < z$ $(y, z \in S)$. Let $C(x) = C(y) = C(z)$. (See Figure 1). It follows that C maps $[0,1]$ onto $[0,1]$.

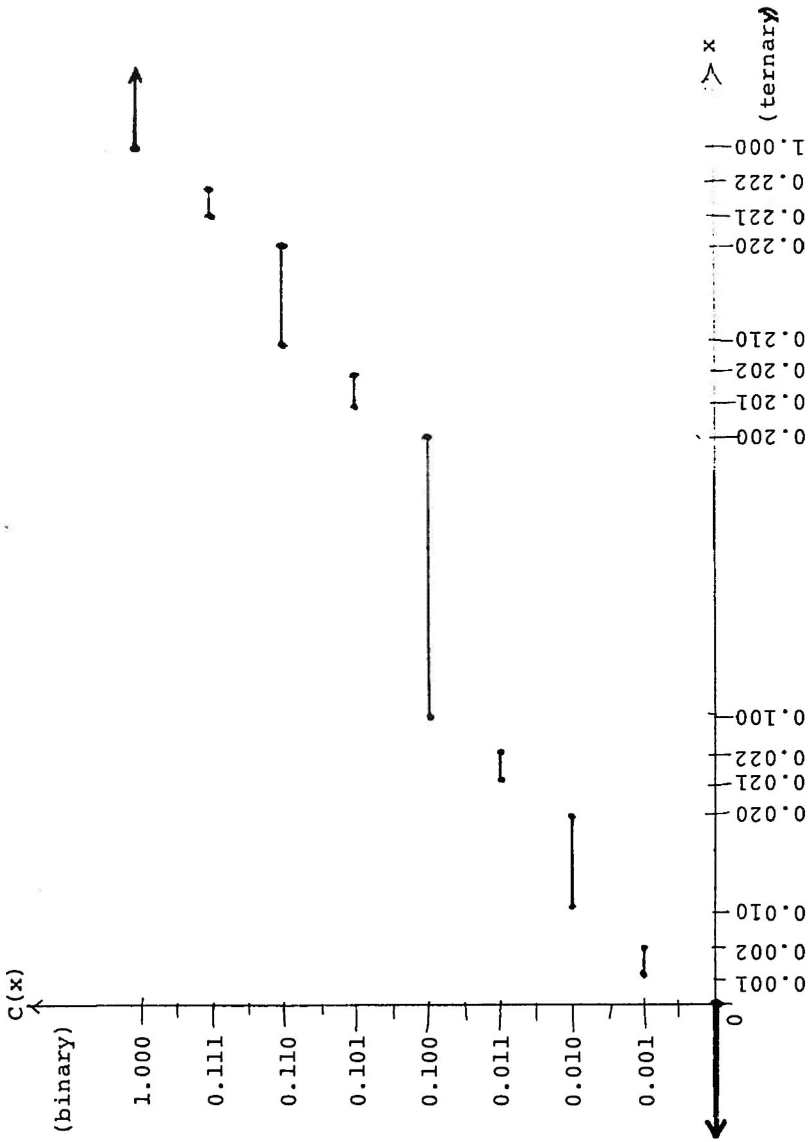
The first step in proving that C is a singular function is to show that C is continuous. Clearly, C is continuous on $(-\infty,0)$ and $(1,\infty)$. If $x \in (0,1) \setminus S$, then x belongs to the interior of a closed interval on which C is constant. Therefore, C is continuous at x.

Now suppose $x \in (0,1) \cap S$ and $\varepsilon > 0$. Then $x = \sum_{i=1}^{\infty} a_i 3^{-i}$ where $a_i \in \{0,2\}$, $i=1,2,\ldots$ and $C(x) = \sum_{i=1}^{\infty} a_i 2^{-(i+1)}$. Choose n large enough that $2^{-n} < \varepsilon$. Let $\delta = \min\{3^{-n}, |x-0|, |x-1|\}$ and suppose $y \in \mathbb{R}$ such that $|x-y| < \delta \leq 3^{-n}$. Then $y \in (0,1)$ and if $y \in S$, then the ternary expansions $x = \sum_{i=1}^{\infty} a_i 3^{-i}$ and $y = \sum_{i=1}^{\infty} b_i 3^{-i}$ $(b_i \in \{0,2\})$ will agree through at least the first n digits $(a_i = b_i$ for $i=1,2,\ldots,n)$. Therefore $a_i 2^{-(i+1)} = b_i 2^{-(i+1)}$ for $i=1,2,\ldots,n$, which implies that $|C(x)-C(y)| < 2^{-n} < \varepsilon$. If $y \notin S$, then there exists some $z \in S$ such that $y < z \leq x$ or $x \leq z < y$ and $C(y) = C(z)$, which implies that $|z-x| < \delta$. Thus the ternary expansions $x = \sum_{i=1}^{\infty} a_i 3^{-i}$ and $z = \sum_{i=1}^{\infty} c_i 3^{-i}$, $c_i \in \{0,2\}$, agree through at least the first n digits. Hence, $a_i 2^{-(i+1)} = c_i 2^{-(i+1)}$ for $i=1,2,\ldots,n$ and $|C(x)-C(z)| < 2^{-n} < \varepsilon$. Since $C(z) = C(y)$, then $|C(x)-C(y)| < \varepsilon$. Similar arguments show that C is continuous at 0 and 1. Therefore, C is continuous for all $x \in \mathbb{R}$.

Next consider the monotonicity of C. Suppose $x \leq y$, $x,y \in [0,1]$. Choose $x' = \sum_{i=1}^{\infty} a_i 3^{-i}$ and $y' = \sum_{i=1}^{\infty} b_i 3^{-i}$ in S (where $a_i, b_i \in \{0,2\}$) such that $x' \leq x \leq y \leq y'$, $C(x) = C(x')$, and $C(y) = C(y')$. Then $C(x) = \sum_{i=1}^{\infty} a_i 2^{-(i+1)} \leq \sum_{i=1}^{\infty} b_i 2^{-(i+1)} = C(y)$. Therefore, C is nondecreasing.

Finally, observe that $C'(x) = 0$ for every x in $[0,1] \setminus S$. But since S is a set of measure zero, then $C'(x) = 0$ almost everywhere on $[0,1]$. Thus this nondecreasing function C is a singular function.

Figure 1.



Other singular functions can be constructed which are strictly increasing. For one based on the Cantor-Lebesgue function consider $f(x) = \sum_{n=1}^{\infty} 2^{-n} C(2^n(x-a_n))$, where $\{a_n\}_{n=1}^{\infty}$ is any countable set that is dense in $\mathbb{R}$. This example can be thought of as adding smaller and smaller copies of the Cantor-Lebesgue function at the points $\{a_n\}$. For a proof of the singularity of f, see [2].

Hellinger Function. Another singular function which is strictly increasing on $[0,1]$ is the Hellinger function [3, pp. 48-49], named for the man in whose doctoral thesis it first appeared in 1907. To construct this function, first let $0<t<1$ and let $F_0(x) = x$, $x \in [0,1]$. Suppose $F_n(x)$ is continuous and linear in the intervals bounded by the points $a_n = k2^{-n}$ and $\beta_n = (k+1)2^{-n}$, $k=0,1,\dots,2^n-1$. Let $F_{n+1}(x) = F_n(x)$ for $x = k2^{-n}$, $k=0,1,\dots,2^n$, and let $F_{n+1}[(\alpha_n+\beta_n)/2] = 1/2(1-t)F_n(\alpha_n) + 1/2(1+t)F_n(\beta_n)$, $k=0,1,\dots,2^n-1$. Then define F_{n+1} to be continuous and linear in the intervals $[a_{n+1},\beta_{n+1}]$. The Hellinger function F is defined by $F(x) = \lim_{n\to\infty}\{F_n(x)\}$, $x \in [0,1]$. For example, choose $t = 1/3$. (See Figure 2). Then

$$F_0(x) = x,\ x \in [0,1],$$

$$F_1[(0+1)/2] = F_1(1/2) = 1/2\,[1-(1/3)]\,F_0(0) + 1/2\,[1+(1/3)]F_0(1)$$
$$= (1/3)(0) + (2/3)(1) = 2/3,$$

$$F_2\ (1/4) = (1/3\ F_1(0) + (2/3)\ F_1(1/2) = (2/3)(2/3) = 4/9,$$

$$F_2\ (3/4) = (1/3)\,F_1(1/2) + (2/3)\,F_1(1) = (1/3)(2/3) + (2/3)(1)$$
$$= 8/9,$$

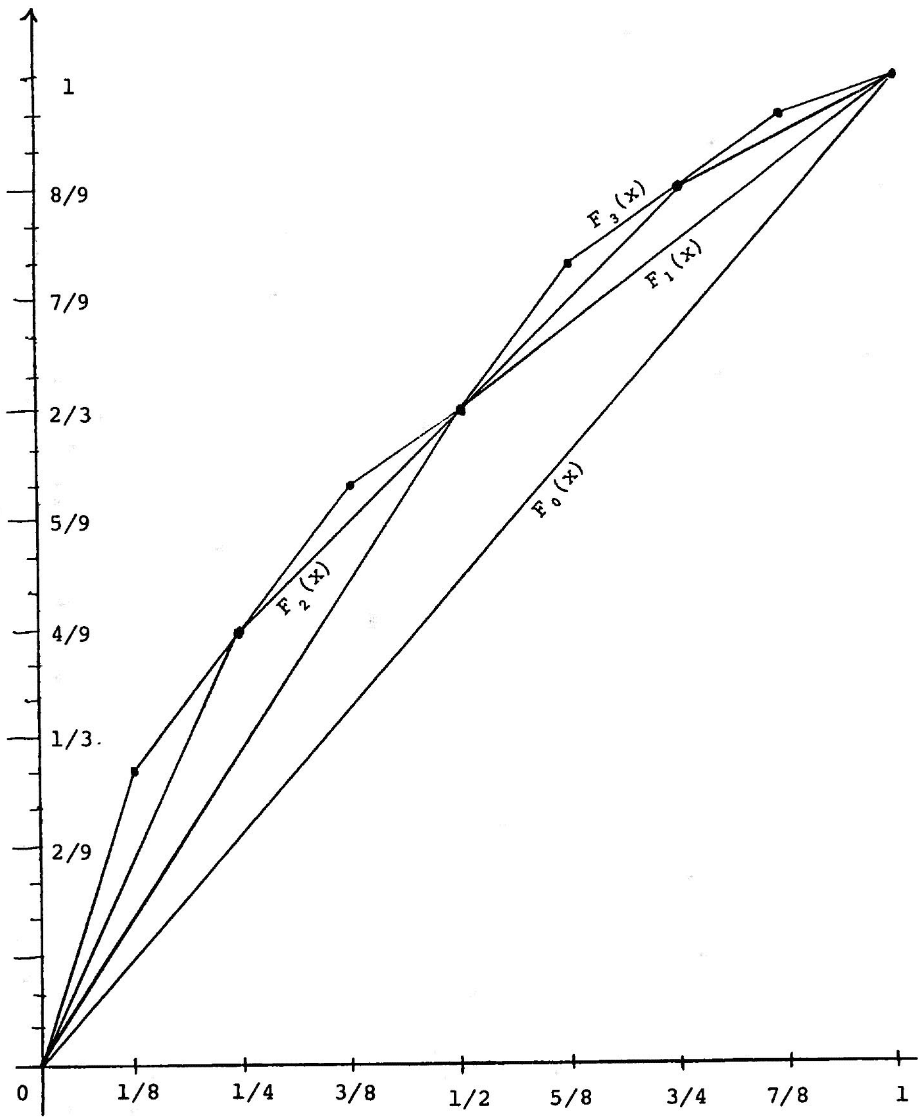
and so forth.

Before considering the singularity of F, we first need to prove several lemmas.

Lemma 1. Let $x \in [0,1]$ and let $\{[\alpha_n,\beta_n]\}_{n=0}^{\infty}$ be a sequence of nested intervals where $\alpha_n = k_n 2^{-n}$ and $\beta_n = (k_n+1)2^{-n}$ with the integer k_n chosen in $[0, 2^n-1]$ so that $x \in [\alpha_n,\beta_n]$. Then $F_{n+1}(\beta_{n+1}) - F_{n+1}(\alpha_{n+1}) = 1/2(1 \pm t)\,[F_n(\beta_n) - F_n(\alpha_n)]$.

Proof. If $\beta_{n+1} = 1/2(\alpha_n+\beta n)$, then $\alpha_{n+1} = \alpha_n$. Thus $F_{n+1}(\beta_{n+1}) = 1/2(1-t)F_n(\alpha_n) + 1/2(1+t)F_n(\beta_n)$ and $F_{n+1}(\alpha_{n+1}) = F_n(\alpha_n)$. So $F_{n+1}(\beta_{n+1}) - F_{n+1}(\alpha_{n+1}) = 1/2(1-t)F_n(\alpha_n) + 1/2(1+t)F_n(\beta_n) - F_n(\alpha_n) = 1/2(1+t)\,[F_n(\beta_n) - F_n(\alpha_n)]$. If $\beta_{n+1} = \beta_n$, then $\alpha_{n+1} = 1/2(\alpha_n+\beta_n)$ and it follows

Figure 2



by a similar argument that $F_{n+1}(\beta_{n+1})-F_{n+1}(\alpha_{n+1}) = 1/2(1-t)[F_n(\beta_n)-F_n$ (a)], and the lemma is proved

Since $F_n(\alpha_n) = F(\alpha_n)$ and $F_n(\beta_n) = F(\beta_n)$ for every n, we have

<u>*Corollary 1*</u>. $F(\beta_{n+1})-F(\alpha_{n+1}) = 1/2(1\pm t)\ F(\beta_n)-F(\alpha_n)$, $n+1\varepsilon I\!N$.

By inductively applying Corollary 1, we have

<u>*Corollary 2*</u>. If $\varepsilon_j = \pm 1$, $j\varepsilon$ IN, and n is a non-negative integer, then $F(\beta n)-F(\alpha_n)= \prod_{j=1}^{n} 1/2(1+\varepsilon_j\ t)$. (In case $n=0$, let $\prod_{j=1}^{0} 1/2(1+\varepsilon_j t) =1$.)

