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CONTENTS

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EDITOR'S NOTE --

In this Issue, you win find the Chapter Reports from 24 of our 270+ Pi Mu Epsilon Chapters. These reports announce 97 papers and talks given to the Chapter meetings (mostly by undergraduates) during 1981-82. Even when this number is added to the 22 papers given at the Summer Pi Mu Epsilon Meeting in Toronto, the 87 papers that can be counted at Regional MAA Meetings, the 19 given at the Ohio Pi Mu Epsilon Meeting plus many more at other Regional Meetings, we see only part of the mathematical activity by undergraduates and beginning graduate students.

The purpose of this Fraternity is to promote mathematics, and the sole purpose of this Journal is to present papers by and for its members. A large number of papers are presented but never sent to the Journal for possible publication. Write them up and send them in! A publication in a national refereed journal will be very impressive on your resume, and even if it is not published, having written a paper and having tried to publish it means a lot to potential employers and graduate schools. Any papers should be sent to Dr. David Ballew, Department of Mathematical Sciences, South Dakota School of Mines and Technology, Rapid City, SD 57701

During 1982-83, we will continue our National Paper Competition. Every paper written by an undergraduate or a graduate student who has not received a Master's Degree at the time of submission is eligible. The winners for 1980-81 ewe:

FIRST PRIZE ($200)	Bradley Strand, "Vector Subspaces of Magic Squares", Department of Mathematics, Carlton College, Northfield MN, 55057 [See this Issue of the Journal]
SECOND PRIZE ($100)	Karen Cunningham, " A Simple Model for Two Interlacing Species and the Principle Competitive Exclusion", Univ. of Texas at Arlington (See the Spring 1983 Issue)
THIRD PRIZE ($50)	Ravi Salgia, "Volume of an N-Dimensional Unit Sphere", Loyola University/Chicago, (See the Spring 1983 Issue)

FIRST PRIZE PAPER
1981-82 Paper Competition

VECTOR SUBSPACES OF MAGIC SQUARES

by Bradley D. Strand
Carleton College

An n^{th} order <u>magic square</u> is an $n \times n$ array of real numbers such that the sum along each of the square's rows, columns, and main diagonals is a constant (called 1, the <u>line-sum</u>). Consider the set of all n^{th} order magic squares whose line-sum is zero, designated $M_o(n)$. It is easy to see that this set is closed, commutative, and associative under addition and real scalar multiplication, contains additive inverses, and in short, satisfies all the conditions necessary for it to be a vector space. Ward [1] has shown the dimension of this vector space to be n^2-2n-1. In this paper, simple techniques of linear algebra will be used to determine the dimension of a certain subspace of $M_o(n)$.

An n^{th} order magic square is said to be <u>skew-symmetric</u> if all pairs of cells symmetric about the square's center have a constant sum, S, called the <u>skew-symmetric sum.</u> Equivalently, $m_{(i,j)}+m_{(n+1-i,n+1-j)} = S$ for all $1 \le i, j \le n$, where $m_{(i,j)}$ represents the entry in the i^{th} row and j^{th} column of M, a general element of $M_o(n)$.

Theorem 1. The subset of $M_o(n)$ consisting of all skew-symmetric zero-sum magic squares of order n, designated $SM_o(n)$, is a vector subspace of $M_o(n)$.

Proof. It suffices to show that $SM_o(n)$ is closed under addition and real scalar multiplication. It is clear that if we have $M_1, M_2 \in SM_o(n)$ with S_1, S_2 as their respective skew-symmetric sums, then $(M_1+M_2) \in SM_o(n)$ with skew-symmetric sum (S_1+S_2). Thus, $SM_o(n)$ is closed under addition. It is also clear that if we have $M \in SM_o(n)$ with skew-symmetric sum S, and $k \in R$, then $kM \in SM_o(n)$ with skew-symmetric sum kS. Thus, $SM_o(n)$ is closed under real scalar multiplication, and hence is a subspace of $M_o(n)$.

We now focus our attention on the problem of determining the dimension of this vector subspace.

Lemma 1. If n is odd, and if $M \in SM_o(n)$ with $S=0$, then M has a zero in its center cell.

Proof. Since n is odd, $n=2k+1$ for some non-negative integer k. Consider the $(k+1)^{st}$ column in M. Since S=O, $m_{(1,k+1)}+m_{(n,k+1)} = 0$. Similarly, each of the first k entries in the $(k+1)^{st}$ column can be paired with its skew-symmetric partner to form a zero-sum pair. There are k such pairs, and since the column-sum is zero (as $M \in SM_o(n)$) we get the equation $k0+m_{(\frac{n+1}{2},\frac{n+1}{2})} = 0$. Thus, the center cell must contain a zero.

Lemma 1 is now used to prove the following theorem.

Theorem 2. If n is odd, and $M \in SM_o(n)$, then M has a zero in its center cell.

Proof. Using Lemma 1, it suffices to show the skew-symmetric sum is zero for any element of $SM_o(n)$. Assume we have an $n \times n$ array of empty cells in which we will describe a general element of $SM_o(n)$. Start in cell (1,1) and enter random numbers across the top row, ending in cell $(1,n-1)$. Call these entries $x_1, x_2, \ldots, x_{n-1}$. Since we are considering a general element of $SM_o(n)$, the row-sums must be zero. This implies cell $(1,n)$ must contain the entry $-\sum_{i=1}^{n-1} x_i$. Since the square is to be skew-symmetric, call its skew-symmetric sum S. Thus, the entry in cell (n,n) is $S-x_1$, the entry in cell $(n, n-1)$ is $S-x_2$, and so on across the bottom row until we get to cell $(n,1)$, whose entry is $S + \sum_{i=1}^{n-1} x_i$. Taking the sum across the bottom row yields the equation $nS=0$, since all the x_i's neatly cancel. Thus, $S=0$, and the proof is complete.

The implication is clear. Not only do all members of $SM_o(n)$ with odd n have a zero in the center cell, but all skew-symmetric pairs of cells in all elements of $SM_o(n)$ contain additive inverses.

Theorem 3. Let n be even and $M \in SM_o(n)$. If two perpendicular bisecting segments each n units long are allowed to divide M into four square $n/2 \times n/2$ "quadrants" as shown in Figure 1, then the sum of the entries in each of the four quadrants must be zero.

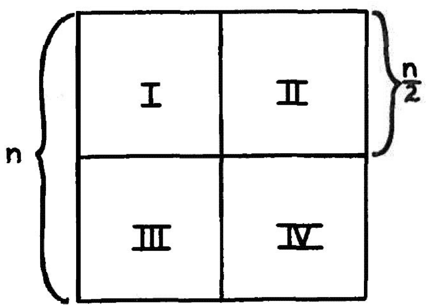

Figure 1

Proof. We can, without loss of generality, consider the entries in quadrant I. Assume the entries in quadrant I have sum A. Since $M \in SM_0(n)$, the skew-symmetry property and Theorem 2 tell us the entries in quadrant IV have sum $-A$. We also know that if we add the entries in quadrants I and II, i.e. the top half of M, we must get zero for a sum, since each individual row has sum zero. But since we know the entries in quadrant I have sum A, the entries in quadrant II must have sum $-A$. Now we add the entries in quadrants II and IV, which must equal zero, and get the equation $-2A=0$. Thus, $A=0$.

This last result implies that when we are constructing even order members of $SM_0(n)$, we may choose no more than $(n^2/4)-1$ entries per quadrant freely.

This leads to the main conclusion of this paper.

Theorem 4. The dimension of $SM_0(n)$ is $(n-1)^2/2$ if n is odd, and is $n(n-2)/2$ if n is even.

Proof. (Odd case) if n is odd, and $M \in SM_0(n)$, then each of M's $2n+2$ line-sums are zero (n row-sums, n column-sums, and 2 diagonal-sums). Moreover, since all skew-symmetric pairs have sum zero (Theorem 2) and since the center square contains a zero (again, Theorem 2), we have $(n^2+1)/2$ additional equations which have sum zero. Thus, we have a system of $(n^2+1)/2 + 2n + 2$ homogeneous equations in the n^2 unknowns m_{ij} $(1 \le i, j \le n)$.

Write these equations in the following order, called the standard odd order: first the $(n^2+1)/2$ skew-symmetric sums beginning with $m_{(1,1)}+m_{(n,n)}=0$, $m_{(1,2)}+m_{(n,n-1)}=0, \ldots, m_{(2,1)}+m_{(n-1,n)}=0, \ldots, m_{\left(\frac{n+1}{2}, \frac{n+1}{2}\right)}=0$ then the n row-sums, in order; followed by the n column-sums, in order; and finally the two diagonals *NW-SE*, and *SW-NE*. The resulting coefficient matrix is an $(((n^2+1)/2 + 2n + 2) \times n^2)$ matrix of 0's and 1's. In the $n=3$ case, it is the 13 x 9 matrix shown in Figure 2, where the elements in the j^{th} column are coefficients of the j^{th} variable in the list $m_{11}, m_{12}, m_{13}, m_{21}, m_{22}, m_{23}, m_{31}, m_{32}, m_{33}$.

100	000	001
010	000	010
001	000	100
000	101	000
000	010	000
111	000	000
000	111	000
000	000	111
100	100	100
010	010	010
001	001	001
100	010	001
001	010	100

Figure 2

It is clear that the first $(n^2+1)/2$ rows of the coefficient matrix are linearly independent since they are in row-echelon form. Consider the next row, which corresponds to the equation that specifies the first row must have sum zero, and consists of n 1's followed by $n(n-1)$ 0's. Using elementary row operations to put this vector into row-echelon form, we get,, in general, $n(n-1)$ 0's followed by n 1's. Thus, it is linearly independent of all previous rows. The next row, corresponding to the equation specifying the second row have sum zero, gives us, upon reduction to row-echelon form, $n(n-2)$ 0's, followed by n 1's, followed by n more 0's. For $n > 3$ this row is also linearly independent of all previous rows. This process continues similarly until we reach the $(n+1)/2^{nd}$ row vector, i.e., row $(n^2+n+2)/2$ of the coefficient matrix. This is clearly a linear combination of rows $(n^2+1)/2 + ((n^2+1)/2 - 1) + \ldots + ((n^2+1)/2 - (n-1)/2)$ of the coefficient matrix, and hence is not linearly independent. Likewise, each of the remaining row vectors, i.e., rows $((n^2+n+2)/2 + 1), ((n^2+n+2)/2 + 2), \ldots, ((n^2+n+2)/2 + (n-1)/2)$ of the coefficient matrix, is linearly dependent, as each is the row-echelon form of rows $((n^2+n)/2), ((n^2+n)/2 - 1), \ldots, ((n^2+1)/2 + 1)$ respectively.

Now consider the vectors correponding to the column-sum equations, beginning with row $(n^2+1)/2 + n + 1$ of the coefficient matrix. When put in row-echelon form, the first non-zero entry occurs in column $(n^2+n)/2$. Thus, this row is linearily independent of all previous vectors. The next column-sum vector, when put in row-echelon form, has its first non-zero entry in column $((n^2+n)/2 - 1)$. For all $n > 3$, this, too, is linearly independent. This pattern continues until we reach the $(n+1)/2^{nd}$ column vector, or row $(n^2+1)/2 + n + (n+1)/2$ in our coefficient matrix. This row is linearly dependent, being the sum of rows $(n+1)/2, ((n+1)/2 + n), \ldots, (n^2+1)/2$ of the coefficient matrix. Similarly, the remaining $(n-1)/2$ column-sum vectors are scalar multiples of the previous column-sum vectors (after they have been reduced to row-echelon form.)

Finally, we must consider the diagonals, but only briefly, for they are always linear combinations of the 1^{st}, $(n+2)^{nd}$, $(2n+3)^{rd}, \ldots,$ $(n^2+1)/2^{nd}$ and of the n^{th}, $(2n-1)^{st}$, $(3n-2)^{nd}, \ldots, (n^2+1)/2^{nd}$ rows of the coefficient matrix for the *NW-SE* and *SW-NE* diagonals respectively. Thus, the matrix has a total of $(n^2 + 2n - 1)/2$ linearly independent rows, and hence has rank $(n^2 + 2n - 1)/2$. By the rank and nullity theorem [2], the dimension of $SM_o(n)$ for odd n, which is the nullity of the coefficient matrix, is $n^2-(n^2 + 2n - 1)/2 = (n^2-2n+1)/2 = (n-1)^2/2$.

(Even case). Here we have $n^2/2$ pairs of skew-symmetric cells which have sum zero, as well as the $2n+2$ row, column, and diagonal sums. In addition, Theorem 3 gives us four more homogeneous equations, since we know each of the four "quadrant-sums" must be zero. Thus, we have a system of $n^2/2 + 2n + 6$ homogeneous equations in the n^2 unknowns m_{ij} $(1 \le i,j \le n)$.

Write these equations in the following order, called the <u>standard even order</u>: first, the $n^2/2$ skew-symmetric sums beginning with $m(1,1)+m(n,n)=0, m(1,2)+m(n,n-1)=0, \ldots, m(n/2,n/2)+m(n/2+1,n/2+1)=0$;

then the four "quadrant-sums," first the upper-left quadrant, followed by the upper-right, lower-left, and lower-right; then the $2n+2$ row, column, and diagonal sums, in exactly the same order as in the odd case. The resulting coefficient matrix is an $((n^2/2 + 2n + 6) \times n^2)$ matrix of 0's and 1's.

In the $n=4$ case, it is the 22 x 16 matrix shown in Figure 3, where the elements in the j^{th} column are the coefficients of the j^{th} variable in the list $m_{11}, m_{12}, m_{13}, \ldots, m_{43}, m_{44}$. Again, it is clear that the first $n^2/2$ rows of the coefficient matrix are linearly independent since they are in row-echelon form. Now we consider the next, i.e. $(n^2/2 + 1)^{st}$ row of the coefficient matrix. which corresponds to the equation specifying the upper-left quadrant-sum be zero, and which consists of $n/2$ consecutive groups of $n/2$ 1's followed by $n/2$ 0's, with the last $n^2/2$ entries being 0's.

1000	0000	0000	0001
0100	0000	0000	0010
0010	0000	0000	0100
0001	0000	0000	1000
0000	1000	0001	0000
0000	0100	0010	0000
0000	0010	0100	0000
0000	0001	1000	0000
1100	1100	0000	0000
0011	0011	0000	0000
0000	0000	1100	1100
0000	0000	0011	0011
1111	0000	0000	0000
0000	1111	0000	0000
0000	0000	1111	0000
0000	0000	0000	1111
1000	1000	1000	1000
0100	0100	0100	0100
0010	0010	0010	0010
0001	0001	0001	0001
1000	0100	0010	0001
0001	0010	0100	1000

Figure 3

In row-echelon form, this reduces to a row with $n^2/2$ 0's followed by $n/2$ consecutive groups of $n/2$ 0's followed by $n/2$ 1's. In other words, the row has been reflected left-to-right. This row has its first non-zero entry in the $n^2/2 + n/2 + 1 = ((n^2+n+2)/2^{nd}$ column, and hence is linearly independent of all previous vectors. The next row to consider is the $(n^2/2 + 2)^{nd}$ row of the coefficient matrix, the one corresponding to the equation specifying the entries in the upper-right quadrant to have sum zero. This row consists of $n/2$ consecutive groups of $n/2$ 0's followed by $n/2$ 1's, followed by $n^2/2$ 0's. Reducing to row-echelon form yields $n^2/2$ 0's, followed by $n/2$ consecutive groups of $n/2$ 1's followed by $n/2$ 0's. This row has its first non-zero entry in the $(n^2/2 + 1)^{st}$ column, and hence it, too, is linearly independent of all previous vectors.

The next two rows of the coefficient matrix, those corresponding to the equations specifying the lower-left and lower-right quadrants have sum zero are clearly multiples of the previous two rows. Of course, this is not surprising when we recall that $M \in SM_o(n)$, and hence, the skew-symmetry property implies that specifying the upper-left and upper-right quadrants completely determines the entries of the lower-right and lower-left quadrants respectively.

Now we consider the n rows corresponding to the equations specifying the row-sums of M be zero. The first row-sum vector contains n 1's followed by $n(n-1)$ 0's. Reducing to row-echelon form, we get another left-to-right reflection. Now we have $n(n-1)$ 0's preceding the n 1's. Since the first non-zero entry in the row is in the $n(n-1)^{st}$ column, we now have another linearly independent vector.

