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PI MU EPSILON JOURNAL
THE OFFICIAL PUBLICATION
OF THE HONORARY MATHEMATICAL FRATERNITY

Editorial correspondence, including books for review, chapter reports, news items and manuscript (two copies) should be mailed to EDITOR, PI MU EPSILON JOURNAL, Mathematics and Computer Science Department, Macalester College, St. Paul, MN 55105. Students submitting manuscripts are requested to identify their college or university and their class or expected graduation date. Others are requested to provide their affiliation, academic or otherwise.

Problems for solution and solutions to problems should be mailed directly to the PROBLEM EDITOR. Puzzle proposals and puzzle solutions should be mailed to the EDITOR.

The PI MU EPSILON JOURNAL is published at Macalester College twice a year—Fall and Spring. One volume consists of five years (10 issues) beginning with the Fall 19x4 or Fall 19x9 issue, starting in 1949. For rates, see inside back cover.

Editor's Note

The Pi Mu Epsilon Journal was founded in 1949 and is dedicated to undergraduate and beginning graduate students interested in mathematics. Submitted articles, announcements and contributions to the Puzzle Section and Problem Department of the Journal should be directed toward this group.

Undergraduate and beginning graduate students are strongly urged to submit papers to the Journal for consideration and possible publication. Student papers will be given top priority.

Expository articles by professionals in all areas of mathematics are especially welcome.

This issue is the first prepared by the newly-elected Editor. It is his hope that he will be able to match the standards set and the quality maintained by his predecessors.

On behalf of the officers, councilors and all members of the Pi Mu Epsilon Fraternity, the Editor extends thanks to Retiring Editor, Dr. David Ballew, for his untiring efforts in editing the Journal during the period Fall 1978 - Spring 1984.

Welcome

The Fraternity extends its welcome to the following newly-installed chapters:

236	Florida Theta	University of Central Florida
237	New York Alpha Beta	LeMoyne College
238	Ohio Omicron	Mount Union College
239	New York Alpha Gamma	Mercy College
240	Massachusetts Zeta	Boston College
241	South Dakota Gamma	South Dakota State University

As of August 1984 the number of chapters was 247, which is more than twice the number of chapters in existence in 1966.

During the 1983-1984 academic year, 3194 new members were initiated into Pi Mu Epsilon. Congratulations to all of you.

A UBIQUITOUS PARTITION OF SUBSETS OF R^n

by Donald John Nicholson
Iowa State University

We are about to embark on an adventure into the beauty of mathematical inquiry. The journey will begin with a simple yet wonderful property of the real numbers, and in the end we will arrive at an elegant result: a ubiquitous partition of subsets of R^n, i.e., a partition in which every element of the partition has a non-empty intersection with each neighborhood (abbreviated nbd) in the space.

The path will lead only through Euclidean spaces, so each space must be understood to be a subspace of R^n. When nbds are mentioned, they refer to nonempty intersections of open subsets of R^n with the space. We will use the symbol $\bar{X}$ to represent the closure in R^n of a set X, X' to denote the set of limit points in R^n of X, and then define $X'' = X' - X$. Finally, $U_r(x_o)$ is the set of all points whose distance from x is less than r, and $B_r(x_o)$ is its closure. Now let us begin.

Let Q and P be the rational and irrational numbers respectively, and let U be a nonempty interval in R. Both $U\cap Q$ and $U\cap P$ are infinite no matter how small U is. Consequently, each point of $U\cap Q$ is a limit point of the set, but there must always be irrational limit points as well. It is this property which we shall generalize in the following definition.

Definition 1. A space X is a Grunowix space* iff $\forall$nbd U in X, $\bar{U} - X \neq \emptyset$.

Proposition 1. If X is a Grunowix space, then X is dense in itself.

Proof. Let x_o be an isolated point in X. Then $\{x_o\}$ is a nbd of x_o in X. But $\{x_o\} - X = \emptyset$, thus X is not a Grunowix space.

* - see excerpt from a letter by the author at end of the paper

If X is a Grunowix space, then $X\subset X'$, but $X \neq X'$. This raises a question: what kind of space is X''? Before answering, let us define a new concept.

Definition 1. Two spaces X and Y are mutually dense iff $\bar{X} = \bar{Y}$. An arbitrary collection of spaces is mutually dense iff they are mutually dense in pairs.

Lemma 1. Let X and Y be disjoint spaces. Then the following are equivalent: (a) $\bar{X} = \bar{Y}$, (b) $X\subset Y''$ and $Y\subset X''$, and (c) $X\subset Y'$ and $Y\subset X'$.

Proof. Suppose $\bar{X} = \bar{Y}$. Then $X\subset\bar{Y}$ and $Y\subset\bar{X}$, and since X and Y are disjoint, $X\subset Y''$ and $Y\subset X''$. Note that $X''\subset X'$ and $Y''\subset Y'$, hence $X\subset Y'$ and $Y\subset X'$. But this implies that $X\subset\bar{Y}$ and $Y\subset\bar{X}$, so $\bar{X}\subset\bar{Y}$ and $\bar{Y}\subset\bar{X}$. This returns us to our original supposition that $\bar{X} = \bar{Y}$.

Lemma 2. If X is a Grunowix space, then $\bar{X} = \overline{X''}$

Proof. By the definition of X'', X and X'' are disjoint, and $X''\subset X'$; therefore, to satisfy Lemma 1 we need only show that $X\subset(X'')'$. Let $x \in X$ and let U be a nbd of x in R^n. Then $\exists\, r>0 \ni U_r(x_o)\subset U$. Now choose $r' \ni 0<r'<r$; this will give us $U_{r'}(x_o)\subset B_{r'}(x_o)\subset U_r(x_o)\subset U$. Observe that $\overline{U_{r'}(x_o)\cap X}\subset B_{r'}(x_o)$, and $U_{r'}(x_o)\cap X$ is a nbd of x in X, so that, since X is a Grunowix space, $\overline{U_{r'}(x_o)\cap X} - X \neq \emptyset$. This implies that $B_{r'}(x_o)$ and thus U contain points in X''; it follows that x is a limit point of X''. Since x is arbitrary, we infer that $X\subset(X'')'$.

Theorem 1. Let X and Y be mutually dense, disjoint spaces. Then X and Y are Grunowix spaces.

Proof. Let $x_o \in X$ and let U be a nbd of x_o in R^n. By Lemma 1 $X\subset Y'$ and $Y\subset X'$, thus $\exists\, y_o \in U\cap Y \ni y_o$ is a limit point of $U\cap X$. This implies that $y_o\in\overline{U\cap X} - X$; since $U\cap X$ is an arbitrary nbd of x_o in X, X is a Grunowix space. By a similar argument Y is a Grunowix space.

Lemma 2 and Theorem 1 answer our question: if X is a Grunowix space, then so is X''; e.g., Q and P are Grunowix spaces. But Theorem 1 has a much more interesting consequence: every collection of spaces which is mutually dense and pairwise disjoint is a collection of Grunowix spaces, and we shall see that such a collection is a ubiquitous partition of its union.

Let us construct such a collection. Let $r, s\varepsilon(0,1)$ and define $r\sim s$ iff $\log_r s\varepsilon Q$. It should be a simple exercise for the reader to show that $\sim$ is an equivalence relation and hence partitions $(0,1)$. Let $[r]$ and $[s]$ be distinct equivalence classes in the partition of $(0,1)$. Then $[r]$ and $[s]$ are disjoint. Furthermore, $\log_r s\varepsilon P$; i.e., $\exists\ p\varepsilon P \ni s=r^P$. Since $p\varepsilon P$, $\exists\ \{q_i\}_{i=1}^{\infty}\subset Q \ni q_i\to p$, thus $r^{q_i}\to r^P=s$. Each $r^{q_i}\varepsilon[r]$, thus s is a limit point of $[r]$ as is every element of $[s]$. Similarly, every element of $[r]$ is a limit point of $[s]$, thus $\overline{[r]}=\overline{[s]}$ by Lemma 1, and $[r]$ and $[s]$ are Grunowix spaces by Theorem 1. Note that each equivalence class is countable, whereas $(0,1)$ is uncountable, thus $\{[r]:r\varepsilon(0,1)\}$ is uncountable.

The above partition is interesting in that the elements are mutually dense; they are like chemically inert gases in a closed container at thermal equilibrium. Just as the molecules of each gas distribute themselves throughout the container, every nbd in $(0,1)$ contains points from every equivalence class; i.e., the partition is ubiquitous.

Let us define a *Grunowix partition* as a partition consisting of mutually dense spaces. By Lemma 1 and the definition of a limit point, this is equivalent to the definition of a ubiquitous partition. Thus every Grunowix partition is ubiquitous, but no other partition is; this is the beauty of the Grunowix space.

DO NOT BLINK! We are about to show how to construct Grunowix partitions of an infinite number of spaces by using only three theorems and our partition of $(0,1)$!

Theorem 2. Let $\{X_a:a\varepsilon A\}$ where A is an indexing set be a Grunowix partition of X, and let $f:X\to Y$ be a continuous bijection. Then $\{f(X_a):a\varepsilon A\}$ is a Grunowix partition of Y.

Proof. The bijectivity condition insures that $\{f(X_a):a\varepsilon A\}$ will be a partition of Y. If $x_o\varepsilon X_a$ and $a\neq b$, $\exists\ \{x_i\}_{i=1}^{\infty}\subset X_b \ni x_i\to x_o$. Since f is continuous, $f(x_i)\to f(x_o)$, thus $f(X_a)$ and $f(X_b)$ are mutually dense. It follows that $\{f(X_a):a\varepsilon A\}$ is a Grunowix partition of Y.

Theorem 3. Let $\{X_i^a:a\varepsilon A\}$ be a Grunowix partition of the i'th factor space of $X=X_1 x X_2 x\dots x X_n$. Then $\{X_1 x\dots x X_i^a x\dots x X_n:a\varepsilon A\}$ is a Grunowix partition of X.

Proof. Let $a,b\varepsilon A$ and $a\neq b$. Then $\overline{X_i^a}=\overline{X_i^b}$. Now $\overline{X_1 x\dots x X_i^a x\dots x X_n}=\overline{X_1}x\dots x\overline{X_i^a}x\dots x\overline{X_n}=\overline{X_1}x\dots x\overline{X_i^b}x\dots x\overline{X_n}=\overline{X_1 x\dots x X_i^b x\dots x X_n}$, thus $\{X_1 x\dots x X_i^a x\dots x X_n:a\varepsilon A\}$ is a mutually dense collection of spaces. Since they are disjoint and their union is X, they form a Grunowix partion of X (we will call this a Grunowix partition of X in the i'th factor space or i'th coordinate).

Theorem 4. Let $\{X_a:a\varepsilon A\}$ be a Grunowix partition of X, and B an open subset of X. Then $\{X_a\cap B:a\varepsilon A\}$ and $\{X_a\cap\overline{B}:a\varepsilon A\}$ are Grunowix partitions of B and $\overline{B}$ respectively.

Proof. Here we will use the equivalence of the definitions of Grunowix and ubiquitous partitions. Now $\forall\ x_o\varepsilon X$, $\forall$ nbd U of x_o in X, and $\forall\ a\varepsilon A$, $U\cap X_a\neq\emptyset$. Since B is open in X, each nbd in B is a nbd in X, thus $\forall\ x_o\varepsilon B$, $\forall$ nbd U of x_o in B, and $\forall\ a\varepsilon A$, $U\cap X_a\neq\emptyset$. But $U\cap X_a=U\cap X_a\cap B$, thus $\{X_a\cap B:a\varepsilon A\}$ is a Grunowix partition of B.

Now let $x\ \varepsilon B''$ and U a nbd of x_o in X. Then $U\cap B\neq\emptyset$, and since B is open in X, $U\cap B$ is open in X, so that $U\cap B\cap X_a\neq\emptyset\ \forall\ a\varepsilon A$. Since $U\cap B\cap X_a\subset U\cap\overline{B}\cap X_a$, $\{X_a\cap\overline{B}:a\varepsilon A\}$ is a Grunowix partition of $\overline{B}$.

These three theorems are our tools for constructing Grunowix partitions. We may construct a Grunowix partition of any open subset of R^n or its closure by collecting the intersections of the set with each element of a Grunowix partition of R^n. A Grunowix partition of R^n may be easily constructed by mapping $(0,1)$ homeomorphically onto one or more factor spaces of R^n and using Theorem 3. Look how many spaces we can ubiquitously partition with a simple equivalence relation on $(0,1)$!

We would like to make one final observation. Constructing a Grunowix partition of R^n by partitioning one or more of its factor spaces imposes a variety of geometric structures on the connected components of the elements of the partition. Since the elements of a Grunowix partition of R are totally disconnected, a Grunowix partition of R^3 in one coordinate will consist of elements whose components are parallel planes, in two coordinates the components will be parallel lines, and in three coordinates the components will be point's. In general the components of the elements of a Grunowix partition of R^n

in m of its coordinates will be $n-m$ dimensional hyperplanes.

There is more. We may use any curvilinear coordinate system in R^n to construct a Grunowix partition so long as each point has unique coordinates. By using spherical or cylindrical coordinates we may have a Grunowix partition of R^3 whose elements consist of concentric spheres or coaxial cones or cylinders. The mysteries of the Grunowix space know no bounds!

The purpose of Figure 1 is to aid in visualizing the Grunowix partition $\{Q^3, R^3-Q^3\}$ of R^3. The element Q^3 is totally disconnected, and since it is countable, R^3-Q^3 is connected. A very rough description of the partition is a countable collection of infinitesimal "boxes" enclosing points. How exquisite! And it all began with the property of the real numbers that every interval contains infinitely many rationals and irrationals.

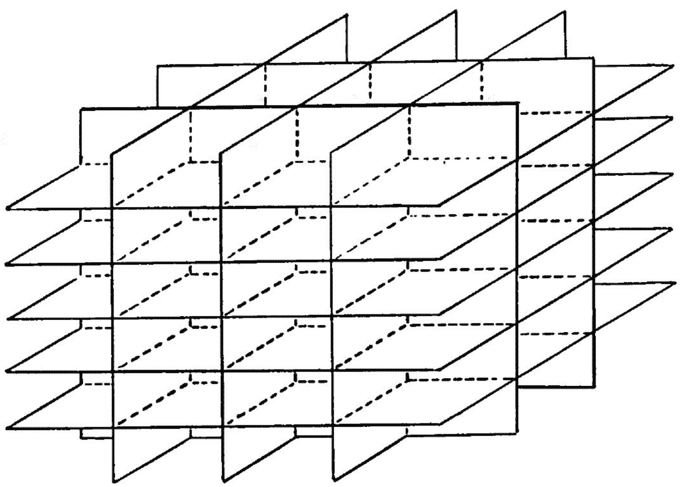

FIGURE 1
Part of R^3-Q^3

REFERENCES

1. Levine, Norman, *On Spaces Nowhere Locally Compact*, Kyungpook Mathematics Journal, 22, No. 2 (December 1982), 167-173.

2. Munkres, James R., *Topology: A First Course*, Prentice-Hall, Inc.,- 1975.

3. Naber, Gregory L., *Topological Methods In Euclidean Spaces*, Cambridge University Press, 1980.

EXCERPT FROM A LETTER

... The. idea of a Grunowix Apace arose from an undergraduate topology homework problem in reference 3. The problem is Exercise 1-28 on page. 25: prove that the. set of rationals as a subspace of the. reals is not locally compact.

*When 1 finally decided what property of the set of rationals prevented **it** from being locally compact, it struck me. en being very fascinating. 1 asked the instructor **if**, this property had a name.. He. knew of none., so 1 generalized the. property and tried to come. up with a **name.** myself. My wife suggested inventing a woad, so we tried constructing words with a scrabble board and letter tiles, and 'Grunowix' was the. first construction we could pronounce.*

*My original definition for the. Grunowix Apace. was formulated differently, but **it** was equivalent to the. present one.. If my papa **it.** devoid of references, it **it.** only because none. of the. math professors 1 was acquainted with coutd steer me. toward any sources that helped me. with this interesting Apace, nor did any searching on my part uncover any useful references. 1 wen forced to develop the. definitions and propositions on my own; it required working manq months avid. wandering down some blind alleys, but 1 am pleased with the. results and what 1 learned from the. experience. It hen made me. love mathematics.*

1 would tike to thank Pi. Donald Sanderson of Iowa State. University for the. multitude of, occasions on which he listened to me. sort out and try to clarify my ideas and for being kind and patient enough to read and constructively criticize five versions of this paper ...

Sincerely,
Donald John Nicholson

Editor's note: This paper was prepared while the author was a senior undergraduate majoring in mathematics and physics at Iowa State University. Donald presented the paper at the National Meeting of Pi Mu Epsilon in Albany, NY in August 1983 and has entered the paper in the Journal's National Paper Competition. The competition is open to students who have not received their master's degree at the time of submission. Papers may be submitted to the Editor at any time.

