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PI MU EPSILON JOURNAL
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Chapter reports, books for review, problems for solution and solutions to problems, should be mailed directly to the special editors found in this issue under the various sections. Editorial correspondence including manuscripts (two copies) and news items should be mailed to THE EDITOR OF THE Pi MU EPSILON JOURNAL. South Dakota School of Mines and Technology. Rapid City, SD 57701 For manuscripts, authors are requested to identify themselves as to their class or year if they are undergraduates or graduates, and the college or university they are attending, and as to position if they are faculty members or in a non-academic profession.

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INTEGRATION: WHY YOU CAN AND WHY YOU CAN'T

by *Rick Miranda*
Colorado State University

At most colleges and universities, a large part of the second semester of calculus is devoted to the arcane subject commonly known as "techniques of integration". The basic problem is to find a closed-form expression for $\int f(x)dx$ where $f(x)$ is a specific function of the variable x. Typically, the following methods are discussed:

- 'forward' substitutions $x = g(u)$
- 'backward' substitutions $u = h(x)$
- integration by parts
- the use of exponentials and logarithms
- trigonometric substitutions
- inverse hyperbolic trig functions
- powers of sines and cosines
- integrals with quadratics
- partial fraction decompositions.

There are two logical reactions to this subject:

a) There is too much material here.

b) There is not enough material here.

For anyone who has taken or taught this course, (a) hardly needs explanation. Firstly, the mastery of all these techniques requires quite a bit of rote memorization of basic integrals, especially for the average student. Secondly, when faced with an integration problem, the 'menu' of possible techniques to try to apply is large enough to make the decision process fairly complicated. Finally, with extensive tables and (lately) computer programs which integrate all functions encountered in this course, the motivation to delve into this subject with one's sleeves rolled up is naturally diminished, and this is made worse by the amount there seem to be to know.

Have you ever heard (b) from a student of this subject? Well,

now you have, and let me explain why. After a good solid course on the techniques of integration, including a thorough discussion of the topics listed above, I could well come away with the following broad classification of integrals:

i) The integrals which I can find.

ii) The integrals which I cannot find.

Statement (b) is one reaction to the existence of the second class.

Most of the integrals encountered in the course are of type (i) (or should be, by the end of the semester). A student, in fact, may never see an integral of type (ii), and may conclude that all integrals are of type (i), for the appropriate choice of "I"; since he (or she) knows in his gut that he can't possibly solve all integration problems, the conclusion is that he is <u>not</u> the appropriate choice for "I", and that the subject is much too complicated for mere mortals to think about.

If an integral of type (ii) <u>is</u> seen in this course, it is usually in one of the "set up but do not evaluate the integral which computes..." problems on an exam; when going over the questions on the next day, the teacher nay make a remark to the effect that "we can't find this integral..." and the subject is embarrassingly dropped. Generally, no attempt is made to explain why some integrals can be found and some can't, and we're back to reaction (b) (on a slightly different level): There is something missing here.

In this article I'd like to discuss why there are integrals of types (i) and (ii), and try to explain the fascinating relationship between this apparently <u>analytic</u> subject and the much more <u>geometric</u> subject of algebraic plane curves.

Let me begin by stating a theorem.

Theorem. Let $R(t)$ be a rational function of the variable t, i.e., $R(t)$ is the ratio of two polynomials. Then

$$\int R(t)\,dt$$

can be found.

(Of course, actually finding a closed-form expression for it involves factoring polynomials and solving linear equations, and is a formidable task in itself -- but I won't address these problems here.)

In my view, it is not unfair to say that, even given the mass of material devoted to integration techniques, this is the <u>only true</u> theorem in this course; the other topics covered are really just methods to use as the occasion arises. This being the case, one would think that this would be the focal point of this course. However, it is hardly ever stated explicitly, and often the details of the process of partial fractions (which is the proof of this theorem) is given much more weight than the simple and obviously powerful statement itself. This is understandable, since carrying out the partial fraction decomposition is a complicated and cumbersome task, even in fairly simple situations, and requires some attention. However, I think it is a mistake not to rise above the fray and drive the point home that here is a large and common class of functions which are all "of type (i)" -- I can integrate them!

If you grant that this is the 'only' theorem of this type, then your mind should naturally turn to the following: can other integrals be brought to this form by clever substitutions, and can this theorem therefore achieve a wider scope of application? The well-known answer to this question is: Sometimes, if you get lucky.

Example. Integrate $\int\sqrt{1+x^2}\,dx$.

Solution. Substitute $x = \dfrac{2t}{(1-t^2)}$. Then $1 + x^2 = \left(\dfrac{(1+t^2)}{(1-t^2)}\right)^2$ and

$dx = \dfrac{(2 + 2t^2)}{(1-t^2)^2}$ so the above integral transforms to $\displaystyle\int\frac{2+4t^2+2t^4}{(1-t^2)^3}\,dt$,

and the theorem applies.

This seemed pretty lucky. What if I try $\int\sqrt{1+x^3}\,dx$? In this case I'm stuck for a clever substitution. What is going on here? In order to fix our attention on a certain general class of functions, consider the following.

Definition. A function $y = y(x)$ is <u>algebraically dependent</u> on x if there is a polynomial $f(x_1,x_2)$ in two variables, such that $f(x,y(x))$ is <u>identically</u> <u>zero</u>.

Examples. $y = \sqrt{x}$ $(f(x_1,x_2) = x_1 - x_2^2)$

$$y = x^{4/5} \quad (f(x_1,x_2) = x_1^4 - x_2^5)$$

$$y = \sqrt{1 + x^3} \quad (f(x_1,x_2) = x_1^3 + 1 - x_2^2)$$

The integrals $\int\sqrt{1 + x^2}\,dx$, $\int\sqrt{1 + x^3}\,dx$, etc., are examples of integrals which involve functions of x which are algebraically dependent on x, and this is the class of functions which I want to focus on. Our general problem can be formulated as follows.

The General Problem of Integration of Algebraic Functions.

Let $R(x_1, x_2)$ be a rational function of two variables.

Let $y = y(x)$ be algebraically dependent on x.

Can $\int R(x, y(x))dx$ be found?

The answer is again: sometimes. But it doesn't have anything to do with luck. Let's try to think about this systematically. If $y = y(x)$ is algebraically dependent on x, then there is this polynomial $f(x_1,x_2)$ such that $f(x, y) \equiv 0$. Now the equation $f(x_1,x_2) = 0$ defines a so-called "algebraic curve" in the (x_1,x_2)-plane, and $(x, y(x))$ always lies on this curve. The properties of this curve should therefore be important in studying $y(x)$. Central for us is the following property.

Definition. Let $f(x_1,x_2)$ be a polynomial in two variables. The curve $C = \{(x_1,x_2)|f(x_1,x_2) = 0\}$ is rationally parametrized if there are rational functions $x_1 = x_1(t)$, $x_2 = x_2(t)$, such that $f(x_1(t), x_2(t))$ is identically zero as a function of t.

In this case the point $(x_1(t), x_2(t))$ will lie on the curve C for all values of t. Let's look at any easy example.

Example. Let $f(x_1,x_2) = x_1^2 + x_2^2 - 1$, so that the curve C is the unit circle. Then C is rationally parametrized by $x_1(t) = \dfrac{(1 - t^2)}{(1 + t^2)}$, $x_2(t) = \dfrac{2t}{(1 + t^2)}$. (Check this!) This is not magic. Note that the point $P = (-1,0)$ is on C. Let L_t be the line through P with slope t; an equation for L_t is $x_2 = t(x_1 + 1)$. For any t, this line L_t will intersect the circle C in two points, one of which is, of course, P. Call the other point P_t. A little algebra will convince you that $P_t = \left(\dfrac{(1 - t^2)}{(1 + t^2)}, \dfrac{2t}{(1 + t^2)}\right)$, giving the explicit parametrization above.

The importance of a rational parametrization for the curve C is demonstrated by the following.

Theorem. Let $R(x_1,x_2)$ be a rational function of two variables and let $y = y(x)$ be algebraically dependent on x, with $f(x, y(x))$ identically zero. Assume that the curve $C = \{(x_1, x_2)|f(x_1,x_2) = 0\}$ can be rationally parametrized. Then $\int R(x, y(x))dx$ can be found.

Proof. Let $x_1 = x_1(t)$, $x_2 = x_2(t)$ be the parametrization of C. Note that $x = x_1(t)$, $y = x_2(t)$ in this case; make this substitution into the integral. One gets $\int R(x_1(t), x_2(t))\left(\dfrac{dx_1}{dt}\right)dt$, which has a rational integrand. We can now apply the theorem.

The above proposition seems to be constructive, too; the only hitch is in parametrizing the curve C. In particular, the immediate question is: Which curves C can be rationally parametrized, and how? If $f(x_1,x_2) = ax_1 + bx_2 + c$, so that the degree of f is one and C is a line, then clearly C may be rationally parametrized; $x_1 = bt + z_1$, $x_2 - at + z_2$, where (z_1,z_2) is any point on C. In this case $y(x) = x_2(x_1) = -(\frac{a}{c})x - (\frac{b}{c})$ is a linear function of x and any rational expression in x and y can be immediately reduced to a rational function of x alone, so the above process is not too enlightening.

Fortunately, there is one other large class of curves which can be parametrized.

Proposition. Any conic C (i.e., defined by $f(x1, x2) = 0$ where $f(x_1,x_2)$ is of degree 2) can be rationally parametrized.

Proof. Let me present two proofs of this statement, one algebraic and one geometric in spirit. The first step of the algebraic proof is to change coordinates from (x_1,x_2) to (x,y) so that $f(x_1,x_2)$ becomes $g(x,y) = \dfrac{x^2}{a^2} \pm \dfrac{y^2}{b^2} - 1$, the "standard form" for a conic. This is a linear change of coordinates, so that if we can parametrize $g(x,y) = 0$ by rational functions, we will be able to transport this parametrization

to $f(x_1,x_2)$. The second step is to explicitly parametrize the standard conic $g(x,y) = 0$. Here is one way.

$$x = a\frac{b^2 \mp a^2t^2}{b^2 \pm a^2t^2}, \quad y = \frac{2ab^2t}{b^2 \pm a^2t^2}.$$

A more geometric proof is afforded by following the hint of the circle example. Pick any point P on the conic C. Parametrize the lines through P by their slopes: if P = (x_0,y_0), let L_t be the line $y - y_0 = t(x - x_0)$ through P with slope t. Now intersect L_t with the conic C; one will get two points, one of which is P, the other is $P_t = (x(t), y(t))$; it is not hard to see that $x(t)$ and $y(t)$ are rational parametrization of the conic C.

Q.E.D.

Note that in the above argument, one might want to use a vertical line sometimes where the slope "is infinity". This leads naturally into some elementary concepts of projective geometry, which I do not wish to discuss at this time.

As promised by our theorem, a proposition about parametrizing curves should give us a nice application to integrals. Here's the result for conics restated for this purpose:

Corollary. For any numbers a, b and c, the integral

$$\int R(x, \sqrt{ax^2 + bx + c})\,dx$$

can be found (where $R(x_1,x_2)$ is a rational expression in two variables).

Proof. If $y = \sqrt{ax^2 + bx + c}$, then y is algebraically dependent on x; $f(x,y) = y^2 - ax^2 - bx - c$ is identically zero. Since $f(x,y)$ has degree 2, the curve $f(x_1,x_2) = 0$ defines a conic, and therefore may be rationally parametrized. Now the theorem applies.

Q.E.D.

In our course on techniques of integration, a lot of time is spent developing methods for handling integrals involving $\sqrt{ax^2 + bx + c}$, but the general result above is very rarely brought out into the open -- I think it should be.

As long as we're here...

Parametrizing conics has been fun for millenia. Let us recall our parametrization of the circle

$$x = \frac{1 - t^2}{1 + t^2}, \qquad y = \frac{2t}{1 + t^2}$$

Note that if t is a rational number, then x and y will both be rational numbers also. So what? Well, write $t = \frac{u}{v}$, with u and v integers. Then, clearing denominators, we see that

$$x = \frac{v^2 - u^2}{u^2 + v^2}, \qquad y = \frac{2uv}{u^2 + v^2},$$

and $x^2 + y^2 = 1$ means that $(v^2 - u^2)^2 + (2uv)^2 = (u^2 + v^2)^2$. In other words, $(v^2 - u^2, 2uv, u^2 + v^2)$ is a Pythagorean triple. Moreover, it is an elementary theorem from number theory that all Pythagorean triples come this way. This very geometric approach to number theory was pioneered by the Greek Diophantus, and has been refined into some amazing results relating the geometry of solutions to equations and the number theory which naturally arises.

But back to integration. Recall the following magic trick for integrating an expression involving $\sin\theta$ and $\cos\theta$: make the substitution $\theta = 2\arctan(t)$. Why does this work? A little trigonometry and differentiation formulas (including the dreaded half-angle formulas) will produce

$$\cos\theta = \frac{1 - t^2}{1 + t^2}, \quad \sin\theta = \frac{2t}{1 + t^2}, \qquad d\theta = \frac{2dt}{1 + t^2}$$

and so this substitution replaces the trigonometric integrand with a rational integrand, and now we use the theorem. From our vantage point, this amazing and ad hoc substitution, which at first glance works "because it works", is seen as exactly substituting the rational parametrization of the circle which we've become quite familiar with for the trigonometric parametrization $x = \cos\theta$, $y = \sin\theta$. Hence we have the following (without any magic!):

Corollary. If $R(x_1,x_2)$ is a rational expression in two variables, then $\int R(\cos\theta, \sin\theta)\,d\theta$ can be found.

Recall the hyperbolic functions $\sinh(x)$ and $\cosh(x)$, so called because they give a parametrization of the hyperbola $x_1^2 - x_2^2 = 1$; $\cosh^2(x) - \sinh^2(x) = 1$ for any x , so $(\cosh(x), \sinh(x))$ always lies on the unit hyperbola. We now know that the unit hyperbola can also be rationally parametrized by

$$x_1 = \frac{1 + t^2}{1 - t^2}, \quad x^2 = \frac{2t}{1 - t^2} \quad .$$

Our main theorem now yields the following immediately.

Corollary. If $R(x_1,x_2)$ is any rational expression in two variables, then $\int R(\cosh(x), \sinh(x))dx$ can be found.
(Using the chain rule it is easy to see that $dx = \frac{2dt}{1 - t^2}$ using the above substitutions for $\cosh(x)$ and $\sinh(x)$.)

This just about exhausts the applications of the existence of rational parametrizations for conics to the theory of integration. Can we proceed to higher degree curves? Well, there are curves which are not conics, but which can still be rationally parametrized:

Example. $y = Sf^{iq}$ satisfies $f(x,y) = y^q - x^p \equiv 0$. This is parametrized by $x = t^q$, $y =$ if. Hence,

Corollary. $\int R(x,x^{p/q})dx$ can be found, where $R(x_1,x_2)$ is any rational expression in two variables.

Example. The lemniscate $f(x,y) = (x^2 + y^2)^2 - (x^2 - y^2) \equiv 0$ (draw this!) has a rational parametrization

$$x = \frac{2t(t + 1)}{4t^2 + 1}, \quad y = \frac{2t(4t^2 - 1)}{(4t^2 + 1)(2t + 1)} \quad .$$

To find this, one intersects the lemniscate C with a circle C_t centered at $(t, -t)$ of radius $\sqrt{2}t$, so that $\underline{0} = (0, 0)$ is on C_t. In fact, $C \cap C_t$ consist of $\underline{0}$ and one other point P_t, which has the above coordinates.

The above example looks like I'm just showing off -- maybe that's right. Finding parametrizations for plane curves is not easy, and in fact most curves $\{f(x,y) = 0\}$ cannot be rationally parametrized! One example is $y^2 - x^3 - 1 = 0$, which defines the algebraic function $y = \sqrt{1 + x^3}$, which I got stuck on earlier. (If you're good with polynomials, you might try to prove that $y^2 - x^3 - 1 = 0$ can't be rationally parametrized.) One corollary of our discussion, then, is that $\int\sqrt{1 + x^3}dx$ can't be expected to be found with our present techniques. In general, the integrals involving the square root of a cubic polynomial in x are classically called elliptic (they arise in computing various quantities associated to an ellipse, e.g., arclength, etc.) and can't be solved in closed form using elementary functions. Now we know why: behind the whole problem lies an unparametrizable curve!