Things get a little more complex, though, when we consider the next row, which consists of n 0's, followed by n 1's, followed by $n(n-2)$ more 0's. When reducing this vector, we see it has its first non-zero entry in column n^2-2n+1. But notice in the $n=4$ case, that $n^2-2n+1=n^2/2 + 1$. Thus, since the vector associated with the equation specifying the upper-right quadrant-sum be zero has, in row echelon form, its first non-zero entry in column $n^2/2 + 1$, we know that we must reduce the vector further. When this is done, we find that the row in question is not linearly independent. In general, this process continues until we reach the $n/2^{nd}$ row-sum vector. (In the $n=4$ case, we have just considered this vector). The $n/2^{nd}$ row-sum vector is always equal to the sum of rows $\frac{n^2}{2} - n + 1$ through $n^2/2$ of the coefficient matrix, minus the row-echelon forms of the upper-left and upper-right quadrant-sum vectors, plus the row-echelon forms of the first $n/2 - 1$ row-sum vectors.

It is clear that the next $n/2$ row-sum vectors are not linearly independent, since by skew-symmetry they are simply the opposite of the first $n/2$ row-sum vectors. Thus, of the n row-sum vectors, only the first $n/2 - 1$ are linearly independent of one another and all previous vectors.

We must now consider the n column-sum vectors. It is lengthy, but not difficult to show that only the first $n/2 - 1$ of the n column-sum vectors are linearly independent of one another and all previous vectors. The $(n/2)^{nd}$ is a linear combination of the sum of rows $(n^2/2 - n/2)$, $(n^2/2 - 3n/2), \ldots, (n/2)$ of the coefficient matrix, plus the $(n/2 - 1)^{st}$ column-sum vector, minus the row-echelon form of the upper-left quadrant-sum vector. The last $n/2$ column-sum vectors are linearly dependent since skew-symmetry tells us they are the opposites of the first $n/2$ column-sum vectors. Thus, like the row-sum vectors, the column-sum vectors give us an additional $n/2 - 1$ linearly independent vectors.

It's also clear that both diagonal-sum vectors are linear combinations of previous vectors. The *NW-SE* diagonal is the sum of the 1^{st}, $(n+2)^{nd}$, $(2n+3)^{rd}, \ldots, (n^2/2)^{nd}$ rows of the coefficient matrix, while the SW-NE diagonal is the sum of the n^{th}, $(2n-1)^{st}$, $(3n-2)^{nd}, \ldots, (n^2/2)^{nd}$ vectors of the coefficient matrix.

Thus, we have the first $n^2/2$ skew-symmetric vectors, two quadrant-sum vectors, $(n/2 - 1)$ row-sum vectors, and $(n/2 - 1)$ column-sum vectors which are linearly independent. Thus, there are in all, $n^2/2 + n$ linearly independent vectors in the original coefficient matrix. By the rank and nullity theorem [2], the dimension of $SM_o(n)$ for even n is $n^2 - (n^2/2 + n) = n(n-2)/2$.

This result seems to correspond nicely to our intuitive geometric interpretation of the system. In the odd order case, we can divide the square array of cells into four $(n-1)/2 \times (n-1)/2$ squares surrounding a "central cross" region one cell thick. (See Figure 4). If we choose the entries in any two adjacent quadrants, skew-symmetry determines the entries in the other two quadrants. The central cross regions are used to make the appropriate row and column-sums zero, and skew-symmetry

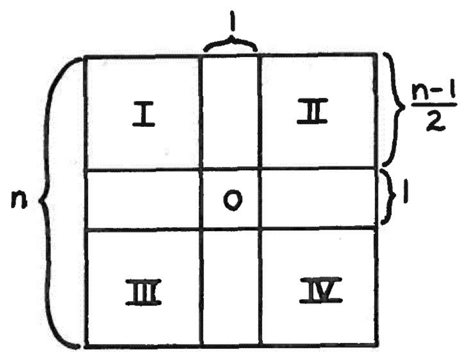

Figure 4

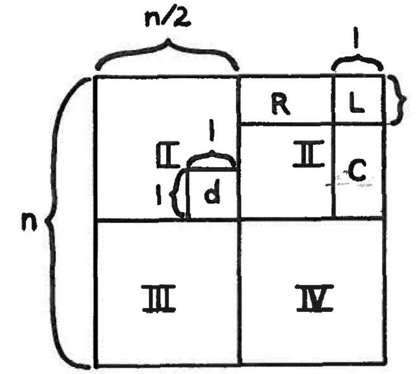

Figure 5

prevents any contradictions. Thus, we are free to choose the entries in exactly two quadrants, or $2((n-1)/2)^2 = (n-1)^2/2$ cells, which is exactly what we expect.

In the even case, we divide the square array into quadrants. (See Figure 5). If we start filling the cells, say in quadrant I, at random, we find we must choose the last cell, labelled d, so that it makes the quadrant-sum zero (Theorem 3). This, by skew-symmetry, completely determines the entries in quadrant IV. We may also fill any of the cells in the region labelled II at random, but that is all. The cells in the regions marked R, C, and L are now pegged so as to make the row and columns-sums zero. But square L seems somehow "strongly pegged," since it must satisfy both row conditions and column conditions. Can this lead to a contradiction? No! Taking the sum in the quadrant yields the equation $L = -(R+C+II)$. (Note, the capital letters stand for the sum of the entries in that particular region). Adding horizontally, we get the equation $L+C = -(I+II+R)$, which implies $L = 1(I+II+R+C)$. Adding vertically we get the equation $L+R = -(IV+II+C)$, which imples $L = -(IV+II+R+C)$. Thus, we have $L = -(R+C+II) = -(I + II + R + C) = -(IV+II+R+C)$. Are these equations always consistent? We see that they are when we remember that $I = IV = 0$, and so all the equations reduce to $L = -(II+R+C)$. Thus, we can choose $(n/2)^2 - 1 + (n/2 - 1)^2 = n(n-2)/2$ cells freely, which again corresponds to the dimension of the subspace.

We can use this result to get an upper bound of the number of regular, i.e. consisting of the numbers $1, 2, 3, \ldots, n^2$, skew-symmetric magic squares of order n. We know the sum of the first n^2 integers is $n^2(n^2+1)/2$, and hence any regular magic square has line-sum $l = n(n^2+1)/2$.

As a result, we can "zero" any regular skew-symmetric magic square by subtracting $1/n$ from each entry in the array. Since choosing the appropriate $(n-1)^2/2$ (odd case) or $n(n-2)/2$ (even case) cells completely determines the magic square, we know that the number of regular n^{th} order skew-symmetric magic squares cannot exceed the number of distinct ways the itegers $1, 2, 3, \ldots, n^2$ can be put into $(n-1)^2/2$ (odd case) or $n(n-2)/2$ (even case) cells. The number of permutations of size k that can be chosen from a set of n elements is $\binom{n}{k}k! = \frac{n!\,k!}{k!(n-k)!} = \frac{n!}{(n-k)!}$.

Hence, when we take into account the eight symmetries of the square, our upper bounds are $\frac{(n^2)!}{8(n^2-(n-1)^2/2)!}$ for odd n, and $\frac{(n^2)!}{8(n^2-n(n-2)/2)!}$ for even n. We get some idea of how crude this upper bound is, however, when we examine the $n=3$ case. Our formula suggests that there exist no more than nine regular skew-symmetric magic squares of order 3, when, in fact, it is easy to show there is only one.

The theorems presented in this paper can be easily extended to magic figures of three and more dimensions. Indeed, the proof that the subspace of odd order skew-symmetric zero-sum magic k-boxes has dimension $(n-1)^k/2$ is especially straightforward, following almost directly from simple generalizations of Theorems 1, 2, and 4.

I close with a question. In this paper I have studied but one subspace of $M_o(n)$. What is the dimension of the subspace spanned by the set of "pandiagonal" zero-sum magic squares? (Pandiagonal squares have the property that all the broken diagonals also have the magic sum). And what about the subspace of squares that are both skew-symmetric and pandiagonal?

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A GEOMETRIC INTERPRETATION OF SYMMETRIC MATRICES

bq *Denis* R. *Floyd*

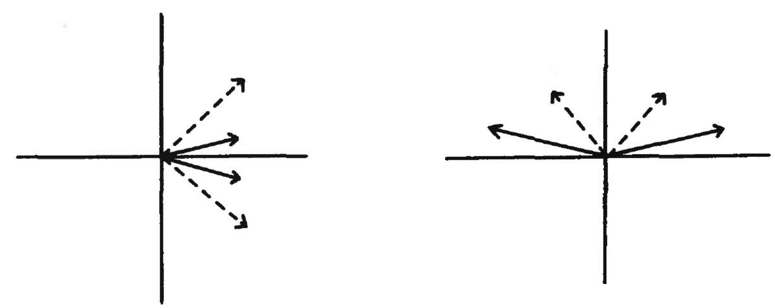
The purpose of this note is to give a geometric interpretation of real symmetric matrices which might be used to motivate the study of this subject in linear algebra courses.

Notation and Preliminary Remarks. Let R^n denote Euclidean n-space. If $B=\{b_1,b_2,\ldots,b_n\}$ is an orthonormal basis for R^n, then for each i, the one-dimensional subspace Rb_i will be called an axis. The statement that vectors v_1 and v_2 are symmetric with respect to one of the axes, say Rb_1, will mean that the coordinates of v_1 and v_2 are of the form $(x_1,x_2,\ldots,x_n)$ and $(x_1,-x_2,-x_3,\ldots,-x_n)$, respectively.

The following is the key definition.

Definition 1. A linear transformation T from R^n to itself will be called symmetric if there exists an orthonormal basis B such that if two vectors v_1 and v_2 are symmetric with respect to one of the axes, then the images $T(v_1)$ and $T(v_2)$ are also symmetric with respect to the same axis.

The definition is illustrated in the diagram below for the two-dimensional case.



The following well-known fact is crucial in the theorem which follows.

Lemma. Let A be a real symmetric matrix (i.e., $m_{ij} = m_{ji}$ for all i,j). Then there exists real orthogonal matrix U such that $U^{-1}AU$ is diagonal.

Proof. See, for example, [1], page 243, Theorem 8.25.

We now state the main result.

Theorem. A linear transformation T from R^n to R^n is symmetric if and only if T is representable by a symmetric matrix with respect to some orthonormal basis B.

Proof. Let T be symmetric, with corresponding orthonormal basis $B = \{b_i\}$. For each i, the choice $v_1 = v_2 = b_i$, Definition 1 implies that $T(b_i) = a_i b_i$, for some scalar a_i. Thus, the matrix of T with respect to B is diagonal, hence symmetric. Conversely, suppose T has symmetric matrix A with respect to orthonormal basis B. By the lemma, there exists real orthogonal matrix U such that $D = U^{-1}AU$ is diagonal. Now D is the matrix of T with respect to an orthonormal basis C. It is then easy to see that T is symmetric, with C the associated orthonormal basis. This concludes the proof.

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LINES AND PARABOLAS IN TAXICAB GEOMETRY

by Joseph M. Moser
and Fred Kramer
San Diego State University

In [1] Reynolds discusses various conic sections in taxicab geometry. She leaves open questions of the meaning of the shortest distance from a point to a line and parabolas.

It is the purpose of this paper to answer these questions. To this end, we state and prove two theorems.

Theorem 1. The shortest distance from a point (x_1, y_1) to a line $Ax + By + C = 0$ in Taxi-cab Geometry is the horizontal distance from the point to the line if $1 < -A/B < \infty$ or $-\infty < -A/B < -1$ and the vertical distance from the point to the line if $0 < -A/B < 1$ or $-1 < -A/B < 0$. If $|-A/B| = 1$, either distance will do. The slope of the line is $-A/B$.

Proof. Case 1. $1 < -A/B < +\infty$ or $0 < -A/B < 1$. Figure 1 indicates that for ℓ_1, $1 < -A/B < +\infty$ and ℓ_2 has the property that $0 < -A/B < 1$. Consider ℓ_1. We wish to show that $k + g < q + k$, or $g < q$. Since $\alpha < 45°$, it is obvious that $k + g < e + c + r$. Again since $\alpha < 45°$, $g < q$, as we wished to show. If ℓ_2 is considered, the same arguments apply in showing the $e + c < e + b$. In this case α' is used.

Case 2. $-\infty < -A/B < -1$ or $-1 < -A/B < 0$. Figure 2 indicates that for ℓ_1, $-\infty < -A/B < -1$ and ℓ_2 has the property that $-1 < -A/B < 0$. The arguments are similar to Case 1.

The next problem is to describe a parabola in "taxicab" geometry. A parabola is the set of points equidistant from a given point (x_0, y_0) called the focus and a given line $y = ax + b$ called the directrix. Since taxicab distances are invariant under translations, we can assume the directrix passes through the origin, that is $b = 0$.

There are essentially four cases we have to consider:

$$\text{(i)} \quad y_0 > ax_0 \qquad |a| < 1$$

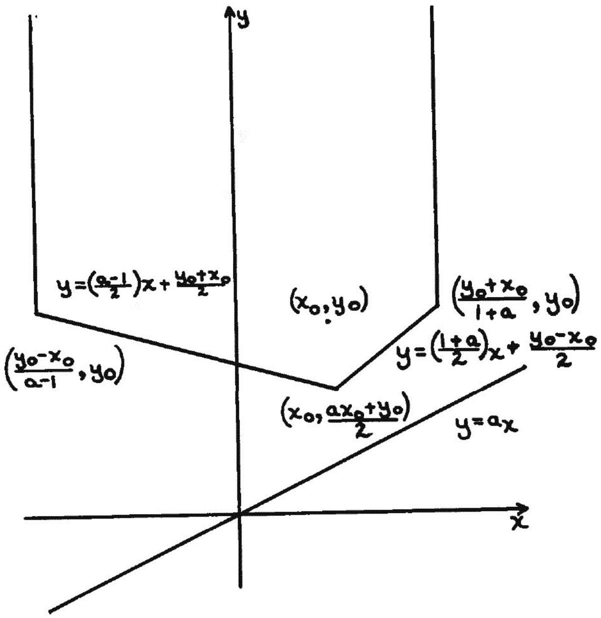

Figure 1.

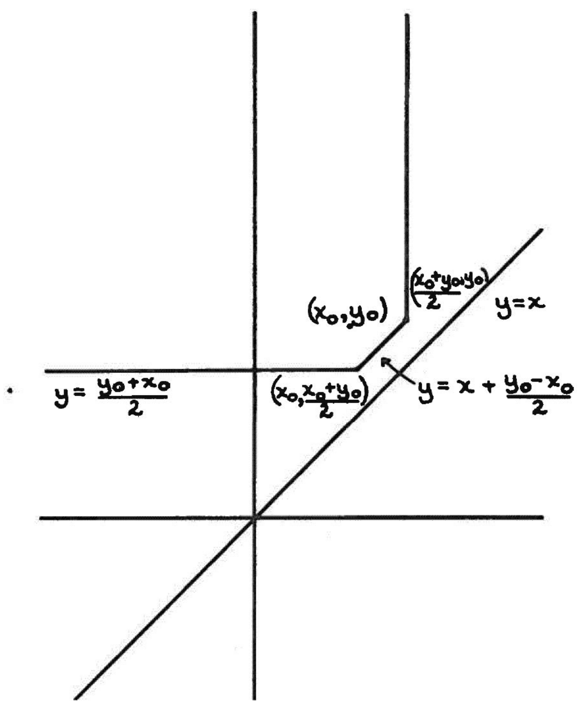

Figure 2.

(ii) $y_o < ax_o \quad |a| < 1$

(iii) $y_o > ax_o \quad |a| > 1$

(iv) $y_o < ax_o \quad |a| > 1.$

Note that we really only need to solve case (i). For cases (ii) - (iv) we can apply the following transformations to redefine the problem as a case (i) problem in a new μ - v coordinate system, use the case (i) solution, and apply the inverse transformation to get the solution defined in terms of the original coordinates. The required transformations are:

case (ii) $\mu = x, \quad v = -y$

case (iii) $\mu : y, \quad v : -x$

case (iv) $\mu = -y, \quad v = -x$

<u>*Theorem 2.*</u>

The parabola in "taxicab" geometry with focus (x_o,y_o) and directrix $y : ax$, with $y_o > ax_o$ and $|a| < 1$, is as described in Figure I.

Proof. Note that although Figure I is drawn with $a > 0$, the solution does not depend on this fact. Note also, that because $|a| < 1$, the taxicab distance between any point (x,y) and the line is just $|y - ax|$, that is, the distance is parallel to the y-axis.

First it is clear that the parabola must be on the same side of the line as the point. So since $y_o > ax_o$, the parabola must be above the line $y = ax$.

However, the parabola cannot be described by a single equation as in the Euclidean case. Therefore, the derivation is done for various regions of the plane (above $y : ax$, of course).