THE SQUARILIC QUADRILATERAL

by Clayton M. Dodge
University of Maine
and Jack Garfunkel
Queensboro Community College

One must admit that the quadrilateral cannot compete with the triangle in producing theorems in plane geometry. In this Journal, (7(1981)317-329 and 7(1982)453-464), we have previously introduced two new quadrilaterals, namely, the equilic quadrilateral and the bow-tie quadrilateral. We now add another new quadrilateral, which we will call squarilic.

Definition 1. Quadrilateral *ABCD* (not necessarily convex) is squarilic if $AB = CD$ and angle B + angle $C = 90^o$. See Figures 1A, 1B and 1C.

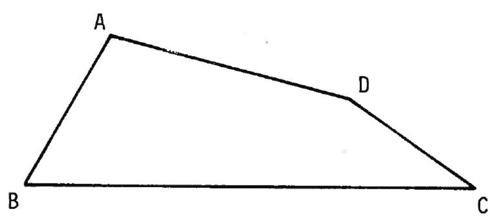

FIGURE 1A

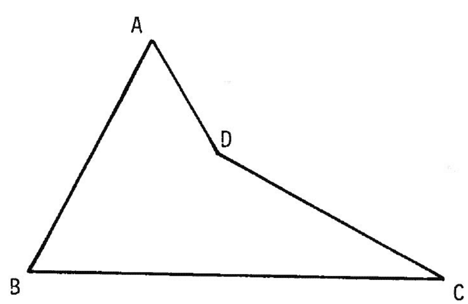

FIGURE 1B

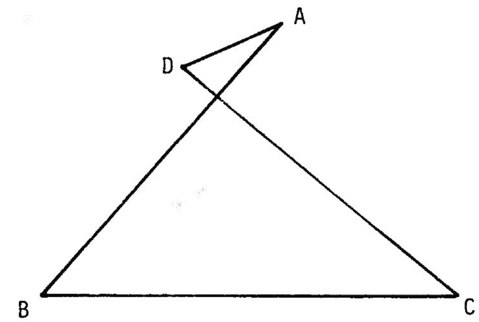

FIGURE 1C

Theorem 1. If a simple quadrilateral is both cyclic and squarilic, it is an isosceles trapezoid with its base angles equal to 45^o.

Proof: The proof is immediate. If the squarilic quadrilateral is inscribed in a circle, *AD* is parallel to *BC* and the theorem follows.

That the restriction to simple quadrilaterals must be included is shown in Figure 2, in which we see a cross cyclic quadrilateral that is not an isosceles trapezoid with base *BC*. Curiously, it is an isosceles trapezoid with bases *BD* and *CA*, as the reader may wish to prove.

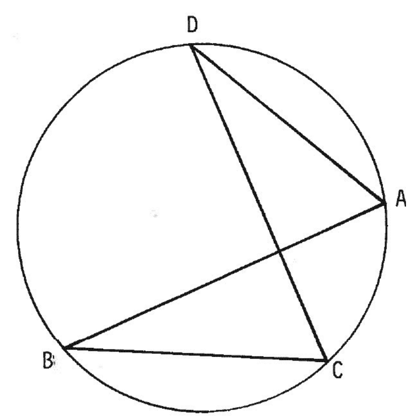

FIGURE 2

Theorem 2. The midpoints of the diagonals and the midpoints of the sides BC and AD of a simple squarilic quadrilateral $ABCD$ are vertices of a square,

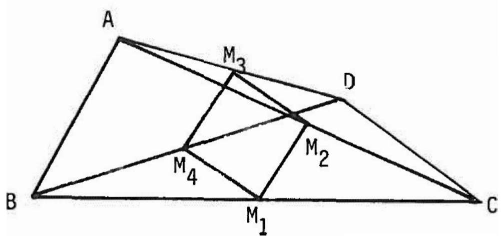

FIGURE 3

Proof. Refer to Figure 3. Let M_1, M_2, M_3, M_4 denote the midpoints of BC, AC, AD, and BD, respectively. Since the line segment joining the midpoints of two sides of a triangle is parallel to the third side and equal to one-half that side, we have,

$$M_3M_4 = \tfrac{1}{2}AB, \text{ and } M_3M_4 \text{ parallel to } AB,$$

$$M_1M_2 = \tfrac{1}{2}AB, \text{ and } M_1M_2 \text{ parallel to } AB.$$

Similarly, we have,

$$M_1M_4 = \tfrac{1}{2}CD, \text{ and } M_1M_4 \text{ parallel to } CD,$$

$$M_2M_3 = \tfrac{1}{2}CD, \text{ and } M_2M_3 \text{ parallel to } CD.$$

Since $AB \equiv CD$, $M_1M_2M_3M_4$ is a rhombus. Furthermore, because AB is perpendicular to CD (extended), $M_1M_2M_3M_4$ is a square.

Question. What figure is obtained in Theorem 2 when the quadrilateral is not simple?

Definition 2. Squares $BCGH$, $ADEF$, $DCJK$, and $BALM$, erected on the sides BC, AD, DC, and BA of squarilic quadrilateral $ABCD$, and squares $ACNP$ and $DBQR$ erected on the diagonals AC and DB are said to have first orientation if they are all similarly oriented and if the interiors of square $BCGH$ and the triangle formed by the sides BC, AB, and CD of the squarilic quadrilateral have a nonempty intersection. If any of these six squares has the opposite orientation, then it is said to have second orientation. Figures 4A, 4B and 5 show these orientations.

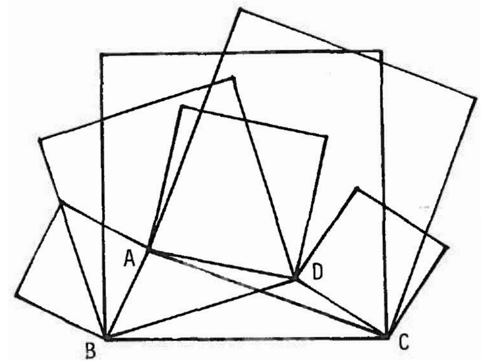

FIGURE 4A

First orientation squares on a simple squarilic quadrilateral

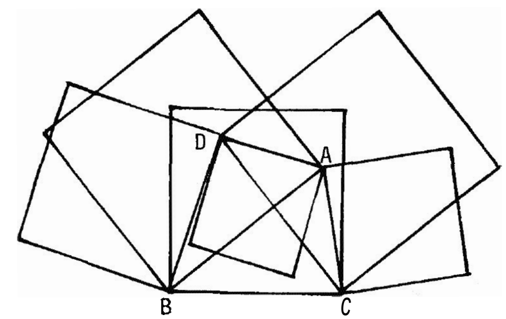

FIGURE 4B

First orientation squares on a cross squarilic quadrilateral

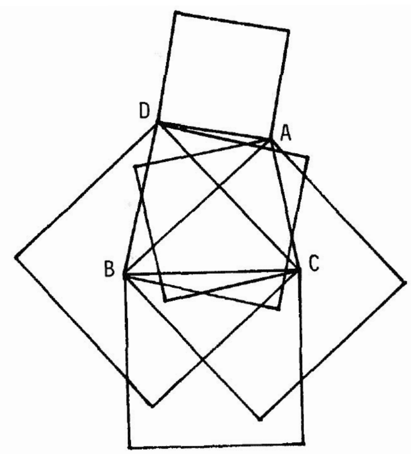

FIGURE 5

Second orientation squares on a cross squarilic quadrilateral

<u>*Theorem 3.*</u> If first oriented squares are constructed on sides ***AD*** and ***BC*** of squarilic quadrilateral ***ABCD,*** then these squares have the same center.

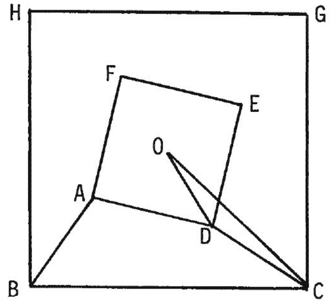

FIGURE 6

Proof. Let O be the center of square ***ADEF*** and O the center of ***BCGH.*** Since a 90^o potation about ***O*** carries ***A*** to ***D*** and since ***AB*** is perpendicular to ***DC,*** the rotation carries ***AB*** to *DC*, as shown in Figure 6. Similarly, a 90^o rotation about O' carries ***B*** to *C*, hence ***AB*** to ***DC.*** Thus these two rotations are identical and $O = O'$.

<u>*Theorem 4.*</u> If first orientation squares are constructed on sides ***AB*** and ***CD*** of squarilic quadrilateral ***ABCD,*** the join of their centers bisects side *AD*, and is parallel and equal to ***BC.***

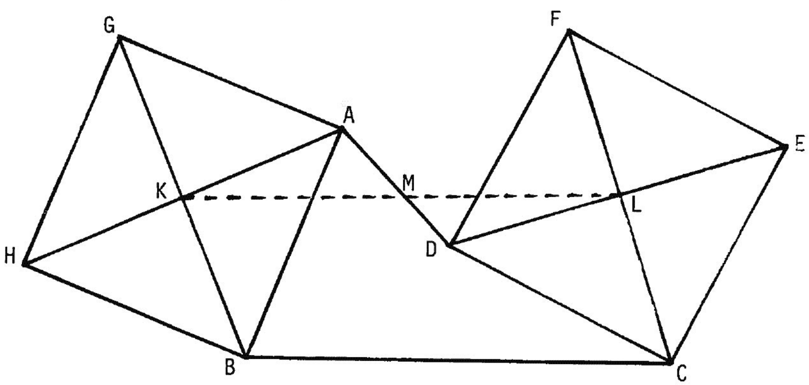

FIGURE 7

Proof. Refer to Figure 7. Since ***AB*** and ***CD*** are equal and perpendicular., squares *AGHB* and ***EFDC*** are congruent with sides parallel. Thus, ***AKDL*** is a parallelogram. Hence its diagonals bisect each other. Since ***BK*** and *CL* are equal and parallel, ***BCLK*** is a parallelogram so that ***BC*** and ***DL*** are parallel and equal.

<u>*Theorem 5.*</u> If second orientation squares ***ACGH*** and ***BDEF*** are erected on the diagonals of squarilic quadrilateral *ABCD*, they will have the same center.

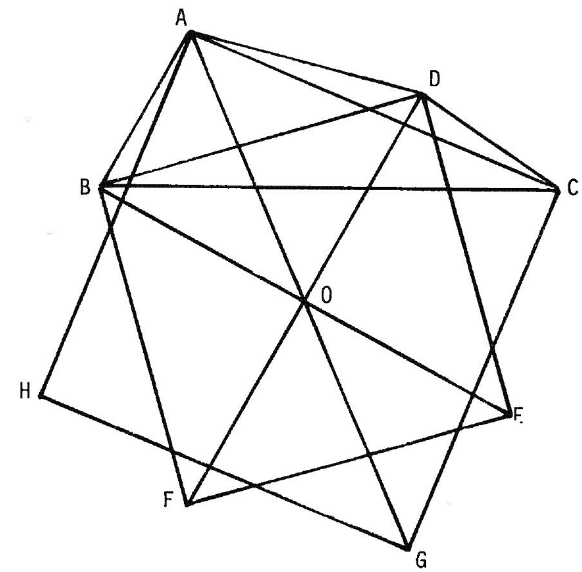

FIGURE 8

Proof. Refer to Figure 8. Let O he the center of the square on ***BD*** and O' the center of the square on *AC*. ***A*** 90^o rotation about O carries ***D*** to ***B,*** hence *DC* to ***BA.*** Similarly, a 90^o rotation about O' carries *C* to ***A,*** hence ***DC*** to ***BA.*** Thus these rotations are identical and $O = O'$.

Theorem 6. If first orientation squares are erected on sides AB and AD and a second orientation square is erected on side DC of squarilic quadrilateral $ABCD$, then the centers of these squares are vertices of an isosceles right triangle with right angle at the center O_1 of the square on AD.

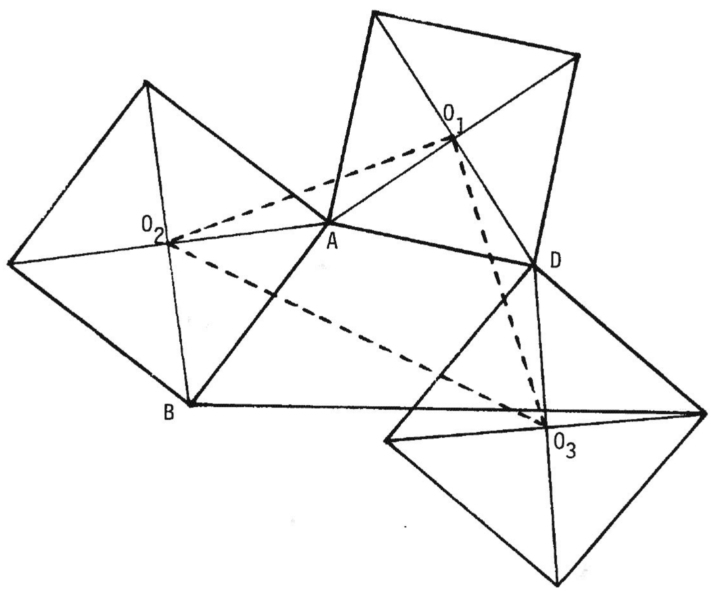

FIGURE 9

Proof. Let the squares on sides AB and CD have centers O_2 and O_3. Since angle BAO_1 = angle CDO_1 and angle O_2AB = angle $O_3DC = 45^o$, then angle O_2AO_1 = angle O_3DO_1. Hence, triangle O_2AO_1 is congruent to triangle O_3DO_1 and a 90^o rotation carries O_2AO_1 to O_3DO_1. Therefore $O_2O_1 = O_3O_1$ and O_2O_1 is perpendicular to O_3O_1.

Before stating the next theorems, we remind our readers of some transformation theory. We let

$$\rho_P, \quad \rho_P^{-1}, \quad \text{and} \quad \sigma_P$$

denote a counterclockwise quarter turn (through 90^o), its inverse clockwise quarter turn, and a halfturn, respectively, each about point P as center. Also let

$$\tau_{AB}$$

denote the translation through vector AB.

We have that

$$\rho_P^{\,2} = \sigma_P \quad \text{and that} \quad \rho_P^{-1}\rho_P \quad \text{and} \quad \sigma_P^{\,2}$$

are each the identity map.

In general, the product of two rotations is a rotation through the sum of the two angles of the given rotations or, if that sum is a multiple of 360^o, a translation. In particular, let $ABCD$ be a counterclockwise square where C is the midpoint of each of the two segments BF and DE, as shown in Figure 10. Then

$$\rho_C\rho_A^{-1}(D) = \rho_C(B) = E, \quad so \quad \rho_C\rho_A^{-1} = \tau_{DE} = \tau_{2DC}.$$

Similarly,

$$\rho_C^{-1}\rho_A = \tau_{BF} = \tau_{2BC}.$$

If two rotations ρ_1 and ρ_2 have the same angles and if there are points A and B such that

$$\rho_1(A) = B \quad \text{and} \quad \rho_2(A) = B$$

and if the centers of the rotations both lie on the same side of line AB, then these rotations have the same center.

Finally, using the ideas above, it is not difficult to show that, if $\rho_P^{-1}\rho_Q = \rho_R\rho_S^{-1}$, then segments PQ and RS are equal in length and perpendicular.

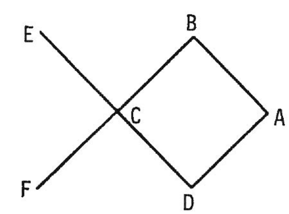

FIGURE 10

Theorem 7. If first orientation squares are erected on side AD (and having center O) and on diagonals AC (center P) and BD (center R) of squarilic quadrilateral $ABCD$, then the centers P, O, and R are collinear, O is the midpoint of PR, and PR is equal to and parallel to one of the diagonals of each of the squares erected on sides CD and AB.