The problem of parametrizing curves actually led to the invention of topology. Assume $\{f(x,y) = 0\}$ is parametrized. This gives a nice continuous function from {t-space} to {solutions to $f(x,y) = 0$}, sending a typical t to $(x(t),y(t))$. There's nothing in all of the above discussion which says that t can't be a complex number instead of just a real number; after all, we went "backward" to rational t's for a number-theoretic application -- why not go "forward" to complex t's? Recall that {complex t-space} is a 2-sphere, if you add the point at ∞ (which, again, we saw earlier was not unreasonable). So the above parametrization can be viewed as a nice continuous function from the 2-sphere to complex solutions (x,y) to $f(x,y) = 0$. Therefore, intuitively, these complex solutions better look pretty much like a sphere. However, in lots of examples, this solutions set doesn't look anything like a sphere. For example, the complex solutions to $y^2 = 1 + x^3$ made up, topologically, a torus. So there seems to be a real topological obstruction here to parametrizing this curve, and the attempt to understand this phenomenon led to the development of modern topology.

It turns out that the general curve of degree at least 3 (i.e., $f(x,y)$ has degree ≥ 3) cannot be rationally parametrized; however, there are special curves which can be, as the examples above illustrate. The general problem of the existence of rational parametrizations of plane curves ultimately led to the flowering of the field of algebraic geometry, and is quite complicated.

Have we then simply substituted one field of ignorance for another? No, not really. I think we have isolated the essential problem, which is one of parametrization, not integration, and along the way elucidated many of the standard results of integration theory, all in terms of one basic idea. This kind of overview can only benefit any student of this subject, can put into its proper perspective the more mundane aspects

of the techniques of integration, and hopefully motivate both student and teacher with a broader picture of the field,

One last highly beneficial side effect to this approach is that, on the horizon of this subject, which seems to some, at first glance, to be a "dead end" mathematically, we see the following topics rising tantalizingly out of the mist:

- the theory of conics
- number theory, and diophantine equations
- topology
- complex variables
- higher analysis
- algebraic geometry.

This is a large part of modern mathematics! Do all hard problems (like why I can't integrate everything) lead to such unexpected, diverse areas? I don't know, but even one example is an occasion for celebration by a lover of mathematics.

REGIONAL MEETINGS

Many regional meetings of the Mathematical Association of America regularily have sessions for undergraduate papers. If two or more colleges and at least one local chapter help sponsor or participate in such undergraduate sessions, financial help is available up to $50. Write to:

Dr. Richard Good
Department of Mathematics
University of Maryland
College Park, Maryland 20742

LITTLE KNOWN COMPUTER LANGUAGES

Author Unknown

PASCAL, FORTRAN, COBOL -- these programming languages are well known and {more or less} well loved throughout the computer industry. There are numerous other languages, however, that are less well known yet still have ardent devotees. In fact, these little known languages generally have the most fanatic admirers. For those who wish to know more about these obscure languages -- and why they are obscure -- we present the following catalog:

--SIMPLE--SIMPLE is an acronym for Sheer Idiots Monopurpose Programming Linguistic Environment. This language, developed at the Hanover College for Technological Misfits, was designed to make it impossible to write code with errors in it. The statements are. therefore, confined to **BEGIN, END** and **STOP.** No matter how you arrange the statements, you can't make a syntax error.

--SLOBOL--- SLOBOL is best known for the speed, or lack of it. Although many compilers allow you to take a coffee break while they compile, **COBOL** compilers allow you to travel to Bolivia to pick the coffee. Three or four programmers are known to have died of boredom sitting at their terminals while waiting for a **SLOBOL** program to compile. Weary **SLOBOL** programmers try to return to a related {but infinitely faster} language, **COCAINE.**

--VALGOL--- From its modest beginnings in Southern California's San Fernando Valley, **VALGOL** is enjoying a dramatic surge of popularity across the industry.

VALGOL commands include **REALLY- LIKE, WELL AND Y'NOW.** Variables are assigned with the **=LIKE** and **"TOTALLY** operators. Other operators include the "California **Booleans," FERSURE** and **NOWAY.** Repetitions of code are

handled in FERSURE loops. Here is a sample VALGOL program.

```
LIKE,Y'NOW {IMEAN} START
IF
A = LIKE BITCHEN AND
B = LIKE TUBULAR AND
C = LIKE GRODY**M4
{FERSURE}**2
THEN
FOR I = LIKE 1 TO OH MAYBE 100
DO WAH + {DITTY**2}
BARF{I} = TOTALLY GROSS{OUT}
SURE
LIKE BAG THIS PROGRAM
REALLY
LIKE TOTALLY {Y'NOW}
```

VALGOL characterized by its unfriendly error messages. For example, when the user makes a syntax error, the interpreter displays the message GAG ME WITH A SPOON!

--LAIDBACK---Historically, VALGOL is a derivative of LAIDBACK, which was developed at the {now defunct} Marin County Center for T'ai Chi, Mellowness and Computer Programming, as an alternative to the more intense atmosphere in nearby Silicon Valley.

The Center was ideal for programmers who liked to soak in hot tubs while they worked. Unfortunately, few programmers could survive there for long, since the Center outlawed pizza and RC Cola in favor of bean curd and Perrier.

Many mourn the demise of LAIDBACK because of its reputation as a gentle and nonthreatening language. For example, LAIDBACK responded to syntax errors with the message: SORRY MAN, I CAN'T DEAL BEHIND THAT.

--SARTRE---Named after the late existential philosopher, SARTRE is an extremely unstructured language. Statements in SARTRE have no purpose, they just are. Thus, SARTRE programs are left to define their own functions. SARTRE programmers tend to be boring and depressed and are no fun at parties.

--FIFTH---FIFTH is a precision mathematical language in which the data types refer to quantity. The data types range from CC, OUNCE, SHOT and JIGGER to FIFTH {hence the name of the language}, LITER- MAGNUM and BLOTTO. Commands refer to ingredients such as CHABLIS, CHARDONNAY, CABERNET, GIN, VERMOUTH, VODKA, SCOTCH and WHATEVERSAROUND.

The many versions of the FIFTH language reflect the sophistication-and financial status of its users. Commands in the ELITE dialect include VSOR and LAFITE, while commands in the GUTTER dialect include HOOTCH and RIPPLE. The latter is a favorite of frustrated FORTH programmers who end up using this language.

--C---This language was named for the grade received by its creator when he submitted it as a class project in a graduate programming class. C- is best described as a "low level" programming language. In fact, the language generally requires more C-statements than machine-code statements to exercise a given task. In this respect, it is very similar to COBOL-

--LITHP---This otherwise unremarkable language is distinguished by the absence of an "S" in its character set. Programmers and users must substitute "TH". LITHP is said to be useful in prothething lithtth.

--DOGO---Developed at the Massachusetts Institute of Obedience Training, DOGO heralds a new era of computer-literate pets. DOGO commands include SIT, STAY, HEEL and ROLL OVER. An innovative feature of DOGO is "puppy graphics," a small cocker spaniel that occasionally leaves a deposit as he travels across the screen.

--FOCUSALL---a language designed to run on small DEC machines with minimal memory. Its only supported distribution is paper tape, for loading in from an ASR-33 teletype. This takes 20 minutes, after which the user is greeted with the message:

CONGRATULATIONS! YOU HAVE JUST LOADED FOCUSALL!

The interpreter is then ready to accept any valid command. The only valid command is:

LOAD FOCUSALL

which causes the system to once again load the interpreter from paper tape.

The power of the language comes from the fact that preceding a command with a statement line causes **it** to be stored as a program line for later execution as in the following example:

```
100 LOAD FOCUSALL
110 LOAD FOCUSALL
150 LOAD FOCUSALL
```

The pronunciation of the name is much more flexible than the language itself. You pronounce **it** according to your mood. Actually, the name came from a combination of DEC FOCAL, a PDPB DELIGHT, and the habit we Optics Lab types in days of yore had of referring to a lead engineer as "Focus Man" {should be "Focus Person"".}. Somebody would then chime out: "and he's gonna focus every chance he gets!"

--PINBOL---PINBOL is best known for the chance involved in making its program run. Three tries at running are allowed, after which the message **"GAME** OVER- INSERT QUARTER AND TRY AGAIN" is displayed.

Some allowable PINBOL instructions and their meanings are:

LEFT FLIPPER	Illogical Left Shift
RIGHT FLIPPER	Illogical Right Shift
SHOOT	Try to Run

PINBOL is known to be extremely addictive. Those who are fluent PINBOL **programmers** are known as PINBOL WIZARDS.

—FASTBOL—commonly known as a QUICKIE. Error messages include: **"COMPUTUS** INTERRUPTUS-" A closely related language is **NOONER.**

--GERITOL---This language is characterized by the habits of its ardent users. Instructions frequently forget their function while executing and conclude with the "I USED TO KNOW THAT" condition code. Loops tend to repeat frequently at sporadic intervals, even when not **intiated.**

ANOTHER DAY AT THE RACES

by Edward Anderson
West Virginia University

In preparing a presentation of "A Day at the Races" by William Tomcsanyi (*Pi Mu Epsilon Journal*, Spring *1982)* for our chapter of Pi Mu Epsilon, West Virginia Alpha, I came across a simplification of the procedure which reduces the calculations in Mr. Tomcsanyi's article.

The impetus for this simplification was Mr. Tomcsanyi's attempt to solve an 8 x 8 matrix representing every horse in an eight-horse race, which if successful would have shown that it is possible to bet on every horse in a given race and come out ahead no matter which one wins. His example produced negative results, indicating no solution, but he stated, "There probably does exist some combination of odds that would somehow yield positive results to the eight equations."

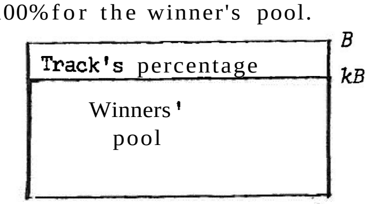
In fact, this conjecture is not true. To prove that it is not, two important relations are used:

I) The SUB of the bets placed on each of the horses is constant throughout the problem:

$$x_1 + x_2 + \ldots + x_8 = S.$$

II) The sum of the fractions of the total pot bet on each horse as computed from the odds is greater than one.

The second of these occurs because the track and the state take a portion of the total amount bet, leaving less than 100% for the winner's pool. Typically, the winners' pool would be about 80% of the total, but to be as general as possible, let us say that the total bet by everyone at the track is B and the winners' pool is US, where k < 1.00. Pictorially, the winners' pool would look like this:



Now, the odds on each horse are computed based on the amount bet on that horse compared to the total winners' pool. If the odds are a to 1, then $a = (kB - h)/h$, where h is the total amount bet on that horse,

or $(a+1) = kB/h$. In an ideal race, where the winners' pool were 100% of the amount bet, we would have $(a+1) = B/h$, or $1/(a+1) = h/B$, the fraction of the total which was bet on this horse. Here, though, we have $1/(a+1) = h/kB$, and the sum of the eight fractions is greater than one:

$$\frac{1}{a_1+1} + \frac{1}{a_2+1} + \cdots + \frac{1}{a_8+1}$$

$$= \frac{1}{kB}(h_1 + h_2 + \cdots + h_8)$$

$$= 1/kB\ (B) = 1/k > 1 .$$

Let us now go back to the 8 x 8 system used by Mr. Tomcsanyi, letting a_i = the odds on horse i and P = the desired profit, as before, but changing x_i to the total dollar amount to be bet on horse i, instead of the number of \$2. bets. The revised matrix is as follows:

$$\begin{bmatrix} a_1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & a_2 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & a_3 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & a_4 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & a_5 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & a_6 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & a_7 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & a_8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = P \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Notice that each row is just an expression for the profit as the amount won minus the amount lost. Since the amount lost is the total amount bet minus the amount bet on the winning horse, a simple means of writing each row equation would be

$$a_i x_i - (S - x_i) = P,$$

or $$(a_i + 1)x_i = P + S.$$

This is easily solved for x_i, giving

$$x_i = \frac{P+S}{a_i+1} .$$

To find S, we use this formula to substitute for each x_i in I:

$$(P+S)\left(\frac{1}{a_1+1} + \frac{1}{a_2+1} + \cdots + \frac{1}{a_8+1}\right) = S.$$

From II, the sum of the $1/(a_i+1)$'s is greater than one. We showed that this number was $1/k$. Thus

$$(P+S)\left(\frac{1}{k}\right) = S, \quad \text{or} \quad P = (k-1)S.$$

Since we have chosen P to be positive and k is less than one, it follows that S must be negative. Thus there is never any way to bet on every horse in a race and come out ahead no matter who wins, even in an ideal situation where the track takes nothing for itself. It is a simple matter to generalize this argument for any size race.

But the steps used in this argument do more than just prove that no solution exists involving every horse in the race; some interesting side benefits fall out along the way. We can now

1) Determine whether there exists a combination of bets yielding a positive solution for any number of horses less than the total;

2) Estimate the total amount which must be bet to produce the desired profit before the individual amounts are computed; and

3) Find the individual bets with a minimum of calculations.

Taken together, these form a three-step process which is as general as the matrix form and, once you get used to it, as simple to compute as the general form of the 3 x 3:

1) You wish to bet on n horses in the race. Compute

$$\frac{1}{a_1+1} + \frac{1}{a_2+1} + \cdots + \frac{1}{a_n+1}$$

If this sum is less than one, there exists a positive solution.

2) Using the sum above as g, plug it into the equation

$P = (\frac{1}{a} - 1)\,S$ and solve for S. You may choose any value for P, but the bigger P is, the bigger S becomes.

3) Once you have chosen P and computed S, compute the amount to bet on each horse by

$$x_i = \frac{P+S}{a_i+1}$$

With this three-step system, it is just as easy to bet on four or five horses as on three, and since the calculations are shorter, it can all be done using later odds.

ON THE DISCRETE LYAPUNOV AND RICCATI MATRIX EQUATIONS

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ABSTRACT

In this note, the inequalities, which are satisfied by the determinants of the positive definite solutions of the discrete algebraic Riccati and Lyapunov matrix equations, are presented. The results give lower bounds for the product of the eigenvalues of the matrix solutions. Also for a discrete Lyapunov equation, we present an algorithm to determine wider what conditions a positive diagonal solution will exist. If all the conditions are satisfied, the algorithm also provides such a diagonal solution.

I. INTRODUCTION

The discrete algebraic Riccati and Lyapunov matrix equations have been used widely in various areas of engineering system theory, particularly in control system theory. The techniques of solving these equations numerically are well-established [4]. Those techniques are mostly iterative algorithms which require making an initial guess of the solution. So if these initial guesses are chosen wisely, one can save a lot of unnecessary computations. Therefore, to obtain precise estimates of the "sizes" of the solutions, we provide here lower bounds for the determinants of the matrix solutions of the two equations. Also for the discrete Lyapunov equations, we address the question of the existence of a positive diagonal solution for this equation and derive an algorithm to provide such a solution if all the conditions are satisfied.

In the following, the notations x^T, $\lambda_i(x)$, $tr(x)$ and $|x|$ denote the transpose, eigenvalue, trace and determinant of the matrix x, respectively. Also for our derivation later, we will make use of the following results [1,3].

i) For any $n\times n$ matrices L and H with $L > 0$

$$tr(L^{-1}HLH^T) \geq \sum_{i=1}^{n} |\lambda_i(H)|^2 \geq \frac{1}{n}[tr(H)]^2 \tag{1}$$

ii) For any real $n\times n$ matrices R and S such that $R = R^T > 0$, $S = S^T > 0$

$$|R|^{1/n} = \min_{|S|=1} \frac{tr(RS)}{n} \tag{2}$$

iii) For any $m\times n$ matrix Y, $n\times m$ matrix Z, $n\times n$ matrix W and $m\times m$ matrix x, we have the following property

$$[W + Zx^{-1}Y]^{-1} = W^{-1} - W^{-1}Z[x + YW^{-1}Z]^{-1}YW^{-1} \tag{3}$$

II. THE RICCATI EQUATION

In this section, we derive a lower bound for the determinant of the discrete algebraic Riccati matrix equation

$$P = A^TPA - A^TPB(I + B^TPB)^{-1}B^TPA + Q \tag{4}$$

where A, P, $Q\varepsilon R^{n\times n}$, $B\varepsilon R^{n\times m}$, $Q = Q^T > 0$. Here we assume $|BB^T| \geq |Q|$ and the matrix A is stable, therefore the solution matrix P is positive definite.