Furthermore, since in Corollary 2, $0<1/2(1+\varepsilon_j t)<1$ for every $j\varepsilon I\!N$, we have

<u>*Corollary 3*</u>. $0<F(\beta_n)-F(\alpha_n) = F_n(\beta_n)-F_n(\alpha_n)\leq 1$, $n=0,1,2,\ldots.$

Let $0<\lambda= 1/2(1+t)<1$.

<u>*Lemma 2*</u>. $0\leq F_{n+1}(x)-F_n(x)\leq \lambda^{n+1}$ for every $x\,\varepsilon I\!R$, $n+1\varepsilon I\!N$.

Proof. Let $x\varepsilon[0,1]$ and choose $\{[\alpha_n,\beta_n]\}_{n=0}^{\infty}$ as in Lemma 1. Let n be a non-negative integer. Choose $y\varepsilon[0,1]$ such that $x= \gamma\alpha_{n+1}+(1-\gamma)\beta_{n+1}$. Since F_{n+1} is linear between α_{n+1} and β_{n+1}, $F_{n+1}(x) = F_{n+1}(\gamma\,\alpha_{n+1}+(1-\gamma)\beta_{n+1}) = \gamma F_{n+1}(\alpha_{n+1})+(1-\gamma)F_{n+1}(6_{n+1})$. Likewise, $Fn(x) =F_n(\gamma\alpha_{n+1}+(1-\gamma\ \beta_{n+1}) =\gamma F_n(\alpha_{n+1})+(1-\gamma)F_n(\beta_{n+1})$. By the way the intervals are nested, either $a_n = \alpha_{n+1}$ or $\beta_n =\beta_{n+1}$. Suppose $\alpha_n = \alpha_{n+1}$. Then $\beta_{n+1} = 1/2(\alpha_n+\beta_n)$ and $F_{n+1}(\alpha_{n+1}) = F_{n+1}(\alpha_n) = F_n(\alpha_n) =F_n(\alpha_{n+1})$ which implies that

$$
\begin{aligned}
F_{n+1}(x)-F_n(x) &= (1-\gamma)[F_{n+1}(\beta_{n+1})-F_n(\beta_{n+1})] \\
&= (1-\gamma)[1/2(1-t)F_n(\alpha_n)+1/2(1+t)F_n(\beta_n)-1/2\,F_n(\alpha_n)-1/2F_n(\beta_n)] \\
&= (t/2)(1-\gamma)[F_n(\beta_n)-F_n(\alpha_n)] \\
&<\lambda[F_n(\beta_n)-F_n(\alpha_n)\ ]= \lambda[F(\beta_n)-F(\alpha_n)] \\
&=\lambda\prod_{j=1}^{n} 1/2(1+\varepsilon_j t)\leq\lambda(\lambda^n) = \lambda^{n+1}.
\end{aligned}
$$

If $\beta_n= \beta_{n+1}$, then it follows similarly that $F_{n+1}(x)-F_n(x)\leq\lambda^{n+1}$. Moreover, by Corollary 3, $F_n(\beta_n)-F_n(\alpha_n) > 0$. Therefore, $F_{n+1}(x)-F_n(x) = (t/2)(1-\gamma)\,[F_n(\beta_n)-F_n(\alpha_n)]$ Â£0and the lemma is proved.

Now we consider the singularity of F. For $x\varepsilon[0,1]$ and $n\,\varepsilon I\!N$, $Fn(x) = x+\sum_{i=0}^{n-1}[F_{i+1}(x)-F_i(x)]$. Since $|F_{i+1}(x)-F_i(x)| = F_{i+1}(x)-F_i(x)\leq\lambda^{i+1}$ for every i and $\sum_{i=0}^{\infty} \lambda^{i+1} = \lambda/(1-\lambda)$, then by the **Weierstrass *M*-test** $\sum_{i=0}^{\infty} [F_{i+1}(x)-F_i(x)]$ converges uniformly on $[0,1]$, which implies that $\{F_n(x)\}_{n=0}^{\infty}$ converges uniformly to $F(x)$ on $[0,1]$. Since F_n is continuous for every n, then F is continuous.

To check the monotonicity of F, suppose $x<y$ and $x,y \in [0,1]$. Choose $\{[\alpha_{n,x}, \beta_{n,x}]\}_{n=0}^{\infty}$ and $\{[\alpha_{n,y}, \beta_{n,y}]\}_{n=0}^{\infty}$ for x and y, respectively, as in Lemma 1. Now choose n large enough that

$$\alpha_{n,x} \le x \le \beta_{n,x} < \alpha_{n,y} \le y \le \beta_{n,y}.$$

If $m \ge n$, then from Corollary 3, it follows that

$$F_m(x) \le F(\beta_{n,x}) = F(\beta_{n,x}) < F(\alpha_{n,y}) = F_m(\alpha_{n,y}) \le F_m(y).$$

Therefore, $F(x) \le F(\beta_{n,x}) < F(\alpha_{n,y}) \le F(y)$. Hence, F is strictly increasing on $[0,1]$.

Finally, we examine the differentiability of F. Let $x \in [0,1]$ with $\{[\alpha_n, \beta_n]\}_{n=0}^{\infty}$ chosen as in Lemma 1; $a_n \to x$ and $\beta_n \to x$ as $n \to \infty$. Suppose $F'(x)$ exists. Then using [5, Exercise #19a, pp. 116-117], $F'(x)$ can be calculated by

$$\lim_{n\to\infty} \frac{F(\beta_n)-F(\alpha_n)}{\beta_n-\alpha_n} = \lim_{n\to\infty} \{(\beta_n - \alpha_n)^{-1} \prod_{j=1}^{n} [(1+\varepsilon_j t)/2]\}$$

$$= \lim_{n\to\infty} \{[(k_n+1)2^{-n} - k_n 2^{-n}]^{-1} [2^{-n}][\prod_{j=1}^{n}(1+\varepsilon_j t)]\}$$

$$= \lim_{n\to\infty} \prod_{j=1}^{n}(1+\varepsilon_j t),$$ which is zero, infinite, or

indeterminate. According to Lebesgue's Theorem, every monotonic function possesses a finite derivative almost everywhere [3, p. 5]. Thus $F'(x) = 0$ everywhere F has a finite derivative, that is, almost everywhere, and therefore F is singular.

Related Results. The reader may be interested to know that $\int_c^x f'(t)\,dt = f(x)-f(c)$ for every $x \in [a,b]$ if, and only if, f is absolutely continuous on $[a,b]$. For a discussion of absolute continuity and this theorem see [4, pp. 104-107].

It can also be shown that every monotonic continuous function can be expressed as the sum of two monotonic continuous functions $g(x)$ and $h(x)$, of which $g(x)$ is singular and $h(x)$ is absolutely continuous [4, Exercise #12a, p. 107].

Lastly, recall that in calculus a student learns that if $f'(x) = g'(x)$ for all x in an interval J (where $f'(x)$ and $g'(x)$ are finite), then $f(x)-g(x) = c$ for some constant c. However, if we permit $f'(x) = g'(x) = \infty$ for some $x \in J$, then $f(x)$ and $g(x)$ do not necessarily differ by a constant. For the construction of such a counterexample see [1, pp. 137-139].

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THE CUBIC EQUATION REVISITED

by *F. Max Stein*
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1. *Introduction.*

When we study the cubic equation we are generally interested in finding its roots. However, there are other ways of looking at the equation; the following discussion shows how straight lines in one plane map into strophoids in another by use of the cubic equation, and the roots of the cubic equation enter into the discussion only casually. We first review some elementary aspects of the cubic polynomial equation to prepare for the discussion of the mapping we wish to consider.

It is assumed as known that the cubic equation in general form is given as

$$At^3 + Bt^2 + Ct + D = 0;$$

it has three roots, at least one of which is real. By the substitution

$$t = x - \frac{B}{3A}$$

the equation can be transformed into the form

$$x^3 + ax + b = 0, \tag{1}$$

the <u>reduced cubic</u> <u>equation</u>. If the roots of (1) can be found, the roots of the original equation can be determined. One method for solving (1), known as <u>Cardan's</u> <u>method</u>, is given in [1] as well as many other places. There are other analytical, as well as numerical, methods for solving (1) which the interested reader can find at various places in the literature.

2. *Geometric Aspects.*

Our interest in the reduced cubic equation (1) lies in another direction; we shall examine some geometric aspects of this equation and its roots. Throughout our discussion we shall assume that a and b in (1) are real; furthermore the coefficient of x^3 in (1) can be considered to be 1 without loss of generality.

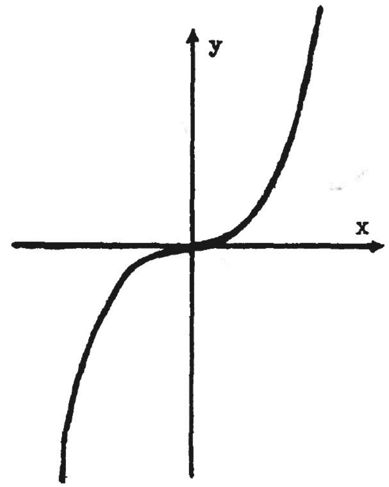


Fig. 1a

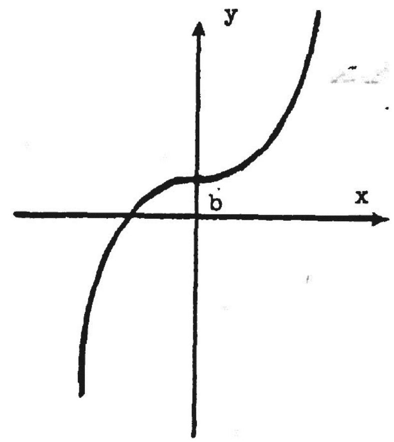


Fig. 1b

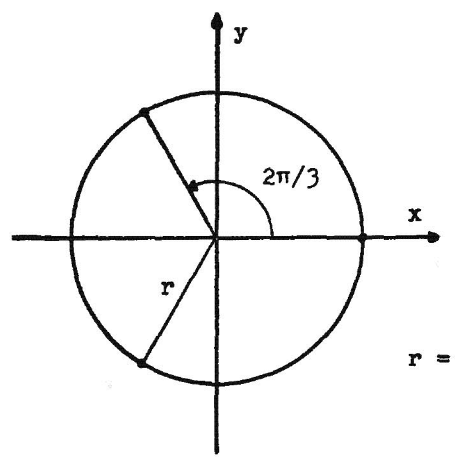


Fig. 2a

$r = \sqrt[3]{|b|}$

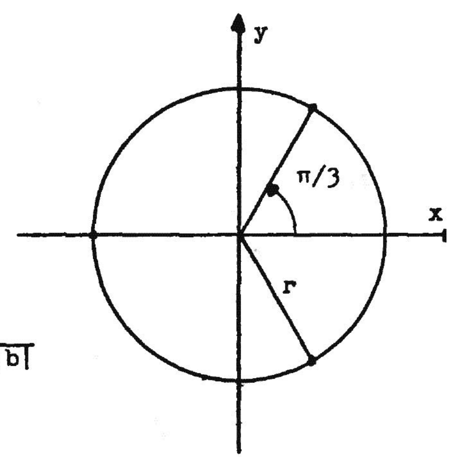


Fig. 2b

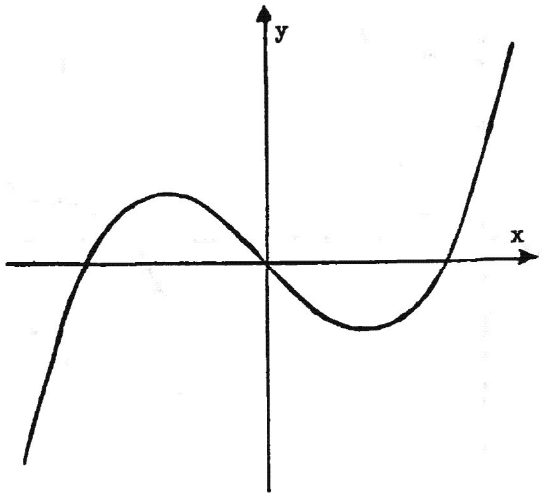


Fig. 3a

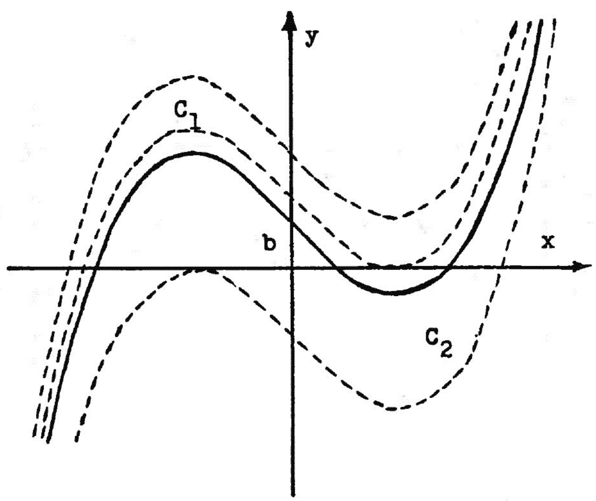


Fig. 3b

We first examine some special cases of (1); to do so, consider the cubic parabola

$$y = x^3, \tag{2}$$

which has the graph given in Fig. 1a. Equation (1) with $a = b = 0$ (Fig. 1a and Fig. 1b) thus has a triple root at $x = 0$. The equation

$$y = x^3 + b \tag{3}$$

merely moves the graph of the curve up (or down) by $|b|$ units, see Fig. 1b. Equation (1) with $a = 0$ and $b \neq 0$ has one real root plus two complex roots – the complex roots are complex conjugates. These three roots can all be obtained from

$$r_k = \sqrt[3]{|b|}\left(\cos\frac{\theta + 2k\pi}{3} + i\sin\frac{\theta + 2k\pi}{3}\right), \quad k = 0, 1, 2. \tag{4}$$

Here $\theta = \pi$ if $b > 0$, and $\theta = 0$ if $b < 0$. From (4) we find that there is a positive real root of $x^3 + b = 0$, $(\sqrt[3]{|b|})$, if $b < 0$ and a negative real root, $(-\sqrt[3]{b})$, if $b > 0$; the other two roots are complex conjugates and can be plotted in the complex plane as in Fig. 2a and Fig. 2b, respectively, see [2].