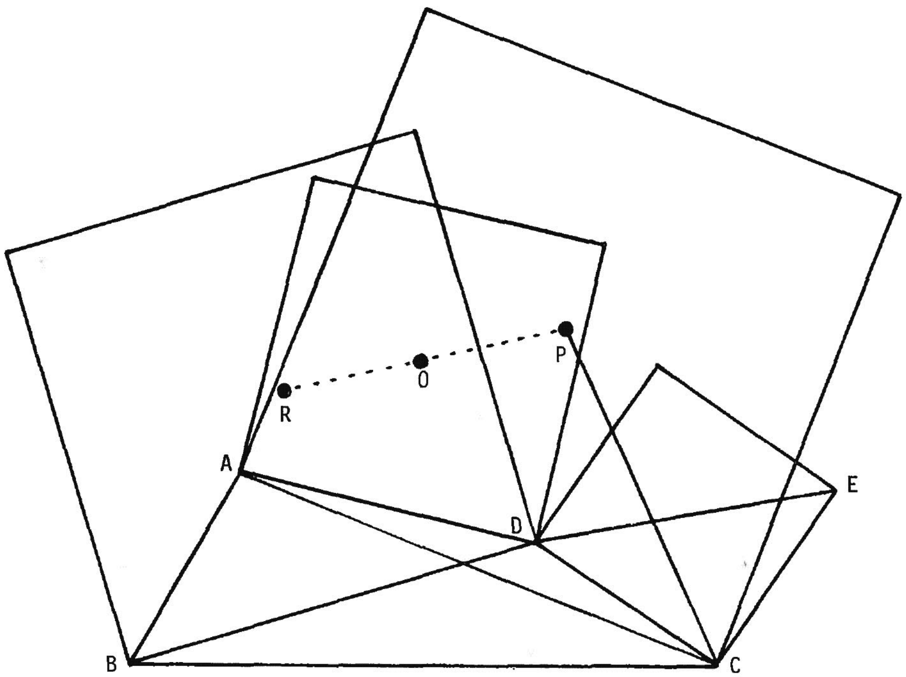

FIGURE 11

Proof. In Figure 11 we see that

$$\rho_O\rho_R^{-1}\rho_O(A) = \rho_O\rho_R^{-1}(D) = \rho_O(B) = C \quad \text{and} \quad \rho_P(A) = C.$$

Now $\rho_O\rho_R^{-1}\rho_O$ is a counterclockwise quarter turn such that O is the midpoint of the segment joining R to its center. It follows that P and that center coincide; that is, O is the midpoint of PR. Since $\rho_O\rho_R^{-1}(D) = C$, then RP is equal and parallel to diagonal DE of the first orientation square on side CD, which square is congruent to and has sides parallel to the squares on side AB.

Theorem 7a. If second orientation squares are erected on sides AD (having center T) and BC (center V) of squarilic quadrilateral $ABCD$, then the midpoint of the segment of centers TV is the common center U

of the two second orientation squares erected on the diagonals AC and BD. Furthermore segment TV is equal to and parallel to the other diagonal (from that of Theorem 7) of the squares on CD and AB.

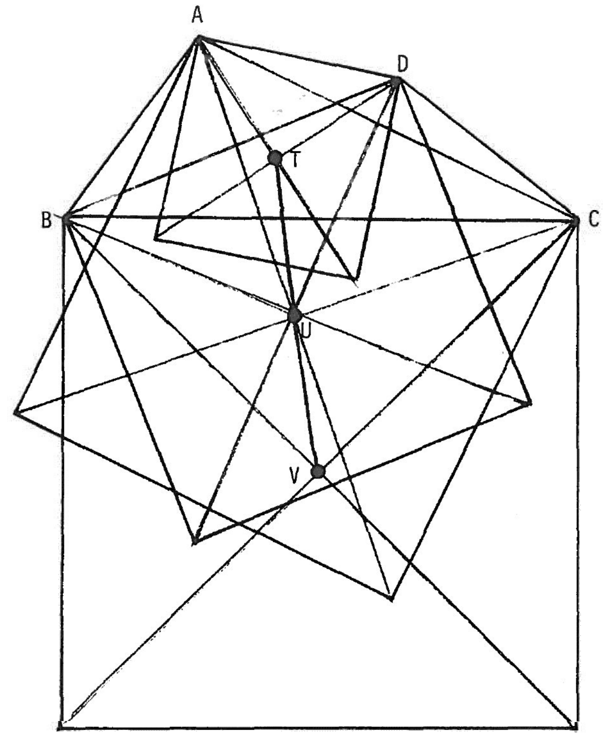

FIGURE 12

Proof. Referring to Figure 12, we have that

$$\rho_V\rho_U^{-1}\rho_T(D) = \rho_V\rho_U^{-1}(A) = \rho_V(C) = B \quad \text{and} \quad \rho_U(D) = B,$$

so U is the midpoint of TV. Also $\rho_V\rho_U^{-1}(A) = B$, so TV is equal to and parallel to a diagonal of the square of side AB, the other diagnonal from that of Theorem 7.

For our final theorem, we could not resist the temptation of going all out and constructing squares on each of the four sides and on both the diagonals.

Theorem 8. If first orientation squares are constructed on each side and on both diagonals of squarilic quadrilateral $ABCD$, the centers of these squares (when joined) form a squarilic quadrilateral.

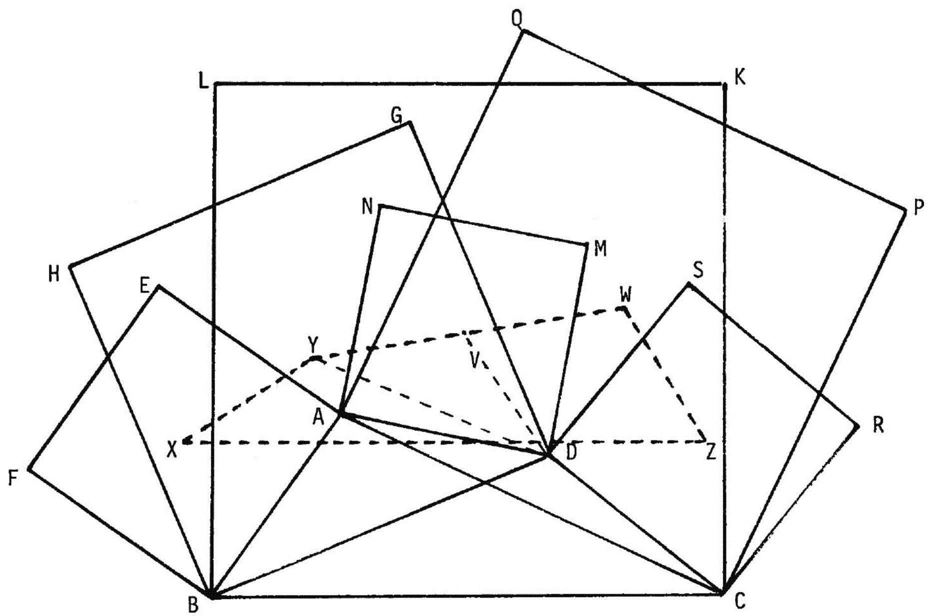

FIGURE 13

Proof. Let the centers of the squares on sides *AB* and *CD* and on diagonals *AC* and *BC* be X, *Z*, *W*, and *Y*, and let the common center of the squares on sides *BC* and *DA* be V, as in Figure 13. The

$$\rho_Z^{-1}\rho_W(A) = D \quad \text{and} \quad \rho_Y\rho_X^{-1}(A) = D,$$

so the translations $\rho_Z^{-1}\rho_W$ and $\rho_Y\rho_X^{-1}$ are equal. Then *WZ* and XY are equal and perpendicular.

Theorem 8a. The centers of the squares erected both ways on *AB* and on *CD* form another squarilic quadrilateral directly congruent to *ABCD* and parallel to it. A halfturn carries one to the other.

Proof. The line of centers of the two squares having *AB* as a side is congruent to *AB* and perpendicular to it. Similarly for the line of centers of the squares on *CD*. It follows that these two lines are the equal sides of a squarilic quadrilateral.

Surely, more properties of this prolific quadrilateral can be found. However, it will be more interesting to consider various degenerations of the quadrilateral. As to be expected, we will obtain theorems about triangles.

Let squarilic quadrilateral *ABCD* degenerate so that angle *ADC* = 180^o, as in Figure 14.

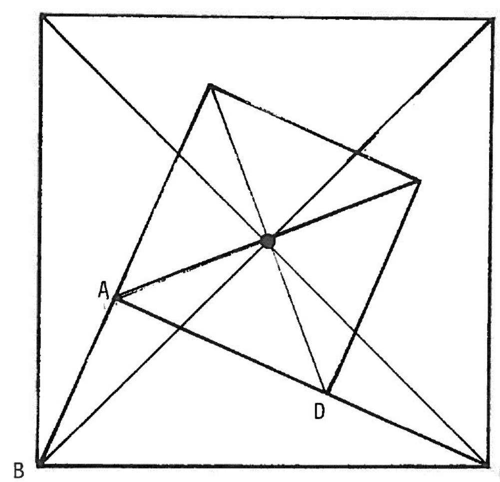

FIGURE 14

Problem 1. In right triangle *ABC*, right angle at *A*, *AC* > *AB*, a point *D* is chosen on *AC* so that *AB* = *CD*. Show that squares erected upward on *AD* and on *BC* have the same center. See Figure 14.

Proof. The above follows directly from Theorem 3.

Using the same degeneration, we have the following:

Problem 2. With the same hypothesis as in problem 1, show that if outward squares are constructed on *AB* and on *DC* the join of their centers bisects *AD*. See Figure 15.

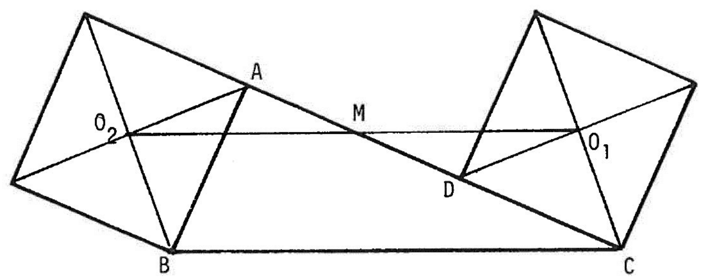

FIGURE 15

Proof. The proof follows from Theorem 4.

Other properties of this right triangle can be found by applying the theorems proved before. We will, however, proceed with another degeneration. Let the squarilic quadrilateral *ABCD* degenerate so that *angle* $BCD = 0^o$. Thus, point *D* is on *BC* and *angle* $B = 90^o$.

***Problem* 3.** In right triangle *ABC*, right angle at *B*, *D* is a point on *BC* such that $AB = CD$. Show that the centers of squares erected on *AB*, *AD*, and on *DC*, as in Figure 16, are vertices of an isosceles right triangle.

Proof. The above follows from Theorem 6.

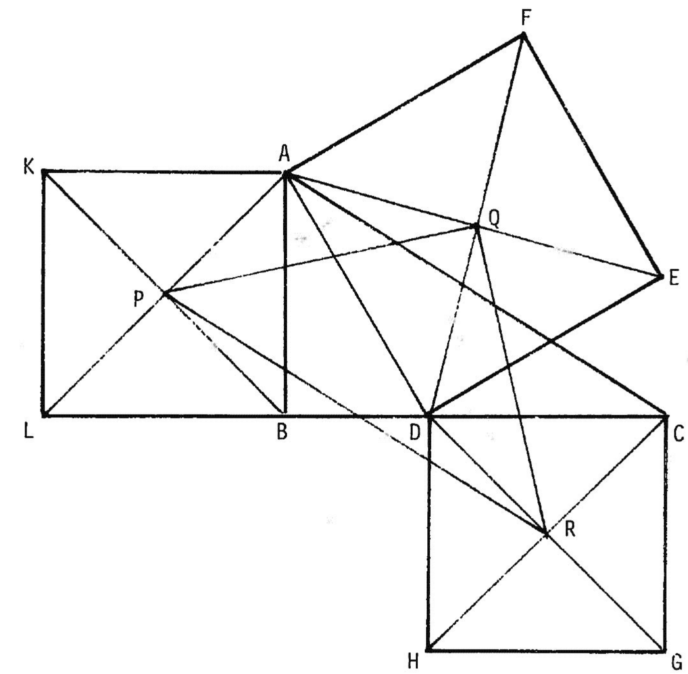

FIGURE 16

Again, we will let the reader discover other properties of this right triangle by applying the theorems proved here. Instead, another degeneration will be considered.

Let the squarilic quadrilateral *ABCD* degenerate so that the point *D* falls on side *AB*. Here *angle* $BDC = 90^o$ and side *AB* equals altitude *CD* of triangle *CAB*.

***Problem* 4.** In triangle *ABC* with altitude $CD = AB$, squares are erected downwardly on *AC* and on *BD*. Show that these squares have the same center. See Figure 17

Proof. The proof follows from Theorem 5.

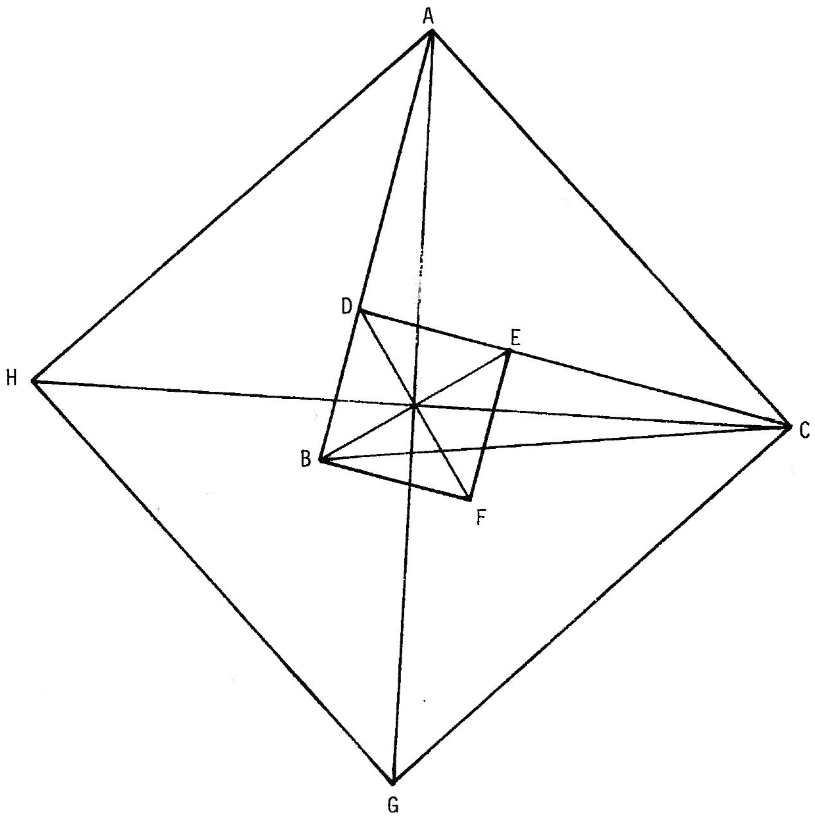

FIGURE 17

Again, we will not belabor the point, but we let the reader discover other propoerties of this triangle by applying others of the theorems proved here.

Finally, we can degenerate our quadrilateral negatively. That is, by making *angle* ***ABC*** $= 90^{o} + \theta$ and *angle* ***BCD*** $= -8$, as in Figure 18. All the properties of the squarilic quadrilateral hold for this negative quadrilateral. For example, from Theorem 7, we get the centers of squares erected on ***BD, AC*** and ***AD*** align.

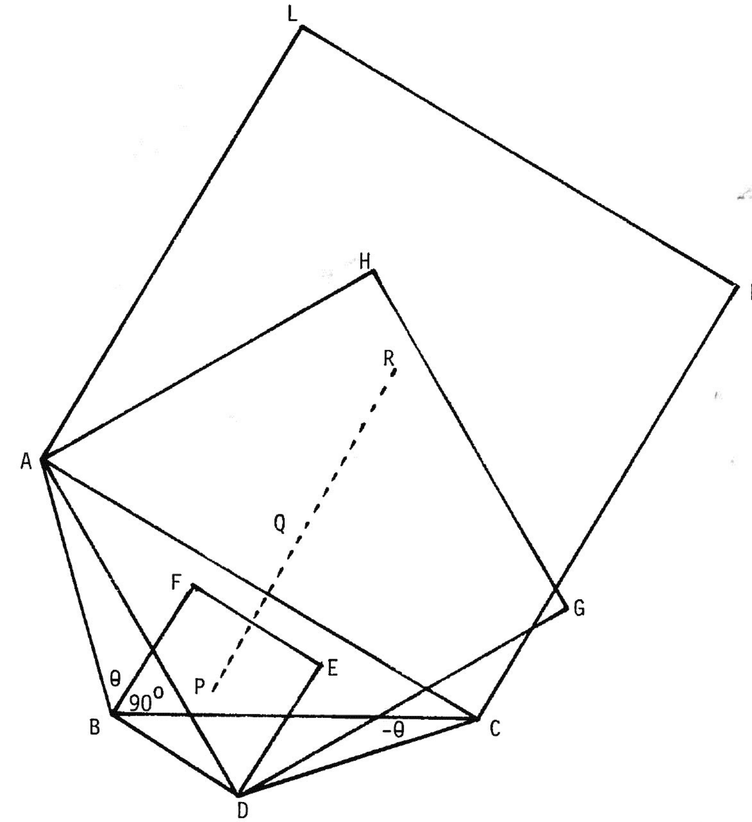

FIGURE 18

Conclusion. Hence. another quadrilateral has been added to the many already existing. This article illustrates a method of discovery. Many high school and college students are looking to do projects in mathematics. Our advice is: create a simple configuration, make accurate sketches, try to discover and then try to prove properties of your configuration.