1. <u>$x = x_o, ax_o < y < y_o$</u>

It is clear that the point on the parabola at $x : x$ must lie halfway between the directrix, $y = ax$ and the focus (x_o, y_o) vertically. If we let d_p represent the distance between (x,y) and the point (x_o, y_o) and d_ℓ the distance between (x,y) and the line $y : ax$, we have

$$d_p = |y_o - y| + |x_o - x|$$

$$= |y_o - y| + |x_o \quad x_o| \qquad \text{since } x = x$$

$$= y_o - Y \qquad \text{since } y > y;$$

and

$$d_\ell = |y - ax|$$

$$= |y - ax_o|$$

$$= y - ax \qquad \text{since } y > ax.$$

The point (x,y) is on the parabola if and only if $d_p = d_\ell$. That is, iff

$$y_o - y = y - ax_o$$

iff $$y = \frac{y_o + ax_o}{2}.$$

2. $x > x_o$ and $ax < y \leq y_o$

In this case we have

$$d_p = |y_o - y| + |x_o - x|$$

$$= y_o - y + x - x_o;$$

and $d_\ell = y - ax$.

so $d_p = d_\ell$

iff $y_o - y + x - x_o = y - ax$

iff $$y = \left(\frac{1+a}{2}\right)x + \frac{y_o - x_o}{2}.$$

This is the equation of a line with positive slope. It is part of the parabola only as long as it is less than or equal to y_o. To find the largest x such that this holds, we solve for x:

$$y_o = \left(\frac{1+a}{2}\right)x + \frac{y_o - x_o}{2},$$

and get $$x = \frac{y_o \quad x_o}{1 + a}.$$

3. $x > \dfrac{y_o + x_o}{1 + a}$

In this case,

$$d_p = |y_o - y| + |x - x_o|$$

$$= |y_o - y| + \left(x - \frac{y_o + x_o}{1 + a}\right) + \left(\frac{y_o + x_o}{1 + a} - x_o\right);$$

and

$$d_\ell = y_o - \left(\frac{y_o + x_o}{1 + a}\right)a - a\left(x - \frac{y_o + x_o}{1 + a}\right) + (y - y_o).$$

Since $\left(\dfrac{y_o + x_o}{1 + a}\right), y_o$ is on the parabola,

$$\frac{y_o + x_o}{1 + a} - x_o = y_o - \left(\frac{y_o + x_o}{1 + a}\right)a.$$

We also have, $y - y_o \leq |y_o - y|$.

and $-a\left(x - \dfrac{y_o + x_o}{1 + a}\right) < x - \dfrac{y_o \quad x_o}{1 + a}$ since

$|a| < 1$ and $x - \dfrac{y_o \quad x_o}{1 + a} > 0$.

Therefore in this region we have

$$d_\ell < d_p.$$

Hence, no points belong to the parabola.

4. $x < x_o$ and $ax < y \leq y$

Here we have

$$d_p = |y_o - y| + |x_o - x|$$

$$= y_o - y + x_o - x$$

and $d_\ell = y - ax$.

so

$$d_p = d_\ell$$

$$\Longleftrightarrow \quad y_o - y + x_o - x = y - ax$$

$$\Longleftrightarrow \quad y = \left(\frac{a-1}{2}\right)x + \frac{y_o + x_o}{2}$$

This is a line with negative slope and is part of the parabola as long as it is less than or equal to y_o. To find the smallest x such that this holds, we solve for x in

$$y_o = \left(\frac{a-1}{2}\right)x + \frac{y_o + x_o}{2}$$

and get

$$x = \frac{y_o - x_o}{a-1}$$

5. $x < \dfrac{y_o - x_o}{a - 1}$

In this case,

$$d_p = |y_o - y| + |x_o - x|$$

$$= |y_o - y| + x_o - \frac{y_o - x_o}{a - 1} + \frac{y_o - x_o}{a - 1} - x \; ;$$

and

$$d_\ell = y_o - \left(\frac{y_o - x_o}{a - 1}\right) a + a\left(\frac{y_o - x_o}{a - 1} - x\right) + y - y_o \; .$$

Since, $\left(\dfrac{y_o - x_o}{a - 1}, y_o\right)$ is on the parabola,

$$\frac{y_o - x_o}{a - 1} - x = y_o - \left(\frac{y_o - x_o}{a - 1}\right) a,$$

We also have $y - y_o \le |y_o - y|$, and $+a\left(\dfrac{y_o - x_o}{a - 1} - x\right) < \left(\dfrac{y_o - x_o}{a -} - x\right)$

since $|a| < 1$ and $\dfrac{y_o - x_o}{a - 1} - x > 0$.

Therefore, in this region we have

$$d_\ell < d_p \; ,$$

and no points belong to the parabola.

6. $\dfrac{y_o - x_o}{a - 1} \le x < x_o$ and $y > y_o$

In this case,

$$d_p = |y - y_o| + |x - x_o|$$

$$= y - y_o + x_o - x \; ;$$

and

$$d_\ell = y - ax.$$

So,

$$d_p = d_\ell$$

iff $\quad y - y_o + x_o - x = y - ax$

iff $\quad x = \dfrac{y_o - x_o}{a - 1}$.

So the points on the parabola in this region are the half-line

$$x = \frac{y_o - x_o}{a - 1} \quad \text{for } y > y_o.$$

7. $x \le x < \dfrac{y_o + x_o}{\cancel{1 + a}}$ and $y > y$

In this case

$$d_p = |y_o - y| + |x_o - x|$$

$$= y - y + x - x_o;$$

and

$$d_\ell = y - ax.$$

So,

$$d_p = d_\ell$$

$$\Longleftrightarrow \quad y - y + x - x = y - ax$$

$$\Longleftrightarrow \quad x = \frac{y_o + x_o}{1 + a} \; .$$

So, the points on the parabola in this region are the **halfline**

$$x = \frac{y_o + x_o}{1+a} \text{ for } y > y .$$

Special Case $a = 1$

The taxicab geometry parabola has a different configuration when the directrix $y = ax$ is such that $|a| = 1$. The description again depends on whether $y > ax$ or $y_o < ax$. These cases can easily be derived as limiting situation of the cases considered above. Figure 2 illustrates the case with $y_o > ax$ and $a > 0$.

REFERENCE

1. Reynolds, Barbara E., "Taxicab Geometry," *Pi Mu Epsilon Journal*, Vol. 7, No. 2, 77-88.

A GUIDE FOR TEACHING MATHEMATICS

Author. Unknown

It is the responsibility of the teacher to actively involve his or her students in the learning process. The most important thing he or she should do is to avoid giving clear, concise, organized lectures. If the presentation of a lesson is too easy to follow, most of the class will not need to learn the new material on their own. They will have a certain degree of confidence in their new knowledge, and this will tend to stifle their intellectual pursuits. If, on the other hand, the lecture is vague, rambling and disorganized, the students will leave with their heads full of questions. In fact, they will be so filled with curiosity that they will try to expand their knowledge on their own.

There are many ways to present a thought provoking lecture. One of the easiest techniques to use is a foreign accent. If the accent is thick enough, even a well organized lecture will produce expressions of intellectual wonder among the students. Effective accents can be acquired in Alabama, China, India, Latin America, New York City, Germany, or any foreign country.

For natives of Kansas, that is, for individuals who cannot speak anything but perfect Midwestern English, this technique may offer difficulties. 'There are two possible solutions: (1) One can teach in a foreign country, or at least in New York or Texas; or (2) One can incorporate a new syllable into one's language. Two very effective syllables to use are "um" and "uh". The chosen syllable should be uttered every second or third word. This reduces the possibility that any coherent concept will be given to the class. For example, one can say, "Um, today, u..m, we will be, um, discussing, um....um, determinants." After a couple of sentences, most of the class will be staring at their watches or out the windows. Very quickly, they will become very anxious to go out and learn the material on their own.

In addition, to being aware of one's own speech patterns, the teacher should also pay close attention to the written word. Effective use of the blackboard should be considered almost a necessity. Illegible handwriting can stimulate a student's interest in new material almost as effectively as incoherent lectures. Often students will meet outside of class to exchange interpretations of lecture notes. Thus illegible handwriting encourages students to work together and share ideas.

Writing illegibly requires a great deal of practice to be effective. If one does not have satisfactory handwriting (that is to say, if one's handwriting is suitable only for formal invitations and eye charts), certain "tricks" can be learned:

1. Write small. For students in the back rows, this is almost as effective as writing illegibly. The disadvantage is that students in the front rows will probably be able to read the board and may possibly learn something without having to spend hours interpreting their notes. Also, the professor who writes small may find that most of his or her class will try to sit near the front of the room, which may be too close for comfort, especially on hot days during summer sessions.

2. Write fast. The faster the teacher writes, the faster the students will have to take notes. Often the teacher can move on to a new subject while his or her students are still trying to copy what is on the board. Students will be so busy during class that they will wait until after class to try to understand the lesson. In addition to spurring students to learn on their own, writing fast allows the professor to cover more material in a given class period.

3. Write something while saying something different. For example, after working out a lengthy problem the instructor tells the class the answer is $x^2 + y$ while writing on the board $y^2 + x$. This forces students to rethink the problem in order to decide which alternative is correct. Students are thus actively involved in problem solving even after the problem is finished.

4. Erase quickly. This technique practically forces those members of the class who take notes to pay constant attention to the lectures. Those who doze off for a few moments will awaken to find nothing to record in their notes on the topics they missed. This technique is especially effective if one uses both hands to write and erase simultaneously.

5. If all else fails, stand in front of what has just been written. By blocking any clear view of the blackboard, the teacher will help improve students' speculative and psychic abilities. Those instructors who are short or underweight may find this procedure extremely difficult.

The above "tricks" may be used separately or combined. It is a good idea to change them occasionally in order to add some variety to the classroom routine.

It is very important that the professor lecture to the blackboard when using it. This helps demonstrate to students how involved the teacher is with the subject. This enthusiasm will most assuredly rub off on the class. Also, by facing the blackboard, one cannot face the class. It is therefore easier to ignore students' questions which tend to interrupt the presentation of topics and make the class period seem to last forever.

There is one last point on teaching technique. It is important that one does not overprepare for lectures. Generally, one should arrive at class a few minutes early, open the book, and glance at the topic for that particular day. Lectures prepared in this manner have a certain freshness and spontaneity that is often missing from those which are more carefully organized. In addition, students will gain a greater appreciation for a correct proof if they see how much time can be spent on a wrong approach.

The first section of this guide has dealt with actual teaching, concentrating on lecturing "tricks", techniques, and preparation. The subject of the last part will be general appearances.

Students tend to have more confidence in an instructor if they believe he or she has a thorough understanding of his or her field. To show a class that one has a thorough understanding of mathematics, it is necessary to appear "spaced-out." Being "spaced-out" implies one is so involved with abstract mathematics that one has lost touch with the real world. There are several ways to project such an image.

1. Dress funny. Old suits, baggy pants, narrow ties, and hairy sweaters are all effective and even more so, when worn together.

2. Don't wash your sweatshirts. Albert Einstein is best remembered for two things -- being a genius and wearing dirty sweatshirts. Even if you are not a genius, you can still wear the sweatshirts. In a matter of weeks, you will gain such a reputation that no one will come near enough to challenge it.

3. Don't comb your hair with anything finer than your left hand.

4. Walk into the wrong room and begin to lecture to whatever class is in

it. (This will help spread your reputation beyond your own students).

5. Walk into the correct classroom and begin lecturing on whatever happens to be left on the blackboard from the previous class.

6. Acquire a facial twitch.

7. Pretend you are deaf if someone asks a question or the bell rings while you are lecturing. Try to keep talking after everyone has left the room.

8. Follow all the guidelines for teaching given above.

By being properly "spaced-out", one will gain the confidence and respect of one's students. This will make it easier to help inspire them in their study of mathematics. Being properly "spaced-out" will also help one to acquire tenure at this or any other reputable college or university.

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This article was found in an old milk bottle on the back steps of the Editors home. We know that it has been published elsewhere and would like to give the proper credit. Please let us know the Author and the original publication.

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SEVENTH CONFERENCE ON UNDERGRADUATE MATHEMATICS

April 15–16, 1983

Oklahoma Beta Chapter of Pi Mu Epsilon will assist the Mathematics Department at Oklahoma State University and the Journal of Undergraduate Mathematics in sponsoring a Conference on Undergraduate Mathematics. Fifteen minute presentations are solicited from undergraduates regarding their mathematical interests: expository papers, discussions of applications of mathematics, reports on their mathematical research. In addition, the conference will feature as invited speakers: George Piranian of the University of Michigan, C. H. Edwards, Jr. of the University of Georgia, Lynn 0. Wilson of Bell Laboratories, Reuben Hersh of the University of New Mexico. Inquiries or abstracts should be sent to Tim Koster, President of Okla. Beta Pi Mu Epsilon, Department of Mathematics, Oklahoma State Univ. Stillwater, OK, 74078

BOW-TIE POLYGONS
A SEQUEL TO THE EQUILIC QUADRILATERAL

by Clayton M. Dodge, University of Maine and Jock Garfunkel, Queen's College, N. Y.

In the January 1981 issue of The Mathematical Monthly, the following problem (E 2866) appeared: Let AKL, AMN, be equilateral triangles. Prove that the equilateral triangles LMX, NKY are concentric**(if Y is chosen on the proper side of NK). Drawing the figure described by the above problem, led us to defining a new quadrilateral. Because of its appearance, we decided to call this quadrilateral the Bow-Tie quadrilateral.

Definition 1: A quadrilateral ABCD is said to be a Bow-Tie quadrilateral if there exists a point P (not necessarily inside the quadrilateral) such that PBC and PDA are similarly oriented equilateral triangles, whose interiors do not intersect. See Figure 1.

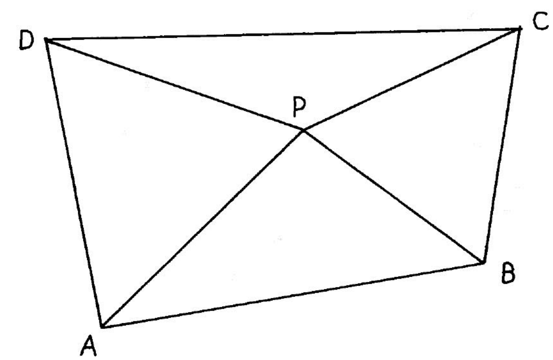


Figure 1.

We prove some basic theorems about this quadrilateral and then investigate some interesting generalizations.

** Two triangles are concentric if they have the same circumcenter.

Theorem 1. *The equilateral triangles PBC and PDA are congruent **if** and only **if** the bow-tie quadrilateral ABCD is an isosceles trapezoid.*

Proof. If the triangles are congruent, we have AD = BC and angle DAB = angle CBA because triangle PAB is isosceles. See Figure 1. Because the sides AD and BC are equal and the base angles at A and B are equal, it follows that the opposite sides AB and DC are parallel. The converse follows readily from the symmetry of an isosceles trapezoid.

Theorem 2. *The diagonals of a bow-tie quadrilateral are equal and meet at an angle of $\pi/3$.*

Proof. Since PB = PC and PD = PA, a rotation about P through angle $\pi/3$ carries triangle PBD to PCA. See Figure 2. The theorem follows.

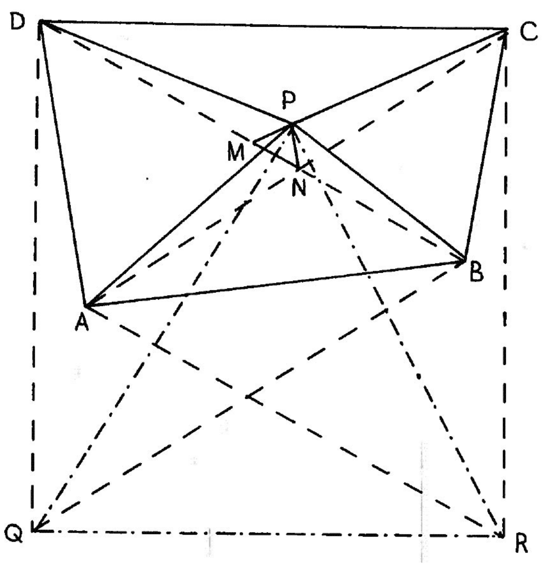


Figure 2

Theorem 3. *If similarly-oriented equilateral triangles QBD and RCA are erected on the diagonals BD and AC of bow-tie quadrilateral ABCD, then PQR is an equilateral triangle.*

Proof. The rotation about P through angle $\pi/3$ carries BD to AC, so it carries triangle QBD to RCA, as shown in Figure 2. Since it also maps PB to PC, it maps triangle PBQ to PCR. Hence, PQ = PR and angle QPR = $\pi/3$, so triangle PQR is equilateral.