<u>*Theorem 1.*</u> The determinant of the positive definite matrix solution P of equation (4) satisfies the following inequality

$$|P| \geq \left[\frac{M + (M^2 + 4n^2|BB^T|^{1/n}|Q|^{1/n})^{1/2}}{2n|BB^T|^{1/n}}\right]^n \tag{5}$$

where $M = \sum_{i=1}^{n} |\lambda_i(A)|^2 + tr(BB^TQ) - n$

Proof. Using (3) with $W = P^{-1}$, $Z = B$, $x^{-1} = I$ and $Y = B^T$, (4) becomes

$$P = A^T[P^{-1} + BB^T]^{-1}A + Q \tag{6}$$

Multiplying (6) by $[P^{-1} + BB^T]$ from the left yields

$$[P^{-1} + BB^T]P = [P^{-1} + BB^T]A^T[P^{-1} + BB^T]^{-1}A + [P^{-1} + BB^T]Q \tag{7}$$

Computing the traces of both sides of (7) and using (1) with $L^{-1} = [P^{-1} + BB^T]$, $H = A^T$ and rearranging terms, we have

$$tr(BB^TP - P^{-1}Q) \geqq M \tag{8}$$

Now using (2) with R = P and note that $|x| = 1/|x^{-1}|$ and $tr(x) = tr(x^T)$, (8) becomes

$$n|BB^T|^{1/n}|P|^{2/n} - M|P|^{1/n} - n|Q|^{1/n} \geqq 0 \tag{9}$$

Solving (9) for $|P|^{1/n}$, we get the inequality (5).

III. THE LYAPUNOV EQUATION

A. Lower Bound for the Solution of the Lyapunov Equation

Setting B = 0 in (4), the result is the discrete algebraic Lyapunov matrix equation

$$P = A^TPA + Q \tag{10}$$

and we present the following theorem.

Theorem 2. The determinant of the positive definite matrix solution P of equation (10) satisfies the following inequality

$$|P| \geqq \left[\frac{n}{n - \sum_{i=1}^{n} |\lambda_i(A)|^2}\right]^n |Q| \tag{11}$$

Proof. Multiplying (10) by P^{-1} from the left and computing the traces of both sides yield

$$n = tr(P^{-1}A^TPA) + tr(P^{-1}Q) \tag{12}$$

Then, using (1) with L = P and H = A and noting that $\lambda_i(A) = \lambda_i(A^T)$, we have

$$tr(P^{-1}Q) \leqq n - \sum_{i=1}^{n} |\lambda_i(A)|^2 \tag{13}$$

Substituting R = P^{-1} and $S = \frac{Q}{|Q|^{1/n}}$ into (2) leads to

$$|P^{-1}|^{1/n} \leqq \frac{tr(P^{-1}Q)}{n|Q|^{1/n}} \leqq \frac{n - \sum_{i=1}^{n}|\lambda_i(A)|^2}{n|Q|^{1/n}} \tag{14}$$

Rewriting (14), we get the inequality (11).

B. Positive Diagonal Solution of the Lyapunov Equation

It is well-known that the solution of the discrete algebraic Lyapunov equation (10) exists if and only if A is a stable matrix, i.e., $|\lambda(A)| < 1$ [5]. However, we are concerned here with the question of whether or not there exists a positive diagonal matrix solution P. In other words, given a real square stable matrix A, we pose the problem: Find the conditions on A such that a positive diagonal matrix P exists where $A^TPA - P < 0$. Such a matrix has been used widely in the stability analysis, control theory and many of its applications [6,7]. In the following sections we present an algorithm to determine under what conditions such a matrix P will exist and if all the conditions are satisfied, the algorithm also provides the value of P.

1. Derivation of the algorithm

The following definitions are needed for our derivation.

a) $X^T = (x_1, x_2, \ldots, x_n)$

b) $P = \text{diag}(x_1, x_2, \ldots, x_n)$

c) $\mathcal{E} = \{e \varepsilon R^n, \; ||e|| = 1\}$

d) $\bar{e} \varepsilon \mathcal{E}$ is an eigenvector corresponding to $\lambda_{max}(A^TPA - P)$

e) $\bar{E}_{(\bar{e})} = \text{diag}(\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n)$

f) $X = \{x \varepsilon R^n, \; 0 \leqq x_i \leqq 1\}$

With the above definitions, let

$$\begin{aligned} h(x) &= \lambda_{max}(A^TPA - P) \\ &= \max_{e \varepsilon \mathcal{E}} e^T(A^TPA - P)e \\ &= \max_{e \varepsilon \mathcal{E}} f(x,e) \end{aligned} \tag{15}$$

We observe that a diagonal matrix $P > 0$ exists such that $A^TPA - P < 0$ if and only if we can find a point $x \varepsilon X$ such that $h(x) < 0$. So the problem now is to determine whether or not such a point $\bar{x}$ exists. Before a search algorithm is derived, we need to point out some properties of $h(x)$.

Let $\bar{x}$ be an interior point of X and

$$\bar{g} \triangleq \nabla_x f(x,e) \Big|_{\substack{x=\bar{x} \\ e=\bar{e}}}$$

$$= \bar{E}_{(A\bar{e})} A\bar{e} - \bar{E}_{(\bar{e})}\bar{e} \tag{16}$$

then it has been shown [2] that

$$h(\bar{x}) = \bar{x}^T \bar{g} \tag{17}$$

and

$$h(x) \geq x^T \bar{g}\ , \text{ for all } x \varepsilon X \tag{18}$$

2. The search procedure

a) Start with $x_1 \varepsilon X$ and compute $h(x_1)$. If $h(x_1) < 0$, stop; otherwise compute g_1 and choose $x_2 \varepsilon X$ such that $x_2^T g_1 < 0$

b) Compute $h(x_k)$. If $h(x_k) < 0$, stop; otherwise compute g_k and choose $x_{k+1} \varepsilon X$ such that $(x_{k+1}^T) g_i < 0$ where $i = 1, 2, \ldots, k$

To choose x_{k+1}, we need to solve a minimax problem

Let

$$S_k \triangleq \min_{x \varepsilon X} \max_{1 \leq i \leq k} \{x^T g_i\} \tag{19a}$$

$$\triangleq \max_{1 \leq i \leq k} \{\bar{x}_k^T g_i\} \tag{19b}$$

Note that

$$S_k < 0$$

$$h(x) \geq x^T g_i,\ i = 1, 2, \ldots, k$$

So

$$h(x) \geq \max_{1 \leq i \leq k} \{x^T g_i\} \geq S_k \tag{20}$$

If $S_k < 0$, stop; otherwise find x_k' where

$$0 = \max_{1 \leq i \leq k} \{(x_k')^T g_i\} \tag{21}$$

Then

$$x_{k+1} = \tfrac{1}{2} x_k' + \tfrac{1}{2} \bar{x}_k \tag{22}$$

To find x_k', let

$$P_k = \bar{x}_k^T g_k < 0,\ q_k = \frac{-P_k}{h(x_k) - P_k} < 1$$

then

$$x_k' = q_k x_k + (1 - q_k)\bar{x}_k \tag{23}$$

So, we can compute x_{k+1} using the following formula

$$x_{k+1} = \tfrac{1}{2} q_k x_k + (1 - \tfrac{1}{2} q_k)\bar{x}_k \tag{24}$$

c) Compute g_{k+1}, replace k by $k+1$ and go back to step b.

The convergence criteria of the above algorithm is established [2] as stated below:

If at any iteration, $h(x_k) < 0$ (or $S_k = 0$), the algorithm stops, indicating definitely that there is a point $x \varepsilon X$ such that $h(x) < 0$ (or no $x \varepsilon X$ exists such that $h(x) < 0$). If neither condition occurs and $k \to \infty$, then the algorithm converges as $S_k \to 0$ implying that $h(x) \geq 0$, for all $x \varepsilon X$.

Also as shown in [2], instead of solving the minimax problem (19), one can obtain the same result by solving the following problem, using linear programming techniques.

Problem

$$\text{Let } S_k = -\max_{x, Z} Z \tag{25a}$$

Subject to

$$0 \leq x_i \leq 1,\ i = 1, 2, \ldots, n \tag{25b}$$

$$0 \leq Z \leq 2\sqrt{n}\ \|A\| \tag{25c}$$

$$x^T g_i + Z \leq 0,\ i = 2, \ldots, k \tag{25d}$$

If $\begin{pmatrix} x^* \\ Z^* \end{pmatrix}$ is the solution of (25), then x^* is the solution of (19).

3. Examples

a) Let $A = \begin{bmatrix} -.1 & .2 \\ -.15 & -.5 \end{bmatrix}$, $x_1 = \begin{bmatrix} .5 \\ .5 \end{bmatrix}$

$$\text{then } A^TPA - P = \begin{bmatrix} -.48375 & .0275 \\ .0275 & -.355 \end{bmatrix}$$

and $h(x_1) = -.35 < 0 \rightarrow$ stop.

A is diagonally stable.

b) Let $A = \begin{bmatrix} 1.9 & -.8 \\ 2.47 & -1 \end{bmatrix}$, $x_1 =$

$$\text{then } A^TPA - P = \begin{bmatrix} 4.35 & -1.99 \\ -1.99 & .32 \end{bmatrix}$$

and $h(x_1) = 5.18 > 0 \rightarrow$ continue.

$$e_1 = \frac{1}{\sqrt{1^2+3^2}} \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} .31 \\ -.94 \end{bmatrix}$$

$$g_1 = \bar{E}_{(Ae_1)} Ae_1 - \bar{E}_{(e_1)} e_1 = \begin{bmatrix} 1.7022 \\ 2.0258 \end{bmatrix}$$

$$x^T g_1 \geqq 0, \text{ for all } x \in X$$

A is not diagonally stable.

CONCLUSION

The inequalities (5) and (11) make it possible to estimate lower bounds for the determinants of the discrete algebraic Riccati and Lyapunov matrix equations. These bounds do not require A to be nonsingular and only require a few matrix computations. These computations can even be further simplified by comparing the first and last terms of (1) instead of the middle term. However, the tightness of the bounds may reduce considerably. Also the conditions for which a positive diagonal solution of the discrete Lyapunov equation exists, are presented. If all the conditions are satisfied, such a solution will also be provided.

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THE FUZZY PLANE

by Christopher Roesmer
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In the early 1930's Lukasiewicz published several papers describing a many-valued logic. [1] A proposition p was allowed to have a truth value [designated $T(p)$] for any real number from zero to one, inclusive. One and zero played the roles of true and false, respectively. The logic can be used to write axioms for the fuzzy plane in which the proposition "point p is on line l" is allowed to vary in truth value from zero to one, inclusive.

Suppose propositions p and q are given with their truth values $T(p)$ and $T(q)$, The negation of $p(\sim p)$, conjunction of p and $q(p\wedge q)$, disjunction of p and $q(p\vee q)$, implication of p to q $(p\rightarrow q)$, and equivalence of p and q $(p\leftrightarrow q)$, have their truth values calculated by the following rules: [1, p. 36]

1. $T(\sim p) = 1 - T(p)$
2. $T(p\vee q) = \max[T(p), T(q)]$
3. $T(p\wedge q) = \min[T(p), T(q)]$
4. $T(p\rightarrow q) = 1$ if $T(p) \le T(q)$

   $= 1 - T(p) + T(q)$ if $T(p) > T(q)$
5. $T(p\leftrightarrow q) = 1 - |T(p) - T(q)|$.

The axioms of the projective plane, which are found in most books on projective geometry, are given below:

1. Given two distinct points, a unique line is on the two points.
2. Given two distinct lines, a unique point is on the two points.
3. There exist four points such that no three of them are collinear.

The axioms can be generalized to the axioms of the fuzzy plane. Letting the proposition "point p is on line l" be designated "$p\,I\,\ell$", the fuzzy

plane is as follows:

1. Given two distinct points p_1 and p_2, a unique line l determines the maximum possible value of $T[(p_1\ I\ \ell)\wedge(p_2 I\ l)]$.
2. Given two distinct lines l_1 and ℓ_2, a unique point p determines the maximum possible value of $T[(p\ I\ \ell_1)\wedge(p\ I\ \ell_2)]$.
3. There exist four points p_1, p_2, p_3, and p_4 and no line ℓ such one or more of the following cases hold:

   i. $[T\ (p_1\ I\ \ell)\wedge(p_2\ I\ \ell)\wedge(p_3\ I\ \ell)] \ge \frac{1}{2}$

   ii. $[T\ (p_1\ I\ \ell)\wedge(p_2\ I\ \ell)\wedge(p_4\ I\ \ell)] \ge \frac{1}{2}$

   iii. $[T\ (p_1\ I\ \ell)\wedge(p_3\ I\ \ell)\wedge(p_4\ I\ \ell)] \ge \frac{1}{2}$

   iv. $[T\ (p_2\ I\ \ell)\wedge(p_3\ I\ \ell)\wedge(p_4\ I\ \ell)] \ge \frac{1}{2}$

As is done to show consistency of the axioms for the projective plane, a model shall be used to show consistency of the axioms for the fuzzy plane. For such a model let one use seven points and lines. The truth value for "$p_i\ I\ \ell_j$" $(i, j = 1, \ldots, 7)$ is given by an incidence matrix in which the real-valued functions ψ_{ij} and ϕ_{ij} are subject to the following restrictions:

i) $0 \le \phi_{ij}(t) < \frac{1}{2}$ for all $t \in \mathbb{R}$

ii) $\frac{1}{2} \le \psi_{ij}(t) \le 1$ for all $t \in \mathbb{R}$

The incidence matrix is as follows:

	p_1	p_2	p_3	p_4	p_5	p_6	p_7
ℓ_1	ψ_{11}	ψ_{12}	ϕ_{13}	ψ_{14}	ϕ_{15}	ϕ_{16}	ϕ_{17}
ℓ_2	ψ_{21}	ϕ_{22}	ψ_{23}	ϕ_{24}	ϕ_{25}	ϕ_{26}	ψ_{27}
ℓ_3	ϕ_{31}	ψ_{32}	ϕ_{33}	ϕ_{34}	ϕ_{35}	ψ_{36}	ψ_{37}
ℓ_4	ψ_{41}	ϕ_{42}	ϕ_{43}	ϕ_{44}	ψ_{45}	ψ_{46}	ϕ_{47}
ℓ_5	ϕ_{51}	ϕ_{52}	ϕ_{53}	ψ_{54}	ψ_{55}	ϕ_{56}	ψ_{57}
ℓ_6	ϕ_{61}	ϕ_{62}	ψ_{63}	ψ_{64}	ϕ_{65}	ψ_{66}	ϕ_{67}
ℓ_7	ϕ_{71}	ψ_{72}	ψ_{73}	ϕ_{74}	ψ_{75}	ϕ_{76}	ϕ_{77}

Let the incidence matrix be labelled D. The following theorem can now be shown.

Theorem. The seven-point plane with incidence matrix D satisfies the axioms of the fuzzy plane for each $t \in \mathbb{R}$

Proof. Axiom 1. Given two points p_i and p_j such that $i \neq j$ and $i, j = 1, \ldots, 7$, a comparison of column vectors for p_i and p_j reveals that $T[(p_i \, I \, \ell_K) \wedge (p_j \, I \, \ell_K)] = \min[T(p_k \, I \, \ell_K), (p_j \, I \, \ell_K)] \geq \frac{1}{2}$ for a unique line $\ell_K (K = 1, \ldots, 7)$.

Axiom 2. Given the two lines ℓ_i and ℓ_j such that $i \neq j$ and $i, j = 1, \ldots, 7$, a comparison of row vectors for ℓ_i and ℓ_j shows that $T[(\ell_i \, I \, p_K) \wedge (\ell_j \, I \, p_K)] = \min[p_K \, I \, \ell_i), (p_K \, I \, \ell_j)] \geq \frac{1}{2}$ for a unique point $p_K (K = 1, \ldots, 7)$.

Axiom 3. Consider the points p_1, p_2, p_3, and p_6.