GRAFFITO

In most sciences one generation tears down what another h a built and what one has established another undoes. In Mathematics alone each generation builds a new story to the. old structure.

H. Hankel (1839-1873)

ASYMPTOTIC SOLUTION OF THE PROBLEMS OF ERDOS AND SIERPIŃSKI CONCERNING M/N

by J. L. Brenner
10 Phillips Road
Palo Alto, CA 94303

Abstract. For each m the rational m/n can be written as the sum $1/x + 1/y + 1/z$ of three unit fractions, except possibly for a set of values of n of asymptotic density 0.

1. *Introduction.* For each $t = 1, 2, \ldots$, an interval containing $1/t$ in its interior is exhibited that contains no rational expressible as the sum of three (or s, preassigned) unit fractions. Yamamoto (1964) proved the existence of such an interval. Stewart and Webb (1966) proved that every interval contains such rationals.

In 1950, Erdös conjectured that every rational $4/n$ $(1 < n)$ (with numerator 4) can be written as the sum of three unit fractions. Sierpiński (1956) made the same conjecture concerning $5/n$ $(2 < n)$. In this article the conjectures of Erdös and Sierpifiski are verified in the asymptotic sense: if there are any exceptional values of n, they exceed 10^8 and have density 0.

2. *Density of Number m/n Expressible as* $\Sigma_1^s 1/x_i$. The results of this section are largely known, though not so explicitly as here outlined.

Let S_k be the set of rationals expressible as a sum $\Sigma_1^k 1/x_i$, $1 \le x_1 \le x_2 \le \ldots \le x_k$; $A_k = S_k \cup \{0\}$.

Thus $A_0 = \{0\}$, $A_1 = \{0, 1, 1/2, 1/3, \ldots\}$. It is easily seen that if $k > 0$ the derived set (of limit points) of A_k is A_{k-1}. Theorem 2.01 follows immediately.

Theorem 2.01. For each k, the set S_k of rationals expressible as the sum of k unit fractions is nowhere dense (in the reals or in the rationals).

This theorem is due to Stewart and Webb (1966) who attribute the argument to H. J. S. Smith (1875).

The interval $(23/24, 41/42)$ contains no rational expressible as the sum of three unit fractions. The proof of Theorem 2.02 is left to the reader.

Theorem 2.02. Define $d_1 = m + 1$, $d_2 = m^2 + m + 1$, $\ldots$. No rational in the interior of the interval

$$(1/m - 1/[md_1d_2\ldots d_s],\ 1/m)$$

can be written as the sum of s unit fractions.

Additional problems are given at the end of this article.

3. *The Conjectures of Erdös and Sierpiński.* In 1965 Yamamoto proved that $4/n = 1/x + 1/y + 1/z$ has a solution when $n \equiv 5 \pmod{24}$. Hence if n has a factor $\equiv 5 \pmod{24}$ there is also a solution. The sieve method shows that the doubtful cases comprise at most $\Pi_{t=1}^{\infty} \left(1 - \frac{1}{24t+5}\right)$ of the integers. Since the infinite product diverges to 0, Erdös' conjecture is asymptotically correct.

Yamamoto has now verified Erdös' conjecture for $1 < n < 10^8$.

In 1964, Stewart proved that $5/n = 1/x + 1/y + 1/z$ has a solution when $n \equiv 13 \pmod{1260}$. A similar sieve shows that the Sierpiński conjecture is asymptotically correct.

Stewart verified Sierpiński's conjecture for $2 < n < 10^9$.

4. *Proof of a Generalization of the Conjectures of Erdös and Sierpiński.*

Theorem 4.01. If m is any fixed positive integer and if n varies through the positive integers, the equation

$$m/n = 1/x + 1/y + 1/z \tag{4.01}$$

has a solution in positive integers x, y, z except possibly for a set of values of n of (asymptotic) density 0 in the set of all integers.

Proof. We consider separately the residue classes $n = ml + t \pmod m$, i.e. $t = 1, 2, \ldots, m-1$. (The case $n = ml$ is trivial, since $1/l = 1/(l+1) + 1/[l(l+1)] = 1/(l+1) + 1/(ll+l+1) + 1/[l(l+1)(ll+l+1)]$.) For each fixed value of t, we shall show that, for almost all values of l (for almost all n) (4.01) has a solution.

Let $p > m$ he any prime in the arithmetic progression $p \equiv -1 \pmod{m-t}$. Set $l = pu + r$, where r is the positive integer $(0<r<p)$ that satisfies $mr + t \equiv 0 \pmod p$. For this value of l, we find $n = mpu + mr + t$. We set $x = pu + r + 1$, and $p + 1 = w(m-t)$. If we choose $y = wnx/p$, $z = wnx$, clearly y, z are integers. Also,

$$\begin{aligned} 1/x + 1/y + 1/z &= 1/x + p/wnx + 1/wnx \\ &= (1/x)(n+m-t)/n \\ &= (1/x)m(pu+mr+1)/n = m/n \end{aligned}$$

At this point, we need a lemma.

Lemma 4.02. (Chandrasekharan) The number of primes p in the arithmetic progression $(m-t)h - 1$ $(h = j, j+1, \ldots)$ is infinite, and the sum of their reciprocals diverges.

Corollary 4.03. The infinite product $\Pi(1-1/p)$ (taken over the primes described in the lemma) diverges to 0.

We return to the main argument. Number the primes in the arithmetic progression of the lemma: $p_1, p_2, \ldots$. For $l = p_1u + r_1$, there is a solution to (4.01), so that at most $1 - 1/p_1$ of the integers n in the residue class $(t \bmod m)$ remain in doubt. Of these remaining values of n, at most $1 - 1/p_2$ of them are in doubt. Thus in the residue class $(t \bmod m)$ the proportion of doubtful values of n is $\Pi(1-1/p_i)$. The corollary above states that this proportion is asymptotically 0, as claimed in the theorem.

It seems a hard problem to prove that for all sufficiently large integers n, the Erdös conjecture (or its generalization) is true. But this conjecture does seem to be correct. (The conjecture has to be verified only for prime values of n.)

5. *Problems*.

5.01. Prove that $2/3 = 1/3 + 1/3 = 1/2 + 1/6$ is the largest rational that can be written in two ways as a sum of two Egyptian fractions. What is the largest rational m/n that can be expressed as $1/x + 1/y$ and also as $1/z + 1/w$, with x, y, a, w all different?

5.02. Note that $3/7 = 1/4 + 1/7 + 1/28 = 1/3 + 1/11 + 1/231$. Prove that $1 = 1/3 + 1/3 + 1/3 = 1/2 + 1/4 + 1/4$ is the largest rational that can be expressed in two ways as the sum of three Egyptian fractions. What is the largest rational with similar properties, but with all six denominators different?

5.03. Note that $3/280 = 1/95 + 1/19\cdot 280 = 1/96 + 1/12\cdot 280$. Find other rationals (if there are any) expressible in both forms $1/x + 1/y = 1/(x+1) + 1/z$, $a > x + 1$.

5.04. Is there a rational expressible simultaneously in the three forms $(u, v, w, t > x + 2)$

$$1/x + 1/y + 1/z = 1/(x+1) + 1/w + 1/t = 1/(x+2) + 1/u + 1/v\ ?$$

5.05. What is the largest rational expressible in three different ways as the sum of two Egyptian fractions (a) if the six denominators must all be different? (b) without this restriction?

5.06. Below the interval $(23/24, 41/42)$, but still inside $(0, 1)$, what is the next entire (longest) interval J such that no rational in J can be written as the sum of three Egyptian fractions?

5.07. What are the longest, second longest, third longest subintervals of $(0, 1)$ that contain no point expressible as the sum of three Egyptian fractions?

5.08. Let $J_1, J_2, \ldots$ be an infinite set of mutually disjoint intervals, all included in $(0, 1)$, such that no rational in any of them can be written as the sum of three Egyptian fractions. Prove that the length of J_v approaches 0 as $v \to \infty$.

5.09. The same as 5.08, but with "three" replaced by "seven."

REFERENCES

1. Chandrasekharan, K., *Introduction to Analytic Number Theory*, Springer, New York, 1968, (See bottom of p. 121.)

2. Erdös, P., *The Solution in Whole Numbers of the Equation* $1/x_1 + 1/x_2 + \ldots + 1/x_n = a/b$, (Hungarian) Matematikai Lapok, 1 (1950), 192-210.

3. Sierpiński, W., *Sur les Decompositions des Nombres Rationels en Fractions Primaires*, Mathesis, 65 (1956), 16-32.

4. Smith, H. J. S., *On the Integration of Discontinuous Functions (1875) Collected Mathematical Papers*, II, New York, 1965, 91-93.

5. Stewart, B. M., *Theory of Numbers, 2nd Edition*, Academic Press, New York, 1964, especially Chapter 28.

6. Stewart, B. M. and Webb, W. A., *Sums of Fractions with Bounded Numerators*, Canadian J. Math., 18 (1966), 999-1003.

7. Vaughan, R. C., *On A Problem of Erdös, Straus, and Schinzel*, Mathematika, 17 (1970), 193-198.

8. Webb, W. A., *On $4/n = 1/x + 1/y + 1/z$*, Proc. Amer. Math. Soc., 25 (1970), 578-584.

9. Yamamoto, K., personal communication. See also, *On a Conjecture of Erdös*, Memoirs of the Faculty of Science, Kyushu University, Series A, Mathematics, vol. 18, no. 2 (1964), 166-167.

________, *On the Diophantine Equation $4/n = 1/x + 1/y + 1/z$*, ibid., 18, no. 2 (1965), 37-47.

Editor's note - Dr. Brenner invites readers to work on the problems which he has proposed. He will be pleased to correspond with students concerning them. He writes that in problem 5.04 "the * just means that the problem is quite a bit harder than the others, and maybe not even suitable for an undergraduate. It acts as a warning."

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ST. JOHN'S UNIVERSITY/COLLEGE OF ST. BENEDICT ANNUAL PI MU EPSILON STUDENT CONFERENCE

March 29 and March 30, 1985

This annual conference is open to all students and teachers - not only to members of Pi Mu Epsilon. The program will consist of several student presentations and two major addresses by the featured speaker, Professor I. N. Herstein of the University of Chicago. Professor Herstein's talks will be for student audiences.

The conference provides an excellent forum in which students who have been working on independent study or research projects can present their work to their peers.

If you have any questions concerning the student paper program or the free on-campus housing arrangements during the conference, contact either Professor Gerald E. Lenz (612-363-3193) or Professor Michael D. Gass (612-363-3192), Department of Mathematics, St. John's University, Collegeville, MN 56321.

Additional information will be available in January.

A THEOREM ON HOMOTHETIC SIMPLEXES

by M. S. Klamkin
University of, Alberta

In a recent note in this Journal (7(1983)518-522) Eisenstein established the following theorem in three different ways:

<u>*Theorem:*</u> Let *AABC* be any triangle.

Let $O = (p,q)$ be an interior point of ΔABC.

Let K be the point of intersection of the medians of *MOB*.

Let L be the point of intersection of the medians of *ABOC*.

Let M be the point of intersection of the medians of ΔAOC.

Then ΔLMK is similar to ΔABC, the ratio of a side of ΔLMK to the corresponding side of *AABC* is 1/3, and the corresponding sides are parallel.

Using vectors, one can obtain a simpler proof which generalizes easily to simplexes.

If $\vec{A}, \vec{B}, \vec{C}, \vec{O}, \vec{K}, \vec{L}, \vec{M}$ denote vectors to the corresponding points, then immediately $3\vec{K} = \vec{A} + \vec{B} + \vec{O}$, $3\vec{L} = \vec{B} + \vec{C} + \vec{O}$, $3\vec{M} = \vec{C} + \vec{A} + \vec{O}$. Since $\vec{K} - \vec{M} = (\vec{B} - \vec{C})/3$, etc., it follows that ΔLMK is homothetic (inversely) to ΔABC with ratio of similitude 1/3. The point O need not lie in the interior of *AABC;* in fact, it need not even be in the plane of *AABC.* Additionally, we can easily find the center of homothecy H of the two triangles. It is the point of concurrency of the lines AL, *BM* and *CK.* Thus

$$\vec{H} = \vec{A} + \lambda\left(\frac{\vec{B} + \vec{C} + \vec{O}}{3} - \vec{A}\right)$$

where λ is an appropriate constant. Since $\vec{H}$ must be symmetric in $\vec{A}$, $\vec{B}$ and $\vec{C}$, $1 - \lambda = \backslash/3$ or $\lambda = 3/4$, giving

$$4\vec{H} = \vec{A} + \vec{B} + \vec{C} + \vec{O}.$$

For simplexes in E^n, let the vertices be $A_0, A_1, \ldots, A$ and let P be an arbitrary point, not necessarily in E^n. Then the simplex whose vertices are the $n + 1$ centroids of the $n + 1$ simplexes determined by each face of the given simplex and P is homothetic to the given simplex with ratio of similitude $1/(n+1)$. Letting C_i, $i = 0, 1, 2, \ldots, n$, denote the vertices of the derived simplex, it follows that

$$\vec{C}_i = (\vec{S} - \vec{A}_i + \vec{P})/(n+1)$$

where $\vec{S} = \vec{A}_0 + \vec{A}_1 + \ldots + \vec{A}_n$.

The rest follows from

$$\vec{C}_i - \vec{C}_j = (\vec{A}_j - \vec{A}_i)/(n+1).$$

This example is a good illustration that one should always be on the lookout for an appropriate representation. Here, the vectorial representation was most appropriate.

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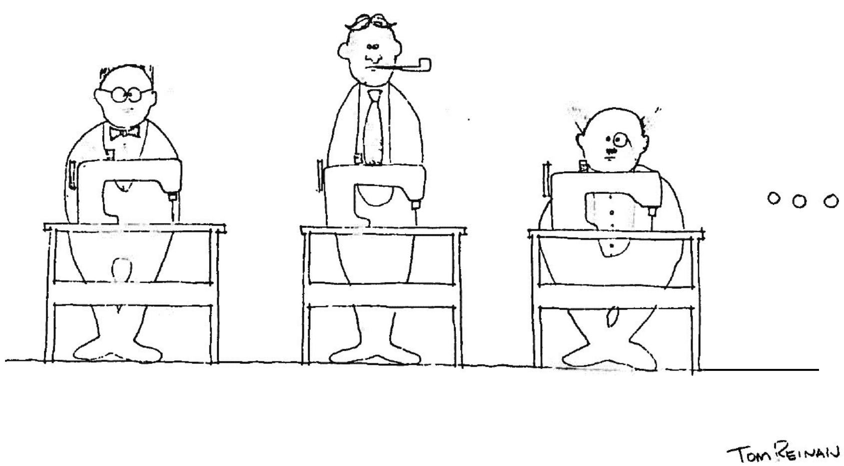


The drawing above, posed as a rebus, is one of several which delighted members of the North Central Section of the Mathematical Association of America at its Fall Meetings at Moorhead State University in Moorhead, MN. Are you able to supply the caption? Decipher the rebus? The drawing above and that on page 32 are the work of Thomas J. Reinan, a senior mathematics major at Moorhead State.

For contributor's caption, see page 39.

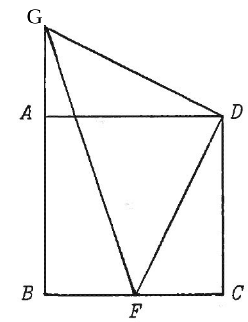
ARCTAN 1 + ARCTAN 2 + ARCTAN 3 = π

by Leon Bankoff
Los Angeles, California

The Fall 1983 issue of the Pi Mu Epsilon *Journal* (page 592) contains a simple trigonometric proof that

$$\arctan 1 + \arctan 2 + \arctan 3 = \pi.$$

An alternate proof based on elementary Euclidean geometry is offered here.

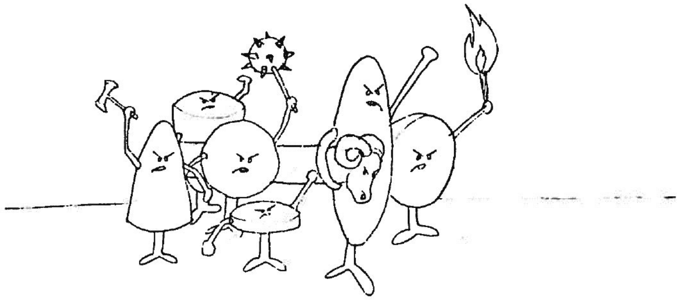


In the figure, F is the midpoint of the side BC of square ABCD. Triangle GAD, congruent to triangle FCD, is described externally on side AD, as shown. It is easily seen that GFD is an isosceles right triangle.