Theorem 4. *Point P, along with the midpoints M of diagonal DB and N of diagonal AC, form an equilateral triangle.*

Proof. Triangle PMS is equilateral because the rotation about P through angle $\pi/3$ carries BD to AC; hence it carries M to S. See Figure 2.

Theorem 5. *If M_1, M_2, M_3, and M_4 are the midpoints of the sides AB, BC, CD, and DA of bow-tie quadrilateral ABCD, then triangles $M_1M_2M_3$ and $M_3M_4M_1$ are congruent equilateral triangles.*

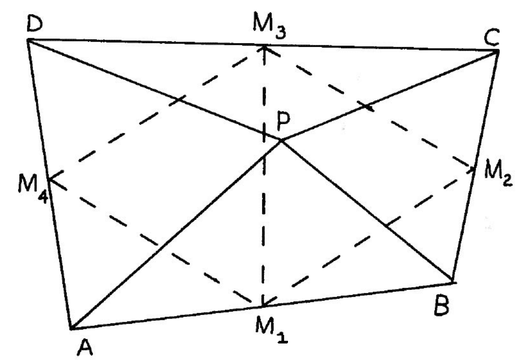


Figure 3

Proof. Refer to Figure 3. From triangle DAC we see that M_4M_3 is half of and parallel to AC. Similarly, M_4M_1 is half of and parallel to BD, so that $M_4M_3 = M_4M_1$ and angle $M_3M_4M_1 = \pi/3$.

Theorem 6. *Erect similarly-oriented triangles RAC and QAB on diagonal AC and side AB of bow-tie quadrilateral ABCD. Then DQR is an equilateral triangle congruent to PBC.*

Proof. The rotation about point A through angle $\pi/3$ carries triangle ABC to AQR and triangle APB to ADQ. Thus it carries triangle PBC to DQR, as shown in Figure 4.

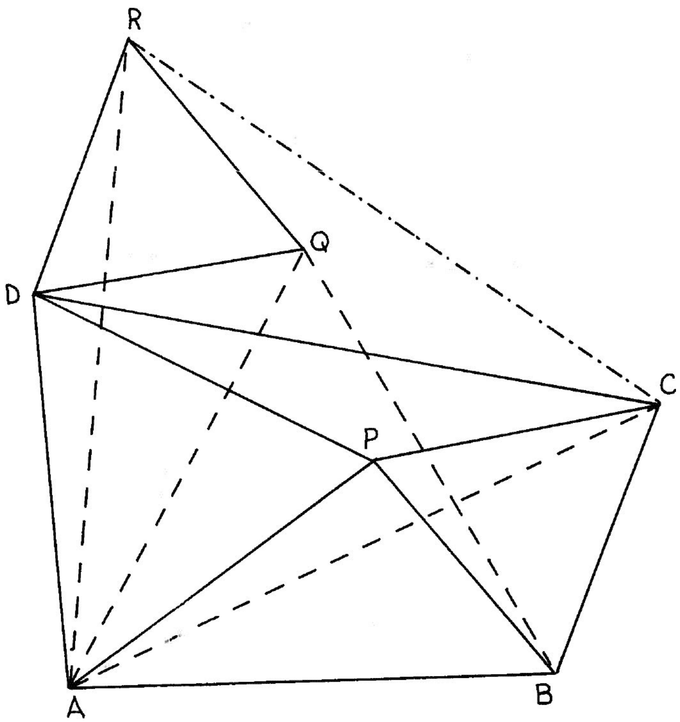


Figure 4

Theorem 7. If counterclockwise equilateral triangles ABQ, CDR, and DSA are constructed on sides AB, CD, and DA of counterclockwise bow-tie quadrilateral ABCD, then triangle QSR is equilateral. Furthermore, triangle QSR is homothetic* to triangle $M_4M_1M_3$ of theorem 5, with ratio 2:1 and center P.

Proof. The rotation about D through angle $\pi/3$ carries triangle DAR to DPC, so $AR = PC$. See Figure 5. It follows that AR and PB are equal and parallel. Hence, PR and AB bisect one another at M_1. Similarly, PQ and DC bisect each other at M_3. Because APDS is a rhombus, PS and DA bisect each other at M_4. Thus P is the center of the homothety of ratio 2:1 that carries $M_1M_3M_4$ to RQS.

* If corresponding sides of 2 similar polygons are parallel, the 2 polygons are said to be similarly placed or homothetic.

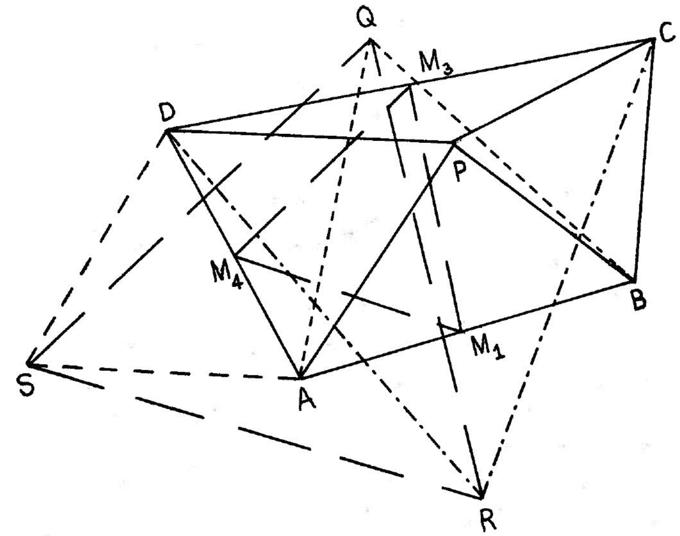


Figure 5

Theorem 8. If counterclockwise equilateral triangles GAB and FCD are erected on sides AB and CD of counterclockwise bow-tie quadrilateral ABCD, then these triangles have the same circumcenter.*

Proof. We use complex coordinates, as in Figure 6. Taking the

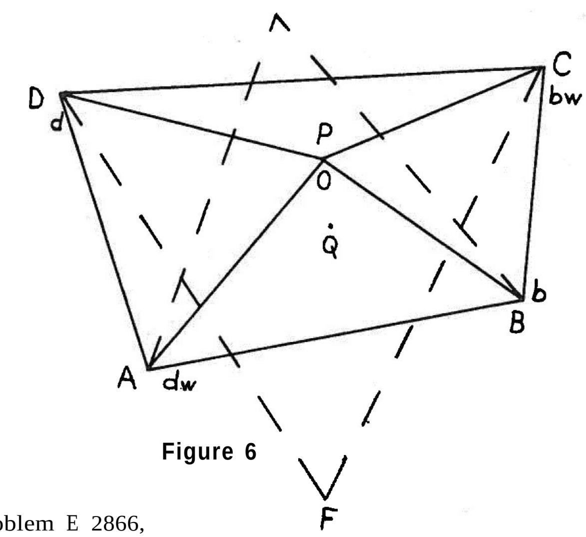


Figure 6

* This theorem is problem E 2866,

origin at P, point D with affix d, point B with affix b, and letting $\omega = \text{cis}(\pi/3)$. Then, $a = d\omega$ and a = bu. Now $g = a + (b - a)\omega = d\omega + (b - d\omega)\omega$, and $f = c + (d - c)\omega = b\omega + (d - b\omega)\omega$. The centroid of triangle ABG has affix

$$t = \frac{a + b + g}{3} = \frac{d\omega + b + d\omega + (b - d\,\omega)\omega}{3} = \frac{b + b\omega + d\,\omega + d}{3},$$

since $\omega - \omega^2 = 1$. The centroid of triangle CDF has the affix

$$u = \frac{c + d + f}{3} = \frac{b\omega + d + b\omega + (d - b\omega)\omega}{3} = \frac{b\omega + b + d + d\omega}{3} = t,$$

and the theorem is established.

We have by no means exhausted the properties of the bow-tie quadrilateral. Surely, the reader will be able to add a few of his own. Before leaving this quadrilateral, it is interesting to note that if the 2 equilateral triangles, PBC, PDA are drawn so their interiors intersect, we have a "western style" bow tie, the simple quadrilateral $DACB$. See Figure 7. The previously proved theorems will hold for this quadrilateral.* Thus, for example, where before we had the diagonals equal, we now have equal sides AC = BD.

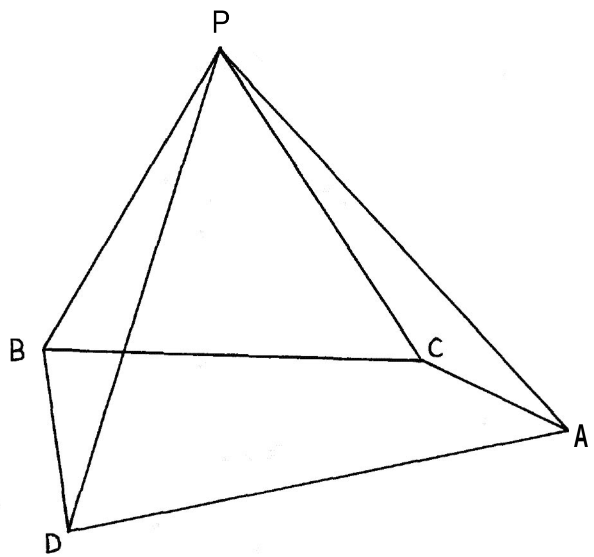


Figure 7

* See "The **Equilic** Quadrilateral", Jack Garfunkel, **Pi Mu Epsilon Journal, Vol. 7,** No. 5, Fall 1981.

Of the many possible generalizations, we outline the case of similar isosceles triangles PBC and PDA. Thus we replace "equilateral triangles" by "similar isosceles triangles having their apex angles at P" in Definition 1 and the theorems that follow. Theorem 1 and its proof are still valid.

Theorem 2 generalizes thus: A simple quadrilateral ABCD has equal diagonals if and only if there is a point P such that triangles PBC and PDA ace directly similar and isosceles. If $\omega/3$ is replaced by the measure of the apex angle of triangle PDA, the stated proof of theorem 2 holds for this generalization, so we need only prove the converse part. So, assume that the diagonals AC and BD are equal and let the perpendicular bisectors of sides AD and BC meet at a point P. Draw PA, PB, PC, and PD. Then triangle APC is congruent to DPB by s.s.s., so angle APC = angle DPB and angle APD = angle CPB. Hence the isosceles triangles APD and BPC are similar.

For Theorem 3 generalization we require only that triangles QBD and RCA be directly similar. Then, letting the rotation be through the apex angle of triangle PDA, the stated proof then shows that triangle PQR is directly similar to PDA.

Generalized Theorem 4 has the conclusion that triangle PUS is similar to PDA.

The conclusion to generalized Theorem 5 is that triangles $M_4M_1M_3$ and $M_2M_3M_1$ are directly similar to triangle PDA.

For the Theorem 6 generalization we let RAC and QAB be directly similar to PBC. The proof then shows triangle DQR directly similar to PBC when the rotations are changed to rotation-homotheties through the base angle of triangle PDA and of ratio PA/DA.

The restated Theorem 7 requires that triangles RCD, QAB, and SAD be directly similar to triangle PDA. Then triangle SRQ is homothetic to $M_4M_1M_3$ with ratio 2:1 and center P and directly similar to PDA.

An appropriate generalization for Theorem 8 states: If PDA and PBC are directly similar isosceles triangles with apex angles at P and of measure ω, then the directly similar isosceles triangles of apex angle IT– ID, erected on AB and CD and oriented the same as PDA share a common apex vertex.

The theorem is seen to be self-inverse and Theorem 8 is the special case where $\omega = \pi/3$. To prove the theorem, note that the rotation about P through angle ω maps triangle PBD to PCA, as in Theorem 2. Let Q be the point of intersection of the perpendicular bisectors of sides AB and CD. Then $QA = QB$, $QC = QD$, and, as we have seen already, $AC = BD$, so triangles QCA and QDB are directly congruent. We also see that the rotation about Q through angle $\pi-\omega$ maps triangle QCA to QDB. Hence, isosceles triangles QAB and QCD have apex angles of measure $\pi-\omega$.

For a different generalization we now replace the two equilateral triangles hinged at the point P with two squares also hinged at point P, obtaining the hexagon of Figure 8. It is interesting to see what properties of the bow-tie quadrilateral carry over.

Definition 2. If $ABCD$ and $AEFG$ are similarly oriented squares, then $BCDEPG$ will be called a **bow-tie hexagon**.

Theorem 9. Diagonals EB and DG of bow-tie hexagon $BCDEPG$ are equal and perpendicular. The proof of this theorem is left to the reader.

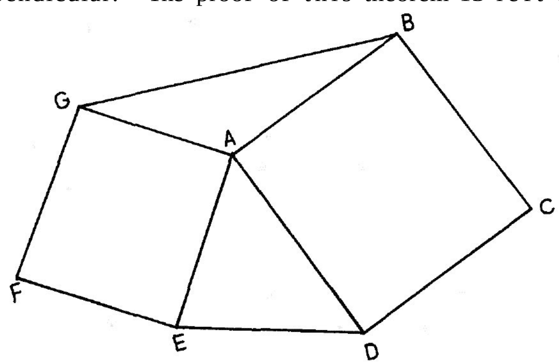


Figure 8

Theorem 10. The three main diagonals FC, EB, and DG of bow-tie hexagon $BCDEFG$ are concurrent.

Proof. Take the origin of the complex plane at A, as in Figure 9. Let G have the affix g and D have the affix d. Then $e = gi$, $f = g + gi$, $b = di$, and $c = d + di$. Line DG has the parametric equation $z = (g - d)t + d$, BE has $z = (g - d)it + di$, and FC has $z = (g + gi - d - di)t + d + di = (1+i)[(g - d)t + d]$. Now three lines $z = a_k t + b_k$ $(k = 1,2,3)$ are concurrent or all parallel iff

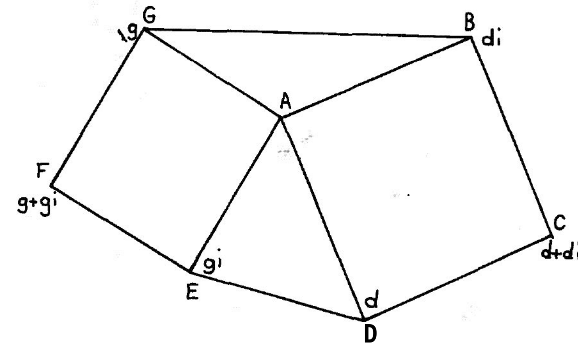


Figure 9

$$\begin{vmatrix} a_1 & \bar{a}_1 & \bar{a}_1 b_1 - a_1\bar{b}_1 \\ a_2 & \bar{a}_2 & \bar{a}_2 b_2 - a_2\bar{b}_2 \\ a_3 & \bar{a}_3 & \bar{a}_3 b_3 - a_3\bar{b}_3 \end{vmatrix} = 0.$$

Applying this theorem to the given lines, we get

$$M = \begin{vmatrix} g - d & \bar{g} - \bar{d} & (\bar{g} - \bar{d})d - (g - d)\bar{d} \\ (g - d)i & -(\bar{g} - \bar{d})i & -(\bar{g} - \bar{d})idi - (g - d)i(-\bar{d}i) \\ (g-d)(1+i) & (\bar{g} - \bar{d})(1 - i) & (\bar{g} - \bar{d})(1 - i)d(1+i) - (g - d)(1+i)\bar{d}(1-i) \end{vmatrix}.$$

It is easy to check that the third row is the sum of the first two rows so that M = 0, and the theorem follows.

Theorem 11. If in the clockwise bow-tie hexagon $BCDEFG$, clockwise squares $DEHK$ and $GBML$ are dwawn, then these two squares have the same center.

Proof. See Figure 10. Consider the bow-tie quadrilateral $GEDB$ and apply the Theorem 8 generalization.

Theorem 12. The centers of the squares $ABCD$ and $AGFE$ and the midpoints of the sides ED and BG are vertices of a square.

Proof. See Figure 11. Theorem 5 generalized applied to $GEDB$ establishes this result.