Case 1. $T[(p_1 \, I \, \ell_2) \wedge (p_3 \, I \, \ell_2)] = \min[\psi_{21}(t), \psi_{23}(t)] \geq \frac{1}{2}$ for each $t \in \mathbb{R}$. But $T(p_2 \, I \, \ell_2) < \frac{1}{2}$. Hence $T[(p_1 \, I \, \ell_2) \wedge (p_3 \, I \, \ell_2) \wedge (p_2 \, I \, \ell_2)] < \frac{1}{2}$.

Case 2. $T[(p_1 \, I \, \ell_1) \wedge (p_2 \, I \, \ell_1)] = \min[\psi_{11}(t), \psi_{12}(t)] \geq \frac{1}{2}$ for each $t \in \mathbb{R}$. But $T(p_6 \, I \, \ell_1) < \frac{1}{2}$. Hence $T[(p_1 \, I \, \ell_1) \wedge (p_2 \, I \, \ell_1) \wedge (p_6 \, I \, \ell_1)] < \frac{1}{2}$.

Case 3. $T[(p_1 \, I \, \ell_2) \wedge (p_3 \, I \, \ell_2)] = \min[\psi_{21}(t), \psi_{23}(t)]$ for each $t \in \mathbb{R}$. But $T(p_6 \, I \, \ell_2) < \frac{1}{2}$. Hence $T[(p_1 \, I \, \ell_2) \wedge (p_3 \, I \, \ell_2) \wedge (p_6 \, I \, \ell_2)] < \frac{1}{2}$.

Case 4. $T[(p_3 \, I \, \ell_7) \wedge (p_2 \, I \, \ell_7)] = \min[\psi_{73}(t), \psi_{72}(t)] \geq \frac{1}{2}$ for each $t \in \mathbb{R}$. But $T(p_6 \, I \, \ell_7) < \frac{1}{2}$. Hence $T[(p_3 \, I \, \ell_7) \wedge (p_2 \, I \, \ell_7) \wedge (p_6 \, I \, \ell_7)] < \frac{1}{2}$.

The seven-point plane with incidence matrix D is an example of a fuzzy plane. The example establishes the consistency of the axioms for the fuzzy plane, a generalization of the projective plane.

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$$\sum_{k=1}^{\infty} \frac{1}{2^k}$$

by Hao-Nhien QUA. Vu, Freshman
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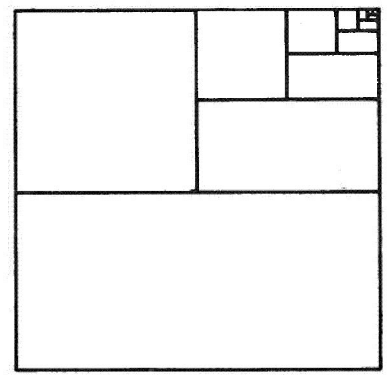
Isn't it beautiful to prove the convergence of the series this way? (The square has area 1)



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AN ANALYSIS OF MONOPOLY

by Thomas Chenier
and Cathy Vanderford
East Carolina University

For as long as games have existed, man has tried to discover ways to consistently win at them. Whether it be backgammon, blackjack, or tic-tac-toe, people have sought systems to help them win a larger proportion of games than might otherwise be expected. This trend has increased with the advent of the computer and its capability to work on large samples of data quickly. Following this lead. we examined the game of Monopoly in order to establish which strategies are more successful as well as the advantages of going first.

The game of Monopoly has been around in one form or another since 1932, when Charles Darrow, an unemployed heating-equipment salesman, drew out the original game. Darrow experimented with the game by playing it with his wife and neighbors. Shortly afterwards. orders for the game started coming in, and Darrow decided to offer it to Parker Bros. for distribution. After initially rejecting Monopoly, Parker Bros. changed its mind and bought the rights in 1934. Since then. over eighty million sets have been sold world-wide, and Parker Bros. now sponsors a world's championship complete with a $5,000 prize.

We were assigned a project to develop a computer program in Pascal which would execute predetermined strategies for the buying, developing, and mortgaging of the various properties in Monopoly. We utilized a records system which kept track of each player's assets, property holdings, which properties were still unowned, etc. After each player's movement, subroutines handled the player's options to buy or to mortgage property as well as whether or not to build houses or hotels. Two other subroutines dealt with the results of drawing either a Chance or Community Chest card (i.e., whether a player paid out or received money and whether or not he moved his token around the board.)

In order to keep the program from becoming too large, we incorporated a few simplifications into the game. For example. if a player went to jail, he automatically paid $50 on his next turn and moved on. If a player drew a "Get Out of Jail Free" card, the bank bought the card for $50 and the card was returned to its respective pile. No trading of property between the players was allowed. Finally, the game was played until one player went bankrupt or until 150 turns had elapsed. If the second condition occurred, the results of the game were examined to determine if the game was a draw of if one of the players had a winning position.

We experimented with four basic strategies. First was the "Bargain Basement" strategy where the player buys any unowned property he lands on so long as he has enough cash on hand to pay for it. The reasoning behind this strategy is to prevent your opponent from obtaining a monopoly while at the same time obtaining one or more properties to develop later. The second strategy was the "Two Corners" strategy. It calls for the purchase of any property (the orange, red and yellow) between Pennsylvania Railroad and the Go to Jail space at any time and the purchase of other properties which can be developed when more than $1,000 is available. Owning these properties means that your opponent should land on at least one of your properties on every trip around the board.

The third strategy was the "Controlled Growth" plan. It calls for the buying of property whenever two conditions were met -- the color group landed on had not yet been split by the two players and $500 was available to the player. This plan allows for growth yet leaves enough capital to develop a monopoly when it is acquired.

The final strategy we tried was the "Modified Two-Corners." It follows the same basic plan as the "Two-Corners" with the added factor of buying Boardwalk-Park Place group. All of the plans involve the properties which could be developed. Additionally, all four called for the purchase of railroads and/or utilities whenever they were landed on.

In actual play the "Two-Corners" led the "Bargain Basement" in a one-hundred game series with a total of 54 wins, 27 losses, and 19 draws. Next the "Controlled Growth" played the "Two-Corners" and won the one-hundred game series with a record of 57-34-9. Lastly, the "Controlled. Growth" strategy played the "Modified Two-Corners" strategy. In two one-hundred game series, the overall record was 88-79-33 in favor of the "Controlled Growth" plan. However, the second series was practically a

draw with only a one win advantage for the leader.

An analysis of some of the games played revealed some of the shortcomings of the strategies involved. The first strategy fell short because it left no money for the building of houses, while the opponent usually had money on hand and developed his property first. The second strategy allowed the opponent too many opportunities to gain a monopoly, while it realistically tried for only three color groups, a task which is relatively easy to block. The last two plans went quite some way in order to minimize these weaknesses and succeeded fairly well.

To measure the desirability of each property, we developed a value function to determine a numerical value of each property's usefulness. We took a frequency distribution of the number of times that each property was landed on and multiplied each frequency by the basic rent of the property. We then divided the product by the cost of the property. The values of the function ranged from 15.82 for Mediterranean Ave. to 123.76 for Water Works; the mean was 59.76, the median 54.33, and the standard deviation was 24.89. After Water Works, the highest property was the Electric Co., with a rounded value of 121. The next three properties were all railroads -- B&O at 93, Reading at 91, and Pennsylvania at 89. Rounding out the top ten properties were Boardwalk at 79, Short Line RR at 70, Illinois, Tennessee, and Pacific Aves. at 64, 57, and 56, respectively. The least valuable properties were Mediterranean Ave., Oriental at 34, Baltic and Vermont Aves. at 35, and Connecticut at 36.

Similar functional values were determined for each of the groups of properties. The highest value belonged to the Orange color group with a value of 970.15. The lowest value of 569.84 belonged to the Dark Purple group. The mean for the groups which could be developed was 760.81, the median was 754.34, and the standard deviation was 129.28. A complete ranking of the groups is as follows: Orange - 970; Lt. Blue - 899; Red - 770; Lt. Purple - 768; Dk. Blue - 740; Yellow - 737; Green - 632; and Dk. Purple - 570. On a similar scale the Railroads had a value of 685 while the Utilities had a value of only 306.

We also investigated the advantage of going first in Monopoly. To test the advantage, each player went first for fifty games out of every one-hundred game series. We then calculated a differential score, that is the number of losses subtracted from the number of wins. The average differential score increased by 6.5 points when a player went first as

opposed to when he went second. Our sample involved four sets of one-hundred game series with two players per game.

In order for everyone here to become Monopoly Moguls, we offer the following suggestions: If your opponent offers you the chance to go first, take it. Buy around the board in a defensive manner (that is at least one property per group). When trading begins, trade for the Orange-Red corner as well as for the Lt. Blue properties. They are landed on most frequently and offer the best return. The railroads and utilities offer a good chance for the buyer to raise some cash with which he may later develop other properties. Finally, whenever your opponent has a hotel on Boardwalk, never, we repeat, never land on it

1984 NATIONAL PI MU EPSILON
MEETING

It is time to be making plans to send an undergraduate delegate or speaker from your Chapter to the Annual Meeting of PI MU EPSILON ***in Eugene, Oregon in August of 1984. Each Speaker who presents a paper will receive travel benefits up to $500 and each delegate, up to $250 (only one speaker or delegate can be funded from a single Chapter, but others can attend.)***

DOES $(a+ib)^{(c+id)}$ EQUAL A REAL NUMBER?

by Ravi Salgia
Loyola University, Chicago

A well known result in complex analysis is that if the pure imaginary number i is raised to a power equal to itself, where $i = \sqrt{-1}$, one obtains an infinite number of solutions. However, when there is a restriction of the argument from $[0, 2\pi)$, it is seen that

$$i^i = exp(-\pi/2)\ .$$

This amazing and intriguing result of an imaginary number raised to its own base gives rise to the question: What condition must two complex numbers, $a+ib$ and $c+id$, meet such that

$$(a+ib)^{(c+id)} = \varepsilon,$$

where a, b, c, and d are all real numbers, and ε is a real number > 0.

To answer this, look at

$$(a+ib)^{(c+id)} = exp[(c+id)\ \ell n(a+ib)] = \varepsilon\ . \qquad (1)$$

From basic complex analysis, it is known that a complex number, $a+ib$, can also be represented in its polar form as $r\ exp(i\theta)$, where $r = \sqrt{a^2 + b^2}$ and $\theta = arctan\ (b/a)$, where $\theta \in [0, 2\pi)$. Thus,

$$\varepsilon = exp[(c+id)\ell n(r\ exp(i\theta))]$$
$$= exp\ (c+id)(\ell n\ r + i\theta)$$
$$= exp[(c\ell nr - \theta d) + i(d\ell nr + c\theta)]$$
$$= [exp(c\ell nr - \theta d)][cos(d\ell nr + c\theta) + isin(d\ell nr + c\theta)]\ .$$

From this,

$$\varepsilon = [exp(c\ell nr - \theta d)][cos(d\ell nr + c\theta)]$$
$$0 = [exp(c\ell nr - \theta d)][sin(d\ell nr + c\theta)]\ .$$

Since $exp(c\ell nr - \theta d) \neq 0$, then $sin(d\ell nr + c\theta) = 0$ or $d\ell nr + c\theta = n\pi$, where n is an integer. Also, since ε is a positive number, and the argument of complex numbers restricted from $[0, 2\pi)$, n must be zero.

Therefore, if the number $(a+ib)^{(c+id)}$ is to be a positive real number, ε, then

$$\frac{d}{2}\ \ell n(a^2 + b^2) + c\ arctan\ (b/a) = 0.$$

Notice, also, that if this occurs, then

$$\varepsilon = exp[\ \frac{c}{2}\ \ell n(a^2 + b^2) - d\ arctan\ (b/a)]\ .$$

To illustrate this formula, consider i^i. In this case $a = c = 0$, and $b = d = 1$, thus

$$\frac{d}{2}\ \ell n(a^2 + b^2) + c\ arctan\ (b/a) = 0$$

which satisfies the condition for equation (1). Thus, i^i is a real number:

$$i^i = exp\ [\ \frac{c}{2}\ \ell n(a^2 + b^2) - d\ arctan\ (b/a)]$$
$$= exp\ [0 - arctan\ (1/0)]$$
$$= exp\ (-\pi/2)\ ,$$

a result which was assumed at the onset of this paper!

<u>Challenge to the Reader</u>: Can you find any other "interesting" $(a+ib)^{(c+id)}$ that is a positive real? If you can, send your results to the Editor and we will publish them in the neat issue.

<u>Acknowledgment</u>

The author wishes to thank Professor Theodore G. Phillips of Loyola University of Chicago, Department of Mathematics, for his kind and helpful assistance in the preparation of this manuscript.

ARCTAN 1 + ARCTAN 2 + ARCTAN 3 = π

by Michael Eisenstein
CBM Educational Center, San Antonio

Consider θ in the right triangle in Figure 1.

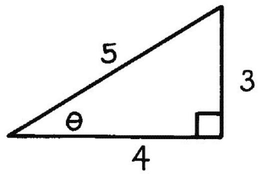


Figure 1

1) $\tan \frac{\pi}{4} = 1$ so Arctan $1 = \frac{\pi}{4}$.

2) $$\tan\left(\frac{\theta + \frac{\pi}{2}}{2}\right) = \frac{\sin\left(\theta + \frac{\pi}{2}\right)}{1 + \cos\left(\theta + \frac{\pi}{2}\right)} = \frac{\cos\theta}{1 - \sin\theta} = 2$$

so Arctan $2 = \dfrac{8 + \frac{\pi}{2}}{2}$

3) $$\tan\left(\frac{\pi - \theta}{2}\right) = \frac{\sin(\pi-\theta)}{1+\cos(\pi-\theta)} = \frac{\sin\theta}{1-\cos\theta} = 3$$

so Arctan $3 = \dfrac{\pi - \theta}{2}$

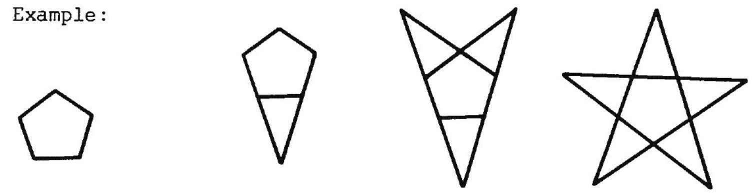
Therefore, Arctan 1 + Arctan 2 + Arctan 3 = $\frac{\pi}{4} + \frac{\theta}{2} + \frac{\pi}{4} + \frac{\pi}{2} - \frac{\theta}{2} = \pi$.

A CURIOUS RATIO OF K-STARS

by Michael Eisenstein
CBM Educational Center, Son Antonio

Theorem. Given a regular polygon of n-sides, $n \geq 5$, let a "k-star" be the polygon together with k triangles added on. (The triangle added on is formed by extending the left and right sides of two consecutive vertices.)

Example:



$n = 5 \quad k = 0$ $\qquad k = 1$ $\qquad k = 3$ $\qquad k = n = 5$

Let P_k and A_k be the perimeter and area of a k-star respectively. Then

$$\frac{P_k}{A_k} = \frac{P_o}{A_o} \quad \text{for } k = 1,2,\ldots,n.$$

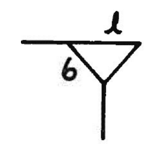
Proof. Let A_Δ be the area of one of the triangles added on. Let b be the length of a side of the polygon.

Then

$$P_o = nb \qquad P_k = (n-k)b + 2kl$$

$$A_\Delta = A_o \qquad A_k = A_o + kA_\Delta$$

Where l is the length of a side of a triangle other than b



We need to show $\dfrac{nb}{A_o} = \dfrac{nb-kb+2kl}{A_o+kA_\Delta}$ is identically true.

The equation is true if and only if

$$nbA_o + nbkA_\Delta = nbA_o - kbA_o + 2klA_o,$$

or

$$nbA_\Delta = (2l-b)A_o,$$

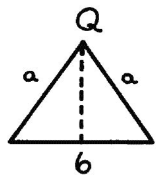
or

I) $$A_o = \frac{nb}{(2l-b)} A_\Delta$$

Consider an interior triangle with point Q at the center of the polygon. The polygon is composed of n of these triangles.