Then angle GFB = *arctan* 3, angle GFD = *arctan* 1, and angle DFC = *arctan* 2, with the result that the sum of the arctangents is equal to π.

Editor's note - Dr. Leon Bankoff was Problem Editor of the Pi Mu Epsilon Journal from Fall 1968 through Fall 1981. In his article "Reflections of a Problem Editor," which appeared in the Fall 1975 issue of this journal, Dr. Bankoff cites desirable qualities of a solution to a problem. His ABCD's of Elegance are A for Accuracy, B for Brevity, C for Clarity and D for Display of Insight, Ingenuity or Imagination. Dr. Bankoff's solution which appears above is an excellent example of how to meet these criteria.

ANOTHER REBUS BY REINAN



TOM REINAN

For contributor's caption, see page 39.

Award Certificates

Your Chapter can make use of the Pi Mu Epsilon Award Certificates available to help you recognize mathematical achievements of your students.

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(1) *Purple on Goldenrod Stock $1.50/dozen*

(2) *Purple on Lavender on Goldenrod .. $2.00/dozen.*

Send requests and orders to Dr. Richard A. Good, Secretary-Treasurer, Department of Mathematics, University of Maryland, College Park, MD 20742

PUZZLE SECTION

Edited by

Joseph D. E. Konhauser

The PUZZLE SECTION is for the enjoyment of those readers who are addicted to working doublecrostics or who find an occasional mathematical puzzle attractive. We consider mathematical puzzles to be problems whose solutions consist of answers immediately recognizable as correct by simple observation and requiring little formal proof. Material submitted and not used here will be sent to the Problem Editor if deemed appropriate for the PROBLEM DEPARTMENT.

Address all proposed puzzles and puzzle solutions to Prof. Joseph D. E. Konhauser, Mathematics and Computer Science Department, Macalester College, St. Paul, Minnesota 55105. Deadlines for puzzles appearing in the Fall Issue will be the next February 15, and for puzzles appearing in the Spring Issue will be the next September 15.

Mathacrostic No. 19

Proposed by Joseph D. E. Konhauser

Macalester College, St. Paul, Minnesota

The puzzle on the following two pages is a keyed anagram. The 219 letters to be entered in the diagram in the numbered spaces will be identical with those in the 25 keyed words at the matching numbers. The key numbers have been entered in the diagram to assist in constructing the solution. When completed, the initial letters of the words will give the name of an author and the title of a book; the completed diagram will be a quotation from that book. For an example, see the solution to the last mathacrostic in the **SOLUTIONS** part of this section.

1 D	2 K		3 O	4 D	5 H		6 V	7 Q	8 H	9 S	10 A		11 T
12 B		13 C	14 R	15 M	16 I	17 Y	18 L	19 V	20 F	21 H	22 R		23 O
24 B	25 W	26 H	27 C	28 G	29 E		30 V	31 Q	32 P		33 F	34 G	35 T
	36 N	37 R	38 A	39 H	40 T		41 U	42 K	43 S		44 T	45 E	46 G
47 M	48 I		49 C	50 H	51 Y	52 W		53 K	54 D		55 U	56 X	57 B
	58 W	59 I	60 O	61 R	62 V	63 D	64 F		65 E	66 L	67 W	68 P	69 O
70 I	71 X	72 B	73 N		74 U	75 O		76 D	77 Y	78 X		79 H	80 E
81 B	82 V	83 O		84 I	85 T	86 Y	87 D		88 K	89 U	90 S	91 G	92 A
93 O	94 Q	95 V	96 Y	97 H	98 I		99 U	100 C	101 H	102 F	103 V		104 P
105 O	106 S	107 M	108 D		109 F	110 V		111 T	112 R	113 A	114 H	115 L	116 I
117 M	118 N	119 Y		120 V	121 D		122 B	123 U	124 E		125 W	126 L	127 R
128 F	129 K	130 I		131 Y	132 R	133 G		134 L	135 Q		136 A	137 J	138 M
139 H	140 C	141 N	142 U	143 W	144 O	145 T		146 U	147 D		148 M	149 L	150 V
	151 D	152 Q	153 J	154 T		155 E	156 I		157 X	158 Q	159 L		160 A
161 F	162 E	163 Q	164 Y		165 P	166 V		167 W	168 F	169 S		170 D	171 M
172 F	173 G	174 V	175 O	176 W		177 H	178 T	179 J		180 R	181 X	182 G	183 Q
184 V	185 P	186 C		187 Y	188 O		189 Q	190 V	191 G	192 R		193 X	194 B
195 Q		196 A	197 Q	198 H	199 I	200 P	201 V		202 K	203 T	204 J	205 W	206 H
207 N	208 L		209 O	210 Y	211 F	212 R	213 W	214 L	215 M	216 B	217 N	218 Q	219 V

DEFINITIONS	WORDS
A. zyqal (comp.)	113 196 160 38 136 92 10
B. a pattern whose regular repetition fills the plane	81 24 194 122 72 57 12 216
C. one alternative to nothing	186 100 13 49 27 140
D. " ____ provides a fiendishly fertile field for famous fruitless follies." Frank Harary, The Four Color Problem ... (2 wds.)	151 108 4 170 76 87 1 147 54 63 121
E. the imaginative projection of a subjective state into an object so that the object appears to be infused with it	45 29 65 162 155 80 124
F. catchword; slogan; platitude; truism	211 168 128 20 109 172 102 161 64 33
G. where Ernest Lawrence Thayer's hero stood (3 wds.)	34 133 182 91 28 173 46 191
H. Church's elegant and powerful symbolism for mathematical processes of abstraction and generalization (comp.)	50 177 198 26 5 114 139 206 39 97 8 21 101 79
I the culmination or highest development of a thing	70 199 16 84 48 130 156 98 59 116
J. a socially inept, foolish or ineffectual person (slang)	153 204 137 179
K. a factory for the production of opera (2 wds.)	53 42 288 202 129
L. $Ca_3Cr_2(SiO_4)_3$; an emerald-green calcium-chromium qarnet	115 18 149 66 134 214 126 208 159
M. controller of current events	171 148 107 138 15 47 215 117
N. all things considered	217 207 141 73 118 36
O. an instrument for determining the concentration or particle size of suspensions by means of transmitted or reflected light	83 105 23 3 144 60 209 93 175 188 75 69
P. said of a cut which divides a line segment in "extreme and mean ratio"	68 165 200 32 185 104
Q. peg and disk puzzle brought out in 1883 by Edouard Lucas (3 wds.)	183 152 189 195 163 218 135 158 94 31 7 197
R. witches' broom	192 37 112 22 14 61 212 180 132 127
S. in relativity, an occurrence at both a specific time and a specific place	169 106 43 9 90
T. the card player who first receives the cards in a deal (2 wds.)	111 203 35 154 145 40 85 11178 44
U. the upside-down bird	123 142 41 146 89 55 99 74
V. in A Study in Scarlet, Watson to Holmes on an article by Holmes on observation and deduction, "What ____ ! I never read such rubbish in my life." (2 wds.)	190 219 110 166 6 30 120 174 62 95 82 19 103 150 184 201
W. On ____ and ____, D'Arcy Thompson's work on the relation of physical and biological principles to mathematical laws (2 wds.)	205 143 67 52 167 58 125 25 213 176
X. Archimedes' work in which he reveals his modus operand!	71 181 157 56 193 78
Y. to the devil as the atheist is to God	86 119 187 77 210 51 17 96 131 164

SOLUTIONS

Mathacrostic No. 18. (See Spring 1984 Issue) *(Proposed by Joseph V. E. Konhauser, Macalester College, St. Paul, Minnesota)*

Words:

A. Automata	J. In a Nutshell	S. Rosetta Stone
B. Gordian Knot	K. Nicholas Cusa	T. Oval
C. Archimedean	L. Finesse	U. Cybernetics
D. Rotatory	M. Icosahedron	V. Ensconced
E. Dudeney	N. Neutral	W. Sunflower
F. Incident With	O. Inductive	X. Sluff and Ruff
G. Neat	P. Threshold	Y. Escalade
H. Ergodic	Q. Enthymeme	Z. Snowball
I. Refutation	R. Pousette	

First Letters: A GARDINER INFINITE PROCESSES

Quotation: *The ideas presented were not all Cauchy's own, but he. selected ~ ~ M a fundamental and carefully formulated idea (such a the. notions of, limit, convergence and continuity in an interval), and used thebe. to construct the framework for analysis on the. basis of rigorous deduction atone.*

Solved by: Jeanette Bickley, Webster Groves High School, MO; Victor G. Feser, Mary College, Bismarck, ND; Robert Forsberg, Lexington, MA; Robert C. Gebhardt, County College of Morris, Randolph, NJ; Allan Gilbertson, Wheaton, MD; Dr. Theodor Kaufman, Nassau Hospital, Mineola, NY; Henry S. Lieberman, John Hancock Mutual Life Insurance Co., Boston, MA; Donald C. Pfaff, University of Nevada-Reno, NV; Robert Prielipp, The University of Wisconsin-Oshkosh, WI; Stephanie Sloyan, Georgian Court College, Lakewood, NJ.

Puzzle Editor's Note: In the definition of word E., the proposer used the dates (1847-1930) which appear in the dedication to Dudeney in the Dover reprint of his ***Amusements in Mathematics.*** Bob Prielipp pointed out that the dates (1857-1931) are given in Eves' ***An Introduction to the History of Mathematics, Fifth Edition.*** According to *Scripta Mathematica,* Volume 1, (1931), Henry Dudeney was born on April 10, 1857, and died on April 24, 1930. May he rest in peace.

COMMENTS ON PUZZLES 1 - 7, SPRING 1984

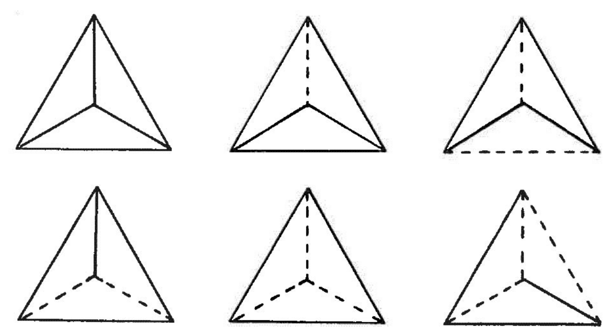
Five readers responded to *Puzzle #1.* Three-space can be separated into sixteen parts by placing a sphere with its center at the center of a cube and such that the sphere is tangent to all sixteen edges of the cube. Responses were received from ten readers for *Puzzle #2.* Samples include $1/2 + 3845/7690 = 1$ from Allan Gilbertson, $13/26 + 485/970 = 16/32 + 485/970 = 31/62 + 485/970 = 1$ from Victor G. Feser and Emil Slowinski, $38/76 + 451/902 = 1$ from Henry Rosche III, $45/90 + 138/276 = 1$ from Edward Aboufadel, $48/96 + 135/270 = 1$ from Marc J. Cochran. The examples, $57/92 + 140/368 = 1$ and $96/102 + 34/578 = 1$, in which the fractions are not both equal to one-half, were submitted by John H. Scott and Paul Barnard, respectively. Four examples of the following type were sent by Robert Prielipp: $(4 + 5)/(1 + 8 + 9) + (0 + 6)/(2 + 3 + 7)$. For *Puzzle #3,* Prielipp submitted

$$(k + 1)^3 + (2k + 7)^3 - (k + 5)^3 - (2k + 6)^3 = 3(2k + 1),$$

which has zero summands for $k = -5, -3$ and -1. A representation for $3(2k + 1)$ which avoids zero summands for these values of k is

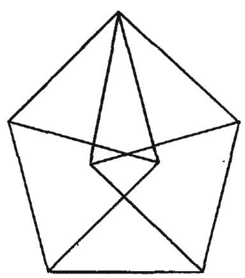
$$(k)^3 - (k - 4)^3 + (2k - 5)^3 - (2k - 4)^3 = 3(2k + 1).$$

Of course, the second representation has zero summands for $k = 0$, 2 and 4, but together the two representations provide a solution. Is there a single representation which solves the problem? Among references sent by Prielipp is a paper by Schinzel and Sierpiński entitled ***Sur les sommes de quatre cubes*** which appeared in *Acta Arith.*, 4 (1958), 20-30. Four readers responded to *Puzzle #4.* Using just two colors, the edges of a regular tetrahedron can be colored in twelve distinguishable ways. Six of the colorings are shown below. The other six can be obtained by switching colors.



All eleven responders to *Puzzle #5* determined that the light signal was on for seven minutes and then off for seven minutes and concluded that at 2:00 P.M. the light signal would be off. In a gentle jibe at the proposer's wording of the puzzle, Robert C. Gebhardt wrote "My students would answer 'Yes'." The proposer found the puzzle in an *NCTM* publication, *Mathematical Challenges*, compiled and annotated by Mannis Charosh.

Only John H. Scott supplied an answer to *Puzzle #6*. He produced a set of twelve points arranged so that three colors are not sufficient for coloring all the points of the plane so that no two points spaced one unit apart are colored alike. A simpler example, shown below, using just seven points, was given by Leo and Willy Moser in the *Canadian Mathematical Bulletin*, Vol. 4, No. 2 (1961) in the sole response to a problem which they had proposed. They proved that every set of six points in the plane can be colored in three colors in such a way that no two points unit distance apart have the same color and that six cannot be replaced by seven.



All twelve respondents to *Puzzle #7* supplied the correct answer - stamp denominations of 1, 4, 7 and 8. *Puzzle #7* is a very special case of a more general, and rather difficult, problem: Given n and m, determine a set of n denominations such that sums of m (or fewer) of these denominations can produce the consecutive integers 1, 2, ... , K, where K is as large as possible. In *Puzzle #7* $n = 4$, $m = 3$ and K = 24. An early reference for the postage stamp problem is R. P. Sprague's *Unterhaltsame Mathematik*, published by Braunschweig in 1960. An English translation by T. H. O'Bierne (1963) was published by Blackie and is now available from Dover. The book contains several very nice problems.

<u>List of solvers:</u> Edward Aboufadel (2, 5), Paul Barnard (1, 2, 5, 7), Marc J. Cochran (2, 7), Mark Evans (5, 7), Victor G. Feser (1, 2, 4, 5, 7), Robert C. Gebhardt (5, 7), Allan Gilbertson (2, 7), John M. Howell (1, 5), Ralph King (5), Brian Kopacz (7), Glen E. Mills (2, 4, 5, 7), Thomas Mitchell (7), Robert Prielipp (2, 3, 7), Henry Rosche III (1, 2, 5, 7), John H. Scott (1, 2, 4, 5, 6, 7), and Emil Slowinski (1, 2, 4, 5).

Answers to rebuses: page 30, Infinite Taylor Series; page 32, Solids of Revolution.

PUZZLES FOR SOLUTION

1. *Proposed by John M. Howell, Littlerock, California.*

a. Using four colors, in how many distinguishable ways can one color the edges of a square if pieces are not permitted to be turned over?

b. By matching edges, form the squares into a rectangle.

c. Omitting the squares with four different colors, assemble the remaining ones into a square.

2. *Proposed by J. P. E. Konhauser, Macalester College, St. Paul, Minnesota..*

It is not true that 3957 = 2648. Without using any mathematical symbols, reposition the eight digits to form a true statement. The odd numbers must remain to the left of the equals sign and the even to the right.

3. *Proposed by J. P. E. Konhauser, Macalester College, St. Pant, Minnesota*

The trio of positive integers {5, 20, 44} has the property that the sum of any two of its members is a perfect square. Can you find a set of four distinct positive integers such that the sum of any three is a perfect square?

4. *Proposed by J. O. E. Konhauser, Macalester College, St. Paul, Minnesota.*

In a certain mathematical system the elements a, b and c satisfy the relations

$$aba = a, \quad bab = b, \quad ab = ba \text{ and } ac = ca.$$

Using only substitution and associativity, show that bc = cb.

5. *Proposed by J. O. E. Konhauser, Macalester College, St. Paul, Minnesota.*

If the numbers 1 through 6 are arranged in a "ring" as shown below then the six sums obtained by adding neighbors are primes.

	1	
6		4
5		3
	2	

What is the largest integer N for which you are able to arrange the numbers 1 through N in a "ring" with the same property?

6. *Proposed by J. V. E. Konhauser, Macalester College, St. Pout, Minnesota.*

Does there exist a set S consisting of six positive integers, none exceeding 24, such that the numbers in the 63 non-empty subsets of S have sums which are different from one another?