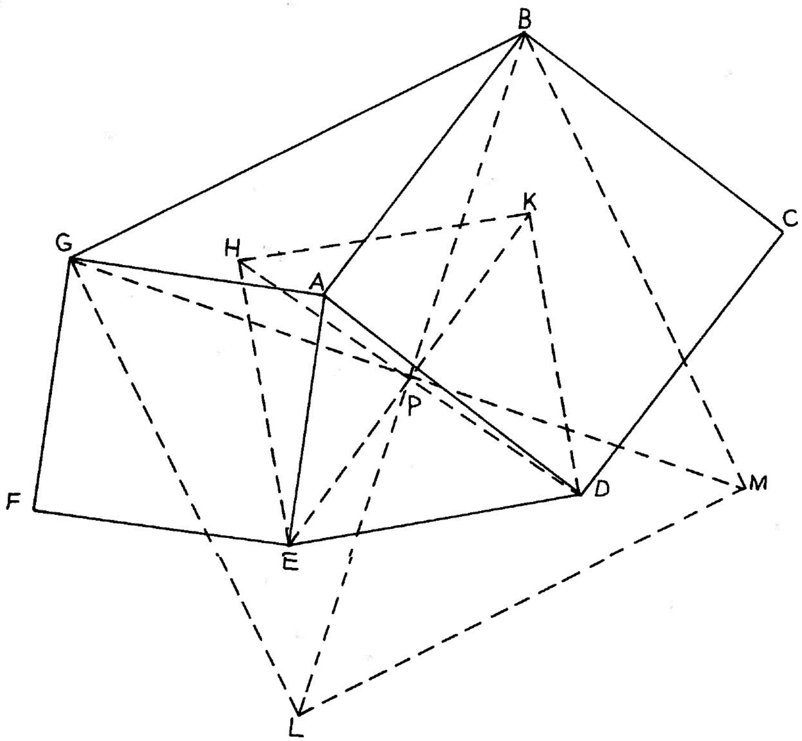


Figure 10

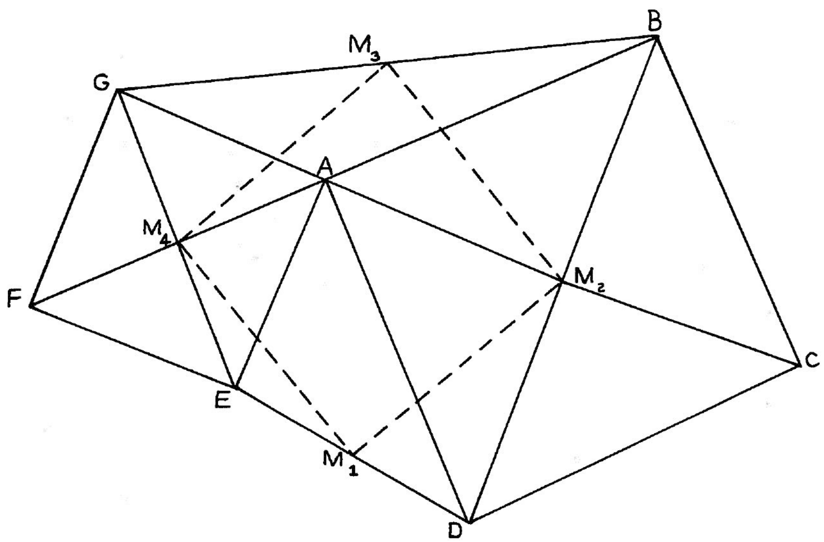


Figure 11

Theorem 13. The centers G', F', and E' of similarly oriented squares *GDUV, FCYZ,* and *EBWX* erected on the diagonals *GD*, *FC*, and *EB* of bow-tie hexagon *BCDEFG,* together with point *A*, are vertices of a square.

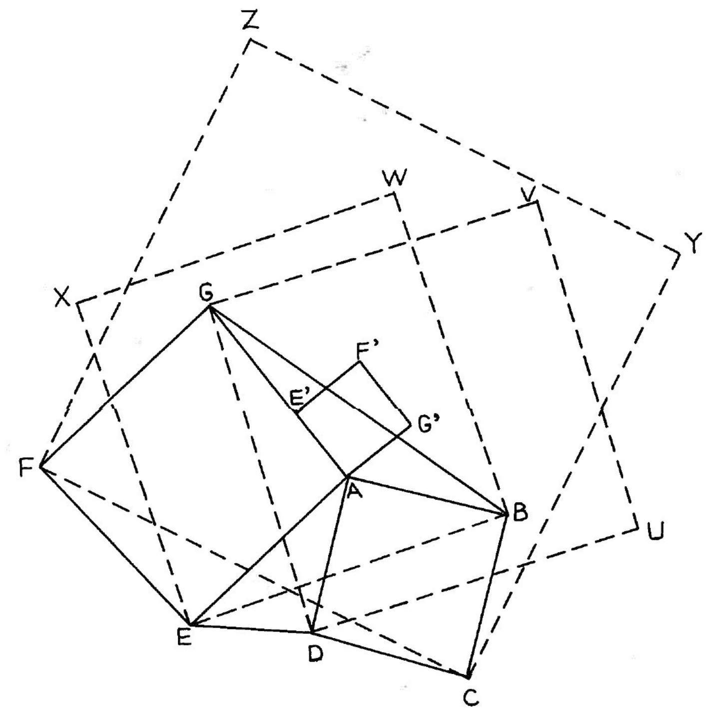


Figure 12

Proof. See **Figure** 12. That triangle $AG'E'$ is similar to AGE follows from the Theorem 3 generalization. Now take the origin of the complex plane at A. Let G and D have affixes g and d. Then $e = gi$, $b = di$, so that $f = g + gi$, and $c = d + di$. Now,

$$f' - a = f = \tfrac{1}{2}(f + y) = \tfrac{1}{2}(g + gi + d + di + i[(d + di) - (g + gi)]$$
$$= \tfrac{1}{2}(g + gi + d + di + di - d - gi + g) = g + di.$$

Since the squares *EBWX* and *DUVG* are congruent and parallel,

$$g' - e' = d - e = d - gi = -i(id + g) = -i(f' - a).$$

Thus AF' and $E'G'$ are equal and perpendicular, completing the proof that $AG'F'E'$ is a square. Of course, if the orientation of the erected squares were reversed, then the diagonal $G'E'$ would have length equal to BG, rather than to ED.

Theorem 14. The midpoints F', E', and G' of the three main diagonals FC, EB, and DG together with point A are vertices of a square.

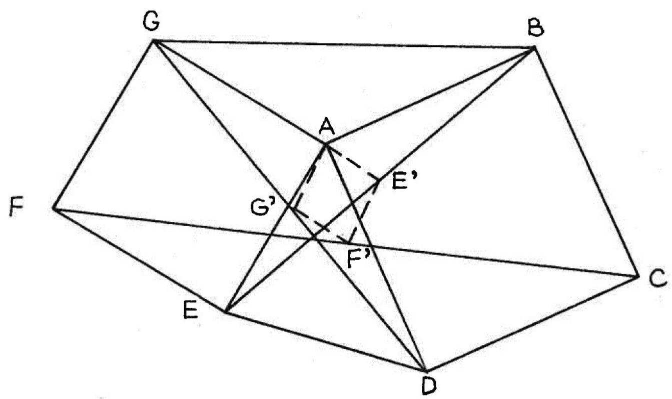


Figure 13

Proof. See Figure 13. This proof parallels that of Theorem 13 and is left for the pleasure of the reader.

No attempt has been made to exhaust the properties of the bow-tie hexagon. Furthermore the reader is invited to examine bow-tie polygons of more than 6 sides. Thus, for example, if instead of squares we use regular pentagons hinged at one point, a most interesting octagon will result.

We are fully aware that the theorems proved here can be generalized even further. Just as we need not have equilateral triangles in the bow-tie quadrilateral, it is not necessary to have squares in the bow-tie hexagon. We could use similar rhombi instead. Since this article is intended as a sequel to the Equilic Quadrilateral, we have emphasized the equilateral triangles and squares. We have, however, included an explanation of how some of the more general theorems can be treated.

VARIATIONS ON THE AM-GM INEQUALITY

by J. L. Brenner
Palo Alto, Calif.

Let $x_1, x_2, \ldots, x_k$ be positive and not all equal.

There are two well-known ways of interpolating means between the arithmetic mean $A = (x_1 + \ldots + x_k)/k$ and the geometric mean $G = (x_1 x_2 \ldots x_k)^{1/k}$. The latter is known to be less than the former ($G < A$). The first interpolation is by power means, M_r, r real, defined by $M_r = [(x_1^r + \ldots + x_k^r)/k]^{1/r}$. The number M_r increases with r [1], and $M_1 = A$; also $\lim_{r\downarrow 0} M_r = G$. (This last result can be proved using calculus.. The M_r clearly constitute a continuous set of means, homogeneous of degree 1.

Another set of means that interpolates between A and G is the set of Maclaurin means. These are $A = L_1$,

$$L_2 = [(x_1x_2 + x_1x_3 + \ldots + x_{k-1}x_k)/C_2^k]^{1/2}$$

$$L_3 = [(x_1x_2x_3 + \ldots)/C_3^k]^{1/3},$$

$$\vdots$$

$$L_k = G.$$

The inequalities $L_1 > L_2 > \ldots > L_k$ $(k > 1)$ are proved in [1]; the first one can be proved by showing that $L_1^2 - L_2^2$ is a sum of squares. (Start with the cases $k = 2,3$.)

The Maclaurin means are based on the elementary symmetric functions; clearly they are defined only for integral values of the parameter r.

Can we interpolate means continuously between L_1 and L_2? Of course we can! Linear interpolation will work; but is there something more delicate? To interpolate between L_1 (which is based on the first elementary symmetric function) and L_2 (which is based on $x_1x_2 + x_1x_3 + \ldots$), let us try

$$L_{1+\varepsilon} = [(x_1x_2^\varepsilon + x_1^\varepsilon x_2 + x_1x_3^\varepsilon + x_1^\varepsilon x_3 + \dots)/(k(k-1))]^{1/(1+\varepsilon)}, \ 0 \le \varepsilon \le 1.$$

$L_{1+\varepsilon}$ takes the value L_1 when $\varepsilon = 0$, and the value L_2 when $\varepsilon = 1$. Also $L_{1+\varepsilon}$ (like all the other means) takes the value x when $x_1 = x_2 = \dots = x_k = x$, and finally, because of the exponent $1/(1+\varepsilon)$, $L_{1+\varepsilon}$ is homogeneous of degree 1:

$$L(\lambda x_1, \dots, \lambda x_k) = \lambda \cdot L(x_1, \dots, x_k).$$

(I have to remark that if two "means" are homogeneous of different degrees, they cannot be comparable--an inequality between them cannot subsist: their ratio is $\gtrless 1$ as the variables approach 0, ∞.)

You can probably infer that we are going to define $L_{2+\varepsilon}$ $(0 \le \varepsilon \le 1)$ by means of the formula

$$L_{2+\varepsilon} = [(x_1x_2x_3^\varepsilon + x_1x_2^\varepsilon x_3 + x_1^\varepsilon x_2x_3 + \dots)/(3C_3^k)]^{1/(2+\varepsilon)} .$$

Now the test: have we interpolated? It can be proved [2] that if $0 < \varepsilon < 1$, these newly invented means do satisfy $L_1 > L_{1+\varepsilon}$, $L_2 > L_{2+\varepsilon}$, . , $L_{k-2} > L_{k-1+\varepsilon}$. It is even true that $L_{k-1+\varepsilon} > L_k = G$. But alas! if $k > 2$, the inequality $L_{1+\varepsilon} > L_2$ is not valid for every choice of $x_1, x_2, \dots, x_k$: the means $L_{1+\varepsilon}$, L_2 are <u>not</u> comparable if $k > 2$. In particular, if $\varepsilon = 1/2$, $x_1 = x_2 = 1$, $x = 0.25$, then $L_1 = 0.721$, $L_{1+\varepsilon} = L_{1.5} = 0.698$, $L_2 = 0.707$, $L_{2.5} = 0.64$, $L_3 = G = 0.63$.

Herman P. Robinson, a retired physicist with a home computer, calculated $L_{1+\varepsilon}$, $L_{2+\varepsilon}$, for a range of values of $\{x_1\}$ and of ε. His calculations suggest that the inequality $L_{1+\varepsilon} > L_{2+\varepsilon}$ always holds. I think I can prove this when $\varepsilon > 0$. This suggests that many natural, interesting inequalities remain to be discovered by you, the reader. The editor helped me to discover the last one, by questioning one of my assertions.

Editor's Note.: The Journal would be interested in further results along these lines; students are encouraged to send their thoughts, results, conjectures and papers.

REFERENCES

1. Hardy, G., Littlewood, J., and Polya, G. *Inequalities*, Cambridge University Press (1932).

2. Brenner, J., *A Unified Treatment of Some Means of Classical Analysis. I. Comparison Theorems.* J. Combinatorics, Information 6 System Sciences 3 (1978), 175-199.

PUZZLE SECTION

Edited by
David Ballew

This Department is for the enjoyment of those readers who are addicted to working crossword puzzles or who find an occasional mathematical puzzle attractive. We consider mathematical puzzles to be problems whose solutions consist of answers immediately recognizable as correct by simple observation and requiring little formal proof. Material submitted and not used here will be sent to the Problems Editor if deemed appropriate for that Department.

Address all proposed puzzles and puzzle solutions to David Ballew, Editor of the Pi Mu Epsilon Journal, Department of Mathematical Sciences, South Dakota School of Mines and Technology, Rapid City, South Dakota, 57701. Deadlines for puzzles appearing in the Fall issue will be the next February 15, and for puzzles appearing in the Spring Issue will be the next September 15.

Mathacrostic No. 15

Submitted by Joseph V. E. Konhouser
Macalester College, St. Paul, Minnesota

Like the preceding puzzles, this puzzle (on the following two pages) is a keyed anagram. The 267 letters to be entered in the diagram in the numbered spaces will be identical with those in the 29 keyed words at matching numbers, and the key numbers have been entered in the diagram to assist in constructing your solution. When completed, the initial letters will give a famous author and the title of his book; the diagram will be a quotation from that book. (See an example solution in the solutions section of this Department.)

Definitions — Words

A. American mathematician (1773–1838), title hero of Horatio Alger's "Nat the Navigator"
165 187 14 48 7 68 88 230

B. black lustrous asphalt occurring especially in Utah
236 20 188 256 163 87 58 208

C. "wild" structure equivalent to the surface of a ball but not bounding a simply-connected region (2 wds.)
42 95 151 185 75 118 266 125 106 197 209 233

D. Walter Winchell's wish to his listeners, " _____ of love."
141 98 161 195 245 179 18

E. show high spirits
73 90 99 143 40 112 214 262 130 226

F. movement of an organism in response to the flow of a current
110 30 147 45 6 59 216 175 21

G. folk epigraph
119 170 4 79 154 181 91 140

H. antiquated synonym for proper divisor (2 wds.)
219 135 2 158 77 264 12 56 31 164 101

I. ointment
9 204 232 146 26 122 60

J. $\phi(t) = 2\sum_{1}^{\infty}(-1)^{n+1}\frac{\sin nt}{n}$ (2 wds.)
67 259 46 128 227 153 183 92 94 144 196 11

K. North American tree with aromatic bark
180 113 22 41 221 138 82 242 257

L. slender and pointed
80 8 123 203 63 250 172 246 35

M. in 1819 he (1781–1848) published a treatise on the kaleidoscope
247 96 34 253 124 23 55 134

N. enzyme catalyst in conversion of sucrose to glucose and fructose
85 36 220 231 104 263 64 3 152

O. reciprocal of the cube
62 83 114 206 239 47 25 218 137 157

P. one interpretation of Einstein's statement "Gott würfelt nicht" to Bohrwhen rejecting quantum theory (5 wds.)
145 89 44 17 178 121 65 234 240 72 109 260
255 136 10 205 225 194

Q. inflammation of the nasal mucous membranes
251 19 107 173 200 155 81 53

R. throwback
133 86 66 207 235 29 120

S. basis of Escher's 1960 lithograph "Ascending and Descending" and 1961 lithograph "Waterfall" (2 wds.)
P E R P E C T U A L M O
190 162 78 127 211 199 39 148 5 169 61 16
T
258 223 182 267

T. now extinct grouse of the north-eastern United States (2 wds.)
254 191 126 217 51 102 156 176

U. of a mapping if distinct elements map into distinct elements
160 76 116 189 139 28 129 202 248

V. most eminent 19th century Russian mathematician (1821–1894)
C H E B Y S H E V
241 1 93 210 171 108 213 150 38

W. α Boötes, "watcher of the bear"
15 192 177 229 57 84 159 237

X. "Contemporary theory of measure still dances to _____'s (1875–1941) tunes," M. Loève, 1965
142 184 49 103 32 222 131 115