Then $b^2 = a^2 + a^2 - 2a^2 \cos \frac{360}{n}$

$$a = \sqrt{\frac{1}{2\left(1-\cos\frac{360}{n}\right)}}\; b$$



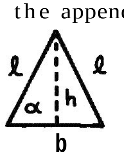
$$t = \sqrt{a^2 - \frac{b^2}{4}}$$

So $$A = \frac{1b}{4}\sqrt{4a^2-b^2}$$, the area,

Thus $$A_0 = \frac{nb}{4}\sqrt{\frac{2}{\left(1-\cos\frac{360}{n}\right)}b^2 - b^2}$$

II) $$A_o = \frac{n}{4} b^2 \sqrt{\frac{1+\cos\frac{360}{n}}{1-\cos\frac{360}{n}}}$$

Now consider the appended triangle



Now $$\alpha = 180 - \frac{(n-2)(180)}{n} = \frac{360}{n}$$

So $$l^2 = l^2 + b^2 - 2lb\cos\frac{360}{n}$$

$$l = \frac{b}{2\cos\frac{360}{n}}$$

$$h = \sqrt{l^2 - \frac{b^2}{4}} = \sqrt{\frac{4l^2-b^2}{2}}$$

So $$A_\Delta = \frac{1}{4} b \sqrt{4\,\frac{b^2}{4\cos^2\frac{360}{n}} - b^2}$$

and

$$AA = \frac{1}{4} b^2 \sqrt{\frac{1}{\cos^2\frac{360}{2}} - 1}$$

Then the right side of I) above is

$$\frac{nb}{2l-b} A_\Delta = \frac{nb}{\left(2\,\frac{b}{2\cos\frac{360}{n}} - b\right)}\; \frac{1}{4} b^2 \frac{\sqrt{1-\cos^2\frac{360}{n}}}{\cos\frac{360}{n}}$$

$$= \frac{n}{4} b^2 \frac{1}{1-\cos\frac{360}{n}} \cdot \sqrt{\left(1+\cos\frac{360}{n}\right)\left(1-\cos\frac{360}{n}\right)}$$

$$= \frac{n}{4} b^2 \sqrt{\frac{1+\cos\frac{360}{n}}{1-\cos\frac{360}{n}}}$$

$$= A_o \; .$$

So I) is established and therefore

$$\frac{P_k}{A_k} = \frac{P_o}{A_o} \quad \text{for } k = 1,2,3,\ldots,n.$$

CHAPTER REPORTS

ALABAMA DELTA (University of South Alabama) The speaker at the induction ceremony was faculty member *Prof. Leon Mattics. Prof. Mattics* presented a lecture on *"The Primitive Roots of Unity"*.

ARKANSAS BETA (Hendrix College) The undergraduate research program was once again very active and several Henrix students attended the Mid-South Mathematics Colloquium at Memphis State on February 18, 1983, and at the Oklahoma-Arkansas MAA meeting at the University of Oklahoma in Norman on March 18-19, *Karen Anderson, Bianca Hearn, and Karen Shirley* presented their papers. Also these three plus *Mike McClurkan* presented their papers at the Conference on Undergraduate Mathematics held in Stillwater, Oklahoma April 15-16. The Annual Hendrix-Sewanee-Southwestern Math Symposium was held at the University of the South in Sewanee on April 29 where *Kmen Anderson, Bianca Hearn, and Karen Shirley* again presented papers.

Several students received awards at the Honors Convocation in May: McHenry-Lane Freshman Math Award was given to *John Crippen* and *Kathy Prunty*; Hogan Senior Math Award was shared by *Bianca Hearn and Karen Shirley*; The Phillip Parker Undergraduate Research Award was given to *Kmen Shirley. Karen Shirley* also received the President's Medal. The Chapter heard the following papers: *Report on the National Pi Mu Epsilon meeting held in Toronto, Canada* by *Dana Payne*; *"Employment and Graduate School Strategies for Math Majors"* by *Dr. Cecil McDermott*; *"What Is Combinatorics?"* by *Dr. Dwayne Collins*; *"Image Processing and Opportunities for Math Graduates Interested in Applied Mathematics"* by *Mark Burton*; *"Interesting Problems in Greek Mathematics"* by *Dr. Robert C. Eslinger*; *"Humanistic Mathematics"* by *Dr. Chris Spatz*; *"The Gamma Function and Log-Convexity"* by *Kmen Anderson*; *"Continued Square Roots"* by *Bianca Hearn*; *"Properties of Separating Point in Continua"* by *Mike McClurkan*; *"Two Fundamental Equations Arising from Notational Ambiguities in Calculus"* by *Kmen Shirley*; *"Introducing the Concept of Integration"* by *Peter Greim*; *"Isaac Newton, The Man"* by *Dr. Billy Bryant*; and *"Forecasting Using the Box-Jenkins Methodology"* by *Dr. Robert Baker*.

CONNECTICUT BETA (University of Hartford) The Chapter heard *Prof. Wistar Comfort* of Wesleyan University speak on *"Some Undecidable Qustions in Mathematics"*. The following students won Departmental Awards: *Antonio Anania, Debra Barberri, Daniel Bowker, Paul Saulnier,* and *Mary Larson*.

GEORGIA BETA (Georgia Tech) *Mary K. Sheffield* won The Outstanding Graduate in Mathematics Award.

ILLINOIS ALPHA (University of Illinois The Chapter organized an information seminar for undergraduates and heard three talks during the 1982-83 school year. *Prof. Kenneth Stolarksy* spoke on *"King Solomon, the Alabama Paradox, and Mathematics Justice." Prof. Lee Rubel* presented a talk entitled *"The Logic of Differential Equations"*. And, *Prof. Heini Halberstam* spoke on *"William Rowen Hamilton and the Beginnings of Modern Algebra"*.

KENTUCKY GAMMA (Murray State University) The Chapter heard talks by *Michele Wilkie, Phil Bryan,* and *Mike Soltys* all of Murray State. *Mr. Soltys* spoke on *"Shortcuts in Multiplication"*.

LOUISIANA KAPPA (Louisiana Tech) sponsored The Annual Calculus Contest which was won by *Georgia Georgious* with second place awarded to *Hassein Sadati*.

MASSACHUSETTS DELTA (University of Lowell) conducted The Annual Mathematics Day for Area High Schools. Approximately 1000 students and teachers attended. In conjunction with this event, the following presentations were given: *"The TAB-FUNCTION in BASIC Language"* by *Dr. Raoul M. Freyre*; *"Four-Color Map Problem"* by *Dr. Joyce Williams*; *"Dynamics of Tennis"* by *Prof. S.J. Bodor*; *"Freshman Calculus at University of Lowell"* by *Prof. P. Condo*; *"Winning at the Racetrack: Luck or Mathematics"* by *Prof. Edward F. Baldyga*; *"Mathematics and Computer Science"* by *Prof. A.W. Doerr*; *" Why We Exist in Three Dimensional Space"* by *Dr. John Brode*; *"Learning Arithmetic in a Foreign Language"* by *Dr. Ken Levasseur*; *"An Easier Way to Graph Polar Curves"* by *Prof. Tom Kudzma*; *"A Logical and Chronological Development of Our Number System"* by *Dr. W.P. Copley*.

MINNESOTA ZETA (Saint Mary's College) had a very active year in which the following presentations were given: *"Interactive Data Analysis to an Air Pollution and Mortality Model"* by *Dr. Gary McDonald*; *"Markov Chains"* by *Sue Blass*; and *"Speculations on the Source or Rigor in Greek Mathematics"* by *David Urion*.

MISSOURI BETA (Washington University) 'he main activity of the Missouri Beta Chapter of Pi Mu Epsilon was a Math Contest for area high school students. Other activities included talks about actuarial careers by representatives from General American Life in the Fall and Spring. Also, at the end of the year a banquet, with elections, was held.

MONTANA BETA (Montana State University) There were two chapter meetings at which invited talks were given. These were: *"Geometric Solutions to Algebraic Problems"* by *Dr. Adrien Hess*; and *"Mathematical Billiards"* by *Dr. Jack Robinson*, Washington State University.

NEW JERSEY THETA (Trenton State College) The Chapter sponsored the following special lectures: *"Einstein's Special and General Theories of Relativity = Historical Perspectives"* by *Dr. John Norton*, Princeton, NJ; *"Artificial Intelligence"* by *Dr. Charles Goldberg*, Trenton State College; *"Topology Can Get Wild"* by *Dr. Edythe Woodruff*; and *"Computation of Circular Areas by the Babylonias and Egyptians"* by *Dr. Siegfried Haenisch*.

NEW YORK PHI (Potsdam) *Dr. Philip Schwartau* spoke at the Fall induction. His talk was entitled *"From Potsdam to Ph.D."*. *Marcia Borden* won the coveted **Clarkson Memorial Award** for the highest four year overall grade point average. This is the sixth consecutive year that this graduating senior award has been won by a member of the chapter. Graduating chapter members fared well outside the discipline of mathematics, also. *Cynthia Pedersen* won the top award in the **Department of Administration and Management**; *Christine Stockschlaeder* in **Chemistry**; *Joan Iannuzzi* in **Computer and Informational Sciences**, and *Sharon Schachter Schoemaker*, chapter secretary, in **Economics**.

NEW YORK OMEGA (Saint Bonaventure) The Chapter was given the following presentations: *"Ramsey Theory"* by *Prof. Jack Graver*, Syracuse University; *"Using Technology to Improve Man - Computer Interaction"* by *Mr. Walter Doherty*, Manager of Systems Performance and Technology Transfer at IBM; *"Selected Problems and Their Methods of Solution"* by *Prof. Ralph King*; and *"The Towers of Hanoi Puzzle - an Application of Math Induction"* by *Prof. Charles Diminnie*. **The Pi Mu Epsilon Award** was presented to *Bernard Sampson*, with honorable mention to *Jane Stolarski*.

NEW YORK ALPHA ALPHA (Queen's College) Heard the following papers: *"Trap-Door Functions and Secret Codes"* and explained how number theory and prime numbers are used in the development of secret codes, by *Dr. Kenneth 8. Kramer*; *"Using the Computer Language BASIC in the Math Classroom"* by *Dr. Ronald I. Rothenberg*. *Linda Hechtman* and *Hal Weinstein* were the recipients of **The 1983 TIME prize** for excellence in Mathematics and service to the NY Alpha Alpha Chapter.

OHIO NU (University of Akron) At the **Initiation and Awards Banquet**, the Chapter presented an award to *Linda Chang* for her mathematics project, *"Chi Squared Predicts the Crab Will"*. *Gerald McCoy*, *Mary Frank*, and *Robert Miller* received the **Samuel Selby Mathematics Scholarship Award for 1982-83.** *Dr. Kenneth Cummins* of Kent State University presented an interesting and very enlightening talk entitled *"How to Know Where You Are in the Middle of Nowhere"*.

OHIO THETA (Xavier University) The recipients of the **Richard J. Wehrmeyer Pi Mu Epsilon Award** are *John Flasophler* and *Steven Kurzhals*. It was-awarded to' them for their excellence in problem solving.

OHIO XI (Youngstown State University) Several members attended the National Meeting held in Toronto, Canada where *Kriss Schueller*, a graduate student, gave a talk. Various speakers on Math related topics visited the club. *Annette Trivolino*, a senior math major from Westminster College spoke about her recent semester spent in France; *Mr. Cyril Mattis* demonstrated construction of geometric models; and *Dr. Bhushan Wadwha* of Cleveland State University gave an entertaining talk on *"Number Theory"*. Other activities included tutoring sessions for area high school students preparing to take a national mathematics test. The chapter helped with the fall Ohio Section M.A.A. meeting held on the Youngstown State University campus. Several members attended the Spring meeting with 2 of our senior members presenting papers.

SOUTH CAROLINA DELTA (Furman university) The Chapter sponsored the **seventh annual Furman University Mathematics Tournament**; 360 students representing 57 high schools participated. *Dr. Ian P. Schagen*, a visiting professor from Loughborough University of Technology, Leicestershire, England, spoke on *"Mathematical Modeling in Oil Field Development"*.

SOUTH DAKOTA BETA (South Dakota School of Mines and Technology) sponsored an undergraduate seminar series on Recreational Mathematics. The Chapter heard papers on *"Factoring Large Numbers"* by *James Sandau*; *"Godel's Theorem"* bq *Dr. Edward Corwin*; and *"Four Jewels of Number Theory"* bq *Dr. David Ballew*. The Chapter sponsored a presentation by industry interviewers on *"The Art of Interviewing"*. Chapter members also gave tutoring sessions for Freshman and helped with the 35th Annual West River Mathematics Contest.

REPORT OF THE 1983 ALBANY MEETING

The Program for the 1983 Meeting of the PI MU EPSILON FRATERNITY was held at SUNY in Albany on August 9 through August 11 of 1983. The program included:

A BASIC Program for the Schredering Equation	*Mary Anne Bromelmeier* *Ohio Delta* *Miami University*
Subset Selection	*David Van Brackle* *Florida Theta* *University of Central Florida*
Two Ancient Greek Construction Problems in Euclidean and Hyperbolic Geometry	*Jack M. Rau* *Oklahoma Beta* *Oklahoma State University*
What Difference Does it Make?	*Dariush Saghafi* *John Carroll University*
The Gamma Function and Log-Convexity	*Karen Anderson* *Arkansas Beta* *Hendrix College*
A Ubiquitous Partition of Subsets of R^n	*Donald John Nicholson* *Iowa Alpha* *Iowa State University*
Leo Moser's Theorem	*Susan McDonald Britt* *North Carolina Delta* *East Carolina University*
Exploratory Data Analysis Using Microcomputers	*Thomas TenHoeve III* *Michigan Delta* *Hope College*
Cryptography -- The Science of Secret Writing	*Denise Vining* *Ohio Delta* *Miami University*

The J. Sutherland Frame Lecture was given by *Prof. Henry L. Alder*, of the University of California, David and entitled *'How to Discover and Prove Theorems: A Demonstration with Partitions.*

PUZZLE SECTION

Edited by
Joseph D.E. Konhauser

This Department is for the enjoyment of those readers who are addicted to working doublecrostics or who find an occasional mathematical puzzle attractive. We consider mathematical puzzles to be problems whose solutions consist of answers immediately recognizable as correct by simple observation and requiring little formal proof. Material submitted and not used here will be sent to the Problems Editor if deemed appropriate for that Department.

Address all proposed puzzles and puzzle solutions to Prof. Joseph Konhauser, Department of Mathematics, Macalester College, St. Paul, Minnesota, 55205. Deadlines for puzzles appearing in the Fall Issue will be the next February 15, and for puzzles appearing in the Spring Issue will be the next September 15.

Mathacrostic No. 17

Submitted by Joseph D.E. Konhauser
Macalester College, St. Paul, Minnesota

Like the preceding puzzles, this puzzle (on the following two pages) is a keyed anagram. The 227 letters to be entered in the diagram in the numbered spaces will be identical with those in the 28 keyed words at the matching numbers. The key numbers have been entered in the diagram to assist in constructing your solution. When completed, the initial letters will give the name of an author and the title of a book; the completed diagram will be a quotation from that book. (See an example solution in the solutions section of this Department.)