7. *Proposed by J. V. E. Konhauser, Macalester College, St. Paul, Minnesota.*

A positive integer $N > 1$ is said to be a balanced number if the number of primes between 1 and N is equal to the number of composites between 1 and N. For example, 10 is a balanced number. Is there a largest balanced number? If so, find it.

GRAFFITO

Problems worthy of, attack prove their worth by hitting back.

Piet Hein

PROBLEM DEPARTMENT

Edited by Clayton W. Dodge
University of, Maine

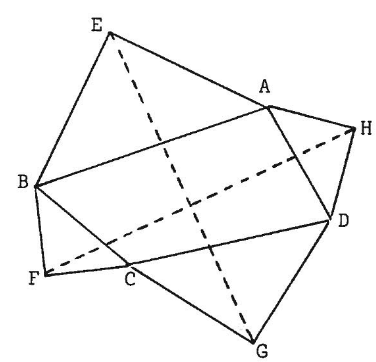
This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

All communications should be addressed to C. W. Dodge, Math. Dept., University of Maine, Orono, ME 04469. Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly identified with name and address. Solutions to problems in this issue should be mailed by July 1, 1985.

Corrections

558. [Fall 1983] *Proposed by Richard I. Hess, Palos Verdes, California.*

Let *ABCD* be a quadrilateral. Let each of the sides *AB, BC, CD, DA* be the diagonal of a square. Let *E, F, G, H* be those vertices of the squares that lie outside the quadrilateral. That is, *EAB, FBC, GCD,* and *HDA* are directly similar isosceles right triangles with apexes E, F, G, H. Prove that *EG* and *FH* are perpendicular. See the figure below. (The correction is that the triangles are similar - Ed.)



Correction to an editorial note. In the Spring 1984 issue following Solution II for problem 518 (page 675) we remarked that Michael Ecker is Problem Editor for *Popular Computing*. He is actually a contributing editor/columist for *Popular Computing* and also for *Byte*. He is Problem Editor for the *AMATYC Review.*

Problems for Solution

574. *Proposed by S. E. Ducer, Rogue Bluffs, Maine.*

Although there are many solutions to this unfortunate base 8 alphametric, there is only one prime *MOOD*. Find that *MOOD*.

$$\begin{array}{r} NOT \\ IN \\ \underline{THE} \\ MOOD \end{array}$$

575. *Proposed by Charles W. Trigg, San Diego, California.*

The sum of the digits of a two-digit integer N is S and the product of the digits is P. One of the differences $N - S$ and $N - P$ is a square and the other is a cube. Find N and show it to be unique.

576. *Proposed by David Iny, Rensselaer Polytechnic Institute, Troy, New York.*

Prove the following for all natural numbers n:

(a) $$1\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \ldots + n\binom{n}{n} = 2^{n-1}n,$$
$$1^2\binom{n}{1} + 2^2\binom{n}{2} + 3^2\binom{n}{3} + \ldots + n^2\binom{n}{n} = 2^{n-2}n(n+1),$$
$$1^3\binom{n}{1} + 2^3\binom{n}{2} + 3^3\binom{n}{3} + \ldots + n^3\binom{n}{n} = 2^{n-3}n^2(n+3).$$

*(b) for each positive integer p there exists a polynomial $q(n)$ of degree p that:

(i) $1^p\binom{n}{1} + 2^p\binom{n}{2} + 3^p\binom{n}{3} + \ldots + n^p\binom{n}{n} = 2^{n-p}q(n)$,

(ii) $q(n)$ has integral coefficients and leading coefficient 1,

(iii) when $p > 1$ is odd, then $q(n)$ is divisible by n^2;

(iv) when $p > 2$ is even, then $q(n)$ is divisible by $n(n+1)$.

*577. *Proposed by David E. Penney, The University of Georgia, Athens.*

In the $3x3x3$ cubical array below, the sum of the eight digits in each of the eight $2x2x2$ corner cubes is a fixed rational multiple (100/13) of the integer in the center. Does there exist such an array of the integers from 1 to 27 in which the eight corner sums are the same *integral* multiple of the integer in the center? [See Problem 504 (Fall 1981) for a similar two-dimensional problem].

Top:			Center:			Bottom:		
24	12	27	2	19	1	25	11	26
10	3	7	17	13	18	9	4	8
20	15	23	6	16	5	21	14	22

578. *Proposed by Emmanuel O. C. Imonitie, Northwest Missouri State University, Maryville.*

Given that x and y have opposite signs, solve the simultaneous equations

$$x + y + xy = -5 \quad \text{and} \quad x^2 + y^2 + x^2y^2 = 49$$

579. *Proposed by R. S. Luthar, University of Wisconsin Center, Janesville.*

Prove that for any positive integer n,

$$4^n(n!)^3 < (n+1)^{3n}.$$

580. *Proposed by Bob Prielipp, University of Wisconsin-Oshkosh.*

Let a, b, and c be the lengths of the sides of a triangle and let s be its semiperimeter. Prove that

$$(a/2)^a(b/2)^b(c/2)^c \geq (s-a)^a(s-b)^b(s-c)^c.$$

581. *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.*

If a triangle similar to a 3-4-5 right triangle has its vertices at lattice points (points with integral coordinates) in the plane, must its legs be parallel to the coordinate axes?

582. *Proposed by Walter Blumberg, Coral Springs, Florida.*

In triangle ABC with sides of lengths a, b, and c, we are given that $bc^2 \cos B = ca^2 \cos C = ab^2 \cos A$. Prove that triangle ABC is equilateral.

583. *Proposed by Joe Pan Austin, Emory University, Atlanta, Georgia.*

An urn contains n balls numbered 1 through n, which are drawn one at a time without replacement. Let x be the first number drawn.

Let y be the first number drawn that is larger than x if $x < n$ and let $y = 0$ if $x = n$. Let N be the draw which gives the y value if $x < n$ and let $N = n + 1$ if $x = n$. Find $E[y]$ and $E[N]$.

584. *Proposed by Jack Garfunkel, Flushing, New York.*

Let ABC be any triangle with base BC. Let D be any point on side AB and E any point on side AC. Let PDE be an isosceles triangle with base DE, oriented the same as ABC, and with apex angle P equal to angle A. Find the locus of all such points P.

585. *Proposed by Victor G. Feser, Mary College, Bismarck, North Dakota.*.

The sum of 17 cents can be made up in exactly six ways: $(0, 0, 17)$, $(0, 1, 12)$, $(0, 2, 7)$, $(0, 3, 2)$, $(1, 0, 7)$, and $(1, 1, 2)$, where (d, n, p) denotes the numbers of dimes, nickels, and pennies, respectively. Find a value of $n > 1$ such that n cents can be made up in exactly n ways and show that that n is unique. You may use pennies, nickels, dimes, quarters, half dollars, and dollars, as needed.

586. *Proposed by Robert C. Gebhardt, Hopatcong, New Jersey.*

For what x does $\sum_{n=0}^{\infty} \frac{x^n}{(2n)!} = 1 + \frac{x}{2!} + \frac{x^2}{4!} + \frac{x^3}{6!} + \frac{x^4}{8!} + \ldots$ converge and what is its sum?

Solutions

239. [Spring 1970, Fall 1983] *Proposed by David L. Silverman, Beverly Hills, California.*

A pair of toruses having hole radius = tube radius = 1 are linked. a) What is the smallest cube into which the toruses can be packed? b) What convex surface enclosing the linked toruses has the smallest volume? c) What convex surface enclosing the linked toruses has the smallest area? d) What is the locus of points in space equidistant from the two links?

Comment by Harry L. Nelson, Livermore, California.

A solution to part (a) is given in the solution by R. Robinson Rowe to Problem *117, *Journal of Recreational Mathematics*, 9(1), 1976-77, 32-36.

403. [Fall 1977, Fall 1983] *Proposed by David L. Silverman, West Los Angeles, California.*

Two players play a game of "Take It or Leave It" on the unit interval (0,1). Each player privately generates a random number from the uniform distribution and either keeps it as his score or rejects it and generates a second number which becomes his score. Neither player knows, prior to his own play, what his opponent's score is or whether it is the result of an acceptance or a rejection. (However, variants based on modifying this condition, either unilaterally or bilaterally, are interesting).

The scores are compared and the player with the higher score wins $1.00 from the other.

a. What strategy will give a player the highest expected score?

b. What strategy will give a player the best chance of winning?

c. If one player knows that his opponent is playing so as to maximize his score, how much of an advantage will he have if he employs the best counter-strategy?

Solution by Harry L. Nelson, Livermore, California.

a. The highest expected score is .625 which arises from the strategy: Take the first number if it is ≥ .5, otherwise take the second. Then the expected score is

$$(1 - x)((1 + x)/2) + x(1/2),$$

where $1 - x$ is the probability that the number will be greater than x, $(1 + x)/2$ is the mean of all such numbers, and $x(1/2)$ is the probability the first number is rejected times the mean value of the number obtained on the second try. By elementary calculus we see that this function maximizes at $x = 1/2$, for which the expected value is .625. Any other strategy will give a lower expectation.

b. The correct strategy for maximizing winning is to reject all numbers less than the expected value that the opponent's strategy gives him. If his strategy is to reject all values less than m, where $m = .61803...$ is the reciprocal of the golden mean $x = 1.61803..$. which is found by solving $1/x = 1 + x$, then there is no better strategy than to use it yourself and no expectation of winning. Should he, however, adopt some other strategy known to you, then you obtain a best winning ratio if you compute his expectation for that strategy and use that number as your rejection criteria.

For example, should he reject all numbers less than .625 (perhaps because he feels you are rejecting all less than .5), your best strategy is to reject all numbers less than .6171875, which is the expectation for his strategy.

c. As stated above, the best strategy against .5 is .625. The advantage gained is less than 1%. In 200 trials you would expect to win 101 and lose 99 times. In a computer simulation of over 2 million trials I won 50.4% of the time.

Also solved by PETER A. GRIFFIN, *California State. University, Sacramento, and* RICHARD I. HESS, *Rancho Palos Verdes, California.*

419. [Spring 1978, Fall 19831 *Proposed by Michael W. Ecker, City University of, New York.*

Seventy-five balls are numbered 1 to 75 and are partitioned into sets of 15 elements each, as follows: $B = \{1, \ldots, 15\}$, $I = \{16, \ldots, 30\}$, $N = \{31, \ldots, 45\}$, $G = \{46, \ldots, 601$, and $Q = \{61, \ldots, 751$, as in Bingo.

Balls are chosen at random, one at a time, until one of the following occurs: At least one from each of the sets B, I, G, O has been chosen, or four of the chosen numbers are from set N, or five of the numbers are from one of the sets, B, I, G, O.

Problem: Find the probability that, of these possible results, four N's are chosen first. (Comment: The result will be approximated by the situation of a very crowded bingo hall and will give the likelihood of what bingo players call "an N game," that is, bingo won with the winning line being the middle column N).

Solution by Van Pru Ving, China, Maine.

We win with an N game if we draw from 0 to 4 B's, from 0 to 4 I's, from 0 to 4 G's, and from 0 to 4 O's, but no more than three of those four letters, and 3 N's, in any order, followed by a final fourth N. Thus we see that an N game can occur in a minimum of 4 draws or a maximum of 16. We represent the various possibilities as ordered triples (x, y, z) of integers with $0 \le z \le y \le x \le 4$. Each such triple represents the selection of x balls from one of the letters B, I, N, G, and O, selecting y balls from a second of those letters, and z balls from a third of those letters. Also 3 n's are selected. These $x+y+z+3$ balls can be rearranged in any order. Finally, the $(x+y+z+4)$th ball must be another N. Hence, selecting

B N I N G B B N G B N

is an N game designated by (4, 2, 1) since there are 4 B's, 2 G's, and 1 I in addition to the necessary 4 N's, one of which terminates the sequence.

The number of ways of choosing an (x, y, z) N game is given by multiplying together:

a) $15 \cdot 14 \cdot 13 \cdot 12 = 32760$ for the four N's;

b) $15 \cdot 14 \cdot \ldots \cdot (16 - x)$, if $x > 0$, for the number of ways of choosing the first x balls;

c) 4, if $x > 0$, since there are four letters (B, I, G, and O) from which to choose the letter for the first x balls;

d) $15 \cdot 14 \cdot \ldots \cdot (16 - y)$, if $y > 0$, for the next y balls;

e) 3, if $y > 0$, since the y balls must be chosen from one of the remaining three letters;

f) $15 \cdot 14 \cdot \ldots \cdot (16 - z)$, if $z > 0$, for the next z balls;

g) 2, if $z > 0$, since there are two letters left to choose from;

h) 1/2, if $x = y$ or if $x = z$, since the order of choosing like numbers of balls is immaterial;

i) 1/3, if $x = z$, for the same reason as (h);

j) $(3 + x + y + z)!/(x!\, y!\, z!\, 3!)$ for the rearrangements (combinations) of the first $3+x+y+z$ balls.

Now it is easy to program a computer or to calculate by hand all (x, y, z) probablilites. The sum of all such probabilities is $0.123493\ldots$, the desired probability. A BASIC program follows.

```
10 T=O
20 FOR Z=O TO 4
30 FOR Y=Z TO 4
40 FOR X=Y TO 4
50 P=5460
60 FOR N=l TO X: P=P*(l6-N)/N: NEXT
70 FOR N=l TO Y: P=P*(l6-N)/N: NEXT
80 FOR N=l TO Z: P=P*(l6-N)/N: NEXT
90 FOR N=l TO 3+X+Y+Z: P=P*N/(76-N): NEXT
100 P=P/(72-X-Y-Z)
110 IF X>0 THEN P=P*4 ELSE 170
120 IF Y>0 THEN P=P*3 ELSE 170
130 IF Z>0 THEN P=P*2
```

```
140 IF X=Y THEN P=P/2: GOTO 160
150 IF Y=Z THEN P=P/2
160 IF X=Z THEN P=P/3
170 PRINT X; Y; Z, P
180 T=T+P
190 NEXT X
200 NEXT Y
210 NEXT Z
220 PRINT "TOTAL ="; T
$
```

Also solved by OSKAR FEICHTINGER *and* WILLIAM A. HALTEMAN *(jointly), University of Maine, Orono, and* RICHARD I. HESS, *Rancho Palos Verdes, California. Partial solution by* ROGER KUEHL, *Kansas City, Missouri. One incorrect solution was also received.*

524. [Fall 1982, Fall 1983] *Proposed by Morris Katz, Macwahoc, Maine.*

Solve this holiday alpametric for a real prime *XMAS*.

$$\begin{array}{r} MERRY \\ \underline{XMAS} \\ DODGE \end{array}$$

71. *Comment by Alan Wayne, Pasco-Hernando Community College, Florida.*

Since *DODGE* = 60647, it is evident that *DODGE* is a prime figure, coming from rational forebears, since *DODGE* is neither the sum nor the product of two squares. [*Not even three spares!* - *Ed.*]

Editorial note. Wayne also pointed out that the usual word for this type of problem is *alphametic* or *cryptarithm*, whereas we have used the term *alphametric* (with an *r*) in this department. This is the editor's own whimsy since *alpha-metric* seems to describe the type of problem quite well.

*525. [Fall 1982, Fall 1983] *Proposed by John M. Howell, Littlerock, California.*

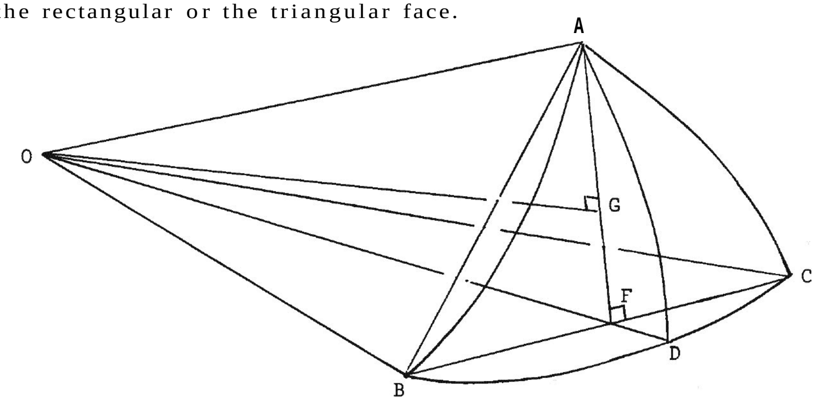
An equilateral triangular prism is used as a die. What must the ratio of the sides be so that the probability of falling on a triangle is the same as falling on a rectangle?

I. *Comment by the proposer.*

I proposed this problem to a summer class some years ago and a student from Colombia came up with a mathematical solution which I have misplaced. Empirical results showed the ratio of the side of the triangle to the other side of the rectangle to be about 1.6.