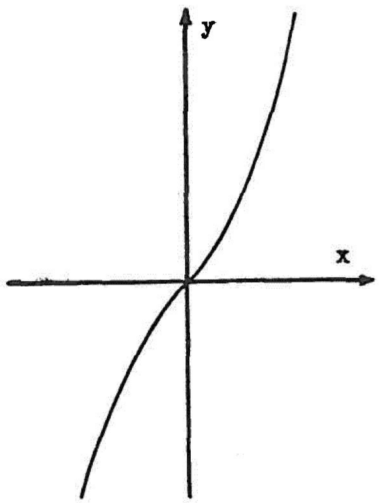
Next we consider cases in which $b = 0$ in (1) – first with $a < 0$. From

$$y = x^3 + ax, \quad y' = 3x^2 + a, \quad \text{and } y'' = 6x \tag{5}$$

we see that the graph of the curve passes through the origin with slope a (recall $a < 0$) and with a point of inflection at the origin, see Fig. 3a. Furthermore there are maximum and minimum points at (Fig. 3a and Fig. 3b) $x = -\sqrt{-a/3}$ and $x = \sqrt{-a/3}$, respectively. Finally note that there are three real roots for $x^3 + ax = 0$ for $a < 0$ – at $x = 0$ and $x = \pm\sqrt{-a}$. Also note that the graph of (5) is symmetric with respect to the origin. Adding a constant term b to the right side of (5) gives

$$y = x^3 + ax + b, \tag{6}$$

which merely moves the graph up (or down) by $|b|$ units, see Fig. 3b. Note, however, that if $|b|$ is large enough there may be one real root and a repeated real root or one real root and two complex conjugate roots of (1), see the dotted graphs in Fig. 3b. Note also that the



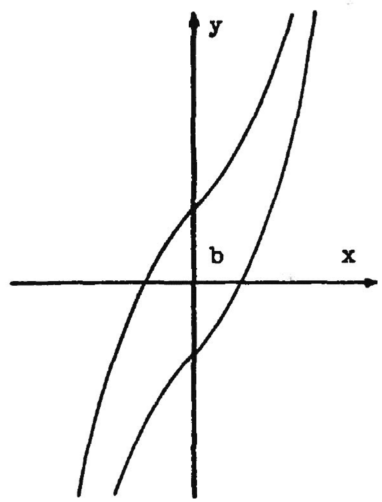


Fig. 4a

Fig. 4b

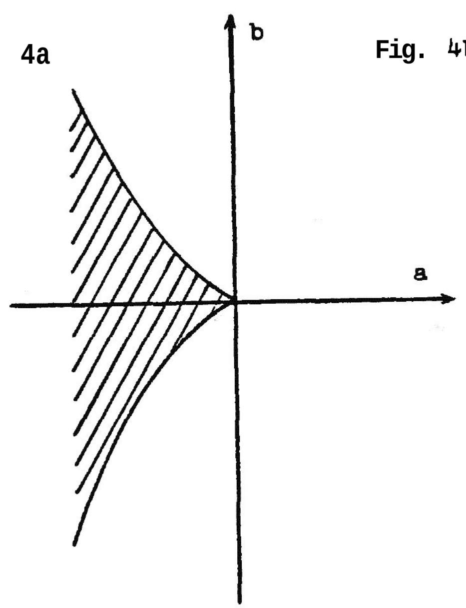


Fig. 5

graph of (6) is symmetric with respect to the point $(0,b)$; i.e., replacing the point $(x,y-b)$ by $(-x,-y+b)$ in (6) yields (6) again.

We will have only two distinct roots of (1) when the maximum (or minimum) point of (6) is on the z-axis; this occurs at $(-\sqrt{-a/3},0)$ or at $(\sqrt{-a/3},0)$, see (5) and C_1 or C_2 in Fig. 3b, and recall that $a < 0$.

On the other hand if $a > 0$ the graph of (5) passes through the origin with slope a and with a point of inflection at the origin as before, only now the slope of the graph is always positive, see Fig. 4a. Hence, there are no maxima or minima for the graph (Fig. 4a and Fig. 4b) Thus $x^3 + ax = 0$ has only one real root - at the origin. Adding a constant term b, as in (6), merely moves the graph up (or down) $|b|$ units, and (1) has only one real root; the other two roots are complex conjugates as before.

3. *Real vs. Complex Roots.*

What is the dividing line between cases for which (1) has all real roots and cases in which there are complex roots? One way to answer this is to find those points where a maximum or minimum point of (6) lies on the x-axis. But these points have already been determined as $(\pm\sqrt{-a/3},0)$. (Recall that $a < 0$.) Substituting these points in (1) we get

$$\pm(-a/3)^{3/2} \pm a(-a/3)^{1/2} + b = 0,$$

which reduces to

$$b^2 = -\frac{4a^3}{27}, \tag{7}$$

a semi-cubical parabola in the a, b-plane, see Fig. 5. It is not difficult to see that, for points to the left of the semi-cubical parabola (the cross-hatched region), (1) will have three real roots (note that $a < 0$ in this region); points on the curve lead to three real roots with two (or three for the origin) being the same; and points to the right of the curve yield a pair of complex conjugate roots. Of course, $a < 0$ may yield some complex roots, depending on the relative sizes of a and b.

4. *The Vector Approach.*

We now write (1) as

$$z^3 + az + b = 0, \quad a \text{ and } b \text{ real}, \tag{8}$$

since we are interested in the complex roots of (1). Here

$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{ie}.$$

We can consider the terms in (8) as vectors, and we write, after rearranging terms,

(9) $\quad b + az + z^3 = be^{i0} + are^{i\theta} + r^3e^{i3\theta} = 0.$

We assume that $b > 0$ for simplicity; cases in which $b < 0$ can be treated in a similar manner as we shall see later,

Since the terms in (9) add to zero, the vectors in the complex plane, or x,y-plane, must form a closed polygon, a triangle, see Fig. 6. Furthermore, since a is real (a > 0 as shown), $r > 0$, and 6 is the argument of the vector az, then the argument of z^3 must be 39. From elementary geometry we can determine that $\alpha = \pi - 29$, and hence $\beta = 39 - \pi$ in Fig. 6.

5. *The Principal Result.*

The major concern of this paper is to examine the locus of the point P in Fig. 6. We want to warn the reader that P is not a complex root of (8). Rather it is the point b + az in the complex plane. The coordinates of $P(x,y)$ can be determined from the parametric equations

(10) $\quad x = b + ar\cos\theta, \qquad y = ar$ sine;

note, however, that we must know r and θ (and hence a root of (8)) to determine x and y completely.

Since we shall be using $\tan\beta = \tan(3\theta - \pi)$, we use some trigonometric identities and write

(11) $$\tan(3\theta - \pi) = \tan 3\theta = \frac{3\tan 9 - \tan^3 9}{1 - 3\tan^2\theta}.$$

The locus of P in the x,y-plane as a varies with b fixed, say b = 1, is the same as the locus of the points of intersection of the lines

(12) $\quad y = (\tan\theta)(x - 1) \quad$ and $y = (\tan\beta)x = (\tan 3\theta)x.$

If we eliminate $\tan\theta$ between these two equations we get

$$\tan\theta = \frac{y}{x - 1},\ x \neq 1, \text{ and } y = \frac{3\tan 6 - \tan^3 9}{1 - 3\tan^2 6}a, \text{ or}$$

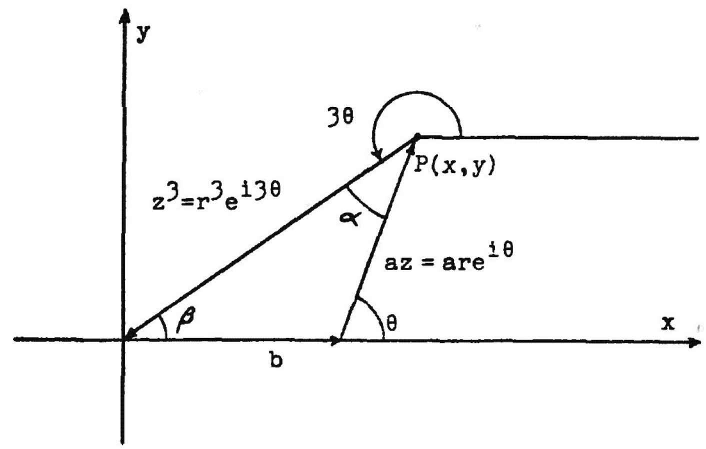


Fig. 6

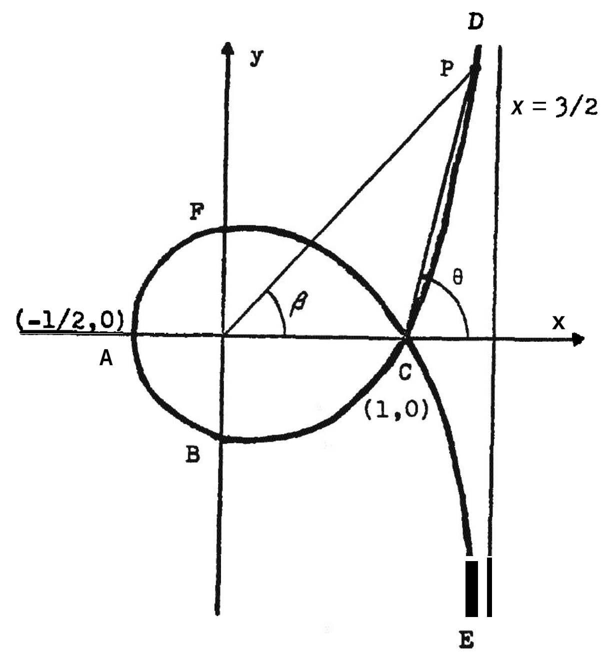


Fig. 7

$$y = \frac{\dfrac{3y}{x-1} - \dfrac{y^3}{(x-1)^3}}{1 - 3\dfrac{y^2}{(x-1)^2}}\, x = \frac{xy[3(x-1)^2 - y^2]}{(x-1)[(x-1)^2 - 3y^2]} .$$

Dividing out y (if $y \neq 0$) and simplifying we get

(13) $$y^2 = \frac{(x-1)^2(2x+1)}{3-2x} = (x-1)^2 \frac{\frac{1}{2}+x}{\frac{3}{2}-x}$$

This is the equation of a strophoid with x-intercepts (1,0) and $(-\frac{1}{2},0)$ and with $x = 3/2$ as an asymptote, see Fig. 7. The lines intersect at (1,0) when $6 = \pi/3$ to remove the restriction that $X \# 1$; also $y = 0$ for $6 = 0$ in both equations in (12). The fact that $X = -1/2$ when $y = 0$ is obtained from (13). [By eliminating y between the two equations in (12), one can obtain

$$x = \frac{\tan 8}{\tan 6 - \tan 3\theta} = \frac{3\tan^2\theta - 1}{2(\tan^2\theta + 1)}$$

Then

$$y = (\tan 3\theta)\, x = \frac{\tan\theta(\tan^2\theta - 3)}{2(\tan^2\theta + 1)}$$

Here we see that when $6 = \pi/3$, $x = 1$ and $y = 0$. Also when $6 = 0$, $x = -1/2$ and $y = 0$.]

6. *The Strophoid.*

We break up the strophoid in Fig. 7 into four parts:

(a) ABC, $0 < \theta < \pi/3$, $a < 0$,

(14) (b) CD $\pi/3 < \theta < \pi/2$, $a > 0$,

(c) EC, $\pi/2 < \theta < 2\pi/3$, $a > 0$,

(d) CFA $2\pi/3 < \theta < \pi$, $a < 0$.

That is, as a in (9) increases from the point (0,1) in Fig. 8, θ increases in Fig. 7 from $\pi/3$ to $\pi/2$, and P varies from C to D. Because of the complex roots appearing as conjugates, we also have for $a > 0$ the portion of the strophoid in Fig. 7 from E to C, only now θ increases

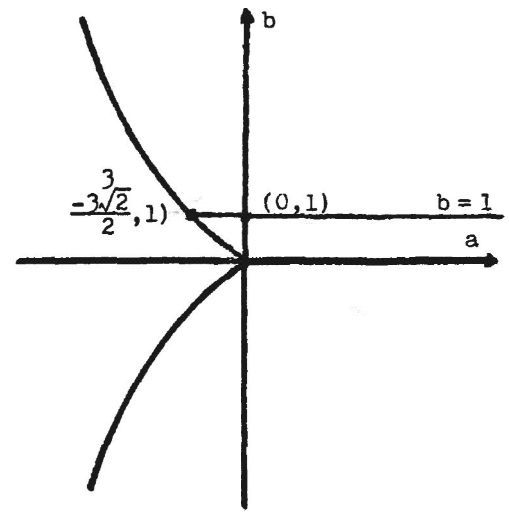


Fig. 8

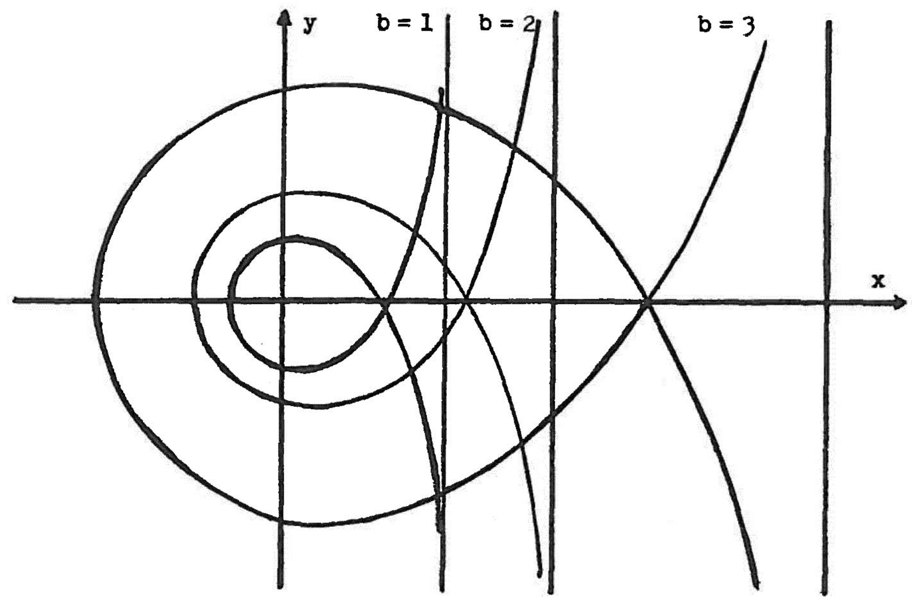


Fig. 9

from $\pi/2$ to $2\pi/3$. Similarly as a in (9) decreases from the point $(0,1)$ to $(-3\sqrt[3]{2}/2,1)$, in Fig. 8, θ decreases from $\pi/3$ to 0, and P traces out the portion of the strophoid CBA in Fig. 7. The complex conjugate root leads to the portion of the strophoid CFA in Fig. 7, only now θ varies from $2\pi/3$ to π. Note that when $a < -3\sqrt[3]{2}/2$ for $b = 1$, we have a point in the cross-hatched region of the a,b -plane in Fig. 5, and thus there are no complex roots of (9).