DEFINITIONS	WORDS
A. a sweet cordial of the Mediterranean region	180 199 77 207 121 34 18
B. latest in an indefinitely numerous sequence	2 189 103 58 175 146 216 29 206
C. double dagger	93 28 76 223 130 184
D. a rabbinical academy	83 64 193 71 116 24 167
E. arsenic trioxide	8 78 70 42 56 132 27 98
F. supposing that not	47 164 192 157 89 39
G. a small grayish European warbler	32 151 50 6 221 147 156 62 20 177
H. mess; predicament (3 wds.)	67 45 179 55 104 172 7 201 82 225 187 128
I. like in quality, nature or status	152 46 171 26 220
J. Hungarian march by unknown composer in honor of a national hero revered by Magyar patriots	110 74 154 191 213 13 88
K. exposure to the rays of the sun	12 69 178 144 99 196 125 43 200 140
L. about 8.669 cubic inches, a British volume measure	173 66 86 118 134 106
M. mirage, especially one seen at the Strait of Messina (2 wds.)	123 10 114 136 211 37 161 183 204 85 48
N. gelatin prepared from the air bladders of sturgeons	181 120 84 95 133 49 148 21 61
O. roughly, one is to a minute as a minute is to 1140 centuries	149 141 203 162 195 60 218 126 35 15
P. a fossil footprint	217 153 22 182 202 79 40
Q. something difficult to dispose of, solve, or decide about	65 101 222 41 169 17
R. mate	52 113 33 129 186 212 11 91 226 155
S. Franklin P. Adams's coinage for a name that sounds like its owner's occupation	105 30 137 163 90 112 188 9
T. a four-bit word	142 119 53 190 87 25
U. irregular short poem or chant	143 92 166 214 5 75 107 131 38
V. a space between regular and normal	124 210 150 224 19 117 139 100 1
W. able to flourish in a salty soil	115 215 208 165 16 59 111 158 31 174
X. pretentious nonsense; clap-trap: drivel	122 57 14 205 94 168 159
Y. aversion to mental work	63 138 81 109 4 185 36 73
Z. judicial investigation, usually before a jury	194 145 170 54 72 97 127
a. umbilic (2 wds.)	135 23 44 160 209 197 176 80 227 108
b. right-handed	96 68 102 51 198 219 3

1 V	2 B	3 b	4 Y	5 U		6 G	7 H	8 E	9 S	10 M	11 R	12 K	13 J
14 X	15 O		16 W	17 Q	18 A	19 V	20 G	21 N		22 P	23 a	24 D	25 T
	26 I		27 E	28 C	29 B	30 S	31 W	32 G	33 R	34 A	35 O	36 Y	
37 M	38 U	39 F	40 P	41 Q	42 E	43 K	44 a	45 H		46 I	47 F	48 M	49 N
50 G	51 b	52 R		53 T	54 Z	55 H		56 E	57 X		58 B	59 W	60 O
	61 N	62 G	63 Y	64 D		65 Q	66 L	67 H	68 b	69 K		70 E	71 D
72 Z	73 Y		74 J	75 U	76 C		77 A	78 E	79 P	80 a	81 Y	82 H	83 D
84 N	85 M	86 L	87 T	88 J		89 F	90 S	91 R	92 U	93 C		94 X	95 N
96 b		97 Z	98 E	99 K	100 V	—	101 Q	102 b	103 B	104 H	105 S	106 L	107 U
108 a	109 Y	110 J	111 W		112 S	113 R	114 M	115 W	116 D	117 V	118 L		119 T
120 N		121 A	122 X	123 M	124 V		125 K	126 O		127 Z	128 H	129 R	
130 C	131 U	132 E	133 N	134 L	135 a	136 M	137 S	138 Y	139 V	140 K		141 O	142 T
143 U		144 K	145 Z	146 B		147 G	148 N	149 O		150 V	151 G	152 I	153 P
154 J		155 R	156 G	157 F	158 W	159 X	160 a	161 M		162 O	163 S		164 F
165 W	166 U		167 D		168 X	169 Q	170 Z	171 I	172 H	173 L	174 W	175 B	
176 a	177 G		178 K	179 H	180 A	181 N	182 P	183 M	184 C		185 Y	186 R	
187 H	188 S	189 B	190 T	191 J	192 F	193 D		194 Z	195 O		196 K		197 a
198 b	199 A	200 K	201 H		202 P	203 O		204 M		205 X	206 B	207 A	208 W
209 a	210 V		211 M	212 R	213 J	214 U	215 W	216 B	217 P	218 O	219 b	220 I	
221 G	222 Q	223 C	224 V	225 H	226 R	227 a							

SOLUTIONS

Mathacrostic No. 16. (See Spring 1983 Issue) *(Proposed by Theodor Kaufman, M.D., Nassau Hospital, Mineola, L.I., New York)*

Words:

A. Rotator	I. Horseshoe	Q. Off and on
B. obbligato	J. Obsequies	R. Floccule
C. Tobacco	K. Ran	S. Equably
D. Harebrained	L. Toothache	T. Great whites
E. Might-be	M. Love-40	U. Acquiescent
F. Acquittal	N. Internecine	V. Loquacity
G. Nannander	O. Frequent	W. Offertory
H. Shasta	P. Exxon	X. Sex-linked

First Letters: (Tony) ROTHMAN: (The) SHORT LIFE OF E(variste) GALOIS - from Scientific American

Quotation: The. *solution to the general quadratic, or second-degree, equation a x + bx + c = 0, known to the. Babylonians, requires the. extraction of, the square loot of a function of the coefficients, namely b^2 - 4ac. Hence, the general quadratic equation is solvable by radicals.*

Solved by: Jeanette Bickley, Webster Grove High School, Missouri; Betsy Curtis, Meadville, Pennsylvania; Victor G. Feser, Mary College, Bismarck, North Dakota; Robert Konhauser, Macalester College, St. Paul, Minnesota; Roger Kuehl, Kansas City, Missouri; Henry S. Lieberman, John Hancock Mutual Life Insurance Co., Boston, Massachusetts; Sister Stephanie Sloyan, Georgian Court College, Lakewood, New Jersey; and The Proposer and The Editor. One unsigned solution was received.

COMMENTS ON PUZZLES 1 - 7 (See Spring 1983 Issue)

The unique answer to *Puzzle #1* has 541, 149 and 216 as rows from top to bottom in that order. Fourteen readers respond to #1. The solution requires just a bit of trial and error. For *Puzzle #2*, only four correct responses were received. All were equivalent to the arrangement of points in Figure 1, which is essentially that of L.M. Kelly, as given by H.M. Croft in *"Incidence Incidents"*, Eureka, October, 1967. A second arrangement is obtainable by drawing the equilateral triangles inwardly on the sides of the square as in Figure 2.

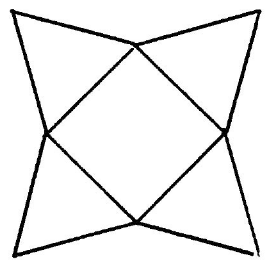


Figure 1

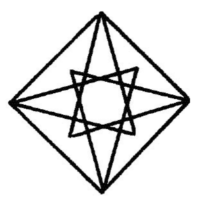


Figure 2

Nine readers responded correctly to *Puzzle #3*. A solution is obtainable without trial and error. For example, Robert C. Gebhardt let M - NUM and n = BER, leading 307m = 692n. Since 307 and 692 are relatively prime, m = 692 and n = 307. *Puzzle #4* drew responses from thirteen readers. The puzzle is well-known and most contributors supplied the two best-known solutions, which are shown in Figure 3. But Jim Gasparri and Phil Shepherd showed that there are infinitely many possible side views. Three of their drawings are reproduced in Figure 4.

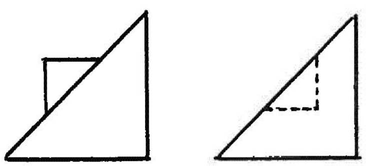


Figure 3

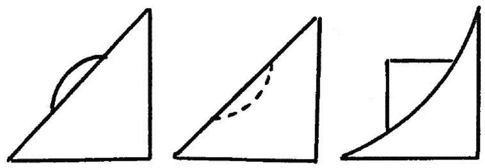


Figure 4

Fourteen responded to *Puzzle #5.* Despite some claims to the contrary, there are two solutions. *Puzzle #6* drew ten responses, several with much detail and a few with some hand-waving. Readers wishing a copy of a detailed solution to #6 should communicate with the Puzzle Editor. *Puzzle #7* drew eight responses. There are only two correct answers, namely (1, 2, 6, 7, 9, 14, 15, 18, 20) and (1, 3, 6, 7, 12, 14, 15, 19, 20). **Victor G. Feser** observed that if one of the sequences is reversed and added termwise to the other, the sums have the common value 21. Surprised? For a proof that there are no other correct answers, see the paper *"Integers, No Three in Arithmetic Progression"* by **Chiang** and **MacIntyre**, Mathematics Magazine, May-June, 1968.

List of Solvers: **Paul Aslanian** (4), **Jeanette Bickley** (1, 2, 3, 4, 5, 6), **David Brady** (7), **Janda S. Cook** (1, 5, 7), **Betsy Curtis** (4), **Victor G. Feser** (1, 3, 4, 5, 6, 7), **Robert Forsberg** (4), **Jim Gasparri** (4), **Robert C. Gebhardt** (1, 2, 3, 4, 5), **David Iny** (1, 3, 5, 6, 7), **Ralph King** (4, 5), **Roger Keuhl** (1, 2, 3, 4, 5, 6, 7), **Glen E. Mills** (1, 3, 5, 6, 7), **Thomas M. Mitchell** (1), **Bus Petrakos** (1), **John H. Scott** (1, 2, 3, 4, 5, 6), **Philip Shepherd** (1, 3, 4, 5, 6), **Emil Slowinski** (1, 3, 5, 6, 7), **Bill Spencer** (4), **Bill Taylor** (4, 5, 6), **David Warland** (1, 5, 7) and **Danny Ying** (1, 5, 6).

PUZZLES FOR SOLUTION

1. *Proposed by I.J. Good, Virginia Polytechnic Institute, Blacksburg, Virginia.*

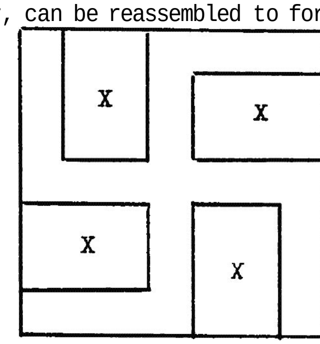
An eight-inch cube has a one-inch cube removed from one corner, and another one-inch cube removed from the opposite corner. Can the resulting body be constructed out of 170 blocks each being one inch by one inch by three inches?

2. *Proposed by Joseph Konhauser, Macalester College, St. Paul, Minnesota.*

Using only standard arithmetical symbols, write the number 4 using three 7's.

3. *Proposed by Joseph Konhauser, Macalester College, St. Paul, Minnesota.*

From the 7 x 7 square, delete the four 2 x 3 rectangles marked X. Dissect the remaining swastika-like region into five pieces which, without being turned over, can be reassembled to form a 5 x 5 square.



4. *Proposed by Joseph Konhauser, Macalester College, St. Paul, Minnesota.*

Find unequal positive integers a, b, and c such that $a^3 + b^3 = c^4$.

5. *Proposed by Joseph Konhauser, Macalester* **Cottage, St. POLL!.,** *Minnesota.*

Sketch a graph (a finite collection of nodes and arcs) such that exactly three arcs terminate at each node and such that it is not possible to color the arcs with three colors so that no two arcs are the same color terminate at the same node.

6. *Proposed by* **Joseph** *Konhauser, Macalester College,* **St.** *Paul, Minnesota.*

On a circle of circumference 21 inches, what is the smallest number of marks which can be located so that for each integer n from 1 to 20 inclusive there are two marks (Not necessarily neighbors) which are separated by n inches measured along the arc of the circle?

7. *Proposed by* **Jobeph** *Konhauser, Macalester College,* **St.** *Paul, Minnesota.*

The eight numbers (2, 3, 4, 6, 9, 14, 22, 31) have sum 91 and the property that taken two at a time the 28 sums obtained are all different. Are you able to find eight positive integers with sum less than 91 with the same property?

NOMINATING COMMITTEE

Elections for the National Officers will be held this Spring. The Nominating Committee is:

J. Sutherland Frame, Chairman
Michigan State University

Richard Andree
University of Oklahoma

E. Allan Davis
University of Utah

The committee solicits recommendations from the membership. Contact any of the above committee members with your suggestions.

PROBLEM DEPARTMENT

Edited *by Clayton W. Vodge*
University of Maine

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel **and** *elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.*

All communications should be addressed to C.W. Dodge, Math. Dept., University of Maine, Orono, ME 04469. Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly identified with name **and** *address. Solutions to problems in this issue should be mailed by July 15, 1984.*

No problem is ever closed. Even when a solution has been published, this department is still interested in new information and will gladly consider any comments you may wish to submit.

From time to time it seems appropriate to publish all problems that remain unsolved, last done the fall of 1968. Currently, through 1982, there are twelve such proposals for which solutions are needed. These are listed below.

120, [Spring 1960, Fall 1968] *Ptopobed by Michael Goldberg, Washington, V.C.*

1. All the orthogonal projections of a surface of constant width have the same perimeter. Does any other surface have this property?

2. A sphere may be turned through all orientations while remaining tangent to the three lateral surfaces of a regular triangular prism. Does any other surface have this property? Note that a solution to (2) is also a solution to (1).

136. [Fall 1961, Fall 1968] *Proposed by Michael Goldberg, Washington, D.C.*

What is the smallest convex area which can be rotated continuously within a regular pentagon while keeping contact with all the sides of the pentagon? This problem is unsolved but has been solved for the square and equilateral triangle. For the square, it is the regular tri-arc made of circular arcs whose radii are equal to the side of the square. For the triangle, it is the two-arc made of equal 60° arcs whose radii are equal to the altitude of the triangle.

144. [Fall 1962, Fall 19681- *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey.*

Find the shape of a curve of length L lying in a vertical plane and having its endpoints fixed in the plane, such that when it revolves about a fixed vertical line in the plane, generates a volume which when filled with water shall be emptied in a minimum of time through an orifice of given area A at the bottom. (Note: The proposer has obtained only the differential equation of the curve.)

190. [Spring 1967] *Proposed by Joseph Arkin, Suffern, N.Y.*

If w, v, t, n, u, q, k, and r are distinct nonzero integers, find infinitely many solutions to the Diophantine equation

$$w^4 + v^4 + t^4 + n^8 = u^4 + q^4 + k^4 + r^8$$

where w, v, u, andq are each a hypotenuse of some Pythagorean right triangle.

239. [Spring 19701 *Proposed by David L. Silverman, Beverly Hills, California.*

A pair of toruses having hole radius = tube radius = 1 are linked. a) What is the smallest cube into which the toruses can be packed? b) What convex surface enclosing the linked toruses has the smallest volume? c) What convex surface enclosing the linked toruses has the smallest area? d) What is the locus of points in space equidistant from the two links?

278. [Spring 19721 *Proposed by Paul Erdös, University of Waterloo, Ontario, Canada.*

Prove every integer $\le n!$ is the sum of $< n$ distinct divisors of $n!$ Try to improve the result for large n; for example, let $f(n)$ be the smallest integer so that every integer $\le n!$ is the sum of $f(n)$ or fewer distinct divisors of n. We know $f(n) < n$. Prove $n - f(n) \to \infty$.

403. [Fall 1977] *Proposed by David L. Silverman, West Los Angeles, California.*

Two players play a game of "Take It or Leave It" on the unit interval (0,1). Each player privately generates a random number from the uniform distribution and either keeps it as his score or rejects it and generates a second number which becomes his score. Neither player knows, prior to his own play, what his opponent's score is or whether it is the result of an acceptance or a rejection. (However, variants based on modifying this condition, either unilaterally or bilaterally, are interesting).

The scores are compared and the player with the higher score wins $1.00 from the other.

a. What strategy will give a player the highest expected score?

b. What strategy will give a player the best chance of winning?

c. If one player knows that his opponent is playing so as to maximize his score, how much of an advantage will he have if he employs the best counter-strategy?

419. [Spring 1978] *Proposed by Michael W. Ecker, City University of New York.*

Seventy-five balls are numbered 1 to 75 and are partitioned into sets of 15 elements each, as follows: $B = \{1, \cdots, 15\}$, $I = \{16, \cdots, 30\}$, $N = \{31, \cdots, 45\}$, $G = \{46, \cdots, 60\}$, and $O = \{61, \cdots, 75\}$, as in Bingo.

Balls are chosen at random, one at a time, until one of the following occurs: At least one from each of the sets B, I, G, O has been chosen, or four of the chosen numbers are from set N, or five of the numbers are from one of the sets, B, I, G, O.

Problem: Find the probability that, of these possible results, four N's are chosen first. (Comment: The result will be approximated by the situation of a very crowded bingo hall and will give the likelihood of what bingo players call "an N game," that is, bingo won with the winning line being the middle column N).

423. [Spring 1978, Spring 1979, Spring 1980, Spring 1981] *Proposed by Richard S. Field, Santa Monica, California.*

Find all solutions in positive integers of the equation $A^D - B^D = C^C$, where D is a prime number.

456.** [Fall 1979, Fall 1980, Spring 1981] ***Proposed by Paul Erdös, Spaceship Earth.