Y. finite in area, infinite in length, nowhere differentiable (2 wds.)
105 27 13 149 228 132 43 97 252 193 265 168

Z. fundamental entity in the universe of Roger Penrose
201 238 111 249 167 37 69 74 261

a. constellation between the hunting dogs, Canis Minor and Canis Major
24 224 166 71 100 54 244

b. trash can (British)
174 117 198 212 33 52 243

c. abominable snowman
50 215 186 70

1 V	2 H	3 N		4 G	5 S	6 F	7 A	8 L	9 I	10 P	11 J		12 H
13 Y	14 A	15 W	16 S	17 P	18 D		19 Q	20 B	21 F		22 K	23 M	24 a
25 O	26 I	27 Y	28 U	29 R		30 F	31 H	32 X		33 b	34 M	35 L	36 N
	37 Z	38 V	39 S	40 E	41 K	42 C	43 Y	44 P	45 F	46 J	47 O	48 A	
49 X	50 c		51 T	52 b	53 Q		54 a	55 M	56 H	57 W	58 B	59 F	60 I
61 S	62 O	63 L		64 N	65 P		66 R		67 J	68 A	69 Z	70 c	71 a
72 P		73 E	74 Y	75 C	76 U		77 H	78 S	79 G	80 L	81 Q	82 K	
83 O	84 W	85 N	86 R	87 B	88 A		89 P	90 E		91 G	92 J	93 V	
94 J	95 C	96 M	97 Y		98 D	99 E		100 a	101 H	102 T	103 X	104 N	105 Y
	106 C	107 Q	108 V		109 P	110 F	111 Z	112 E	113 K	114 O	115 X		116 U
117 b	118 C	119 G	120 R	121 P	122 I	123 L	124 M		125 C	126 T	127 S	128 J	129 U
130 E	131 X	132 Y	133 R	134 M	135 H	136 P		137 O	138 K		139 U	140 G	141 D
142 X	143 E	144 J	145 P	146 I	147 F	148 S		149 Y	150 V	151 C	152 N		153 J
154 G	155 Q	156 T	157 O		158 H	159 W	160 U	161 D	162 S		163 B	164 H	165 A
166 a	167 Z	168 Y	169 S	170 G	171 V		172 L	173 Q	174 b		175 F	176 T	177 W
178 P	179 D	180 K	181 G	182 S	183 J	184 X	185 C	186 c		187 A	188 B	189 U	
190 S	191 T	192 W	193 Y	194 P	195 D	196 J	197 C	198 b		199 S	200 Q	201 Z	
202 U	203 L	204 I	205 P	206 O		207 R	208 B	209 C	210 V	211 S		212 b	213 V
214 E		215 c	216 F	217 T	218 O	219 H	220 N	221 K	222 X	223 S	224 a	225 P	226 E
	227 J	228 Y		229 W	230 A	231 N		232 I	233 C	234 P	235 R	236 B	237 W
	238 Z	239 O	240 P		241 V	242 K	243 b	244 a	245 D	246 L		247 M	248 U
	249 Z	250 L	251 Q	252 Y		253 M	254 T	255 P	256 B		257 K	258 S	259 J
260 P	261 Y	262 E		263 N	264 H		265 Y	266 C	267 S				

SOLUTIONS

Mathacrostic No. 14. (See Spring 1982 issue) (*Proposed by Joseph D. E. Konhauser*)

Definitions and Key:

A. Inchworm	H. Synkaryon	O. Arrows theorem	V. Throwing stick
B. Lily-white	I. Polya	P. Null	W. Aida
C. Athwart	J. Rin-tin-tin	Q. Dichotomy	X. The two eyes
D. Klein Bottle	K. Otherwise	R. Ramanujan	Y. Idiot savant
E. Analytic	L. Out-of-the-way	S. En suite	Z. Odyssey
F. Thorny	M. Funicular	T. Fifty-fifty	a. Nightshade
G. Obnubilate	N. Swatch	U. Union suit	b. Shroud

First Letters: I. Lakatos Proofs and Refutations

Quotation: *If you want to know the normal healthy body, study it when it is abnormal, when it is ill. If you want to know functions, study their singularities. If, you want to know ordinary polyhedra, study their lunatic fringe. This is how one can carry mathematical analysis into the very heart of the subject.*

Solved by: Jeanette Bickley, Webster Groves High School, Missouri; Pat Collier and Bob Prielipp, University of Wisconsin-Oshkosh; Victor Feser, Mary College, Bismark; Robert Gebhardt, New Jersey; Theodor Kaufman, Brooklyn; Roger Kuehl, Kansas City; Henry S. Lieberman, John Hancock Mutual; D. C. Pfaff, University of Nevada, Reno; Sister Stephanie Sloyan, Georgian Court College; Patricia and Allan Tuchman, University of Illinois; The Proposer and The Editor.

THE 1982 MEETING IN TORONTO

The Program for the 1982 Meeting of the PI MU EPSILON FRATERNITY was held at the University of Toronto on August 21 through 26 of 1982. The program included:

Forward and Backward Eigenvalues	Kriss Schueller Ohio Xi Youngstown State University
Application of the Golden Number to Fibonacci Algebra	Duane Cooper Georgia Delta Atlanta University Center
The Influence of the Math Meet on Recruiting	Maria Spetseris South Carolina Gamma College of Charleston
The Volume of the N-Dimensional Sphere	Ravi Salgia Illinois Theta Loyola University
A Mathematical Model for Paired-Associate Learning	Dore Ann Celentano New Jersey Epsilon St. Peter's College
An Analysis of Monopoly Strategies	Thomas Chenier and Cathy Vandiford North Carolina Delta East Carolina University
Population Competition and Crop Yield	Donna Ford Ohio Delta Miami University
Approximating Partial Sums of the Harmonic Series	Karim K. Carter Arkansas Gamma University of Arkansas, Pine Bluff
Application of Stochastic Control to Battlefield Replacements	Andrew Stumpff Missouri Beta Washington University
Effects of Air Pollution on Pulmonary Functions of Children	Deborah Pennell Montana Alpha University of Montana
A Geometric Analysis of the Relative Sizes of Classical Orthogonal Polynomials	Arthur W. Mifflin Illinois Delta Southern Illinois University at Carbondale

The Stone - Cecil Compactification in the Structure Space of a Distributive Lattice	Christopher Brislawn California Epsilon Pomona College
A Mathematical Approach to Improving Your Backgammon Skills	Joan Hart Ohio Delta Miami University
Estimating the Inclusion Radius of Polynomial Zeros	William Somsky Iowa Alpha Iowa. State University
Creating Kaleidoscopes	Mary Anne Bromelmeier Ohio Delta Miami University
A Derivation of a Fifth Order Predictor - Corrector Method	Eric D. Stutz New Jersey Gamma Rutgers University, Camden
Computer Graphics in Numerical Analysis	Daniel John Pierce Illinois Epsilon Northern Illinois University
Peppermint Patty's Dilemma -- Math Anxiety	Cathie Spino Ohio Delta Miami University
Love and Psycho-Mathematics	Richard Porter New Jmey Beta Douglass College
Investigations of Maxfield's Theorem	Laura Southard Oklahoma Alpha. University of Oklahoma.
Alternative Tic-Tac-Toes	Kevin Saylors Ohio Delta Miami University

The J. Sutherland Frame Lecture was given by Professor Israel Halperin and entitled *"The Changing Face of Mathematics"*.

CHAPTER REPORTS

ARKANSAS BETA (Hendrix College) The Undergraduate Research Program was again very active; the members attended the Mid-South Mathematics Colloquium at Memphis State, The Oklahoma/Arkansas MAA Meeting, and the Annual Hendrix-Sewanee-Southwester Math Symposium.

The McHenry-Lane Freshman Award was given to Kim Riley; The Hogan Senior Mathematics Award was shared by Jerry Coker and Carol Smith; The Phillip Parker Undergraduate Research Award was presented to Carol Smith.

The Chapter heard the following papers: *"A Report on the Summer Meetings in Pittsburg"*, by Carol Smith; *"Logic out of Topology"*, by Dr. Zeev Barel; *"Some Ways to Finesse a Computer, Though Not at Bridge"*, by Dean John Merril; *"Mathematics in the History of Philosophy"*, by Dr. John Churchill; *"The Anatomy of Computer Science"*, by Dr. Carol Ziegler, Univ. of Arkansas at Little Rock; *"Hidden Musical Numbers and. the Music of the Spheres: The Relationship Between Mathematics and Music in the Remissonce"*, by Dr. David Taylor; *"Program HIPLOT: High Resolution Plotting on the Commodore Printer"*, by Tim Best; *"Mathematical Modeling and the Wine Cellar"*, by Dennis Meridith; *"The Evaluation of Travel Time: An Example in the Application of Queuing Theory"*, by Wade. Nixon; *"Taxicab Relativity"*, by Ben Schumacher; *"The Past and Present Employment and Graduate School Status for Hendrix Mathematics Graduates"*, by Dr. Cecil McDermott; *"Economist or Computers?"*, Dr.. Thomas Fomby, Southern Methodist University; *"The Axioms for a Group"*, Dr. Bernard Madison, University of Arkansas; *"Computer Simulation of a Markov Chain"*, by Jerry Coker; *"Geodesics on Surfaces; A BASIC Approach"*, by Benjamin Schumacher; *"Lines in a Metric Space"*, by Carol Smith.

FLORIDA EPSILON (University of South Florida) The Chapter paper presentations included: *"Egyptian Fractions"*, by Deborah Wallis; *"Information Theory for Those Who Don't Know Any"*, by Dr. Jacob Wolfowitz; *"Circles, Squares and Lattics"*, by Dr. Frank Cleaver (Chapter Advisor); *"Some Algebras of Differential Operators--From Euclidean Geometry to Electromagnetism"*, by Dr. Ernest Thieleker; *"Doing Calculus with Infinitesimals"*, by Dr. Rohit Parikh of Boston University; *"Maya Arithmetic"*, by Victor Arean; *"Age Dependent Population Models"*, by Dr. Glenn Webb of Vanderbilt University; *"Environmental Mathematics"*, by Dr. Ben Fusaro of the Univ.

of Florida in Gainesville; *"Voyager I Views Saturn"*, by *Dr. Joseph Carr*; *"The Integration of $Sec^3x\ dx$"*, by *Michael Walloga*; *"The QR Algorithm for Finding Eigenvalues"*, by *Mark Bateh* of the University of Florida; *"Buffman Enfolding"*, by *Van Sharpe* of Valdosta State College; *"Perfect and Duffinian Numbers"*, by *Jose Stoute*; *"Creativity in Mathematics"*, by *Dr. Dwight Benjamin Goodness* *"Some Aspects of Fibonacci Numbers"* by *Craig Hubbard*; and *"Stonehenge and the Views of Saturn"* by *Jobeph Carr*.

The Chapter attended the Mathematical Association of America Meeting at Bethune-Cookman College.

Craig Hubbard won the Award as **The Outstanding Scholar of Pi Mu Epsilon for 1981.**

ILLINOIS ALPHA (University of Illinois) heard *"Pegboard Solitaire"* by *Prof. Hugh L. Montgomery* of the University of Michigan and *"The Vibrating String Controversy"* by *Prof. Heini Halberstam*.

IOWA ALPHA (Iowa State University) presented **The Pi Mu Epsilon Scholarship Award** to *Kirk Pruhs* and *Michael Kabala*, **The Outstanding Achievement Award** to *William Somsky*, and **The Dio Lewis Awards** to *Barbara Rus* and *Steven Seda*. Other Departmental awards were: **The Gertrude Herr Adamson Awards for Demonstrated Ingenuity in Mathematics** were given to *William Somsky, Pamela Ehlers, Allen Farquhar, Kurtis Ruby, Nazanin Imani, Fhed Adams*, and *Philip McKinley*; **The Marian Daniels Memorial Scholarships** were presented to *William Somsky, Steven Seda, Nazanin Imani, Bonnie. Weller, Fred Adams, Unda Parkes* and *Steven Breed*.

The Chapter heard the following papers: *"Group Theory and the Size of Animals"* by *Prof. James Murdock* and *"The History and Strategy of Monopoly"* by *Prof. Irvin Hentzel*.

MINNESOTA EPSILON (St. Cloud State University) had programs consisting of: *"Problem Solving Strategies in Mathematics"* by *Dr. Robert Earles*; *"Some Ideas about Metric Spaces"* and *"Son-standard Metrics on the Cantor Set"* by *Dr. Harold Martin*; *"Job Placement in the Mathematical Sciences"* by *Walter Larson*; *"The Alabama Paradox: A Serious Game of Musical Chairs"* by *Ralph Carr*; *"Trends in the Information Sciences"* by *Susie B. Kugel*; *"What is an Actuary?"* by *Dr. Louis Friedler* of the College of St. Scholastics; *"Microprocessor Based Product Design"* by *Dr. Robert W. Johnson*;

"Continued Fractions" by *Mark Anderson*; and several students spoke about their experiences interning in the twin cities area. Further, the Chapter sponsored an area high school mathematics contest that attracted 1600 high school students.

MINNESOTA DELTA (Saint John's Unversity) sponsored a Regional Pi Mu Epsilon Conference in April of 1982. This coincided with the Spring Meeting of the North Central Section of the MAA. The featured speaker was *Dr. A. B. Wilcox* who presented *"Mathematics: Where is it Going?"*.

MINNESOTA ZETA (Saint Mary's College) heard *Dr. Richard Jarvinen* speak on *"What is the Future of Core Mathematics? Where is the Balance Between Traditional Mathematics and the New Areas?"*; *Barbara Burn* discuss *"An Introduction to Stability Theory"*; and *Ronald Karwoski* on *"Mathematical Models of Epidemics"*.

MONTANA BETA (Montana State University) held three meetings at which were given the following talks: *"The Less Well-Known Methods (Legal and Illegal) to do Operations and to Solve Equations in Arithmetic and Algebra"*, by *Adrien Hess*; *"Women in Mathematics"* by *Dr. Byron McAllister*; and *"Mathematical Magic"* by *Dr. Jean Abel*.

NEW JERSEY EPSILON (Saint Peter's College) sponsored a panel discussion on careers: *"What Can You Do With a Major in Mathematics and/or Computer Science?"*, with participants *Donald Bunda, Daniel Crifo, Thomas Murphy* and *Janet Seaman Sullivan*. Other presentations included: *"A Mathematical Model, for Paired-Associate Learning"* by *Vohe Ann Celentano*; *"Simpson's 'Paradox"* by *Prof. Philip Ambrosini*; and *"Some Remarkable Mew Sphere Packings"* by *Dr. Neil Sloane* of Bell Laboratories.

NEW JERSEY THETA (Trenton State College) heard *Dr. Edward Conjura* and heard *Dr. Siegfried Haenisch* on *"A History of the Organization"* at the Annual Induction Ceremony.

NEW YORK OMEGA (Saint Bonaventure University) The Chapter held six business meetings and presented the following three programs: *"Can Chrysler Survive? -- With the Help of Calculus"* by *Prof. Philip Church* of Syracuse

University; *"Infinite Cardinal Numbers -- How to Count Past Infinity"* by *Professor Douglas Cashing;* and presented **The Pi Mu Epsilon Award** to *Valerie Heeter.*

NEW YORK PHI (SUNY at Potsdam) had a very active year with a fund-raising raffle, several open houses, parties and other events. The Chapter heard *Dr. Stephens* speak on "A *Successful Mathematics Program"* and presented **The Pi Mu Epsilon Awards** to *Nancy Offslager and Lydia Hardy.*

PENNSYLVANIA NU (Edinboro State College) sponsored Open Houses, parties and the program *"Fibonacci Numbers" by Dr. Richard Reese.*

NEW YORK ALPHA ALPHA (CUNY-Queen's College) heard *"The Sometimes Terrible Consequences of Being a Math Major" by Dr. Martin Braun; "Professional Opportunities in Actuarial Mathematics" by Dr. Orin Linden* of the Home Insurance Company; and *"Euclid Revisited: A New Look at Some Simple Geometric Constructions". Sidney Gottesman* was the recipient of **The Pi Mu Epsilon Prize For Excellence in Mathematics.**

OHIO DELTA (Miami University) began its activities with the Pi Mu Epsilon Eight Annual Student Conference. Nineteen students including ten Miami students contributed papers to the Conference. During the remainder of the year the Chapter heard the following papers and talks: *"Multiple Comparisons and Model Selection" by Dr. Bababhai Patel; "Fourier Series and Boundary Value 'Problems" by Dr. Thomas Bengtson; "Games for the Holiday Season" by Dr. Richard Laatsch;* and *"Issues and Trends in School Mathematics (1890-1980)" by Dr. Ron Harkins.* Further, the Chapter sponsored a panel discussion on *"What One Can Expect In Graduate School"* with panelists *Dr. Edward Bolger, Dr. Thomas Bengtson, Dr. Robert Schaefer, Dr. Donald Byrkett, Michael Selmon, Beth Snyder and Dr. Huneke.*

The Chapter sponsored the **Annual Pi Mu Epsilon Sophomore Mathematics Examination,** an examination composed of ten calculus and linear algebra problems. The winners were *Jann Cook and Lee Ann Shollenberger.* The Chapter sponsored several other activities including films and talks by former students who are now in industry.