In the preceding discussion we have assumed that $b = 1$. For $b > 0$ in general we have that the equations of the intersecting lines which determine P are

(15) $$y = (\tan\theta)(x-b) \text{ and } y = (\tan 3\theta)x,$$

similar to the equations in (12). Upon substituting for $\tan 3\theta$ from (11) into the second equation in (15) and then for $\tan\theta$ from the first, we get

$$y = \frac{\dfrac{3y}{x-b} - \dfrac{y^3}{(x-b)^3}}{1 - \dfrac{3y^2}{(x-b)^2}}\, x$$

Upon simplifying and solving for y^2 as before, we get

(16) $$y^2 = (x-b)^2\,\frac{\frac{b}{2}+x}{\frac{3b}{2}-x}\,.$$

From this equation we observe that as b increases from 0 we get a family of strophoids which have x-intercepts at $x = -b/2$ and at $x = b$. Furthermore the corresponding asymptotes are $x = 3b/2$, see Fig. 9. That is, we have a mapping of the lines $b = b_0$ in the a,b-plane into strophoids in the x,y-plane.

When $b = 0$ in (9) we have $z^3 + az = z(z^2 + a) = 0$ which has roots at $z = 0$ and $z = \pm\sqrt{-a}$. If $a > 0$ (which is necessary for complex roots) then the point P is at $(0, \pm\sqrt{|-a|})$, and the strophoid degenerates to the y-axis as a varies.

When $b < 0$ we merely obtain the reflection of Fig. 9 in the y-axis. This is readily seen by letting b be negative in (16).

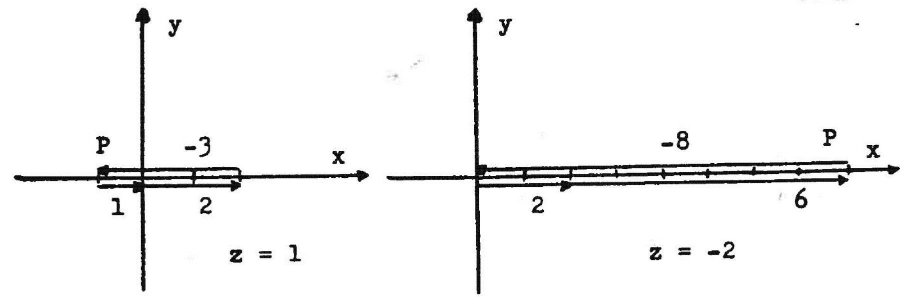


Fig. 10

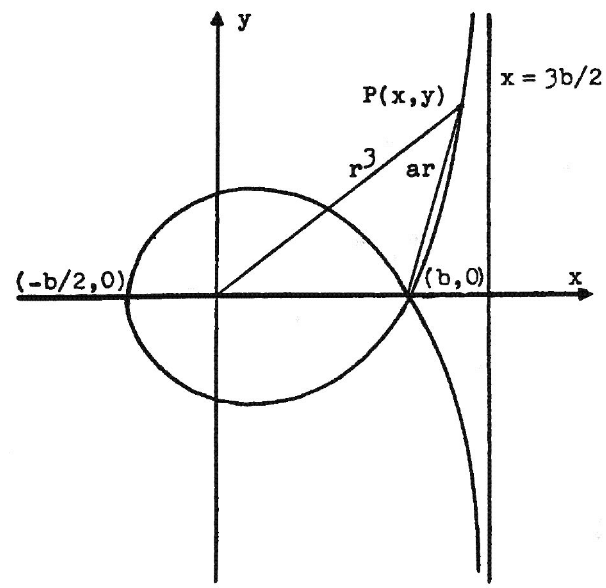


Fig. 11

7. *Real Roots.*

Thus far we have considered only those cases in which a root z of (9) is complex. The discussion, starting with (9) could apply equally well for real roots z; in this case the strophoids lie entirely on the x-axis. That is, $b + az + z^3 = 0$ can be considered as the sum of three vectors starting and ending at the origin and all lying on the x-axis. For example in $2 - 3z + z^3 = 0$ we see that the roots are 1, 1, and -2 (and hence (7) is satisfied). The vector graphs for 1 and -2 appear as in Fig. 10; the points P always lie on the x-axis. Recall that the points (a,b) must lie in the cross-hatched region or on the semi-cubical parabola in Fig. 5 for real roots, and a is always negative (or zero) there.

8. *The Case a is Constant.*

For any point P on the strophoid (16) with coordinates (x,y) we have that the distance from the origin to P in the x,y-plane is

$$r^3 = (x^2 + y^2)^{1/2}, \tag{17}$$

the cube of the magnitude of a complex root of (9), see Fig. 11. Also, for $a > 0$ the distance from the point $(b,0)$ to P in the x,y-plane is ar. Hence, we have

$$r = (x^2 + y^2)^{1/6} \quad \text{and} \quad ar = a(x^2 + y^2)^{1/6}$$

which leads to

$$b = x \pm [a^2(x^2 + y^2)^{1/3} - y^2]^{1/2} : \tag{18}$$

Now if we replace b in (16) by (18) we get

$$y^2 = [a^2(x^2 + y^2)^{1/3} - y^2]\frac{3x - [a^2(x^2 + y^2)^{1/3} - y^2]^{1/2}}{x - 3[a^2(x^2 + y^2)^{1/3} - y^2]^{1/2}}. \tag{19}$$

By assigning values to a in (19) we get the image curves in the x,y-plane of the straight lines $a = a_0$ in the a,b-plane with b entirely arbitrary in (19). Since a simple interpretation of (19) is not readily available, we do not proceed further in this direction.

9. *Conclusions.*

By looking at the cubic equation (1) in a different light, we encounter an interesting mapping. While the usual study of the cubic

equation involves finding its roots or in graphing it, here we encounter a family of strophoids. A study such as this gives rise to additional problems which include a thorough examination of the mapping of lines $a = a_0$ in (19), the mapping of regions in the a,b-plane into regions in the x,y-plane, and the mapping of lines, curves, and regions lying in the cross-hatched region of Fig. 5 similar to that done for the quadratic equation in [3].

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A DAY AT THE RACES

by William T. Tomcsanyi

Horse racing is known as the sport of kings. Why? Because kings are the only ones who can really afford to participate in the sport. Those who can't afford to participate are the ones you'll see jamming the racecourses each day.

And everybody thinks he can make a fortune because he either feels lucky or he has a "system".

"I have a system". That phrase has probably rolled off more handicappers' tongues than any other. What's more, every gambler intuitively knows that the only sure thing is that there is no sure thing. Or, is there?

The fact that the odds-on favorite doesn't always win the race (if it did, we'd all be millionaires) indicates that the art of betting on horses is as much concerned with people as with horses. Any race-goer who fails to appreciate this will lose. Any system-builder who fails to recognize this will wind up with a worthless system. And any sociologist who has never gone to the races is missing a spectacular display of human behavior.

The aim of any system is to make the odds more favorable to the bettor. Some systems are incredibly complex, involving details such as post position, weather, race distance, size of purse money and even the integrity of the jockey or driver.

What's needed is a straightforward, logical and simple system that neglects all the factors of a horse race except the odds.

The Re-Equine *System*

Given a field of eight horses in a race, can you pick two of them such that you are positive one will win the race? It's easier to pick two than just one. After all, by picking two horses you've just increased your chances of winning from 1/8 to 1/4.

Now, assuming the above, is there a way to bet on both of them such that no matter which horse wins you make a desired profit?

Let x_1 = the number of bets on horsey

x_2 = the number of bets on horse$_2$

a = the odds on horse$_1$ (a to 1 odds)

b = the odds on horse, (b to 1 odds)

P = the desired profit

where one bet = \$2 (minimum bet); the fiscal return is calculated by (2) (odds) + 2, and the profit is (2) (odds).

If horse$_1$ wins: $2ax_1 - 2x_2 = P$

If horse$_2$ wins: $-2x_1 + 2bx_2 = P$

These equations represent the two winning cases.

We now have two simultaneous equations with two unknowns. What they mean is if horse$_1$ wins, then the fiscal return on that horse minus the investment on horse$_2$ equals the chosen profit. If horse$_2$ wins, then the return on that horse minus the investment on horse$_1$ equals that same chosen profit.

A general form of these equations can be arrived at through operations of linear algebra by setting up a matrix equation of the form $Ax = b$.

$$2ax_1 - 2x_2 = P$$

$$-2x_1 + 2bx_2 = P$$

$$\begin{pmatrix} 2a & -2 \\ -2 & 2b \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} P \\ P \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{4(ab-1)}\begin{pmatrix} 2b & 2 \\ 2 & 2a \end{pmatrix}\begin{pmatrix} P \\ P \end{pmatrix}$$

$$= \frac{1}{4(ab-1)}\begin{pmatrix} (2b+2)P \\ (2a+2)P \end{pmatrix}$$

$$= \frac{2P}{4(ab-1)}\begin{pmatrix} b+1 \\ a+1 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{P}{2(ab-1)}\begin{pmatrix} b+1 \\ a+1 \end{pmatrix} \qquad \text{General Form}$$

Note: The product of a and b must not equal one or the system will have no solution. In other words, if both horses are even money shots, do not bet the race. The same is true if their respective odds are multiplicative inverses of each other.

Example

Let's say you have chosen to make $50 profit and have picked two horses in a race that are going off at 5 to 1 and 3 to 1, respectively.

$P = 50$

$a = 5$

$b = 3$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{P}{2(ab-1)} \begin{pmatrix} b+1 \\ a+1 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{50}{2((5)(3)-1)} \begin{pmatrix} 3+1 \\ 5+1 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{50}{28} \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} (50/28)(4) \\ (50/28)(6) \end{pmatrix}$$

$x_1 = 7.14$

$x_2 = 10.7$

$x_1 \approx 7$

$x_2 \approx 10.5$

x_1 and x_2 must be rounded off to make whole dollar bets. This truncation will affect P slightly.

Amount of money on horse $_1$ = $14

Amount of money on horse2 = $21

Total Investment = $35

If horse $_1$ wins: $2(5)(7)-2(10.5)$ = $49 profit

If **horse** $_2$ wins: $-2(7)+2(3)(10.5)$ = $49 profit

Racing Matrix Theory

If it works for two horses, then why won't **it** work for three? In fact, will this system work if every horse in a race is taken into consideration? This system could be the greatest breakthrough in the history of gambling if **it** allows you to bet each horse in a race such that no matter which horse wins a desired profit is attained.

Could this finally be the first "sure thing?"

The. Octal-Equine Matrix

There are usually eight or nine horses in a standardbred race and that number varies a little more in the thoroughbred version. However, let's take eight horses to constitute a typical field. The eight simultaneous equations would look like

$$\begin{aligned}
2a_1x_1-2x_2-2x_3-2x_4-2x_5-2x_6-2x_7-2x_8 &= P \\
-2x_1+2a_2x_2-2x_3-2x_4-2x_5-2x_6-2x_7-2x_8 &= P \\
-2x_1-2x_2+2a_3x_3-2x_4-2x_5-2x_6-2x_7-2x_8 &= P \\
-2x_1-2x_2-2x_3+2a_4x_4-2x_5-2x_6-2x_7-2x_8 &= P \\
-2x_1-2x_2-2x_3-2x_4+2a_5x_5-2x_6-2x_7-2x_8 &= P \\
-2x_1-2x_2-2x_3-2x_4-2x_5+2a_6x_6-2x_7-2x_8 &= P \\
-2x_1-2x_2-2x_3-2x_4-2x_5-2x_6+2a_7x_7-2x_8 &= P \\
-2x_1-2x_2-2x_3-2x_4-2x_5-2x_6-2x_7+2a_8x_8 &= P
\end{aligned}$$

where a_i = odds on horse; (a_i to 1 odds)

x_i = number of bets on horse;

P = desired profit

In the form of $A\vec{x} = \vec{b}$, the equations become the following system

$$\begin{bmatrix}
a_1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & a_2 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & a_3 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & a_4 & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 & a_5 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 & -1 & a_6 & -1 & -1 \\
-1 & -1 & -1 & -1 & -1 & -1 & a_7 & -1 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & a_8
\end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix}
= \frac{P}{2}
\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Solving this system of equations by hand would be an absurd task that could easily take days. A standard hand calculator will take about ten minutes to invert an eight by eight matrix and spit out the solutions to such a system.

Since horses rarely go off at integer odds, the next example utilizes realistic odds taken from a typical race.

Example

Input	Output
a : 37-1	x_1 = -$11.15701428
a : 17.2-1	x_2 : -$23.29486498
a : 15.9-1	x_3 : -$25.08677767
a = 11.2-1	x : -$34.75135596
a = 8.9-1	X = -$42.82490330
a : 4.4-1	x_6 : -$78.51232272
a_7 = 3.4-1	x = -$96.35603243
a_8 = 1-1	x_8 : -$211.9832713

where the x_i are converted directly into dollars by the programmable calculator, and P is set at $100.

Problem. The machine came up with negative solutions. Unfortunately, there is no way to bet negative dollars on a horse, and this means that this particular combination of horses cannot be bet simultaneously to yield a desired profit.

Maybe it was just a bad combination of odds? Yes, that's true. However, in several actual racetrack situations no odds were encountered that would yield a positive solution vector. There probably does exist some combination of odds that would somehow yield positive results to the eight equations (Can you find one?).

But let's not give up yet. Just because the system's not guaranteed to work for eight horses doesn't mean it can't make the bettor's odds more favorable with fewer horses. First, however, it's necessary to examine the basic problems of this system.

Six *Fundamental Problems.*

1) Latest **Odds.** No matter how many horses are considered in a given field, the accuracy of this system relies on the latest available odds. A horse may be listed in a program at 5 to 1 odds but may actually go off at 3 to 2. At the track, for instance, it's best to bet as late as possible so that the betting calculations can be made with the most recent odds as read off the tote board.