Is there an infinite path on visible lattice points avoiding all (u, v) where both u and v are primes? (The proposer offers twenty-five dollars for a solution).

"Let me restate problem 456. I want a path on visible lattice points (with relatively prime coordinates) which does not pass through a point (p, q) where both coordinates are primes and where both coordinates tend to infinity. Explanation: (u, v) has four neighbors, $(u+1, v)$, $(u-1, v)$, $(u, v+1)$, $(u, v-1)$, and a point can be joined only to one of its neighbors.

'I offer 50 dollars for a path which goes through visible lattice points and avoids (p, q) and moves monotonically away from the origin, i.e., (u, v) can be joined only to $(u+1, v)$ or $(u, v+1)$. The start of the path can be any $(u, v) = 1$. I pay also for a non-existence proof. I do not know the solution and I apologize for the unclearly and incorrectly stated problem 456. My old age and stupidity is, I believe, adequate explanation and excuse."

493. [Spring 1981] ***Proposed by Kenneth M. Wilke, Topeka, Kansas.***

Determine the greatest power which divides $n!$. Prove that for $n > 21$ it is a square. (This is a restatement of problem 467 [Spring 1980].)

525.** [Fall 1982] ***Proposed by John M. Howell, Littlerock, California.

An equilateral triangular prism is used as a die What must the ratio of sides be so that the probability of falling on a triangle is the same as falling on a rectangle?

CORRECTION

536. [Spring 1983] ***Proposed by Martha Matticks, Veazie, Maine.***

A recent alphametric in Crux Mathematicorum [1982: 77, problem 721] asks one to show that, in base ten,

TRIGG is three times *WRONG*.

In defense of the Dean of Numbers, solve these alphametrics independently of each other:

(a) *TSIGG* × *3* = *RIGHT* in base eight where the digit *3* can be reused,

(b) *TRIGG* = *3* × *RIGHT* in base ten where the digit **3** can be reused, and

(c) *TRIGG* × *7* = *SIGHT* in base seventeen.

[Part (a) was erroneously listed with a wrong base.]

Problems for Solution

547. ***Proposed by Morris Katz, Macwahoc, Maine.***

Solve this musical alphametric.

$$\begin{array}{r} SING \\ IS \\ THE \\ WAYNE \end{array}$$

548. ***Proposed by Paul A. McKlueen, Charlotte, North Carolina.***

Arrange the ten digits in a row, e.g.

$$d_1d_2d_3d_4d_5d_6d_7d_8d_9d_{10},$$

so that the following conditions are satisfied: the number $d_2d_3d_4$ is divisible by 2, $d_3d_4d_5$ is divisible by 3, $d_4d_5d_6$ by 5, $d_5d_6d_7$ by 7, $d_6d_7d_8$ by 11, $d_7d_8d_9$ by 13, and the number $d_8d_9d_{10}$ divisible by 17.

549. ***Proposed by R. S. Luthar, University of Wisconsin Center, Janesville.***

If a, b, c are positive numbers, prove that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} > \frac{10\pi}{21}.$$

[For an interesting related problem see Problem 356 in *The Pentagon*, Spring 1983, p. 120].

550. ***Proposed by I, R. Hess, Washington, D.C.***

How many different Pythagorean triples have a side or hypotenuse equal to 1040?

551. ***Proposed by Robert C. Gebhardt, Hopatcong, New Jersey.***

If k is the largest odd integer not exceeding the positive integer n, $n \geq 2$, prove that

$$\cos^2 \frac{\pi}{2n} + \cos^2 \frac{3\pi}{2n} + \cos^2 \frac{5\pi}{2n} + \cdots + \cos^2 \frac{k\pi}{2n} = \frac{n}{4}.$$

552. ***Proposed by Albert White, St. Bonaventure University, New York.***

Let $a_1 = 1$ and $a_n = 2a_{n-1} + (-1)^n$ for $n > 1$. Find

$$\lim_{n\to\infty} \frac{a_n}{2^{n+1}}$$

553.** ***Plopobed by Jack Garfunkel, Faking, New Yohk.

Given a triangle ABC erect equilateral triangles BAP and ACQ outwardly on sides AB and CA. Let R be the midpoint of side BC and let G be the centroid of triangle ACQ. Prove that triangle PRG is a 30°-60°-90° triangle.

554. ***Proposed by Charles W. Trigg, San Diego, California.***

The S.P.F.A. (Society for Persecution of Feline Animals) established a

P U R R
F R E E
A R E A

at its headquarters.

In the word square each letter uniquely represents a decimal digit, and each word and acronym represents a square integer. What are these squares?

555. ***Proposed by Richard V. Stratton, Colorado Springs, Colorado.***

Eighteen toothpicks can be arranged to form six congruent equilateral triangles. Rearrange the toothpicks to form sixteen congruent equilateral triangles each of the same size as the original six.

556. ***Proposed by Richard 1. Hess, Palos Vehdu, California.***

A normal pair of unbiased dice give a total of 2 through 12 according to the distribution 1, 2, 3, 4, 5, 6, 5, 4, 3, 2, 1. How should you change the spots on the dice so that the sums 2 through 12 and only those sums still occur but with as uniform a distribution as possible? (Minimize the sum of the squares of the deviations from completely uniform).

557.** ***Proposed by Pauvre Fish, Seal Beach, California.

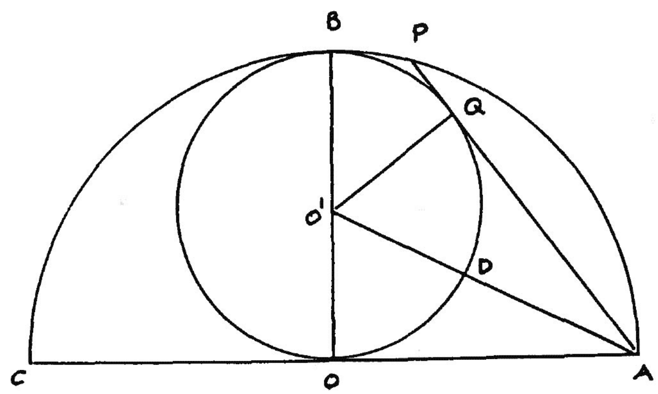
It is known and easy to show with elementary calculus that

$$\int_1^0 \frac{x^4(1-x)^4}{1+x^2}\,dx = \frac{22}{7} - \pi .$$

Find a definite integral whose value is $\frac{193}{71} - e$, where e is the base of natural logarithms.

558. ***Proposed by Richard I. Hess, Palos Verdes, California.***

Let $ABCD$ be a quadrilateral. Let each of the sides AB, BC, CD, DA be the diagonal of a square. Let E, F, G, H be those vertices of the squares that lie outside the quadrilateral. That is, EAB, FBC, GCD, and HDA are directly congruent isosceles right triangles with apexes E, F, G, H. Prove that EG and FH are perpendicular. See the figure below.



559. ***Proposed by Sidney Penner, Bronx Community College, New York.***

"This is quite amazing," said B. "My bingo card does not contain a *BINGO*, but if I cover one more square, regardless of its location, then I will have a *BINGO*."

a) What is the maximum number of covered squares on B's card?

b) What is the minimum number?

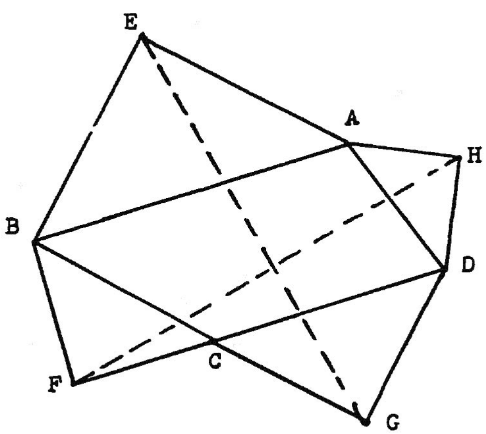
Recall that a bingo card is a 5×5 matrix with the center square

already covered at the start of the game. A *BINGO* can occur in 12 ways, by covering the 5 squares of any row, column, or diagonal.

560. *Proposed by Leon Bankoff, Los Angeles, California.*

Two proofs of a Problem **10713** appeared in the **1891** (pp. **34-35**) **1892** (p. **79**) issues of the *Educational Times*. Unfortunately, neither proof is valid. The problem and its supposed proofs are stated below with wording somewhat modernized for clarification. Find all errors.

Problem **10713.** *Proposed by W. J. Greenstreet, M.A.* In a given circle the radii OA and OB are perpendicular. Let the **circle** on OB as diameter have center O' and let $O'A$ cut this new circle in point D. Then AD is the length of the side of a regular decagon inscribed in the given circle. Also, let the tangent AQ to the new circle cut the given circle again at P. See the diagram below. Then AP is the length of the side of a regular pentagon inscribed in the given circle.



I. *Solution by R. Knowles, M.A., Prof. Zerr, and others.*
Take OA and OB as coordinate axes. Then the equations for the circles are

$$x^2 + y^2 = c^2 \quad \text{and} \quad x^2 + y^2 - cy = 0.$$

Now $(AO')^2 = 5c^2/4$ and

$$AD = AO' - O'D = (\sqrt{5} - 1)c/2,$$

which is equal to the side of a regular inscribed decagon. Let

$hx + ky = a^2$ be a chord of circle (O) that is tangent to circle (O') and equal to the side of the inscribed pentagon. Because it is a tangent, we have

$$h^2 + k^2 = (k - 2c)^2.$$

The condition that this chord equals the side of the pentagon is

$$(k - 2c)^2 = 2c^2(3 - \sqrt{5}),$$

whence

$$k - (3 - \sqrt{5})c \quad \text{or} \quad k = (1 + \sqrt{5})c\ .$$

The latter value makes h impossible. Therefore there is only one real chord of circle (O), tangent to circle (O'), which is equal to the side of the inscribed pentagon.

II. *Solution by the Proposer.*
Let $OA = a$. Then $OO' = c/2$ and

$$AD = AO' - O'D = c\sqrt{1 + \tan^2 OAO'} - \frac{c}{2} = \frac{c}{2}(\sqrt{5} - 1)$$
$$= \frac{2c}{4}(\sqrt{5} - 1) = 2c \sin 18^o,$$

so AD is a side of the inscribed decagon. Now $AP^2 = AD^2 + c^2$ [Casey's *Euclid*, iv. 10, Prob. **6**]. Therefore

$$AP^2 = \frac{a^2}{4}(6 - 2\sqrt{5}) + a^2,$$

$$AP = \frac{1}{2}c\sqrt{10 - 2\sqrt{5}} = 2c \sin 36^o\ ,$$

so AP is the side of the regular inscribed pentagon.

Solutions

522. **[Fall 1982]** *Proposed by Charles W. Trigg, San Diego, California.*

Arrange nine consecutive digits in a 3-by-3 array so that each of the six three-digit integers in the columns (read downward) and rows is divisible by 17.

Amalgam of solutions submitted independently by Bob Prielipp, University of Wisconsin-Oshkosh, and Kenneth M. Wilke, Topeka, Kansas.

First list all three-digit multiples of 17 that do not contain a repeated digit: 017, 034, 051, ..., 986. We arbitrarily select a number from the list for the top row and another with the same initial digit and no other common digits for the left column. To avoid dupli-

cation of effort we take the column number greater than the row number. A systematic search then produced the following arrays:

2 0 4	3 0 6	4 2 5	9 1 8
3 5 7	7 8 2	7 3 1	3 0 6
8 1 6	4 5 9	6 8 0	5 2 7 .

The arrays formed by interchanging rows and columns also satisfy our initial conditions. Since **051** and **085** are actually two-digit numbers, the first two arrays are eliminated. The second and fourth arrays are eliminated because they do not contain nine consecutive integers. Hence the unique solution is the third array (and its transpose). The others are "near misses."

Also solved bq RHONDA L. AULL, *Clemson University, SC,* VICTOR G. FESER, *Mary College, Bismarck, NU,* ROBERT C. GEBHARDT, *Hopatcong, NJ,* JIM GOEKE, *SJ, St. Louis, MO,* DAVID INY, *Rensselaer Polytechnic Institue, Troy, NY,* RANDY ISTVANEK, *Kenosha, WI,* TIMOTHY C. KEARNS, *Catharpin, VA,* ROGER KUEHL, *Kansas City, MO,* ROBERT G. LOGAN, *Middletown, NJ,* PAUL A. McKLUEEN, *Charlotte, NC,* LINDA J. MILLER, *Hope College, Holland, MI,* NATHAN RUD, *St. Olaf College Problem Solving Group, Northfield, MN,* JANE E. STOLARSKI, *St. Bonaventure University, NY,* THOMAS F. SWEENEY, *Russell Sage College, Troy, NY, and the* PROPOSER. *"Near Miss" solutions were found bq* KATHLEEN GRECO, *Rancho Palos Verdes, CA,* GLEN E. MILLS, *Pensacola Junior College, FL,* ELIZABETH A. SWIFT, *California State University, Long Beach, and.* THEODORE G. ZAVALES, *Rutgers University, New Brunswick, NJ.*

523. [Fall 1982] *Proposed by Stanley Rabinowitz, Digital Equipment Cow,, Merrimack, Nw Hampshire.*

Let $ABCD$ be a parallelogram. Erect directly similar right triangles ADE and FBA outwardly on sides AB and DA (so that angles ADE and FBA are right angles). Prove that CE and CF are perpendicular.

I. Solution by Leon Bankoff, LoA Angeles, California.

$ED/AD = AB/BF \rightarrow ED/BC = DC/BF$. $\measuredangle EDC = \measuredangle CBF$. Therefore $\Delta EDC \sim$ ACBF. $(ED \perp BC$ A $DC \perp FB) \rightarrow EC \perp FC$. w^3*.

*Why waste words?

II. Solution by the Proposer.

We prove a more general result. If ADE and FBA are directly similar triangles and $ABCD$ is a parallelogram, then triangle FCE is directly similar to triangles ATE and FBA .

Identify vectors in the plane with complex numbers. Let A be at the origin and let points B and D be represented by the complex numbers b and d respectively. Let k be the complex representation of the stretch-rotation that carried $\vec{AD}$ to &. Then & $= kd$. Then $\vec{FB} = b/k$ and $\vec{FC} = \vec{FB} + \vec{BC} = b/k + d$. Also $\vec{EC} = \vec{ED} + \vec{DC} = kd + b$. Now $k\,\vec{FC} = b + kd = \vec{EC}$, so triangle FCE is directly similar to the two given triangles.

Also solved by EDWARD S. DOLAN, *Secaucus, NJ,* JACK GARFUNKEL, *Flushing, NY,* EMMANUEL, O.C. IMONITIE, *Northwest Missouri Studs. University, Maryville,* DAVID INY, *Rensselaer Polytechnic Institute, Thoq, NY,* ROGER KUEHL, *Kansas City, MO,* HENRY S. LIEBERMAN, *John Hancock Mutual Life Ins. Co., Boston, MA,* BOB PRIELIPP, *University of Wisconsin-Oshkosh,* CHARLES W. TRIGG, *San Diego, CA, and the.* PROPOSE!? *[second solution]. Solutions to special cases were submitted by* RALPH KING, *Saint Bonaventure University, NY, and* QUYEN NGUYEN, *Akron University, OH.*

524. [Fall 1982] *Proposed by Morris Katz, Macwahoc, Maine*

Solve this holiday alphametric for a real prime XMAS.

$$\begin{array}{r} MERRY \\ \underline{XMAS} \\ DODGE \end{array}$$

Sotlition by David Iny, Rensselaer Polytechnic Institute, Troy, Nw Yolk.

We note that $D = M + 1$ and $R + M +$ (possible carry) $= 10 + d$. Hence R = 1 and the "possible carry" is 0. Now S is 3, 7, or 9, and G is $A + 1$ or $A + 2$. Using this information it is easy to eliminate all pairs (E, Y) except (8, 7). Two possibilities result: 47118 + 6409 = 53527 which is eliminated because $6409 = 13 \cdot 17 \cdot 29$, and the unique solution

$$\begin{array}{r} 57118 \\ \underline{3529} \\ 60647 \end{array}$$

Also solved by VICTOR G. FESER, *Mary College, Bismarck, ND,* JIM GOEKE, *SJ, St. Louis, MO,* RANDY ISTVANEK, *Kenosha, WI,* ROGER KUEHL, *Kansas City, MO,* GLEN E. MILLS, *Pensacola Junior College, FL, St. Olaf Problem Solving Group, St. Olaf College, Northfield, MN,* CHARLES W.