OHIO NU (University of Akron) presented **The Samuel Selby Mathematics Award** to *Sue Smith, Yvonne Zubovic, Tim Davis, Brenda Krager and Rosalie Hiebel.* Other activities included an award at the Arkon Regional Science Fair, help with the Ohio Council of Teachers of Mathematics Math Contest, movies and picnics. The Chapter heard a panel discussion *"Mathematics Beyond the Classroom II"* and the following talks: *"Cryptography" by Dr. V. Frederick Rickey* and *"Some Historical Tibits" by Dr. Kenneth Cummins.*

The 1982-83 winners of the **Samuel Selby Mathematics Award** will be *Thomas Batchik, Ken Gole and William Lenzotti.* Other talks included: *"Stalking the Gerrymander" by Dr. Phil Schmidt; "History of Mathematics" by Dr. Robert Carson; "Patterns in Mathematics" by Dr. Milko Jeglic* of the University of Notre Dame; and *"Research in Biomedical Engineering" by Dr. Robert Herron.*

OHIO XI (Youngstown State) has the goal of at least one speaker per quarter and, hopefully, one per month. The Chapter has an active program tutoring area high school students for the National MAA Mathematics Exam.

OKLAHOMA GAMMA (Cameron University) sponsored six students in a Collegiate Mathematics Examination at the University of Texas at Arlington. The Chapter heard *Judy Thompson* on *"Chisombop: The Art of Korean Finger Counting".*

SOUTH CAROLINA BETA (Clemson University) sponsored a intramural softball team, a skiing trip, a student delegate to the Southeast Regional Meeting of the MAA, and a Science Day for high school students. In addition, the Chapter heard: *"The Boom in Micro- and Mini- Computers" by Dr. Ed Page; "Amazing MAZES" by Dr. Doug Shier; "Job Interviews and Resumes" by Lucy Reddick;* and *"The Gambler's Ruin" by Dr. Joel Brawley.*

SOUTH CAROLINA GAMMA (College of Charleston) sponsored the programs: *"Games an Graph" by Joel Brawley; "The Buffon Needle" by Fred Morgan and Jim Reneke* of Clemson University; and "A *Modal of 'Probability" by Dr. Robert Norton.* The Chapter further sponsored the Fifth Annual Mathematics Meet, compiled a book of old departmental calculus finals for sale to students, and lost to the faculty in the Annual Softball Game.

SOUTH DAKOTA BETA (South Dakota School of Mines and Technology) had a great year due to its fine Faculty Advisor. The Chapter heard talks by *Dr. Ron Weger* on *"Extending Newton's Method"*: *Janet Potts and Linda Plower* on *"Summer Employment Opportunities"*, *Debbie Wentzel* on *"Algebra and Energy"*, *Colleen Quatier* on *"Why Should a Mathematician Worry About Copywrite Policy?"*; *Dean Mogck* on *"Newton's Method For Complex Variables"*; and *Brian Bunsness* on *"Programming the Khachiyan Algorithm"*. Further, the Chapter the Annual South Dakota Collegiate Mathematics Contest, helped sponsor the Western South Dakota High School Mathematics Contest, several open houses, picnics and parties.

TEXAS BETA (Lamar University) announces that *Gregory Dyess, Glenn Loupe, Sarah Guidri and Chau Minh Dang* were winners in **The Annual Homer Dennis Freshman Mathematics Contest.**

TEXAS DELTA (Stephen F. Austin State University) heard the following speakers: *"Card Tricks and Latin Squares"* by *Harold Bunch*; *"The Geometric Solution to the Quadratic Equation"* by *Doyle Alexander*; *"Generalization in Mathematics"* by *R. G. Dean*; *"What Numerical Analysts Do"* by *Thomas Atchison*; *"Swiss Cheese and Other Things"* by *Dr. W. T. Guy* of the University of Texas; and *"Inequalities -- Some Methods of Attack, Fighting in the Trenches, and Graceful Retreat"*.

VIRGINIA GAMMA (James Madison University) sponsored *Carol Nesslein* who spoke on *"Job Searching and Careers Related to Mathematics"* and three statistics talks given by *Dr. Harold Reider* of the University of North Carolina at Charlotte, *Dr. Robert Silber* of North Carolina State University, and *Dr. John Mandell* of the National Bureau of Standards. The Chapter presented **The Outstanding Senior Mathematics Award** to *Steve Conley*.

Is your Chapter's Report here? Why not? Send Chapter Reports to Dr. Richard Good, Dept. of Mathematics, University of Maryland, College Park, Maryland and to Dr. David Ballew, Dept. of Mathematical Sciences, SDSM&T, Rapid City, South Dakota 57701

PROBLEM DEPARTMENT

Edited by Clayton W. Dodge
University of Maine

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

All communications should be addressed to C. W. Dodge, Math Dept., University of Maine, Orono, ME 04469. Please submit each proposal and solution on a separate sheet (one side only) properly identified with name and address. Solutions to problems in this issue should be mailed by July 15, 1983.

Problems for Solution

522. *Proposed by Charles W. Trigg, San Diego, California.*

Arrange nine consecutive digits in a 3-by-3 array so that each of the six three-digit integers in the columns (read downward) and rows is divisible by 17.

523. *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Merrimack, New Hampshire.*

Let $ABCD$ be a parallelogram. Erect directly similar right triangles ADE and FBA outwardly on sides AB and DA (so that angles ADE and FBA are right angles). Prove that CE and CF are perpendicular.

524. *Proposed by Morris Katz, Macwahoc, Maine.*

Solve this holiday alphametric for a real prime XMAS.

$$\begin{array}{r} MERRY \\ XMAS \\ DODGE \end{array}$$

***525.** *Proposed by John M. Howell, Littlerock, California.*

An equilateral triangular prism is used as a die. What must the ratio of sides be so that the probability of falling on a triangle is the same as falling on a rectangle?

526. *Proposed by Morris Katz, Macwahoc, Maine.*

Solve this alphametric in base twelve, with apologies to J.A.H. Hunter.

$$\begin{array}{r} SUE \\ \underline{EIGHT} \\ PUTTY \end{array}$$

527. *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg Pennsylvania.*

Find the volume of the largest rectangular parallelepiped with upper vertices on the surface and lower vertices on the xy-plane that can be inscribed in the elliptic paraboloid $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2h - 2z$.

528. *Proposed by Alan Wayne, Pasco-Hernando Community College, Florida.*

In the set of natural triangles--that is, the set of triangles with side lengths that are integers--consider, for instance, the trio: (19, 24, 35), (15, 29, 34) and (14, 31, 33). Call this trio a "size triplet", because the three triangles have the same perimeter and the same area. Since the common area is least, this is the smallest size triplet. What is the next larger size triplet?

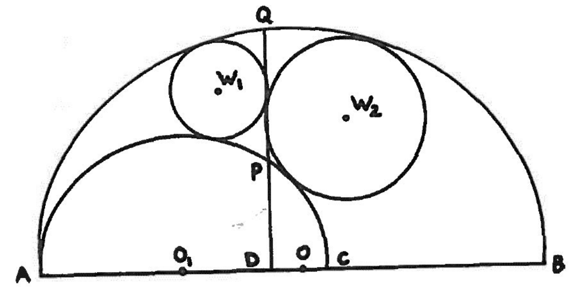
529. *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Merrimack, New Hampshire.*

Show that there is no "universal field" that contains an isomorphic image of every finite field.

530. *Proposed by Lean Bankoff, Los Angeles, California.*

In the accompanying diagram, AB $(= 2r)$ is the diameter of circle (O) and AC $(= 2r_1)$ the diameter of circle (O_1), D is a point on diameter AC, and the half-chord DQ perpendicular to AC cuts the circle (O_1) at P. The circles (W_1) of radius ρ_1 and (W_2) of radius ρ_2 are tangent to circles (O) and (O_1) and touch PQ on opposite sides. Show that

$\rho_1/\rho_2 = r_1/r$.



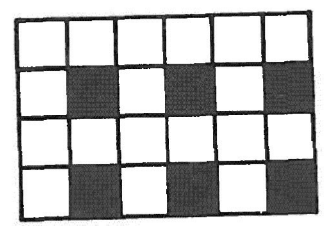
531. *Proposed by Robert E. Megginson, University of Illinois.*

Prove, without using mathematical induction, that

$2\cdot 6\cdot 10\cdot 14 \ldots (4n - 2) = (n + 1)(n + 2) \ldots (2n)$.

532. *Proposed by Morris Katz, Macwahoc, Maine and Charles W. Trigg, San Diego, California.*

From a square grid of side 17, alternate squares are removed to form a sieve. Dissect this sieve into fewer than a dozen pieces and reassemble them into a square of side 15. See problem 491 [Spring 1982, page 421].



533. *Proposed by D.O. Fantus, Alexandria, Virginia.*

It is known that cancelling the sixes in the proper fraction 16/64 yields the equivalent fraction 1/4 in lowest terms (problem E24, September 1933, ***The American Mathematical Monthly***). Find or characterize all proper fractions having 3-digit numerators and 3-digit denominators that reduce to lowest terms by cancelling the same digit from numerator and denominator.

Solutions

498. [Fall 1981] *Proposed by R.S. Luthar, University of Wisconsin, Janesville.*

Find the general solution of

$$x^3 + y^3 + 3xy = 1.$$

Solution by Peter Szabaga, Woodside, New York.

The equation $x^3 + y^3 + 3xy = 1$ may be written as

$$(x^3 + 3x^2y + 3xy^2 + y^3) - (3x^2y + 3xy^2 - 3xy) - 1 = 0, \text{ or}$$

$$((x + y)^3 - 1) - 3xy((x + y) - 1) = 0,$$

which has $x + y - 1$ as a factor. Then

$$((x + y) - 1)((x + y)^2 + (x + y) + 1 - 3xy) = 0,$$

which reduces to the two equations:

(1) $x + y - 1 = 0$, or $y = -x + 1$; (2) $(x + y)^2 + (x + y) + 1 - 3xy = 0$, which may be rewritten as $y^2 - (x - 1)y + (x^2 + x + 1) = 0$. This equation may be solved as a quadratic in y:

$$y = \frac{(x - 1)\pm\sqrt{(x - 1)^2 - 4(x^2 + x + 1)}}{2} = \frac{x - 1 \pm (x + 1)\sqrt{-3}}{2}$$

Also solved by JOHN M. HOWELL. HENRY S. LIEBERMAN. VINCENT A. MILLER. STANLEY RABINOWITZ. KENNETH M. WILKE. *and the* PROPOSER. All real solutions, namely $y = 1 - x$ and the isolated point $(-1, -1)$. were found by VICTOR G. FESER. DAVID INY, DAVID E. PENNEY. *and* BOB PRIELIPP.

499. [Fall 1981] *Proposed by Victor G. Feser, Mary College, Bismarck, North Dakota.*

The array below is defined by the following properties:

i) The entries are distinct positive integers.

ii) In each column, the entries are consecutive integers, top to bottom.

iii) In each row, each integer (except the first one, of course) is a multiple of the integer at its left.

1	7	511
2	8	512
3	9	513

a) Find a fourth column for this array.

b) Find the minimal fourth column for this array, and show it is minimal.

c) Construct an array of 4 rows and 4 columns with the same properties. Is it minimal?

Solution by W.C. Igips, Danbury, Connecticut.

a) One possible fourth column is 134217727
134217728
134217729.

b) Let A be the first number of the fourth column. Then we must have $A \equiv 0 \pmod{511}$, $A + 1 \equiv 0 \pmod{512}$, and $A + 2 \equiv 0 \pmod{513}$. This means that $A \equiv 511 \pmod{511}$, $A \equiv 511 \pmod{512}$, and $A \equiv 511 \pmod{513}$. So $A \equiv 511 (\text{mod } LCM(511, 512, 513)) \equiv 511 \pmod{134217216}$. Then the minimum nontrivial value of A is $511 + 134217216 = 134317727$.

c) Using the same strategy as in part (b), we let A be the first number in column 2 (where column 1 contains 1, 2, 3, and 4). Then $A \equiv 0 \pmod 1$, $A + 1 \equiv 0 \pmod 2$, $A + 2 \equiv 0 \pmod 3$, $A + 3 \equiv 0 \pmod 4$ lead to $A = 13$. Then let B be the first number in column 3. The same strategy, applied again, leads to $B = 21853$. And, with C the first number in column 4, we eventually find that

$$C \equiv 21853 (\text{mod } LCM(21853, 21854, 21855, 21856))$$

$$= 21853 + 2^5 \times 3 \times 5 \times 7^2 \times 13 \times 31 \times 41^2 \times 47 \times 223 \times 683$$

$$= 21853 + 114{,}060{,}035{,}298{,}125{,}280$$

$$= 114{,}060{,}035{,}298{,}147{,}133.$$

Also solved by ROBERT C. GEBHARDT, ROGER KUEHL. JEFF LOVELAND, BOB PRIELIPP. KENNETH M. WILKE, *and the* PROPOSER.

500. [Fall 1981] *Proposed by Chuck Allison and Peter Chu, San Pedro, California.*

A condemned prisoner is given a chance to escape execution. He is given two boxes capable of holding sixteen bottles each, and is required to place eight bottles of water and eight bottles of clear poison in those boxes leaving no box empty. He will then summon the guard who will then pick one box at random and then select a bottle from that box which the prisoner must drink. How should the prisoner arrange the bottles in the two boxes to maximize his probability of survival, and what is that probability?

Essentially similar solutions were submitted by JEANETTE BICKLEY, *St. Louis, Missouri*, MICHAEL W. ECKER, *Scranton, Pennsylvania*, MARK EVANS, *Louisville, Kentucky*, VICTOR G. FESER, *Mary College, Bismarck, North Dakota*, ROBERT C. GEBHARDT, *Hopatcong, New Jersey*, JOHN M. HOWELL, *Littlerock, California*, ROGER KUEHL, *Kansas City, Missouri*, HENRY S. LIEBERMAN, *Newton, Massachusetts*, JEFF LOVELAND, *North Logan, Vermont*, ALEX MCKALE, *Swarthmore College, Pennsylvania*, ROGER MEGGINSON, *Bemont, Illinois*, KENNETH M. WILKE, *Topeka, Kansas, and the* PROPOSER.

The prisoner should place only one bottle of water in one box; he should place the other fifteen bottles in the other box. His probability of survival is $1/2(1) + 1/2(\frac{7}{15}) = \frac{11}{15}$.

One incorrect solution was received. This problem is certainly not new. Ecker and Wilke pointed out that it appeared as problem 325 in *The Pentagon* (*XL*, 1981, pp. 111-113). Its origin is much earlier.

501. [Fall 1981] *Proposed by Robert C. Gebhardt, Parsippany, New Jersey.*

A rectangle is inscribed inside a circle. The area of the circle is twice the area of the rectangle. What are the proportions of the rectangle?

Solution by Henry S. Lieberman, Newton, Massachusetts.

Let d be the diameter of the circle and x and y be the sides of the inscribed rectangle. Then

$$x^2 + y^2 = d^2 \text{ and } \pi\frac{d^2}{4} = 2xy$$

so that

$$\frac{x^2 + y^2}{xy} = \frac{d^2}{\frac{\pi d^2}{8}} = \frac{8}{\pi},$$

or

$$\frac{x}{y} + \frac{y}{x} = \frac{8}{\pi}.$$

Let $t = \frac{x}{Y}$ so that

$$t + \frac{1}{t} = \frac{8}{\pi} \text{ or } t^2 - \frac{8}{\pi}t + 1 = 0.$$

Solving for t we get:

$$t = \frac{1}{2}\left(\frac{8}{\pi} \pm \sqrt{\frac{64}{\pi^2} - 4}\right) \text{ or } t = \frac{1}{\pi}\left(4 + \sqrt{16 - \pi^2}\right).$$

Hence the ratio of the longer side to the shorter side of the rectangle is

$$\frac{1}{\pi}\left(4 + \sqrt{16 - \pi^2}\right).$$

Most of the following "also-solvers" used the same method: JEANETTE BICKLEY, MARTIN J. BROWN, DAVID DEL SESTO, MARCO A. ETTRICK, MARK EVANS, VICTOR G. FESER, JOHN M. HOWELL, RALPH KING, ROGER KUEHL, JEFF LOVELAND, DOUG MATLOCK, ROBERT MEGGINSON, VINCENT A. MILLER, STANLEY RABINOWITZ, GEORGE W. RAINEY, WADE W. SHERARD, PETER SZABAGA, CHARLES W. TRIGG, *and the* PROPOSER.