II. *Solution adapted from that submitted by Richard I. Hess, Rancho Palos Verdes, California.*

If tumbling is assumed to have no effect, then the condition to be met is that each face subtend $4\pi/5$ solid angle as seen from the center of gravity. The true effect of tumbling or rotational velocity is not easy to determine and it is not clear whether it would favor the rectangular or the triangular face.



In the equilateral spherical triangle *ABC* of the accompanying figure, let the sides (arcs) *BC*, *CA*, *AB* intercept central angles of measure a, b, c, and let the angles of the triangle have measures A, B, C. Let D be the midpoint of arc *BC*, so spherical triangle *ABD* has a right angle at D and so that $\angle A = \angle B/2$. Then triangle *ABD* must have spherical excess 72° $(=2\pi/5)$. That is,

$$B + B/2 + 90^\circ - 180^\circ = 72^\circ,$$

so that $B = 108^\circ$ and $B/2 = 54^\circ$. Now, by Napier's rules,

$$\cos\frac{a}{2} = \frac{\cos(B/2)}{\sin B} = \frac{\cos(B/2)}{2\sin(B/2)\cos(B/2)} =$$

$$= \frac{1}{2\sin(B/2)} = \frac{1}{2\sin 54^\circ} = .618034,$$

so $a = 103.65459^\circ = \angle BOC$.

Let *OD* meet *BC* at F and let G be the foot of the perpendicular from O to *AF*. Then G is the centroid of plane triangle *ABC*, *OG* is half the width of the rectangle, and BF is half its length. Without loss of generality we let the sphere have radius 1. Then, from triangle OBF,

$$BF = OB \sin BOF = \sin 51.827292^{\circ} = .7861514.$$

Since

$$AG = \frac{2}{3} AF = \frac{2}{3} (BF \sqrt{3}) = .9077694,$$

then

$$OG = \sqrt{OA^2 - AG^2} = \sqrt{1 - .8240454} = \sqrt{.1759546} = .4194695.$$

Therefore the ratio of the length to the width of the rectangle

$$\frac{.7861514}{.4194695} = 1.8741564.$$

Editorial note. Howell presented a table of empirical results from 100 tosses of dice with ratios from 1.04 to 2.00, which showed a solution somewhere between 1.5 and 1.6 and another solution at about 1.87.

536. [Spring 1983, Fall 1983] *Proposed by Martha Mattics, Veazie, Maine.*

A recent alphametric in Crux Mathematicorum [1982: 77, problem 721] asks one to show that, in base ten,

TRIGG *is three times* WRONG.

In defense of the Dean of Numbers, solve these alphametrics independently of each other:

(a) TRIGG X 3 = RIGHT *in base eight where the digit* 3 *can be reused,*

(b) TRIGG = 3 X RIGHT *in base ten where the digit* 3 *can be reused, and*

(c) TRIGG X 7 = RIGHT *in base seventeen.*

Solution by Glen E. Mills, Pensacola Junior College, Florida.

(a) We see that $T = 1$ or 2. If $T = 2$, then $G = 6$ and $3G + 2 \equiv H \pmod 8$ implies $H = 4$. Now $3I + 2 \equiv G \pmod 8$ demands that $I = 4$. Since $H = 4$, we have a contradiction. Therefore $T = 1$. Now $G = 3$, $H = 2$, $I = 6$, and $R = 4$. The restoration then is

$$14633 \times 3 = 46321 \text{ base } 8.$$

(b) Here $R = 1$, 2, or 3. If $R = 3$, then $T = 9$ but then we must carry more then 0 to the last column, contradicting $T = 9$. If $R = 1$, then $T = 3$, 4, or 5. If $R = 1$ and $T = 3$, then $G = 9$, so $H = 3$, a contradiction. For $R = 1$ and $T = 4$ we have $G = 2$, so $H = 7$ and $I = 8$, whence $R = 4$, another contradiction. We cannot have $T = 5$ since then $G = 5$ also. Hence we have $R = 2$ and $T = 6$, 7, or 8. For $R = 2$ and $T = 6$ we have $G = 8$, whence $H = 9$ and $I = 6$, a contradiction. For $R = 2$ and $T = 8$ we have $G = 4$, so $H = 4$, again a contradiction. Thus we must have $R = 2$ and $T = 7$, in which case $G = 1$, $H = 3$, and $I = 4$. The restoration is

$$72411 = 3 \times 24137 \text{ base ten}$$

(c) In this alphametric we have $T = 1$ or 2. Otherwise R is greater than a single digit. We construct a table of multiples of 7 in base seventeen and note that each possible last digit is obtainable uniquely as a digit times 7. Then, for $T = 2$ we must have $G = 10$, so $H = 6$, $I = 13$, and $R = 6$. But then the last product gives $R = 16$, so we cannot have $T = 2$. Hence $T = 1$, $G = 5$, $H = 3$, $I = 15$, and $R = 11$. The restoration is

$$1\ 11\ 15\ 5\ 5 \times 7 = 11\ 15\ 5\ 3\ 1 \text{ base } 17.$$

Also solved by JEANETTE BICKLEY, *Webster Groves High School, MO,* MARK EVANS, *Louisville, KY* VICTOR G. FESER, *Mary College, Bismarck, ND,* RICHARD I. HESS, *Rancho Palos Verdes, CA,* DAVID INY, *Rensselaer Polytechnic Institute, Troy, NY,* ROGER KUEHL, *Kansas City, MO,* SUSAN SADOFSKY, *Brockton, MA,* MICHAEL J. TAYLOR, *Indianapolis Power & Light Co., IN,* CHARLES W. TRIGG, *San Diego, CA,* HAO-NHIEN QUI VU *(part b only), Lafayette, IN,* KENNETH M. WILKE, *Topeka, KS, and the* PROPOSER.

547. [Fall 1983] *Proposed by Morris Katz, Macwahoc, Maine.*

Solve this musical alphametric.

SING
IN
THE
WAYNE

Solution by Charles W. Trigg, San Diego, California.

Immediately $W = 1$, $S = 9$, and $A = 0$. Then

$$G + N = 10,$$
$$H + I + 1 = 10,$$
$$I + T + 1 = Y + 10 \text{ and } 2 \le Y \le 6.$$

Now $\{G, N\} = \{2, 8\}$, $\{3, 7\}$ or $\{4, 6\}$ and $\{H, I\} = \{2, 7\}$, $\{3, 6\}$ $\{4, 5\}$ Now we examine the possibilities for (Y, T, I, H), eliminating $(6, 8, 7, 2)$, $(5, 8, 6, 3)$, $(4, 8, 5, 4)$, $(4, 6, 7, 2)$, $(3, 8, 4, 5)$, (2, 8, 3, 6), (2, 7, 4, 5), and (2, 5, 6, 3). The four remaining possibilities, (4, 7, 6, 3), (3, 7, 5, 4), (3, 5, 7, 2), and (2, 6, 5, 4), yield eight solutions:

9682	9582	9764	9573
68	58	76	57
735	746	528	648
10485	10386	10368	10278

and the four sums obtained by interchanging the values of G and N.

Also solved by MARK EVANS (partial solution), Louisville, KY, VICTOR G. FESER, Mary College, Bismarck, ND, RICHARD I. HESS, Rancho Palos Verdes, CA, GLENN E MILLS, Pensacola Junior College, FL, BOB PRIELIPP, University of Wisconsin - Oshkosh, KENNETH M. WILKE, Topeka, KS, and the PROPOSER.

548. [Fall 1983] *Proposed by Paul A. McKlueen, Charlotte, North Carolina.*

Arrange the ten digits in a row, e.g.

$$d_1d_2d_3d_4d_5d_6d_7d_8d_9d_{10},$$

so that the following conditions are satisfied: the number $d_2d_3d_4$ is divisible by 2, $d_3d_4d_5$ is divisible by 3, $d_4d_5d_6$ by 5, $d_5d_6d_7$ by 7, $d_6d_7d_8$ by 11, $d_7d_8d_9$ by 13, and the number $d_8d_9d_{10}$ divisible by 17.

Solution by Glen E. Mills, Pensacola Junior College, Florida.

Since 5 divides $d_4d_5d_6$, then $d_6 = 0$ or 5; but if $d_6 = 0$, then $d_7 = d_8$ since 11 divides $d_6d_7d_8$. Therefore $d_6 = 5$. Now list all multiples of 17, of 13, and of 7 between 010 and 990 that consist of three distinct digits. Comparing these numbers we find that 357289 and 952867 are the only possibilities for $d_5d_6d_7d_8d_9d_{10}$. The former sequence yields four solutions 4160357289, 1460357289, 4106357289, and 1406357289, and the latter produces two more solutions 4130952867 and 1430952867.

Also solved by MARK EVANS (1 solution), Louisville, KY, VICTOR G. FESER (2 solutions), Mary College, Bismarck, NU, ROBERT C. GEBHARDT, Hopatcong, NJ, RICHARD I. HESS, Rancho Palos Verdes, CA, STEPHANIE SLOYAN, Georgian Court College, Lakewood, NJ, CHARLES W. TRIGG, (4 solutions) San Diego, CA, KENNETH M. MILKE, (4 solutions) Topeka, KS, and the PROPOSER (4 solutions).

549. [Fall 1983] *Proposed by R. S. Luthar, University of Wisconsin Center, Janesville.*

If a, b, c are positive numbers prove that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} > \frac{10\pi}{21}$$

[For an interesting related problem see Problem 356 in *The Pentagon*, Spring 1983, p. 120.]

I. *Solution by Walter Blumberg, Coral Springs, Florida.*

The following relation is known: Let $S = \sum_{i=1}^{n} m_i$, where $n \geq 2$ and all $m_i > 0$. Then $\sum_{i=1}^{n} \frac{m_i}{S - m_i} \geq \frac{n}{n-1}$, with equality if and only if all m_i are equal. Let $n = 3$, $m_1 = a$, $m_2 = b$, and $m_3 = c$ to get the inequality

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2},$$ with equality iff $a = b = c$.

Finally we note that $3/2 > 10\pi/21$.

II. *Solution by Bill Olk, Clintonville, Wisconsin.*

If u and v are positive numbers, then it is well known that

$$\frac{u}{v} + \frac{v}{u} \geq 2.$$

Letting $p = b + c$, $q = c + a$, and $r = a + b$, we apply the above inequality to p and q, to q and r, and to r and p, obtaining

$$\frac{q+r}{p} + \frac{r+p}{q} + \frac{p+q}{r} \geq 6,$$

$$\frac{q+r-p}{2p} + \frac{r+p-q}{2q} + \frac{p+q-r}{2r} \geq \frac{3}{2},$$

which reduces to

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2} > \frac{10\pi}{21}.$$

III. *Solution by Edwin M. Klein, University of Wisconsin, Whitewater.*

We prove the stronger inequality that the left side of the given inequality is greater than or equal to 3/2. Clearing of fractions and simplifying, we get the equivalent inequality

$$2a^3 + 2b^3 + 2c^3 \geq a^2b + ab^2 + a^2c + ac^2 + b^2c + bc^2.$$

This inequality follows from three applications of

$$x^3 + y^3 \geq xy(x + y),$$

which in turn is proved by multiplying each side of the inequality by

$$(x - y)^2 \geq 0$$

$x + y$.

Also solved by LEON BANKOFF, *Los Angeles, CA*, JACK GARFUNKEL, *Flushing, NY*, MURRAY S. KLAMKIN, *University of Alberta, Edmonton, Canada*, HENRY S. LIEBERMAN, *John Hancock Mutual Life Insurance Co., Boston, MA*, BOB PRIELIPP, *University of Wisconsin - Oshkosh*, KENNETH M. WILKE (*two solutions*), *Topeka, KS, and the* PROPOSER.

BANKOFF found the inequality in V. A. Krechmar, *A Problem Book in Algebra*, tr. by Victor Shiffer, Mir Publishers, 1974, Solution to Problem 27, pp. 406-7. GARFUNKEL gave the reference: O. Bottema, *Geometric Inequalities*, p. 15. WILKE used Exercise 25 on page 51 of Chrystal, *Algebra*, 7th ed., 1964, part 11.

550. [Fall 1983] *Proposed by I. R. Hess, Washington, D. C.*

How many different Pythagorean triples have a side or hypotenuse equal to *1040?*

Solution by Edwin M. Klein, University of Wisconsin - Whitewater.

In n has prime factorization $n = 2^{a_0} p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k}$, then the number of Pythagorean triangles having n as a leg is

$$L = \frac{(2a_0 - 1)(2a_1 + 1)(2a_2 + 1)\ldots(2a_k + 1) - 1}{2}$$

while if $n = 2^{a_0} q_1^{a_1} q_2^{a_2} \ldots q_u^{a_u} r_1^{b_1} r_2^{b_2} \ldots r_v^{b_v}$, where the q's are primes of the form $4s + 3$ and the r's are primes of the form $4s + 1$, then n is the hypotenuse of

$$H = \frac{(2b_1 + 1)(2b_2 + 1)\ldots(2b_v + 1) - 1}{2}$$

such triangles. See Albert H. Beiler, *Recreations in the Theory of Numbers*, Dover (1964), pp. 116-117. Thus $1040 = 2^4 \cdot 5 \cdot 13$ is the leg of 31 such triangles and the hypotenuse of another 4, for a total of 35 triangles.

Also solved by VICTOR G. FESER (*partial solution*), *Mary College, Bismarck, ND*, RICHARD I. HESS, *Rancho Palos Verdes, CA*, JOHN M. HOWELL (*partial solution*), *Littlerock, CA*, GLEN E. MILLS, *Pensacola Junior College, FL*, TOM MOORE (*partial solution*), *Bridgewater State College, MA*, I. PHILIP SCALISI (*partial solution*), *Bridgewater State College, MA*, KENNETH M. WILKE, *Topeka, KS, and the* PROPOSER.

551. [Fall 1983] *Proposed by Robert C. Gebhardt, Hopatcong, New Jersy.*

If k is the largest odd integer not exceeding the positive integer n, $n \geq 2$, prove that

$$\cos^2 \frac{\pi}{2n} + \cos^2 \frac{3\pi}{2n} + \cos^2 \frac{5\pi}{2n} + \ldots + \cos^2 \frac{k\pi}{2n} = \frac{n}{4}.$$

Solution by Murray S. Klamkin, University of Alberta, Edmonton, Canada.

Considering the parity of n, we have to show that

$$(1) \quad \sum_{j=1}^{n} \cos^2 \frac{(2j - 1)\pi}{4n} = \frac{n}{2} \qquad \text{for } n \text{ even},$$

$$(2) \quad \sum_{j=1}^{n} \cos^2 \frac{(2j - 1)\pi}{2(2n - 1)} = \frac{2n - 1}{4} \qquad \text{for } n \text{ odd}.$$

Using $2\cos^2 x = 1 + \cos 2x$, the above equations reduce to

$$(1') \quad \sum_{j=1}^{n} \cos \frac{(2j - 1)\pi}{2n} = 0,$$

$$(2') \quad \sum_{j=1}^{n} \cos \frac{(2j - 1)\pi}{2n - 1} = -\frac{1}{2}.$$

Both results follow immediately from the formula

$$\sum_{j=1}^{n} \cos (2j - 1)x = \frac{\sin 2nx}{2 \sin x},$$

which is easy to prove by clearing of fractions and replacing each resulting left side product using the formula

$$2 \sin u \cos v = \sin(u + v) - \sin (v - u).$$

The resulting sum of sine terms collapses to $\sin 2nx$.

Also solved by FRANK P. BATTLES, *Massachusetts Maritime Academy, Buzzards Bay*, RUSSELL EULER, *Northwest Missouri State University, Maryville*, RICHARD I. HESS, *Rancho Palos Verdes, CA*, BOB PRIELIPP, *Hopatcong, NJ*, KENNETH M. WILKE, *Topeka, KS, and the* PROPOSER.