TRIGG, *San Diego*, CA, and the PROPOSER. Two erroneous solutions were submitted.

526. [Fall 1982] *Proposed by Morris Katz, Macwahoc, Maine.*

Solve this alphametric in base twelve, with apologies to J.A.H. Hunter.

SUE
EIGHT
PUTTY

Solution by Charles U. Trigg, San Diego, California.

Represent the "digits" ten and eleven by X and L, respectively, and the base twelve by B. Immediately we have $I = L$, $U = 0$, $E + 1 = P$, $H + 1 = T$, $E + T = Y + B$, and $S + G = T + B$. Then $E \leq 9$, $T \geq 4$, and $T \leq 7$. Now tabulate the possibilities for H, T, E, P, and Y. In each case, among the unassigned digits there must be two such that $S + G = T + B$. This is possible in only one case, except for the interchange of X and 7, shown below:

X08
8L745
90551.

Also solved by MARK EVANS, *Louisville*, KY, VICTOR G. FESER, *Mary College, Bismarck, ND*, DAVID INY, *Rensselaer Polytechnic Institute, Troy*, NY, ROGER KUEHL, *Kansas City*, MO, GLEN E. MILLS, *Pensacola Junior College*, FL, BOB PRIELIPP, *University of Wisconsin-Oshkosh*, KENNETH M. WILKE, *Topeka, KS*, and the PROPOSER.

527. [Fall 1982] *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, Pennsylvania.*

Find the volume of the largest rectangular parallelepiped with upper vertices on the surface and lower vertices on the xy-plane that can be inscribed in the elliptic paraboloid $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2h - 2z$.

Solution by Henry S. Lieberman, John Hancock Mutual Life Ins. Co., Boston, Massachusetts.

The upper vertices of the parallelepiped form a rectangle inscribed in an ellipse and with sides parallel to the coordinate axes. Let (x, y, z) be the coordinates of the upper vertex in the first octant. Then the volume of the parallelepiped is $V = 4xyz$. Since we have $z = h - (x^2/a^2 + y^2/b^2)/2$, then

$$V = 4xy\left(h - \frac{x^2}{2a^2} - \frac{y^2}{2b^2}\right).$$

Then

$$\frac{\partial V}{\partial x} = 4y\left(h - \frac{x^2}{2a^2} - \frac{y^2}{2b^2}\right) + 4xy\left(-\frac{x}{a^2}\right) = 4y\left(h - \frac{3x^2}{2a^2} - \frac{y^2}{2b^2}\right)$$

and similarly

$$\frac{\partial V}{\partial y} = 4x\left(h - \frac{x^2}{2a^2} - \frac{3y^2}{2b^2}\right).$$

Setting these first partials equal to zero we see that the nontrivial critical points (with neither x nor y equal to zero) derive from

$$h = \frac{3x^2}{2a^2} + \frac{y^2}{2b^2} \quad \text{and} \quad h = \frac{x^2}{2a^2} + \frac{3y^2}{2b^2}.$$

Now subtract three times each equation from the other to solve for x and y, obtaining

$$x = \frac{a}{2}\sqrt{2h} \quad \text{and} \quad y = \frac{b}{2}\sqrt{2h}.$$

Since at the two extremes $z = 0$ and $z = h$ we have $V = 0$ and since V is clearly positive for intermediate values of z, V attains its maximum at an interior critical point, the point found above. Hence the maximum volume must be

$$V_{max} = 4xyz = 4\left(\frac{a}{2}\sqrt{2h}\right)\left(\frac{b}{2}\sqrt{2h}\right)\left(h - \frac{1}{2}\left(\frac{h}{2} + \frac{h}{2}\right)\right) = abh^2.$$

Also solved by DAVID DELSESTO, *No. Scituate, RI*, ROBERT C. GEBHARDT, *Hopatcong, NJ*, TIMOTHY C. KEARNS, *Catharpin, VA*, and the PROPOSER.

528. [Fall 1982] *Proposed by Alan Wayne, Pasco-Hernando Community College, Florida.*

In the set of natural triangles--that is, the set of triangles with side lengths that are integers--consider, for instance, the trio: (19, 24, 35), (15, 29, 34) and (14, 31, 33). Call this trio a "size triplet", because the three triangles have the same perimeter and the same area. Since the common area is least, this is the smallest size triplet. What is the next larger size triplet?

Solution by David Iny, Rensselaer Polytechic Institute, Troy, New York.

The next larger size triple is (24, 25, 41), (17, 33, 40), and (15, 37, 38), found by computer. These first two size triples are of

even perimeter and smallest in terms of perimeter as well. The smallest size double is (15, 8, 8) and (14, 14, 3), but in terms of perimeter is (4, 11, 11) and (7, 7, 12).

Furthermore, by a slight modification of Foster and Robins' solution to problem E 2872 [*The American Mathematical Monthly*, vol. 89, no. 7, August-September 1982, pp. 499-5001, we can construct ten triangles of equal perimeter and area, (1242700, 830280, 579020), (1246032, 752250, 653718), (1245675, 765765, 640560), (1182675, 1101360, 367965), (1186770, 1093950, 3712801, (1206660, 1047540, 397800). (1219920, 1001130, 430950), (1233180, 928200, 490620). (1236495, 901680, 5138251, and (1246440, 729300, 676260).

Also solved by the PROPOSER.

529. [Fall 19821 *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Merrimack, Nan Hampshire.*

Show that there is no "universal field" that contains an isomorphic image of every finite field.

Solution by Tom Moore, Bridgewater State College, Massachusetts.

If such a "universal field" existed, then its unity 1 is nonzero and is the unity element for all the subfields. By assumption there are subfields of characteristic 2 and 3. Hence we have both $1 + 1 = 0$ and $1 + 1 + 1 = 0$, so $1 = 0$, a contradiction.

Also solved by MICHAEL W. ECKER, *Pennsylvania State University, Worthington Scranton Campus, and the* PROPOSER.

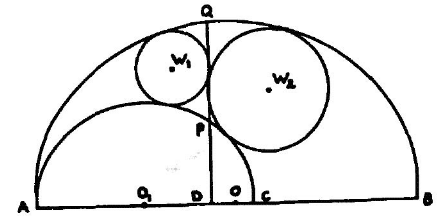
530. [Fall 19821 *Proposed by Leon Bankoff, Lob Angeles, California.*

In the accompanying diagram, AB $(= 2r)$ is the diameter of circle (O) and AC $(= 2r_1)$ the diameter of circle (O_1), D is a point on diameter AC, and the half-chord DQ perpendicuar to AC cuts the circle (O_1) at P. The circles (W_1) of radius ρ_1 and (W_2) of radius ρ_2 are tangent to circles (O) and (O_1) and touch PQ on opposite sides. Show that $\rho_1/\rho_2 = r_1/r$.

I. *Solution by Henry S. Lieberman, John Hancock Mutual Life Ins. Co., Boston, Massachusetts.*

For $i = 1$ and 2 we apply the law of cosines to triangles W_iO_1O to get

$$(*) \qquad (OW_i)^2 = (O_1W_i)^2 + (O_1O)^2 - 2(O_1W_i)(O_1O)\cos W_iO_1O.$$



Since the projections of W_1 and W_2 are at distances ρ_1 and ρ_2 from D, we have that

$$O_1W_2 \cos W_2O_1O - O_1W_1 \cos W_1O_1O = \rho_1 + \rho_2 .$$

Also $O_1O = r - r_1$, $OW_i = r - \rho_i$, and $O_1W_i = r_1 + \rho_i$, $i = 1, 2$.

Now subtract equations $(*)$ to get

$$(OW_2)^2 - (OW_1)^2 = (O_1W_2)^2 - (O_1W_1)^2 - 2(O_1O)(\rho_1 + \rho_2),$$

$$(r - \rho_2)^2 - (r - \rho_1)^2 = (r_1 + \rho_2)^2 - (r_1 + \rho_1)^2 - 2(r - r_1)(\rho_1 + \rho_2),$$

which reduces to

$$2r\rho_1 = 2r_1\rho_2, \quad \text{whence} \quad \frac{r}{r_1} = \frac{\rho_2}{\rho_1} .$$

II. *Solution by the Proposer.*

Let d_1 and d_2 denote the distances of W_1 and W_2 from the radical axis of circles (O) and (O_1) (their common tangent at A). Let CB $(= 2r_2)$ be the diameter of the circle (O_2).

It is known [Casey, *Sequel* to *Euclid*, p. 118] that if a variable circle touches two fixed circles, its radius has a constant ratio to the perpendicular from its center onto the radical axis. So

$$\frac{\rho_1}{d_1} = \frac{\rho_2}{d_2} = \frac{r_2}{AO_2} = \frac{r_2}{r + r_1} = e ,$$

the eccentricity of the ellipse whose foci are O and O_1 and whose major axis is AO_2. Then

$$d_1 - d_2 = \rho_1 + \rho_2 = ed_1 + ed_2, \text{ or } d_1(1 - e) = d_2(1 + e) .$$

Hence we have

$$\frac{\rho_1}{\rho_2} = \frac{d_1}{d_2} = \frac{1 + e}{1 - e} = \frac{1 + r_2/(r + r_1)}{1 - r_2/(r + r_1)} = \frac{r}{r_1} ,$$

the required result.

Also solved by Eric C. Nummela, *Nw England College, Henniker, NH.*

531. [Fall 1982] *Proposed by Robert E. Megginson, University of Illinois.*

Prove, without using mathematical induction, that

$2\cdot 6\cdot 10\cdot 14 \ldots (4n - 2) = (n + 1)(n+2)\ldots(2n).$

I. Amalgam of essentially similar solutions by VICTOR E. BLOOMFIELD, *Bellevue, Washington,* DAVID DELSESTO, *No. Scituate, Rhode Island,* CHARLES R. DIMINNIE, *St. Bonaventure University, NY,* MICHAEL W. ECKER, *Pennsylvania State University, Worthington Scranton Campus,* RUSSELL EULER, *Northwest Missouri State University, Maryville,* JACK GARFUNKEL, *Flushing, New York,* ROBERT C. GEBHARDT, *Hopatcong, NJ,* JOHN M. HOWELL, *Littlerock, California,* TIMOTHY C. KEARNS, *Catharpin, Virginia,* HENRY S. LIEBERMAN, *John Hancock Mutual Life Ins. Co., Boston, MA,* PETER A. LINDSTROM, *North Lake Cortege, Irving, TX,* QUYEN DINH NGUYEN, *University of Akron, OH,* BOB PRIELIPP, *University of Wisconsin-Oshkosh,* HARRY SEDINGER, *St. Bonaventure University, NY,* KENNETH M. WILKE, *Topeka, KS, and the* PROPOSER.

We have

$$\begin{aligned}
&2 \cdot 6 \cdot 10 \cdot 14 \cdot\cdots\cdot (4n - 2)\\
&\quad = (2 \cdot 4 \cdot 6 \cdot\cdots\cdot (4n - 2)(4n))/(4 \cdot 8 \cdot 12 \cdot\cdots\cdot (4n))\\
&\quad = 2^{2n}\,(2n)!/4^n(n!) = (2n)!/n! = (n + 1)(n + 2)\cdots(2n).
\end{aligned}$$

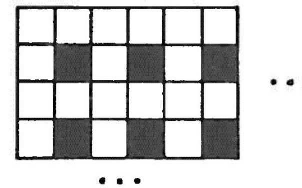
II. Solution by Paul A. McKlueen, Charlotte, North Carolina.

We have

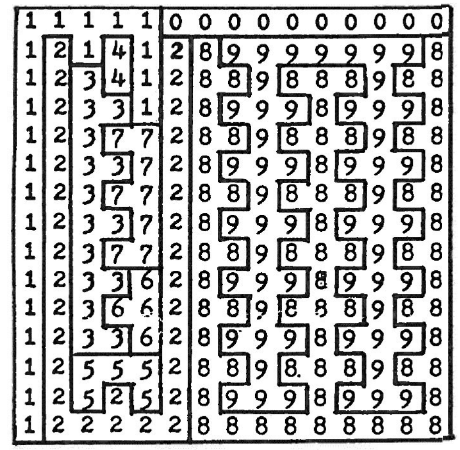
$$\begin{aligned}
(n + 1)(n + 2) \cdots (2n) &= \frac{n!(n + 1)(n + 2) \cdots (2n)}{n!}\\
&= \frac{(2n)!}{n!}\\
&= \frac{1 \cdot 2}{1} \cdot \frac{3 \cdot 4}{2} \cdot\cdots\cdot \frac{(2n - 1)(2n)}{n}\\
&= 2 \cdot 6 \cdot 10 \cdot 14 \cdot\cdots\cdot (4n - 2)\,.
\end{aligned}$$

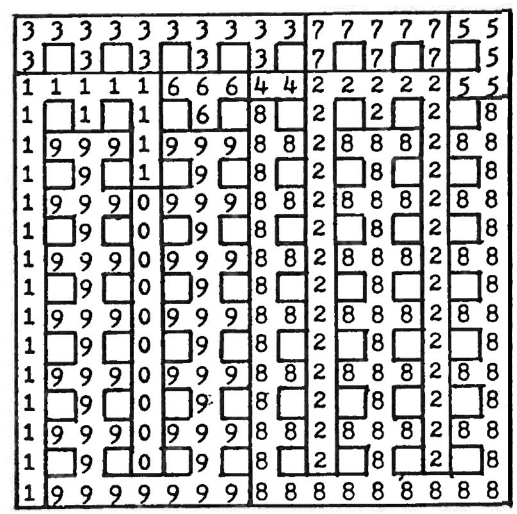
532. [Fall 1982] *Proposed by Morris Katz, Macwahoc, Maine, and Charles W. Trigg, San Diego, California.*

From a square grid of side 17, alternate squares are removed to form a sieve. Dissect this sieve into fewer than a dozen pieces and reassemble them into a square of side 15. See problem 491 [Spring 1982, page 421].



Solution using 10 pieces by David Iny, Rensselaer Polytechnic Institute, Troy, New York.





An eleven-piece solution was provided by the Proposers.

533. [Fall 1982] *Proposed by D. O. Fantus, Alexandria, Virginia.*

It is known that cancelling the sixes in the proper fraction 16/64 yields the equivalent fraction 1/4 in lowest terms (problem E24, September 1933, ***The American Mathematical Monthly***). Find or characterize all proper fractions having 3-digit numerators and 3-digit denominators that reduce to lowest terms by cancelling the same digit from numerator and denominator.

I. *Solution by David Tny, Rensselaer Polytechnic Institute, Troy, New York.*

There are two classes of solutions:

(1) $\frac{abc}{She} = \frac{ac}{de}$ where $b = a + c = d + e$ and ac/de is a proper fraction in lowest terms. In this case a factor of 11 is divided out, and

(2) $\frac{ab0}{de0} = \frac{ab}{de}$ where ab/de is a proper fraction in lowest terms. Here a factor of 10 is divided out.

TI. Comment by Bob Prielipp, University of Wisconsin-Oshkosh.

Here are some references related to this problem:

1. R.P. Boas, "Anomalous Cancellation," pp. 113–129 of ***Mathematical Plums***, edited by Ross Honsberger. The Mathematical Association of America, 1979. (Seven additional references to this problem are given on page 129).

2. Charles W. Trigg, Solution to Problem 434, ***Mathematics Magazone***, 1961, pp. 367-368.

3. Charles W. Trigg, Solution to Problem 365, ***Pi Mu Epsilon Journal***, 1977, pp. 372–374. ***[The editor of this department really should have spotted this one! - ed.]***

Also solved by GLEN E. MILLS, *Pensacola Junior College, FL,* CHARLES W. TRIGG, *San Diego, CA, and the* PROPOSER. *Trigg supplied the additional reference:*

4. W. E. Buker, Solution to Problem 1317, ***School Science and Mathematics***, 34 (April 1934), pp. 432–3.

Late solution to Problem 518 *by* DOUGLAS FRIEDMAN, *University of Pennsylvania, Philadelphia. Late solution to Problem* 520 *by* JOHN BAILEY, *Millsaps College, Jackson, MS.*

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