502. [Fall 1981] *Proposed by Robert C. Gebhardt, Parsippany, New Jersey.*

Consider $2^k + 2^k = 1^k + 3^k$ for $k = 1$,

and $2^k + 2^k + 2^k = 1^k + 1^k + 1^k + 3^k$ for $k = 1, 2$.

Complete the equations

$2^k + 2^k + 2^k + 2^k = ?$ for $k = 1, 2, 3$,

and $? = ?$ for $k = 1, 2, 3, 4$,

where the left side is a function of 2^k only, and the right side is a function of 1^k and 3^k only.

I. Solution by Stanley Rabinowitz, Digital Equipment Corp., Merrimack, New Hampshire.

We have

$$2^k + 2^k + 2^k + 2^k = 1^k + 1^k + 1^k + 1^k + 1^k + 1^k + (-1)^k + 3^k$$

$$\text{for } k = 1, 2, 3$$

and

$$2^k + 2^k + 2^k + 2^k + 2^k + (-2)^k = 1^k + 1^k + 1^k + 1^k + 1^k + 1^k + 1^k + 1^k + 1^k + 1^k + (-1)^k + (-1)^k + (-1)^k + (-1)^k + (-1)^k + 3^k$$

$$\text{for } k = 1, 2, 3, 4.$$

We arrived at these formulas by the following method:

Let $a, b, \ldots, c \overset{n}{=} d, e, \ldots, f$ mean that

$$a^k + b^k + \ldots + c^k = d^k + e^k + \ldots + f^k \quad \text{for } k = 1, 2, \ldots n$$

where there are the same number of terms on both sides of the equal signs (add extra 0's if necessary). Then a theorem of Tarry says that if

$$a,b,\ldots,c \stackrel{n}{=} d,e,\ldots,f$$

and if x is arbitrary, then

$$a,b,\ldots,c,\ (d+x),\ (e+x),\ \ldots,\ (f+x) \stackrel{n+1}{=}$$
$$d,\ e,\ \ldots,\ f,\ (a+x),\ (b+x),\ \ldots,\ (c+x).$$

We applied this theorem using $x = -1$. See Dickson, *History of the Theory of Numbers*, Vol. II, Chelsea Publishing Company, 1971, page 710.

2. *Solution by Robert E. Megginson, University of Illinois.*

By a little experimentation, it is easy to see that a solution to $2^k + 2^k + 2^k + 2^k = ?$ must involve more than a sum of terms of the form 1^k and 3^k , but it is not explicitly stated what sort of unary or binary functions of 1^k and 3^k are allowed. Thus, I feel perfectly justified in noting that:

$$2^k = (1^k + 1^k)(\ln(3^k))/\ln(1^k + 1^k + 1^k)$$

for any positive integer k, which gives obvious solutions to both parts of the problem.

Also solved by VINCENT A. MILLER, *who found polynomials in k as coefficients of 3^k, and by the* PROPOSER, *who intended a solution of Type 1.*

503. [Fall, 1981] *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, Pennsylvania.*

Find the equation of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ with minimum volume which shall pass through the point $p(r,s,t)$, $0 < r < a$, $0 < s < b$, $0 < t < c$.

Solution by Robert C. Gebhardt, Hopatcong, New Jersey.

The point (r,s,t) is on the ellipsoid, so

$$(\tfrac{r}{a})^2 + (\tfrac{s}{b})^2 + (\tfrac{t}{c})^2 = 1.$$

Solving, get $\quad c^2 = a^2b^2t^2/[a^2b^2 - b^2r^2 - a^2s^2].$

The volume of an ellipsoid is known to be $V = (4/3)\pi abc$. To minimize V, minimize V^2 by setting $\partial V^2/\partial a$ and $\partial V^2/\partial b$ to 0.

$$\partial V^2/\partial a = (\tfrac{4}{3}\pi)^2 \cdot \frac{(a^2b^2 - b^2r^2 - a^2s^2)(4a^3b^4t^2) - (a^4b^4t^2)(2ab^2 - 2as^2)}{(\ldots.)^2} = 0.$$

Setting the numerator to zero, get $a^2b^2 - 2b^2r^2 - a^2s^2 = 0.$ (I)

Similarly, by setting $\partial V^2/\partial b = 0$, get $a^2b^2 - b^2r^2 - 2a^2s^2 = 0.$ (II)

Subtracting (I) from (II) and rearranging the result, get $r/a = s/b$. Similarly get $r/a = t/c$. Putting these results in the equation for the ellipsoid, get $a = r\sqrt{3}$, $b = s\sqrt{3}$, $c = t\sqrt{3}$. This is hardly a surprise. since, if $r = s = t$, the ellipsoid would become a sphere of radius $r\sqrt{3}$, as expected.

Also solved by HARRY S. LIEBERMAN, VINCENT A. MILLER, *and the* PROPOSER.

504. [Fall 1981] *Proposed by Charles W. Trigg, San Diego, California.*

In the square array of the nine non-zero digits

9	2	6
4	1	7
8	3	5

the sum of the digits in each 2-by-2 corner array is 16. Find another arrangement of the nine digits in which the sum of the digits in each corner array is five times the central digit.

Solution by Victor G. Feser, Mary College, Bismarck, North Dakota.

If the central digit is 1 or 2, then the 9, wherever it is placed, will cause one corner to overflow. If the central digit is 5 or greater, then the 1, wherever it is placed, will cause one corner to fall short. If the central digit is 3: if the 9 is placed at the side, it's easy to see there's no solution; if it's placed at the corner, there are a number of cases, but none leads to a solution.

So the central digit is 4. If the 9 is in the corner, there are a number of cases, but none works out. So the 9 is on the side; again, there are a number of cases--and some of them work out.

Ignoring rotations and reflections, there are exactly two solutions:

6	1	7		1	6	7
9	4	8		9	4	3
2	5	3		2	5	8

Both solutions were also found by DAVID INY, ROGER KUEHL, HENRY S. LIEBERMAN, VINCENT A. MILLER, DAVID E. PENNEY (with several extensions to the problem), BOB PRIELIPP, STANLEY RABINOWITZ, KENNETH M. WILKE, , *and the* PROPOSER. JEANETTE BICKLEY, CAROL DIMINNIE, MARK EVANS, ROBERT C. GEBHARDT, SAMUEL GUT, BRUCE KING, JEFF LOVELAND, KIRK MAHONEY, ALEX McKALE, *and* ROBERT MEGGINSON *each discovered one solution.*

505. [Fall 1981] *Proposed by John M. Howell, Littlerock, California.*

A baseball team has all .300 hitters. They never steal a base, get picked off base or hit into a double play. And men on base advance only one base when there is a hit.

a) What is the probability of this team getting one or more runs in an inning?

b) What is the expected number of runs scored by this team per inning?

Solution by Roger Kuehl, Kansas City, Missouri.

a) The probability of a team getting no runs in an inning is the sum of the probabilities of

1) three outs, $(.7)^3 = .343$,

2) two outs and one hit (in any order), followed by an out, $3(.7)^2(.3)^2\cdot(.7) = .3087$,

3) two outs and two hits (in any order), followed by an out, $6(.7)^2(.3)^2\cdot(.7) = .18522$, and

4) two outs and three hits (in any order), followed by the third out, $10(.7)^2(.3)^3\cdot(.7) = .09261$,

that is,

$$.343 + .3087 + .18522 + .09261 = .92953.$$

Thus the probability of getting at least one hit is

$$1 - .92953 = .07047.$$

b) Exactly one run is scored when four hits are made before three outs accumulate. This can occur in

$$\frac{7!}{3!4!} - \frac{6!}{3!3!} = \frac{6!}{4!2!} = 15$$

ways, so the probability of 1 run is

$$15(.3)^4(.7)^3 = .0416745.$$

The desired expected number of runs per inning is the sum of each number of runs times its probability of occurrence. We have

$$E = \sum_{n=1}^{\infty} n\cdot\binom{n+5}{2}(.3)^{n+3}(.7)^3$$

$$= .0416745 + .035066 + .0210039 + \ldots$$

$$= .117334.$$

Also solved by CAROL DIMINNIE, MARK EVANS, ROBERT C. GEBHARDT, DAVID INY, JEFF LOVELAND, *and the* PROPOSER.

506. [Fall 1981] *Proposed by Morris Katz, Macwahoc, Maine.*

"The addition cryptarithm IN + THE = MOOD is not difficult, but the solution cannot be unique because N and E can be interchanged, and so can I and H."

"Even taking account of those interchanges," his friend replied, "there are still many different solutions."

"That is so," agreed the first, "but let me tell you the value of one of those four letters."

I could not hear the letter and the value he whispered to his colleague but the reply was quite clear. "Ah, now the solution is unique except, of course, for the interchange of the two letters of the other pair, and it used every digit that is an odd prime, too."

Solution by Robert E. Megginson, University of Illinois.

It is obvious that $T = 9$ and $MO = 10$. We will construct the possible solutions, first where the addition $N + E$ does not result in a carry, then where a 1 is carried.

Case 1: $N + E \le 9$. Then $I + H = 10$. Since 0, 1, and 9 have been eliminated, here are the possible values for the pair (I,H):

$$(2,8),\ (3,7),\ (4,6) \text{ and interchanges; e.g., } (8,2).$$

The possible values for the triple (N,E,D) are:

$$(2,3,5),\ (2,4,6),\ (2,5,7),\ (2,6,8),\ (3,4,7),\ (3,5,8)$$

and interchanges of N and E. If we take all possible matchings of (N,E,D) with (I,H), we obtain these possible values for (N,E,D,I,H):

$$(2,3,5,4,6),\ (2,4,6,3,7),\ (2,5,7,4,6),$$
$$(2,6,8,3,7),\ (3,4,7,2,8),\ (3,5,8,4,6),$$

along with the solutions differing only in interchanges of N with E and I with H.

Case 2: $N + E \ge 10$. Now $1 + I + H = 10$. Except for the obvious interchanges, here are the possibilities:

(I,H): $(2,7)$, $(3,6)$, $(4,5)$.

(N,E,D): $(4,8,2)$, $(5,7,2)$, $(5,8,3)$, $(6,7,3)$, $(6,8,4)$, $(7,8,5)$.

(N,E,D,I,H): $(4,8,2,3,6)$, $(5,7,2,3,6)$, $(5,8,3,2,7)$,

$$(6,7,3,4,5),\ (6,8,4,2,7),\ (7,8,5,3,6).$$

It is easy to check that if the value of N or of E is specified we do not obtain a unique (up to interchange) solution. Thus, either I or H was specified. Again, it is easy to check that requiring I (or H) to be 2, 3, 4, 6, or 7 does not result in uniqueness. If $I = 5$ we obtain the unique solution $(6,7,3,4,5)$, while requiring $I = 8$ gives only $(3,4,7,2,8)$.

Since the solution uses every digit that is an odd prime. the only possibility is that I (or H) was specified to be 5, giving the solution $(N,E,D,I,H) = (6,7,3,4,5)$. The cryptarithm must have been: $46 + 957 + 1003$, except for the possible interchanges.

Also solved by MARK EVANS, VICTOR G. FESER, ROGER KUEHL, VINCENT A. MILLER, CHARLES W. TRIGG, *and the* PROPOSER.

FESER noted that the problem "seems to be badly stated . . . . It says there's a unique solution, and--as a sort of bonus, a fact not needed for solution--'it uses every digit that is an odd prime.' In fact this condition is necessary for finding a unique solution." KATZ replied to FESER'S comment by stating that "clearly the friend needed only the value given to him and, if we had heard that comment, it, too, would have been enough for us. Since we could not hear it, it should be no surprise that other information might be necessary."

507. [Fall 1981] *Proposed by Herbert R. Bailey, Robe Polytechnic Institute, Terre Haute, Indiana.*

A unit square is to be covered by three circles of equal radius. Find the minimum necessary radius.

Solution edited from that submitted by David Iny, Rensselaer Polytechnic Institute, Troy, New York.

In the accompanying figure, H is the midpoint of side BC. The three covering circles pass through $ADFE$, $FCHG$, and $HBEG$, so the square of their diameter is given by

$$d^2 = CH^2 + HG^2 = HB^2 + BE^2 = AE^2 + AD^2$$
$$= \frac{1}{4} + BE^2 = \frac{1}{4} + BE^2 = (1 - BE)^2 + 1^2.$$

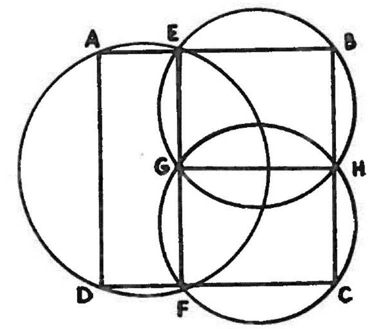
Solving for BE, we get

$$\frac{1}{4} + BE^2 = 1 - 2BE + BE^2 + 1,$$
$$2BE = \frac{7}{4},$$

so

$$d^2 = \frac{1}{4} + \frac{49}{64} = \frac{65}{64} \text{ and } d = \frac{\sqrt{65}}{8}$$

The desired radius is therefore $\sqrt{65}/16 \approx .50389$. Since $AE = 1/8$ is less than $1/3$, the configuration in the accompanying figure is best.



Also solved by the PROPOSER. *Thee incorrect solutions were submitted.*

508. [Fall 1981] *Proposed by Bruce W. King, Burnt Hills, New York.*

When Professor Umbugio asked his calculus class to find the derivative of y^2 with respect to x^2 for the function $y = x^2 - x$, his nephew Socrates Umbugio found $\frac{dy}{dx} \cdot \frac{y}{x}$ and obtained the correct answer, Help the professor to enlighten his nephew about taking derivatives.

Solution by Stanley Rabinowitz, Digital Equipment Corporation, Merrimack, New Hampshire.

The formula used is actually correct, for by the chain rule we have:

$$\frac{d(y^2)}{d(x^2)} = \frac{d(y^2)}{dx} / \frac{d(x^2)}{dx} = (2y\frac{dy}{dx}) \;/\; (2x) = \frac{dy}{dx} \cdot \frac{y}{x} .$$

RABINOWITZ receives three gold stars. Only he and the PROPOSER gave a complete solution. Credit for partial solutions goes to each of the following persons, who agreed the answer was correct for the stated function or for some special class of functions only: MARK EVANS, VICTOR G. FESER, RALPH KING, ROBERT E. MEGGINSON, VINCENT A. MILLER and PETER SZABAGA.

509. [Fall 1981] *Proposed by Jack Garfunkel, Queens College, New York.*

Given a triangle ABC with its incircle I, touching the sides of the triangle at points L, M, N. Let P, Q, R be the midpoints of arcs NL, LM, and MN respectively. Form triangle DEF by drawing tangents to the circle at P, Q, and R. Prove that the perimeter of triangle DEF $\le$ perimeter of triangle ABC.

Solution by David Iny, Renssalaer Polytechnic Institute, Troy, New York.

Let r be the radius of the incircle (I). The bisectors of angles A, B, and C of triangle ABC all pass through I. Then the perimeter of triangle ABC is

$$2r\cot\frac{A}{2} + 2r\cot\frac{B}{2} + 2r\cot\frac{C}{2}.$$

Similarly, the perimeter of triangle DEF is

$$2r\cot\frac{D}{2} + 2r\cot\frac{E}{2} + 2r\cot\frac{F}{2}$$

$$= 2r\cot\frac{B+C}{4} + 2r\cot\frac{C+A}{4} + 2r\cot\frac{A+B}{4}$$

since $D = (B + C)/2$, etc., because the angle formed by two tangents to a circle is measured by half the difference of the intercepted arcs. The theorem follows because $\cot\theta$ is convex for $0 < \theta < \pi/2$, so

$$2\cot\frac{D}{2} = 2\cot\frac{A+B}{4} \le \cot\frac{A}{2} + \cot\frac{B}{2}, \text{ etc.} \tag{1}$$

Clearly, equality holds if and only if ABC is equilateral.

Also solved by M.S. KLAMKIN, JEFF LOVELAND, ROBERT E. MEGGINSON, *and the* PROPOSER. KLAMKIN pointed out that inequality (1) and more general results are obtained in his paper "Inequalities for a Triangle Associated with n Given Triangles," *Publications de la Faculte' D'Electrotechnique de L'Universite a Belgrade,* No. 330 (1970) pp. 3-7.

Late solution to problem 466 by FRED GALVIN.

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