2) ***Stability of Odds.*** This is really a function of getting the most recent odds on a horse. The stability this system is concerned with is that of the odds at which you bet versus the odds the horses really go off at. What is also important to realize is that if you bet a large amount on any combination of horses, the odds on those horses will change.

3) *The Malady of Truncation.* Racetracks only accept integer dollar amounts. In other words, you cannot bet $2.58 on a horse even though one of the solutions given by the equations might be x : 1.29. You are forced to round off the answer to the nearest integer or to the nearest half. For instance, 1.29 is closer to 1.5 than it is to 1.0. Therefore, the bet would call for $3 as opposed to $2. Also, if you are using a programmable calculator or computer, the machine will truncate numbers as it calculates the inverse of a matrix.

4) *Efficiency.* The efficiency of this system can be defined as what is put in versus what one gets back. As the number of horses taken into account increases, so does the investment that must be made in order to achieve the same profit. For example, to make $50 profit on two horses may only require an investment of $24. But making that same profit on eight horses may require a multi-hundred dollar investment. This kind of investment, as noted before, will also affect the stability of the odds.

5) *Degree of Complexity.* Calculating bets for two horses can be done very quickly. Unfortunately, calculation time increases quickly as more horses are taken into account. The eight-horse equations, for instance, can easily take over ten minutes for a calculator to solve and there are only a maximum of sixteen minutes between races. This means that you can't feed accurate odds into the calculator and it will yield radically different results than with the latest odds before post time.

Although the eight-horse system doesn't seem to work, the same time constraints still apply for more than three horses. Additionally, when considering more than three horses, the stiffness of the equations becomes a factor and so do conditions on odds that insure positive solutions. Stiffness can be defined as an adverse reaction of the equations if one of the odds changes slightly. In other words, if the solutions yielded by a set of odds change dramatically when just one of the odds changes slightly, you may end up on welfare.

6) *Mob Interest.* This isn't necessarily a drawback of this system but rather a drawback of successful gambling. Mob interest is two-

fold: there will be a mob of people at the track interested in how you are winning all the time, and then there will be the other mob interested in putting you out of business before you put one of their bookies out of business. Stay away from bookies with a system that works.

The 3 by 3 -- *A* System That *Works*

Can you handicap any race and come up with the three best horses in that race? Part of the beauty of this system is that you don't need to know anything at all about horse racing to win. All you have to be able to do is read the three best (lowest) odds off the tote board at the track and substitute them into a general formula.

Automatically, your worst chances of winning become 3/8. Also, it is safe to estimate that at the big-name racetracks like Roosevelt and Aqueduct, one of the three favorites in each race wins about seventy percent of the time.

What's more, the 3 by 3 system doesn't encounter all those problems like stiffness, ineffiency, and complexity. The substitution of numbers into the general formula leads to quick and easy solutions and makes the method very practical.

Derivation

The three simultaneous equations for this method are

$$\begin{aligned} 2ax_1 - 2x_2 - 2x_3 &= P \\ -2x_1 + 2bx_2 - 2x_3 &= P \\ -2x_1 - 2x_2 + 2cx_3 &= P \end{aligned}$$

where a, b and c are the odds on horses one, "two and three and X_i are the number of bets on each horse, and P is the desired profit.

In the form of Ax = b, we have

$$\begin{bmatrix} a & -1 & -1 \\ -1 & b & -1 \\ -1 & -1 & c \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{P}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

A general formula is arrived at by inverting the matrix A and multiplying both sides of the equation by its inverse. By method of the Classical Adjoint, the inverse of A is easily calculated:

$$A = \begin{bmatrix} a & -1 & -1 \\ -1 & b & -1 \\ -1 & -1 & c \end{bmatrix}, \quad \det(A) = abc -(a +\&+a)-2$$

Classical Adjoint $C_{ij} = (-1)^{i+j}\det(B_{ij})$

$$A^{-1} = (1/\det(A))C^T$$

$$C = \begin{bmatrix} bc-1 & c+1 & b+1 \\ c+1 & ac-1 & a+1 \\ b+1 & a+1 & ab-1 \end{bmatrix}$$

The matrix C is symmetric and so it's equal to its transpose, and the inverse of A is as follows

$$A^{-1} = \frac{1}{abc-(a+b+c)-2}[C]$$

After multiplying both sides of the equation in the form $Ax = b$ by A inverse, the general formula of the 3 by 3 system is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{P}{2\det(A)}\begin{bmatrix} (bc+b+c)+1 \\ (ac+a+c)+1 \\ (ab+a+b)+1 \end{bmatrix}$$

where det(A) = abc-(a+ b+a)-2

The remainder of the process is choosing the three horses with the best odds, the amount of profit you want to make and simple multiplication and addition.

Example

Horse $_1$: 3-1, a = 3
Horse $_2$: 3-2, b = 1.5
Horse $_3$: 4-1, c = 4
Profit (P) = $50

det(A) from the formula = 7.5

in dollars

$$\begin{cases} x_1 = \frac{50}{2(7.5)}[(1.5)(4) + 1.5 + 4 + 1] \approx \$83 \\ x_2 = (3.33)\ [(3)(4) + 3 + 4 + 1] \approx \$133 \\ x3 = (3.33)\ [(3)(1.5) + 3 + 1.5 + 1] = \$67 \end{cases}$$

If horse$_1$ wins: $(83)(3) + 83 - 83 - 133 - 67 = \$49$
If horse2 wins: $(133)(1.5) + 133 - 83 - 133 - 67 = \$49.5$
If horse$_3$ wins: $(67)(4) + 67 - 83 - 133 - 67 = \$52$

Now, the total investment for this race would be $283 and the total return would be about $333.

The System's Advantages

The first advantage is that you don't need any knowledge of horse racing at all to work this system. Secondly, even if the odds fluctuate slightly from the time you bet, you'll probably still wind up making a decent profit. Thirdly, having three or more horses run for you in the same race is a comfortable feeling. And, finally, the winning percentage of the 3 by 3 is high, almost seventy percent.

There still may be no such thing as a sure thing and if there were, then it wouldn't be gambling, would it?

MATCHING PRIZE FUND

If your Chapter presents awards for Outstanding Mathematical Papers or Student Achievement in Mathematics, you may apply to the National Office to match the amount spent by your Chapter. For example, $30 of awards can result in your Chapter receiving $15 reimbursement from the National Office. These funds may also be used for the rental of Mathematics Films. Write to:

Dr. Richard Good
Secretary-Treasurer, Pi Mu Epsilon
Department of Mathematics
The University of Maryland
College Park, Maryland 20742

BE SURE AND SEND THE NAMES OF LOCAL AWARD WINNERS TO THE JOURNAL FOR PUBLICATION IN THE NEXT POSSIBLE ISSUE.

PARTIAL DIFFERENTIATION OF FUNCTIONS OF A SINGLE VARIABLE

by Richard Katz and Stewart Venit
California State University

In this note we will give a technique (or "rule") for differentiating certain combinations of functions. This rule includes as special cases the standard ones for differentiating sums, differences, products and quotients of functions. Moreover, it also provides an alternative method for determining the derivative of the function $f(x)^{g(x)}$.

The technique presented here was inspired by two types of errors frequently encountered when students are asked to differentiate a function of the form $f(x)^{g(x)}$. They either view this function as an exponential one and use the formula

$$\frac{d}{dx}(a^u) = a^u(\ln a)\,\frac{du}{dx},$$

or as a power function and use

$$\frac{d}{dx}(u^n) = n(u^{n-1})\,\frac{du}{dx}.$$

As a result, one often sees (incorrect) answers to this problem of the form

$$g(x)\left[f(x)^{g(x)-1}\right]f'(x) \quad \text{or} \quad f(x)^{g(x)}\left[\ln f(x)\right]g'(x).$$

It is at first surprising to notice that the correct answer (obtained by, say, logarithmic differentiation) is the <u>sum</u> of these two incorrect ones! However, as we shall see, this is not a coincidence but a special case of the following general principle.

<u>Partial Differentiation Rule:</u> Let $h(x)$ be a combination of the functions $f(x)$ and $g(x)$ which can be written as $h(x) = H(f(x),g(x))$. Then $h'(x)$ can be obtained by first differentiating $H(x)$ treating $f(x)$ as a constant, then differentiating $H(x)$ treating $g(x)$ as a constant, and finally adding the two results.

Proof. In the function $H(f(x), g(x))$, change the arguments of f and g to s and t, respectively, obtaining the function $F(s,t)$. Then $h(x) = F(x,x)$. Now the partial derivatives of $F(s,t)$ evaluated at the point (x,x), namely $F_s(x,x)$ and $F_t(x,x)$, are the derivatives that one obtains by differentiating $H(x)$ considering, respectively, $g(x)$ and $f(x)$ to be constant. Finally by the chain rule $h'(x) = F_s(x,x) + F_t(x,x)$ as desired.

It is easy to see that the rules for differentiating sums, differences, products and quotients of $f(x)$ and $g(x)$ are special cases of this partial differentiation rule. For example, to differentiate $h(x) = f(x)g(x)$ we successively hold $f(x)$ and $g(x)$ constant differentiating the other function and add the results to obtain

$$h'(x) = f(x)g'(x) + f'(x)g(x)$$

which is of course the "product rule." The most interesting special case of our rule is its application to the function $f(x)^{g(x)}$. This gives, as indicated at the beginning of the note, a simple alternative method (there are no complicated formulas or algorithms to memorize.) for differentiating this function.

We should remark in closing that this technique does not apply to the composition of functions. This is easily seen by choosing $f(x) = g(x) = x$. Then $f(g(x)) = f(x) = x$, so its derivative is 1. However, applying the partial differentiation rule we would get

$$0 + 0 = 0.$$

%%

CHAPTER REPORT

MISSOURI DELTA (WESTMINSTER COLLEGE) At the Annual Initiation of new members, *Carol Williams* spoke on "Experimental Physics Projects". The advisor is *Dr. Michael Z. Williams*.

%%

PUZZLE SECTION

Edited by
David Ballew

This Department is for the enjoyment of those readers who are addicted to working crossword puzzles or who find an occasional *mathematical puzzle* attractive. We consider mathematical puzzles to be problem whose solutions consist of answers immediately *recognizable* as correct by simple *observation* and requiring little formal proof. Material *submitted* and not *used here* will be sent to the *Problems* Editor if deemed appropriate for that Department.

Address all proposed pussies and puzzle solutions to David *Ballew*, Editor of the Pi Mu Epsilon Journal, Department of Mathematical Sciences, South Dakota School of Mines and Technology, Rapid City, South Dakota, 57701. Deadlines for puzzles appearing in the *Fall* issue will be the next February **15**, and puzzles *appearing* in the Spring issue will be due the next September **15**.

Mathacrostic No. 14

submitted by Joseph V. E. Konhouser
Macalester College, St. Paul, Minnesota

Like the preceding puzzles, this puzzle (on the following two pages) is a keyed anagram. The 239 letters to be entered in the diagram in the numbered spaces will be identical with those in the 28 keyed words at matching numbers, and the key numbers have been entered in the diagram to assist in constructing your solution. When completed, the initial letters will give a famous author and the title of his book; the diagram will be a quotation from that book. (See an example solution in the solutions section of this Department.)

1 T	2 L		3 E	4 I	5 P		6 K	7 W	8 J	9 a		10 S	11 O
	12 D	13 R	14 Z	15 L		16 T	17 a	18 X		19 U	20 Y	21 C	22 A
23 H	24 G		25 N	26 S	27 O	28 B	29 Q	30 F	31 X		32 G	33 b	34 Y
35 L		36 V	37 J	38 S	39 W	40 I		41 K	42 O		43 B	44 V	45 Z
46 D		47 U	48 Y		49 T	50 K		51 M	52 G	53 a	54 H	55 F	56 R
57 C	58 B		59 O	60 K	61 D	62 P		63 W	64 J		65 E	66 Y	
67 V	68 I	69 E		70 J	71 T		72 Z	73 O	74 R		75 X	76 E	77 J
78 T		79 D	80 Q		81 H	82 R	83 K	84 V		85 M	86 G	87 U	88 A
89 F	90 S	91 L	92 Y	93 a		94 X	95 B	96 b	97 Z	98 T		99 N	100 C
101 O	102 V	103 J		104 H	105 U	106 E	107 a	108 M	109 D	110 Y	111 O	112 G	113 L
114 Q	115 K	116 N		117 D	118 T		119 Z	120 Q	121 M		122 N	123 Y	124 V
125 C		126 D	127 X		128 V	129 J	130 L	131 A		132 D	133 M	134 a	135 Y
136 S	137 R	138 O	139 B		140 I	141 U	142 P	143 F	144 B	145 L	146 Q	147 b	148 R
	149 Z	150 X	151 L	152 b	153 T		154 G	155 b	156 K	157 Y	158 R		159 M
160 U	161 A	162 N	163 C	164 B	165 Q		166 T	167 H	168 a	169 F	170 V	171 G	
172 K	173 X	174 A	175 O		176 J	177 Z		178 L	179 F	180 C		181 G	182 M
183 X		184 N	185 E	186 H		187 E	188 a	189 V	190 O	191 H		192 Q	193 W
194 E	195 A	196 D	197 O	198 I	199 V	200 J	201 M	202 R	203 P		204 C	205 G	206 L
207 D	208 Q	209 S	210 B	211 U		212 M	213 H	214 V	215 A		216 U	217 O	218 X
	219 Y	220 a	221 K	222 H		223 Q	224 B	225 G	226 A	227 L		228 V	229 T
	230 Y	231 a	232 O		233 b	234 U	235 D	236 R	237 S	238 V	239 X		

Definitions	Words
A. a looper	174 161 88 195 131 215 226 22
B. pure; irreproachable (comp.)	58 210 28 139 43 144 164 95 224
C. at right angles to the centerline	204 125 100 180 57 21 163
D. A mathematician named _____ Thought the Möbius band was divine. Said he, 'If you glue The edges of two You'll get a weird _____ like mine.' (2 wds.)	12 109 196 117 46 235 132 79 126 207 61
E. holomorphic; regular	185 106 76 69 3 194 65 187
F. full of difficulties	89 30 179 55 169 143
G. overcloud	181 52 205 86 32 112 24 225 154 171
H. a cell nucleus formed by the fusion of two pre-existing nuclei	104 222 186 81 23 167 191 54 213
I. exponent of the craft of discovery in mathematics, b. 1887	140 4 68 40 198
J. German Shepherd film star (1918-1932) of the 1920's and early 1930's (comp.)	103 70 8 37 200 129 64 176 77
K. in another state or condition	83 172 60 115 221 6 41 50 156
L. off the beaten track (comp.)	130 151 227 91 2 113 178 145 15 206 35
M. cable railway in which an ascending car counterbalances a descending car	85 121 182 212 201 108 159 51 133
N. a collection of samples	116 122 162 99 184 25
O. the proposition "The only decision procedure that satisfies certain elementary principles of social welfare is a dictatorship." (2 wds.)	27 111 138 73 59 175 42 217 232 11 190 101 197
P. sometime modifier of element, set, space, sequence	62 5 203 142
Q. one of the paradoxes of Zeno	146 114 165 223 120 29 80 192 208
R. of this Indian mathematician (1887–1920) G. H. Hardy said 'The limitations of his knowledge were as startling as its profundity.'	158 202 56 148 13 74 236 137 82
S. in an integrated functional unit (2 wds.)	237 136 209 38 90 10 26
T. half pro and half con (comp.)	71 1 118 78 98 166 49 229 16 153
U. an undergarment with top and bottom in one piece (2 wds.)	234 19 105 141 87 211 160 47 216
V. a precursor of the die (2 wds.)	199 44 189 228 84 67 124 170 36 214 102
W. an 1871 Verdi opera by which Egypt should not have been judged	193 63 39 7 238 128
X. " of exact science are logic and mathematics." De Morgan (3 wds.)	150 173 218 239 75 127 183 31 18 94
Y. a generally mentally defective person with an unusual aptitude in some special field (2 wds.)	157 34 135 20 230 66 110 219 123 92 48
Z. a long wandering	14 97 72 149 177 45 119
a. the deadly one is a weed with white or bluish flowers and black berries	53 168 107 231 9 93 17 188 134 220
b. a winding-sheet	233 155 147 33 96 152

SOLUTIONS

Mathacrostic No. 13. (See Fall **1981** issue) (*Proposed by J.D.E. Konhauser*)

Definitions and Key:

A. Enarthrosis
b. Rehash
C. Non-standard
D. Sweet pea
E. Thoughts
F. Tschirnhausen
G. Heuristic
H. Epsilontics
I. Majority
J. Aha insight
K. Glow
L. Intuitions
M. cusp
N. Middle third
O. .Internist
P. Replication
Q. Rep-tile
R. Open-and-shut
S. Rubik's cube
T. Opposite
U. Fiddle-faddle
V. Monge
W. Chop
X. Elephant
Y. Stevin
Z. Concepts
a. Hein
b. Ends with ideas
c. Revehent

~~First Letters~~: ERNST THE MAGIC MIRROR OF M C ESCHER

Quotation: *Drawing is decep-fcton. On the one hand Escher haWed to reveal this decep-fcton in various prints, and on the other hand he has perfected it and turned it into superillusion, conjuring up with it impossible things, and this with such suppleness, logic, and clarity that the. impossible makes perfect sense.*

Solved by: Jeanette Bickley, Webster Groves High School, Missouri; Louis H. Cairoli, Kansas State University; Victor Feser, Mary College; Robert Forsberg, Lexington, Mass.; Robert Gebhardt, Hopatcong, NJ.; Henry S. Lieberman, John Hancock Mutual; Robert Prielipp, Univ. of Wisc-Oshkosh; Chris Thomas, Ann Arbor, Mich; The Proposer and The Editor.

Cross Word Puzzle. (See Fall **1981** Issue) (*Proposed by Alex Mehaffey Jh. and Curt Olson.*)

Solved by: Victor Feser, Mary College (partially); Roger Kuehl, Kansas City; The Proposers and The Editor.

1 P	I		2 M	3 U		4 E	5 P	6 S	I	7 L	O	8 N
A			9 E	G		10 D	A	Y		E		O
11 R	I	12 S	E		13 A			14 M	E	T	15 E	R
A		16 I	T	E	M			M		17 T	B	M
18 B	A	R			19 S	20 P	H	E	21 R	E		
O			22 N	23 U		A		24 T	O	R	25 U	26 S
L		27 T	O	P		28 S	U	R	D		29 N	I
30 A	31 B	E	L			C		Y			32 I	X
	33 R	N		34 M	35 A	A			36 S	37 E	T	
38 R	I		39 K	A	R	L		40 T	R	E	Y	
41 E	D	G	E		T		42 O	R				43 G
	G		P				44 D	E		45 I	46 D	O
47 T	E	E			48 D	E	D	E	K	I	N	D

PROBLEM DEPARTMENT

Edited by Clayton W. Dodge
University of Maine

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

All communications should be addressed to C. W. Dodge, Math Dept., University of Maine, Orono, ME 04469. Contributors desiring acknowledgment of their offerings are requested to enclose a self-addressed postcard or envelope. Please submit each proposal and solution on a separate sheet (one side only) properly identified with name and address. Solutions to problems in this issue should be nailed by December 15, 1982.

Problems for Solution

510. ***Phopobed by Charles W. Trigg? San Diego, California.***

A hexagonal number has the form $2n - n$. In base nine, show that the hexagonal number corresponding to an n that ends in 7 terminates in 11.

511. ***Proposed by Erwin Just and Norman Schaumberger, Bronx Community College, New Yohk.***

If $a > 0$ and $\beta \geq 1$, prove that

(A) $\left(\frac{e}{\alpha}\right)^{\beta\alpha^{\beta}} \leq e^{e^{\beta}-\alpha^{\beta}}$ and

(b) $\left(\frac{\alpha}{e}\right)^{\beta e^{\beta}} \leq e^{\alpha^{\beta}-e^{\beta}}$

512. ***Proposed by Jack Garfunkel, Flushing, New Yohk.***

Denote the number of ways a positive integer n can be partitioned into 3 positive integers by $P_3(n)$. Thus, for example, $P_3(7) = 4$, since we have

$1 + 1 + 5$, $1 + 2 + 4$, $1 + 3 + 3$, and $2 + 2 + 3$ each equal to 7.

Prove the **following:** If a, b, c are positive integers and $a^2 + b^2 = c^2$, then

$$P_3(a) + P_3(b) = P_3(c).$$

513. ***Proposed** by **Ronald E. Shiffler, Georgia** State **University, Atlanta, Georgia.***

Our old friend Prof. Euclide Pasquale **Bombasto** Umbugio, eminent retired numerologist from Guayazuela, has been delving into statistics of late in an effort to prove that his retirement salary is so laughably low that he should be given food stamps in addition to his good conduct pass to the 1986 baton twirlers semifinals. He has checked several distributions involving real numbers and in every case, the average deviation $(a.d.)$ is less than or equal to the standard deviation σ, where

$$a.d. = \frac{1}{n}\sum_{i=1}^{n} |x_i - \bar{x}| \text{ and } \sigma^2 = \frac{1}{n}\sum_{i=1}^{n} (x_i - \bar{x})^2.$$

Of course, $\bar{x}$ is the data mean

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i.$$

He conjectures that $a.d. \leqq \sigma$ is always true. Help the professor to prove his conjecture.

514. ***Proposed** by Raymond **E. Spaulding, Radford University, Radford, Virginia.***

Let $A_1A_2A_3 \cdots A_n$ be a regular polygon where $A_{n+j} = A_j$ and $A_iA_{i+1} = 1$. Let B_i be a point on the segment A_iA_{i+1} where $A_iB_i = x$. Let C_i be the point where A_iB_{i+1} intersects $A_{i+1}B_{i+2}$. Find the area of a regular polygon $C_1C_2C_3 \cdots C_n$ in terms of n and x.

***515.** ***Proposed** by Jack **Garfunkel, Flushing,** New Yolk.*

Given a sequence of concentric circles with a triangle ABC circumscribing the outermost circle. Tangent lines are drawn from each vertex of ABC to the next inner circle, forming the sides of triangle A', B', C'. Tangents are now drawn from vertices A', B', C' to the next inner circle and they are the sides of triangle A'', B'', C'', and so on. Prove that the angles of triangle $A^{(n)}B^{(n)}C^{(n)}$ approach $\pi/3$.

516. ***Proposed** by J. L. **Brenner, Palo Alto, California.***

Prove, for a, b, c positive, that $\frac{1}{3}(a + b + c) \geqq \sqrt{\frac{1}{3}(ab + bc + ca)}$ with equality if and only if $a = b = c$. Does this generalize to

$$\frac{1}{4}(a + b + c + d) \geqq \sqrt{\frac{1}{4}(ab + bc + cd + da)}\ ?$$

517. ***Proposed** by **Charles W. Trigg,** San **Diego, California.***

The nine non-zero digits are arranged to form three three-digit primes with a sum that is divisible by 11. Find the primes and their sum.

518. ***Proposed** by **Michael W. Ecker, Pennsylvania** State **University, Worthington Scranton Campus.***

A baseball player gets a hit and observes that his batting average rises by exactly 10 points, i.e, by .010, and no rounding is necessary at all, where batting average is ratio of number of hits to times at bat (excluding walks, etc.). If this is not the player's first hit, how many hits does he now have?

519. ***Proposed** by **Charles** W. **Trigg, San Diego, California.***

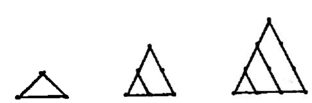
Solve the equation

$$3^{2x} - (34)15^{x-1} + 5^{2x} = 0.$$

520. ***Proposed** by **Chuck Allison, Huntington** Beach, **California.***

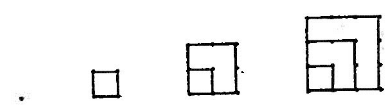
The following diagrams describe the first few polygonal or k-gonal numbers:

Triangular; $k = 3$: $P(n, 3) = \frac{n(n+1)}{2}$



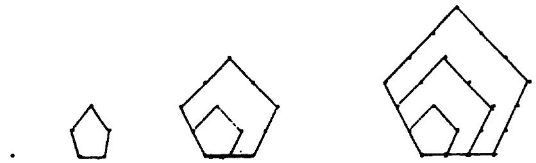
1 3 6 10 15 21 ...

Square; $k = 4$: $P(n, 4) = n^2$



1 4 9 16 25 36 ...

Pentagonal; k = 5: $P(n, 5) = \frac{n(3n-1)}{2}$



1 5 12 22 35 51 ...

where the numbers represent the number of dots shown, and each figure is an extension of its predecessor. The nth number of each sequence is given by the above formulas. Find a general formula for the nth k-gonal number $P(n, k)$.

521. *Proposed by Morris Katz, Macwahoc, Maine.*

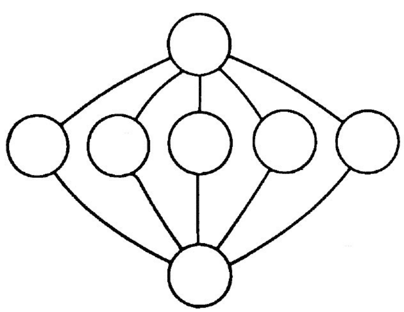
I was told, when I first saw that alphamatic, that a particular value for K produced a unique solution, but I have forgotten what the value is. So find the unique solution where DAILY is prime.

```
   WE
   DO
  WEE
 WORK
DAILY
```

Solutions

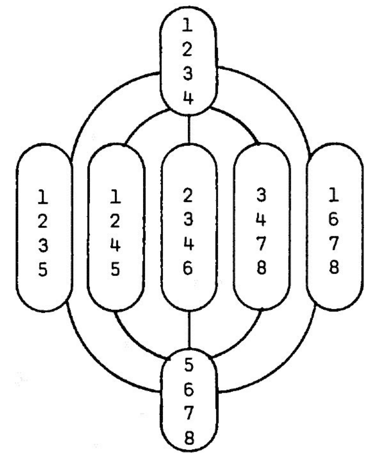
466 [Spring 1980, Spring 1981] *Proposed by Herbert Taylor, South Pasadena, California.*

Let the adversary put four distinct symbols in each box (node) of this graph. Prove or disprove: No matter what pattern of symbols he puts, we can choose two symbols from each box in such a way that adjacent boxes have disjoint chosen 2-sets.



Solution by the Proposer.

Disproof: The adversary could put symbols in boxes in the pattern shown.



476. [Fall 1980, Spring 1981] *Proposed by Jack Garfunkel, Queens College, Flushing, New York.*

If A, B, C, D are the internal angles of a convex quadrilateral, that is if $A + B + C + D = 360^\circ$, then $\sqrt{2}[\cos(A/2) + \cos(B/2) + \cos(C/2) + \cos(D/2)] \le [\cot(A/2) + \cot(B/2) + \cot(C/2) + \cot(D/2)]$, with equality when $A = B = C = D = 90^\circ$.

Solution by M. S. Klamkin, University of Alberta.

Since $\cos x$ is concave down for $0 \le x \le 90^\circ$,

$$\Sigma \cos A/2 \le 4 \cos 45^\circ = 2\sqrt{2}.$$

Since $\cot x$ is concave up for $0 \le x \le 90^\circ$,

$$\Sigma \cot A/2 \ge 4 \cot 45^\circ = 4.$$

Thus,

$$\Sigma \cot A/2 \ge \sqrt{2}\, \Sigma \cos A/2.$$

The latter inequality is not valid if the quadrilateral is non-convex.; just let $A = B = C = 30^\circ$, $D = 270^\circ$.

Also we similarly have

$$\Sigma \cot A/2 \ge \sqrt{2}\, \Sigma \sin A/2 ,$$

$$\Sigma \tan A/2 \geq \sqrt{2}\,\Sigma \cos A/2\,,$$
$$\Sigma \tan A/2 \geq \sqrt{2}\,\Sigma \sin A/2\,.$$

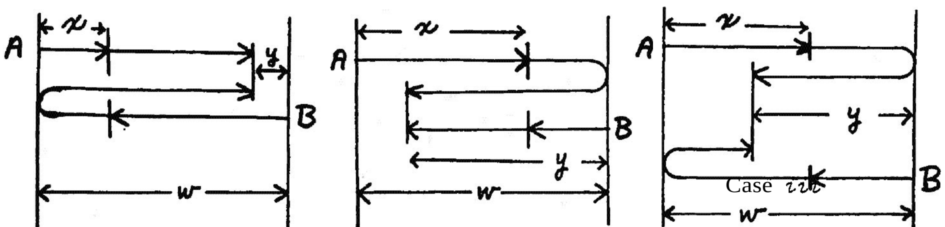
Also solved by RALPH KING and *the* PROPOSER.

486. [Spring 1981] *Proposed by Chuck Allison and Peter Chu, San Pedro, California.*

Swimmers A and B start from opposite sides of a river and swim to their corresponding opposite sides and then back again, each swimming at his own constant rate. If on the first pass they meet each other x feet from A's starting side, and on the second pass they meet at a point y feet from B's starting side, how wide is the river in terms of x and y?

Solution by Kevin Theall, Essex Falls, New Jersey.

Consider the following three cases, shown in the diagrams below.



Case *iii*

Case *ii*

Letting v_a and v_b be the respective swimming rates for A and B, we have case *i* when $2v_a \leq v_b$. Then, considering the distances travelled between meetings, we have

$$\frac{v_b}{v_a} = \frac{w - x}{x} = \frac{w - y + x}{w - x - y}\,,$$

whence

$$w = \frac{3x + y + \sqrt{(x-y)^2 + 8x^2}}{2}.$$

If $v_a \geq 2v_b$, then case *ii* occurs and we have

$$\frac{v_b}{v_a} = \frac{w - x}{x} = \frac{y}{w + y}\,, \quad \text{so} \quad w = \frac{x - y + \sqrt{(x-y)^2 + 8yx}}{2}$$

Finally, case *iii* occurs when $4v_a > 2v_b > v_a$. Then

$$\frac{v_b}{v_a} = \frac{w - x}{x} = \frac{2w - y}{w + y}\,, \qquad w = 3x - y.$$

Case iii was also solved by LEONOR M. ABRAIDO-FANDIÑO, ARA BASMAKIAN, MIKE BEACH, DAVID DEL SESTO (*cases ii* and *iii*), MARK EVANS (who remarked that the quicker swimmer must not be more than twice as fast as the slower swimmer), VICTOR G. FESER (who recognized a second possibility), ROBERT C. GEBHARDT, JOHN M. HOWELL, RALPH KING, HENRY S. LIEBERMAN, BOB PRIELIPP, DOUGLAS RALL (*cases* A *and iii*), ANITA REED, KENNETH M. WILKE, BRENT WRASMAN (*cases ii* and *iii*), *and the* PROPOSER.

487. [Spring 1981] *Proposed by Solomon W. Golomb, University of, Southern California.*

We know that 1/7 = .142857... repeating with period 6. With A = 142 and B = 857, the first and Second halves of the period, respectively, we observe that A + B = 999, and B = 6A + 5. Prove this generalization:

If p is prime, and the decimal expansion of $1/p$ has period $2t$, where A and B are the first and second halves of the period, then A + B consists of "all 9's", and when B is divided by A, there is a quotient of p − 1 with a remainder of p − 2.

Can you also generalize from the relation 14 + 28 + 57 = 99? Finally, what happens if the expansions are in base b and p is merely relatively prime to b? (Note: In base b > 1, b is always equal to 10, but not necessarily equal to ten.)

I. *Partial solution by Bob Prielipp, University of, Wisconsin-Oshkosh.*

It can be established that the sum of two halves of the period will always turn out this way when the period belongs to the fraction a/p whose denominator p is a prime, provided that the period has an even number of digits. (For a proof of this fact, see Rademacher and Toeplitz, The Enjoyment of Mathematics, Princeton University Press, 1957, pp. 158-160. Another proof may be found in W. G. Leavitt, "A Theorem on Repeating Decimals," The American Mathematical Monthly, June-July 1967, pp. 669-673). This property is sometimes called the nines-property.

Next we demonstrate that if p is a prime number and

$$1/p = \overline{a_1a_2\ldots a_ta_{t+1}a_{t+2}\ldots a_{2t}}$$

(so $1/p$ is an infinite repeating decimal with a period of even length) then

$$a_{t+1}a_{t+2}\ldots a_{2t} = (p-1)(a_1a_2\ldots a_t) + (p-2).$$

To conform to the notation of the statement of the problem, let $A = a_1a_2\ldots a_t$ and let $B = a_{t+1}a_{t+2}\ldots a_{2t}$. Then

$$(*) \qquad 10^{2t} = (A \cdot 10^t + B)p + 1.$$

Using the nines-property, we have that

$$10^t/p = A + .\overline{BA} = A - .\overline{AB} + .\overline{AB} + .\overline{BA} = A - 1/p + .\overline{9},$$

so $10^t = Ap + (p-1)$. Substituting $Ap + (p-1)$ for 10^t in (*) and simplifying yields $B = A(p-1) + (p-2)$.

II. Partial solution by Kenneth M. Wilke, Topeka Kansas.

Let p be a prime > 5 such that the decimal expansion of $1/p$ has period 3t. Divide the period of $1/p$ into three groups of t digits each and denote the first group of t digits on the left by A_1, the next group by A_2 and the third group by A_3. We shall show that $A_1 + A_2 + A_3 = 10^t - 1$.

Proof. Since the period of $1/p$ is $3t$, then 3t is the smallest exponent r such that $p \mid 10^r - 1$. Thus $p \nmid 10^t - 1$ so we must have $p \mid 10^{2t} + 10^t + 1$. But $10^t \equiv r \pmod p$ where $0 \le r < p$. Hence we have

$$p \mid r^2 + r + 1 \text{ since } 10^{2t} + 10^t + 1 \equiv r^2 + r + 1 \pmod p.$$

Hence $r^2 + r + 1 = pk$ for some integer k. Now $10^t = A_1p + r$ so that

$$\frac{10^{3t} - 1}{p} = A_1 \cdot 10^{2t} + A_2 \cdot 10^t + A_3.$$

Now since

$$\frac{10^{3t} - 1}{p} = \frac{10^{3t} - 10^{2t} + 10^{2t} - 10^t + 10^t - 1}{p}$$

and since $A_1 = [10^t/p]$, we have $A = [r \cdot 10^t/p]$ where [a;] is the greatest integer in x. Thus

$$\frac{r \cdot 10^t}{p} = \frac{r(A_1p + r)}{p} = rA_1 + \frac{r^2}{p} = rA_1 + \frac{pk - r - 1}{p}.$$

Then since $[(pk - r - 1)/p] = k - 1$, we have $A_2 = rA_1 + k - 1$. Also $[(pk - r - 1)/p] = k = 1$ because $p(k - 1) + (p - r - 1) = pk - r - 1$ and $0 \le p - r - 1 < p$ since $0 \le r < p$.

Finally the division process yields $A = \frac{10^t(p - r - 1) - 1}{p}$. But this is equivalent to

$$A = 10^t - \frac{(r + 1)(A_1p + r) - 1}{p} = 10^t - A_1(r + 1) - k$$

which is an integer. Then

$$A_1 + A_2 + A_3 = A_1 + (rA_1 + k - 1) + (10^t - A_1(r + 1) - k) = 10^t - 1$$

as required.

KENNETH M. WILKE *also solved the first parts of the problem.*

488. [Spring 1981] *Proposed by Herb Taylor, South Pasadena, California.*

Take the numbers from 1 to 24 and put them into 8 disjoint 3-sets $[a,b,c]$ such that in each 3- set, $a + b = c$.

Summary of solutions submitted by SUZANNE CRISCIONE, *Providence College, Rhode Island,* DAVID DEL SESTO, *North Kingstown, Rhode Island,* MARK EVANS, *Louisville, Kentucky,* JAMIE GREGORY, *Terre Haute, Indiana,* KATHLEEN HENRY, *New Rochelle, New York,* JOHN M. HOWELL, *Littlerock, California,* MARJORIE HSU, *St. Olaf College, Northfield, Minnesota,* ROBERT KELLY, *Providence, Rhode Island,* RALPH E. KING, *Saint Bonaventure, University, New York,* JEAN LANE, *Union College, Cranford, New Jersey,* PAUL A. MCKLUEEN, *Raleigh, North Carolina,* TAGHI REZAY-GARACANI, *Oklahoma State University, Stillwater, Oklahoma,* JUDY A. SCHULTZ, *Providence College, Rhode Island,* CHARLES W. TRIGG, *San Diego, California, and the* PROPOSER.

We have (11, 13, 24), (7, 16, 23), (10, 12, 22), (6, 15, 21), (2, 18, 20), (5, 14, 19), (8, 9, 17), (1, 3, 4) is a solution utilizing the largest six elements in separate sets, whereas (1, 13, 14), (2, 22, 24), (3, 20, 23), (4, 17, 21), (5, 10, 15), (6, 12, 18), (7, 9, 16), (8, 11, 19) places all 8 smallest integers separately. According to Trigg, there are some 2000 solutions. No general methods were given, and only those solutions submitted by Howell and Hsu were duplicates, specifically (1, 2, 3), (4, 17, 21), (5, 13, 18), (6, 14, 20), (7, 15, 22), (8, 16, 24), (9, 10, 19).

489. [Spring 1981] *Proposed by Michael U. Ecker, Pennsylvania State University, Worthington Scranton Campus.*

Let k and n be positive integers with $k < n$. Two players take turns choosing, on each turn, a positive integer $\le$ k. A running total is kept, and the player to achieve n as the sum is the winner.

State and prove winning strategy results for the game. (The game with $n = 50$ and k = 6 has been used as a teaching tool, with modest popularity, at the elementary and secondary school levels).

Solution by Kenneth M. Wilke, Topeka, Kansas.

Given integers n and k with $n > k$, there are integers a and r such that $n = a(k+1)+r$ with $0 \le r < k+1$. Now if a player is to play from a total t, his winning strategy is to add the number $y < k+1$ which makes $t + y \equiv r \pmod{k+1}$. This strategy always wins unless $t \equiv r \pmod{k+1}$ already because if $t \equiv r \pmod{k+1}$ and you select the number y to add, your opponent, who knows the best strategy will select $k+1-y$ so that you will be playing from a new total $t+k+1$. Now if $t \equiv r \pmod{k+1}$, then $t+k+1 \equiv r \pmod{k+1}$ also and you are playing from a losing position because whatever number y you choose, the opponent counters with $k+1-y$ leaving you again in a losing position. E.G. for $n = 50$, $k = 6$, $50 = 7 \cdot 7 + 1$ so that the first player to achieve one of the totals 1, 8, 15, 22, 29, 36, 43 or 50 wins by following this strategy.

Also solved by JOHN M. HOWELL and *the* PROPOSER.

490. [Spring 1981] ***Proposed by Joyce W. Williams, North University of, Lowell.***

The function $f(n)$ is to be constructed to give the number of days in a year through the nth month for $n = 0, 1, \ldots, 12$. That is, $f(0) = 0$, $f(1) = 31, \ldots, f(12) = 365$. Leap year is to be ignored. What is the simplest solution?

I. ***Solutions by Victor G. Feser, Mary College, Bismarck, North Dakota.***

One person's simple is another's complex. Here are three solutions. The first one might be called "simple-minded." Simply define proper values for $x = 0, 1, \ldots, 12$; for all other x, let $f(x)$ = anything.

The "simplest" kind of function is linear, so define a continuous, piecewise linear function on the intervals $(-\infty, 1], [1, 2], \ldots, [11, +\infty)$.

A continuous, smooth function can also be defined. Just construct a polynomial having the correct values at $x = 0, 1, \ldots, 12$.

II. ***Solution by David Sutherland, North Texas State University, Denton.***

Starting with the thirteen statements $f(0) = 0$, $f(1) = 31$, $f(2) = 59$, $\ldots$, $f(12) = 365$, we simplify by reducing the number of statements to three:

$$\begin{aligned} f(n) &= 31n & &\text{for } n = 0, 1 \\ &= 31n - [\tfrac{n}{2} + 2] & &\text{for } n = 2, 3, 4, 5, 6, 7, 9, 11 \\ &= 31n - [\tfrac{n}{2} + 1] & &\text{for } n = 8, 10, 12 \end{aligned}$$

where the brackets indicate the greatest integer function.

III. ***Solution by the Proposer.***

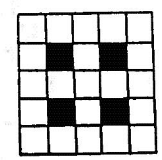
One solution is the function

$$f(x) = [(0.5)(61n-1) + (1.1)|n-2| - |n-1|],$$

where the brackets indicate the greatest integer function.

491. [Spring 1981] ***Proposed by Charles W. Trigg, San Diego, California.***

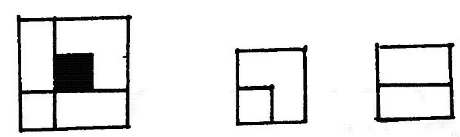
From a square grid of side $2n+1$ alternate squares are removed to form a sieve. (a) What is the smallest sieve that can be dissected and the parts assembled into two squares with integer sides? (b) What is the smallest number of pieces into which the sieve must be cut to accomplish this assembly?



I. ***Solution by Victor G. Feser, Mary College, Bismarck, North Dakota.***

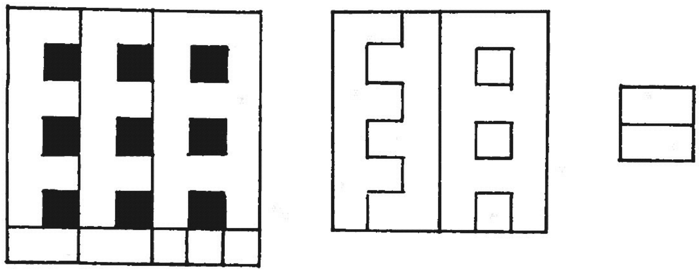
If $n = 0$, we have a trivial solution: either make a sieve by removing no squares, make no cuts, and reassemble into a square of side 1 and a square of side 0; or make a sieve by removing one square, make no cuts and reassemble into two squares of side 0.

For $n = 1$, make two cuts to get four pieces which reassemble to produce two squares of side 2, as shown in the figure below.



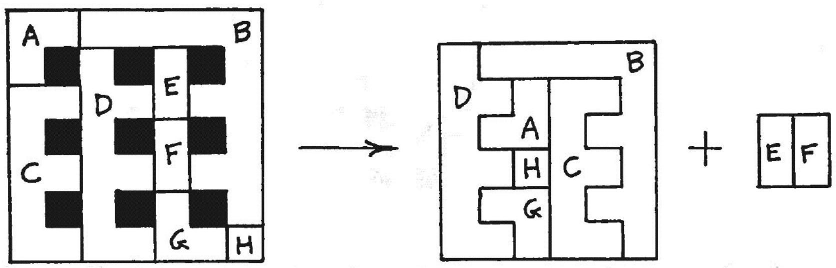
For $n = 2$ no solution exists since the sieve has 21 squares, which does not yield 2 squares, although it does yield 3.

An interesting case arises for $n = 3$. The sieve has 40 squares, and this gives two unequal squares, with 8 pieces, as shown below.



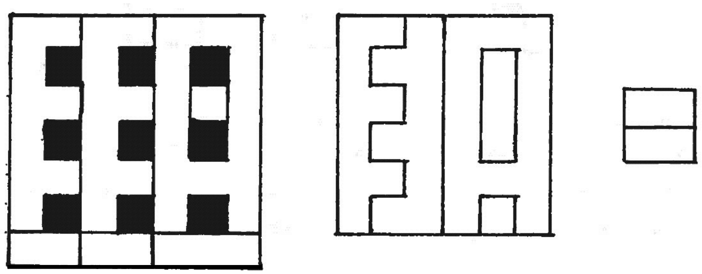
II. Solution by the Proposer.

[In addition to comments similar to those of Solution I] the 7 x 7 sieve is the smallest that can be dissected and reassembled into two unequal squares. It can be done with 8 pieces as shown below.



We further note that $f(2) = 21 = 1 + 4 + 16$, so a 5 x 5 sieve is the smallest one that can be disected and assembled into three squares. Also, $f(4) = 65 = 1 + 64 = 16 + 49 = 4 + 25 + 36 = 4 + 9 + 16 + 36$, so a 9 x 9 sieve is the smallest one that can be dissected and assembled into four, three, and two squares; in the latter case, in two ways. Then, $f(5) = 96$, $f(6) = 133$. $f(7) = 176$, and $f(8) = 225$. Thus a 17 x 17 sieve is the smallest one that can be dissected and assembled into one square.

Editorial comment. Feser's 7 x 7 solution can be reduced to 7 pieces, as shown below.

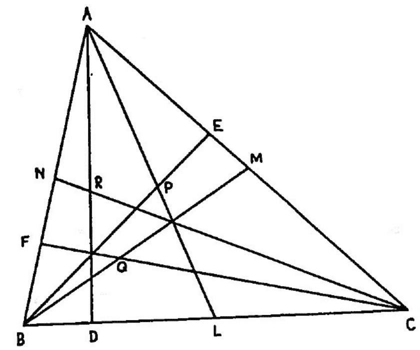


492. *[Spring 1981] Proposed by Jack Garfunkel, Queens College, Flushing, N.Y.*

Given an acute triangle ABC with altitudes denoted by h_a, h_b, h_c and medians by m_a, m_b, m_c to sides a, b, c respectively. The points P, Q, R are determined by the intersections $m_a \cap h_b$, $m_b \cap h_c$, and $m_c \cap h_a$, respectively. Prove:

$$\frac{AP}{PL} + \frac{BQ}{QM} + \frac{CR}{RN} \geq 6,$$

where L, M, N are the feet of the medians.



Solution by the Proposer.

Denote angle BPL by θ, and let $2R = 1$, where R is the circumradius of triangle ABC. We have in triangle ABP, since $c = 2R \sin C$,

$$\frac{\sin(90^\circ - A)}{\sin\theta} = \frac{AP}{c} = \frac{AP}{\sin C}, \text{ so } AP = \frac{\sin C\cos A}{\sin\theta}.$$

In triangle BLP, we have, because $BL = \frac{1}{2}a = \frac{1}{2}(2R)\sin A$,

$$\frac{\sin(90^\circ - C)}{\sin\theta} = \frac{PL}{\frac{1}{2}\sin A}, \text{ so } PL = \frac{\frac{1}{2}\sin A\cos C}{\sin\theta}$$

Hence,

$$\frac{AP}{PL} = \frac{2\sin C\cos A}{\sin A\cos C} = 2\cot A\tan C.$$

Similarly,

$$\frac{BQ}{QM} = 2\cot B\tan A, \text{ and } \frac{CR}{RN} = 2\cot C\tan B.$$

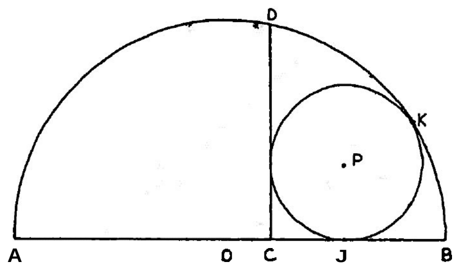
We have to show that

$$2\ \Sigma \cot A \tan C \geq 6 \text{ or } \Sigma \frac{\cot A \tan C}{3} \geq 1.$$

Since the geometric mean of these three products, $\cot A \tan C$, etc., is 1, and since the geometric mean is less than or equal to the arithmetic mean, this inequality follows.

494. [Spring 1981] *Proposed by Zelda Katz, Beverly Hills, California.*

In the annexed figure CD is a half-chord perpendicular to the diameter AB of the semicircle (O), and the inscribed circle (P) touches AB in J and the arc DB in K. Show by elementary plane geometry, without using inversion, that $AD = AJ$.



Solution by Henry S. Lieberman, Boston, Massachusetts.

The key observation is that O, P and K are collinear. This is so because the circle P is tangent to circle O at K.

Let r be the radius of circle P and R that of circle O. Then $OP = R - r$ and from right triangle OPJ we get

$$(R-r)^2 = r^2 + (r+OC)^2.$$

Also, $R^2 = OC^2 + DC^2$ from the right triangle ODC. Combining these two equations we get:

$$DC^2 = r^2 + 2rOC + 2rR.$$

But

$$AJ = R + r + OC.$$

Therefore,

$$AJ^2 = r^2 + 2r(R+OC) + (R+OC)^2 = DC^2 + AC^2.$$

Since $AD^2 = DC^2 + AC^2$, then $AJ = AD$.

Also solved bq JACK GARFUNKEL, RALPH KING, *and* the PROPOSER.

495. [Spring 19811 *Proposed by Richard Hess, Palos Verdes, California.*

A regular pentagon is drawn on ordinary graph paper. Prove that no more than two of its vertices lie on grid points.

Solution by the Proposer.

Assume that 3 vertices do fall on grid points. Whichever 3 they are they include a 36° angle in the triangle they form. There are only two distinct cases.

Translate coordinates so that the origin (0, 0) is at the 36° vertex and let the other two vertices have coordinates (a, b) and (c, d). Then a, b, c, and d are rational.

By the law of cosines applied to this triangle we get that

$$\cos 36^\circ = \frac{1+\sqrt{5}}{4} = \frac{ac+bd}{\sqrt{a^2+b^2}\cdot\sqrt{c^2+d^2}}$$

Now square both sides to get

$$\frac{3+\sqrt{5}}{8} - \frac{(ac + M\ l)^2}{(a^2+b^2)(c^2+d^2)}$$

The right side of this equation is rational, but the left side is irrational, a clear contradiction.

496. [Spring 1981] *Proposed by Donald Canard, Anaheim, California.*

P is any point within a triangle ABC, whose sides are a, b, c, whose semiperimeter is s and whose orthocenter is H. Let x denote the distance from P to BC and let R denote the circumradius of triangle ABC. Show that

$$PA^2 = PH^2 + b^2 + c^2 - 4R^2 - \frac{ax}{2s}(b^2 + c^2 - a^2).$$

Solution bq Mickey Souris, Orlando, Florida.

The denominator in the stated equation should be $2rs$, not $2s$, where r is the inradius of the triangle. Then the following equations are known:

$$AH = 2R \cos A, \qquad h_a = 2R \sin B \sin C,$$

where h_a is the altitude to vertex A.

$$a = 2R \sin A, \quad b = 2R \sin B, \quad c = 2R \sin C$$

by the law of sines,

$$b^2 + c^2 - a^2 = 2bc \cos A, \quad \text{and } abc = 4Rrs.$$

Now let y be the length of the perpendicular PK from P to the altitude AH. From right triangles PAK and PHK, we get

$$(h_a - x)^2 + y^2 = PA^2 \text{ and } (h_a - AH - x)^2 + y^2 = PH^2,$$

so

$$(h_a - x)^2 - (h_a - AH - x)^2 = PA^2 - PH^2 ,$$

which we solve for PA to get

$$\begin{aligned} PA^2 &= PH^2 - AH^2 + 2h_a \cdot AH - 2x \cdot AH \\ &= PH^2 - 4R^2\cos^2 A + 4R \sin B \sin C \cdot 2R\cos A - 2x \cdot 2R\cos A \\ &= PH^2 - 4R^2 + 4R^2\sin^2 A + 2bc \cos A - 2x \cdot 2R\left(\frac{b^2+c^2-a^2}{2bc}\right) \\ &= PH^2 - 4R^2 + a^2 + (b^2+c^2-a^2) - \frac{4axR}{2abc}(b^2+c^2-a^2) \\ &= PH^2 - 4R^2 + b^2 + c^2 - \frac{4axR}{8Rrs}(b^2+c^2-a^2) \\ &= PH^2 - 4R^2 + b^2 + c^2 - \frac{ax}{2rs}(b^2+c^2-a^2). \end{aligned}$$

Also solved by the PROPOSER, except for overlooking the r. He inadvertently said the area of the triangle was equal to s instead of rs, but we shall duck the question of pronouncing an appropriate punishment.

497. [Spring 1981] *Proposed by Scott Kim, Artificial Intelligence Laboratory, Stanford University.*

Three drummers are positioned at the corners of a large equilateral triangle, say 1 mile on a side. Each drummer beats his drum at a constant rate r, with the time between beats being equal to the time it takes for the sound to travel the length of one side of the triangle. The drums are synchronized so that a listener standing in the center of the triangle would hear all three beats simultaneously. This means that it seems to each drummer that the other two drums are in synch with his own drum (actually they are delayed by one beat).

Problem: Where else can a listener stand (besides the center and corners) and hear all three drums in synchronization?

Unsolved (untried): What if the drummers beat at a rate of nr, for $n = 2,3,4,\ldots$?

Solution by Mark Evans, Louisville, Kentucky.

For $n = 1$ there are no solutions other than those mentioned by the proposer. For, if ABC is the equilateral triangle of side 1 mile, then any listener L must be located so that, for example $LA - LB = d$ is an integer. Since LAB is a triangle, we can have $d = 0$ if L lies on the perpendicular bisector of AB or $d = \pm 1$ if L lies on AB extended. By the triangle inequality, no other possibilities exist.

For $n > 1$, place triangle ABC in the Cartesian plane so $A(-1/2, 0)$, $B(0, \sqrt{3}/2)$, and $C(1/2, 0)$. Let $L(x, y)$ be a solution point, where all three drums are heard synchronously. Let r denote the number of beats delay (or lead) of the signals from A to those from B, and s and t the delays between the signals from A and C and from B and C. Then

$$r = n(LA) - n(LB), \text{ etc.}$$

Let

$$f(x,y) = \frac{r}{n}.$$

For points on line AC (the x-axis) we have

$$f(x, 0) = x + \frac{1}{2} - \left(x^2 + \frac{3}{4}\right)^{1/2} ,$$

so

$$\frac{\partial f(x, 0)}{\partial x} = 1 - x\left(x^2 - \frac{3}{4}\right)^{-1/2}.$$

Then $f(x, 0)$ has no minimum or maximum on the interval $(0, \infty)$. Since $f(0, 0) = (-\sqrt{3} + 1)/2$ and

$$\lim_{x \to \infty} f(x, 0) = \frac{1}{2} ,$$

then

$$\frac{-\sqrt{3}+1}{2} \leqq \frac{r}{n} \leqq \frac{1}{2} .$$

Similarly, let $g(x,y) = s/n$ and $h(x,y) = t/n$. Then $g(x, 0) = 2x$ for $0 \leqq x \leqq 1/2$ and $g(x, 0) = 1$ for $x \geqq 1/2$, and $h(x, 0) = g(x, 0) - f(x, 0)$.

For $x \geqq 1/2$ we have

$$f(x, 0) = x + \frac{1}{2} - \left(x^2 + \frac{3}{4}\right)^{1/2} = \frac{r}{n} ,$$

which we solve by isolating the radical and squaring to get

$$x = \frac{n^2 + 2nr - 2r^2}{2n^2 - 4rn} \quad \text{for} \quad 0 \leqq \frac{r}{n} < \frac{1}{2} .$$

Thus we have $[(n-1)/2]$ new solutions on the positive x-axis for each $n \geqq 3$.

For $0 \leqq x \leqq 1/2$, by a similar argument we find that

$$x = \frac{n^2 + 2nr - r^2}{2n^2 - 4nr} ,$$

provided $3nr/(n - 2r)$ is an integer and $(1-\sqrt{3})/\sqrt{2} \leqq r/n \leqq 0$.

For points on the y-axis (the altitude through B), we have

$$f(0, y) = (y^2 + \tfrac{1}{4})^{1/2} - \left|y - \frac{\sqrt{3}}{2}\right| = \frac{r}{n},$$

and a similar argument leads to

$$y = \frac{4r(r^2 - n^2) + n(n^2 - 2r^2)\sqrt{3}}{2n(3n^2 - 4r^2)} \quad \text{for } \frac{\sqrt{3}}{2} \leqq \frac{r}{n} \leqq 1$$

and

$$y = \frac{4r(r^2 - n^2) + n(2r^2 - n^2)\sqrt{3}}{2n(4r^2 - 3n^2)} \quad \text{for } \frac{-\sqrt{3}}{2} < \frac{r}{n} \leqq \frac{\sqrt{3}}{2}$$

Solutions to the first part were also submitted by ROBERT KUEHL *and the* PROPOSER.

Kuehl remarked that a listener could "stand" at any point on a line perpendicular to the plane of the triangle through the point where the medians intersect.

A CARD OF THANKS

Starting with the Fall 1968 issue a Los Angeles dentist took the reins of this problem department and ably lead it for more than a dozen years, increasing the number of problems per issue from 8 to 12 while maintaining a high standard of excellence. His primary interest is geometry, but the other branches of mathematics have also been fully represented in the more than 300 problems he has included in these pages. Now Dr. Leon Bankoff is retiring from this post, and the vacancy he leaves will be a difficult one to fill. We extend ow deep appreciation and warm thanks to Leon Bankoff for the great time and effort he has spent serving this department and its readers. His strong leadership will continue to influence the Problem Department for years to come.

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