552. [Fall 1983] *Proposed by Albert White, St. Bonaventure University, New York.*

Let $a_1 = 1$ and $a_n = 2a_{n-1} + (-1)^n$ for $n > 1$. Find

$$\lim_{n\to\infty} \frac{a_n}{2^{n+1}}.$$

I. *Solution by Edward M. Klein, University of Wisconsin-Whitewater.*

We show by mathematical induction that

$$(*) \qquad a_n = \frac{2^{n+1} + (-1)^n}{3}.$$

This equation is clearly true for $n = 1$. Assuming $(*)$ holds for a_n, we have

$$a_{n+1} = 2a_n + (-1)^{n+1} = \frac{2^{n+2} + 2(-1)^n + 3(-1)^{n+1}}{3}$$

$$= \frac{2^{n+2} + (-1)^n(2-3)}{3} = \frac{2^{n+2} + (-1)^{n+1}}{3},$$

so $(*)$ holds also for a_{n+1} and our proof of $(*)$ is complete. Hence

$$\lim_{n\to\infty} \frac{a_n}{2^{n+1}} = \lim_{n\to\infty} \left(1 + \frac{(-1)^n}{2^{n+1}}\right)/3 = \frac{1}{3}.$$

II. *Solution by Murray S. Klamkin, University of Alberta, Edmonton, Canada, and Tom Moore, Bridgewater State College, Massachusetts, (independently).*

Set $a_0 = 1$. Then $a_n - 2a_{n-1} = (-1)^n$ for $n > 0$. Let

$$f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots,$$

so that we have

$$f(x) - 2xf(x) = 1 - x + x^2 - x^3 + \ldots + (-1)^nx^n + \ldots$$

Therefore

$$(1 - 2x)f(x) = \frac{1}{1+x}.$$

By partial fractions we have

$$f(x) = \frac{2/3}{1-2x} + \frac{1/3}{1+x}$$

$$= \frac{2}{3}\sum_{n=0}^{\infty} 2^nx^n + \frac{1}{3}\sum_{n=0}^{\infty} (-1)^nx^n = \sum_{n=0}^{\infty} \left(\frac{2^{n+1} + (-1)^n}{3}\right)x^n.$$

This shows that $a_n = (2^{n+1} + (-1)^n)/3$ for $n \geq 1$. Thus we find that

$$\lim_{n\to\infty} \frac{a_n}{2^{n+1}} = \frac{1}{3}\lim_{n\to\infty} \frac{2^{n+1} + (-1)^n}{2^{n+1}} = \frac{1}{3}.$$

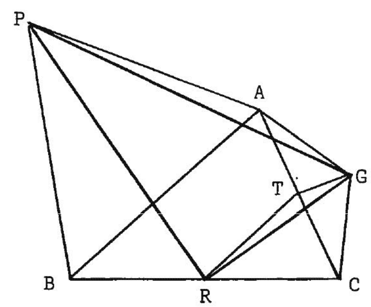
Also solved by SYLVAIN BOIVIN, *Université du Québec à Chicoutimi, Canada,* RUSSELL EULER, *Northwest Missouri State University, Maryville,* MARK EVANS, *Louisville, KY,* MARTIN P. GELFAND, *University of Pennsylvania, Philadelphia,* RICHARD I. HESS, *Rancho Palos Verdes, CA,* HENRY S. LIEBERMAN, *John Hancock Mutual Life Insurance Co., Boston, MA,* BOB PRIELIPP, *University of Wisconsin-Oshkosh,* J. PHILIP SCALISI, *Bridgewater State College, MA,* HARRY SEDINGER, *St. Bonaventure University, NY,* HAO-NHIEN QUI VU, *Purdue University, Lafayette, IN,* KENNETH M. WILKE, *Topeka, KS, and the* PROPOSER.

*553. [Fall 1983] *Proposed by Jack Garfunkel, Flushing, New York.*

Given a triangle ABC erect equilateral triangles BAP and ACQ outwardly on sides AB and CA. Let R be the midpoint of side BC and let G be the centroid of triangle ACQ. Prove that triangle PRG is a 30^o-60^o-90^o triangle.

Solution by Leon Bankoff, Los Angeles, California.

Let T be the midpoint of side AC. We have $AG = 2(GT)$ with $\measuredangle AGT = 60^o$. Also $PA = BA = 2(RT)$ with $\measuredangle(PA,AB) = \measuredangle(PA,TR) = 60^o$. Hence $\measuredangle PAG = \measuredangle RTG$ and, since $PA/RT = AG/TG = 2$, it follows that triangles RTG and PAG are similar. Thus the corresponding sides of triangles RTG and PAG are inclined at a 60^o angle, with the result that $\measuredangle PGR = 60^o$. Since $PG = 2(RG)$, triangle PRG is a 30^o-60^o-90^o triangle. See the accompanying figure. Furthermore this proof and its notation are valid for all species of triangles. When $\measuredangle BAC = 90^o$, the proof is trivial.

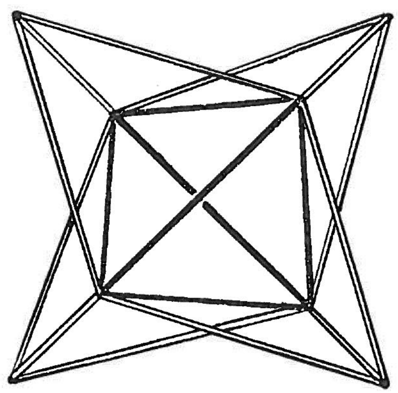


Also solved by LEON BANKOFF (*second solution*), *Los Angeles, CA*, RICHARD I. HESS, *Rancho Palos Verdes, CA*, HENRY S. LIEBERMAN, *John Hancock Mutual Life Insurance Co., Boston, MA*, MURRAY S. KLAMKIN, *University of Alberta, Edmonton, Canada*, BILL OLK, *Clintonville, WI*, *and* WILLIAM H. PEIRCE, *Stonington, CT.*

555. [Fall 1983] *Proposed by Richard D. Stratton, Colorado Springs, Colorado.*

Eighteen toothpicks can be arranged to form six congruent equilateral triangles. Rearrange the toothpicks to form sixteen congruent equilateral triangles each of the same size as the original six.

Amalgam of solutions submitted independently by VICTOR G. FESER, *Mary College, Bismarck, North Dakota*, ROBERT C. GEBHARDT, *Hopatcong, New Jersey*, GORDON D. GLENN, *Eastern Washington University, Cheney*, RICHARD I. HESS, *Rancho Palos Verdes, California*, JOHN M. HOWELL, *Littlerock, California*, GLEN *and* PATRICK MILLS, *Pensacola, Florida*, HARRY SEDINGER, *St. Bonaventure University, New York*, HAO-NHIEN QUI VU, *Purdue University, Lafayette, Indiana, and the* PROPOSER.



The figure is a stellated regular tetrahedron, a regular tetrahedron constructed with six toothpicks with another regular tetrahedron constructed of three additional toothpicks on each of its four faces. See the figure (by Glenn) in which the original tetrahedron is shown with solid edges.

556. [Fall 1983] *Proposed by Richard I. Hess, Palos Verdes, California.*

A normal pair of unbiased dice give a total of 2 through 12 according to the distribution 1, 2, 3, 4, 5, 6, 5, 4, 3, 2, 1. How should you change the spots on the dice so that the sums 2 through 12 and only those sums still occur but with as uniform a distribution as possible? (Minimize the sum of the squares of the deviations from completely uniform).

I. *Amalgam of solutions submitted independently by Mark Evans, Louisville, Kentucky, and John M. Howell, Littlerock, California.*

Leave one die unchanged and let the other die have three faces numbered 1 and three faces numbered 6. Then the distribution will be three of each sum except 7 and six 7's. Since the average frequency is 36/11, the sum of the squares of the deviations is

$$10(3/11)^2 + 1(30/11)^2 = 990/121 = 8.1818\ldots$$

II. *Solution by the Proposer.*

Let the first die have sides numbered 1, 2, 3, 6, 7, 8 and for the second die 1, 1, 2, 3, 4, 4. The resulting frequencies are 2, 3, 4, 4, 3, 4, 3, 4, 4, 3, 2 for the sums 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 respectively. The sum of the squares of the deviations is

$$2(14/11)^2 + 4(3/11)^2 + 5(8/11)^2 = 6.1818\ldots$$

Editorial note. Although the proposer's solution is the best of those submitted, none of the solvers even attempted to prove his solution minimal. Howell suggested that the distribution would be perfectly uniform if one die were labelled 1, 1, 1, 6, 6, 6, and the second die had just 5 sides labelled 1, 2, 3, 4, 5. This second die can be realized by using an ordinary die and tossing over again whenever a 6 appears.

*557. [Fall 1983] *Proposed by Pauvre Fish, Seal Beach, California.*

It is known and easy to show with elementary calculus that

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx = \frac{22}{7} - \pi.$$

Find a definite integral whose value is $\frac{193}{71} - e$, where e is the base of natural logarithms.

Summary of solutions by FRANK P BATTLES (*a*), *Massachusetts Maritime Academy, Buzzards Bay*, RUSSELL EULER (*b*), *Northwest Missouri*

State University, Maryville, MARK EVANS (c), *Louisville, Kentucky,* ROBERT C. GEBHARDT (c), *Hopatcong, New Jersey,* BILL OLK (c), *Clintonville, Wisconsin,* BOB PRIELIPP (d), *University of Wisconsin-Oshkosh, and* HARRY SEDINGER (b) *and* (e), *SX. Bonaventure University, New York.*

Each of the following integrals is equal to $193/71 - e$:

(a) $$\int_0^{193/71-e} dx,$$

(b) $$\int_0^1 \left(\frac{193}{71} - e\right) dx,$$

(c) $$\int_0^1 \left(\frac{122}{71} - e^x\right) dx,$$

(d) $$\int_0^{\ln(193/71)} e^x\, dx, \text{ and}$$

(e) $$\int_0^1 (122x^{70} - e^x)\, dx.$$

Editorial note. It was hoped that some delightful integral such as the given one for $22/7 - \pi$ would be found. Perhaps some clever reader will still discover an elegant integral for the desired $193/71 - e$.

559. [Fall 1983] *Proposed by Sidney Penner, Bronx Community College, New York.*

"This is quite amazing," said B. 'My bingo card does not contain a *BINGO*, but if *I* cover one more square, regardless of its location, then I will have a *BINGO*."

a) What is the maximum number of covered squares on *B's* card?

b) What is the minimum number?

Recall that a bingo card is a 5×5 matrix with the center square already covered at the start of the game. *A BINGO* can occur in *12* ways, by covering the *5* squares of any row, column, or diagonal.

Solution by Edwin M. Klein, University of Wisconsin, Whitewater.

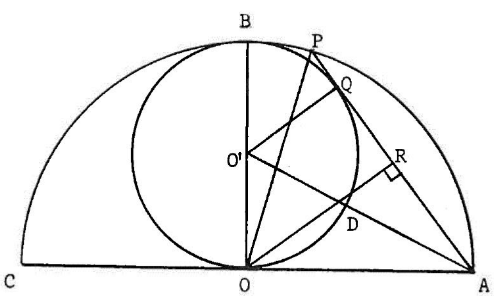
a) There can be at most *20* covered squares, since each row must have at least one uncovered square. If the bingo card is represented by a 5×5 matrix C, then the 20-square maximum can be attained by uncovering only elements C_{11}, C_{24}, C_{32}, C_{45}, and C_{53}.

b) If 10 or more squares are left uncovered, then there must be two rows (and also two columns) with at least two elements of each uncovered. Then only eight of the twelve *BINGOs* can be realized by covering one more square, contradicting the fact that at least ten must be attainable, since covering different squares produces different *BINGOs*. Therefore at most nine squares can be uncovered and at least *16* must be covered. This minimum case is illustrated by uncovering the entire first row and first column of the card.

Also solved by MARK EVANS, *Louisville, KY,* RICHARD *I.* HESS, *Rancho Palos Verdes, CA,* GLEN E. MILLS *(partial solution), Pensacola Junior College, FL, and the* PROPOSER.

560. [Fall 1983] *Proposed by Leon Bankoff, Lob Angeles, California*

Two proofs of a Problem 10713 appeared in the 1891(pp. 34-35) 1892 (p. 79) issues of the *Educational* Times. Unfortunately, neither proof is valid. The problem and its supposed proofs are stated below with wording somewhat modernized for clarification. Find all errors.



Problem 10713. *Proposed by W. J. Greenstreet, M.A.* In a given circle the radii OA and OB are perpendicular. Let the circle on OB as diameter have center O' and let $O'A$ cut this new circle in point D. Then AD is the length of the side of a regular decagon inscribed in the given circle. Also, let the tangent AQ to the new circle cut the given circle again at P. See the diagram above. Then AP is the length of the side of a regular pentagon inscribed in the given circle.

I. Solution by R. Knowles, M.A., Professor Zerr, and others.

Take OA and OB as coordinate axes. Then the equations for the circles are

$$x^2 + y^2 = c^2 \qquad \text{and} \qquad x^2 + y^2 - cy = 0.$$

Now $(AO')^2 = 5c^2/4$ and

$$AD = AO' - O'D = (\sqrt{5} - 1)c/2,$$

which is equal to the side of a regular inscribed decagon. Let $hx + ky = c^2$ be a chord of circle (O) that is tangent to circle (O') and equal to the side of the inscribed pentagon. Because it is a tangent, we have

$$h^2 + k^2 = (k - 2c)^2.$$

The condition that this chord equals the side of the pentagon is

$$(k - 2c)^2 = 2c^2(3 - \sqrt{5}).$$

whence

$$k = (3 - \sqrt{5})c \text{ or } k = (1 + \sqrt{5})c.$$

The latter value makes h impossible. Therefore there is only one real chord of circle (O), tangent to circle (O'), which is equal to the side of the inscribed pentagon.

II. Solution by the Proposer.

Let $OA = c$. Then $OO' = c/2$ and

$$AD = AO' - O'D = c\sqrt{1 + \tan^2 OAO'} - \frac{c}{2} = \frac{c}{2}(\sqrt{5} - 1)$$

$$= \frac{2c}{4}(\sqrt{5} - 1) = 2c \sin 18^\circ,$$

so AD is a side of the inscribed decagon. Now $AP^2 = AD^2 + c^2$ [Casey's *Euclid*, iv. 10, Prob. 6]. Therefore

$$AP^2 = \frac{c^2}{4}(6 - 2\sqrt{5}) + c^2,$$

$$AP = \frac{1}{2}c\sqrt{10 - 2\sqrt{5}} = 2c \sin 36^\circ,$$

so AP is the side of the regular inscribed pentagon.

I. Solution by the Proposer.

We first dispose of Solution I by R. Knowles. In the given figure we have that $\tan \angle OAO' = 1/2$ and $\tan \angle OAP = \tan 2\angle OAO' = 4/3$. Since $\angle APC = 90^\circ$, then APC is a $3:4:5$ right triangle. Since AP/AC is rational, then AP cannot be the side of a regular in-pentagon, which is known to be irrational with respect to the diameter AC.

For Solution II we note that $\angle OAO' = \text{arc} \tan 1/2 = 26.565..^\circ$. Then $\angle OAP = 53.130..^\circ$ and $\angle POA = 73.34..^\circ$. If PA is the side of a regular in-pentagon, however, then $\angle POA$ should be equal to 72°.

II. Solution by Morris Katz, Macwahoc, Maine.

Let $x = \sin 18^\circ$. Then

$$\cos 18^\circ = \sqrt{1 - x^2}, \quad \sin 36^\circ = 2x\sqrt{1 - x^2},$$

$$\cos 36^\circ = \cos^2 18^\circ - \sin^2 18^\circ = 1 - 2x^2,$$

and

$$\sin 72^\circ = 4x\sqrt{1 - x^2}(1 - 2x^2) = \cos 18^\circ = \sqrt{1 - x^2}$$

Because $x^2 \neq 1$, we must have

$$4x(1 - 2x^2) = 1, \quad 4x - 8x^3 = 1, \quad \text{and } 8x^3 - 4x + 1 = 0,$$

which has roots $(\sqrt{5} - 1)/4$, $1/2$, and $(-\sqrt{5} - 1)/4$, the first of which is $\sin 18^\circ$. If we take $c = OA = 1$, then the side of the regular inscribed decagon is $2 \sin 18^\circ = (\sqrt{5} - 1)/2$ and that of the pentagon is

$$2 \sin 36^\circ = 4 \sin 18^\circ \cos 18^\circ = (\sqrt{10 - 2\sqrt{5}})/2$$

since $(\sqrt{5} - 1)(\sqrt{5} + 1) = 4$.

Now $OA = 1$, $OO' = 1/2$, $O'A = \sqrt{5}/2$, and $AD = (\sqrt{5} - 1)/2$, so AD is indeed the side of the regular inscribed decagon. Now drop perpendicular OR to line AP and draw OP. Since triangles AOO and AQO' are congruent, then $\angle OAR = 2\angle OAO'$. We have that

$$\cos OAR = \cos^2 OAO' - \sin^2 OAO' = \frac{4}{5} - \frac{1}{5} = \frac{3}{5}.$$

Because $OA = OP$, then triangle OAP is isosceles and $AP = 2AR = 6/5$ and not the side of the regular pentagon.

The question asks us to find all errors. So far we have uncovered the truth. Now we must uncover the errors. A check shows that Solution I, as stated, is *almost* true; every statement is correct, except that by symmetry there is another such chord symmetric to OO. In fact the positive value of h is $2\sqrt{\sqrt{5}} - 2$. The error is that the line $hx + ky = 1$ does not pass through $(1, 0)$, that is, point A. This occurs only when $h = c$, so it is a different chord-tangent that has the desired length. In solution II, the equation $AP^2 = AD^2 + c^2$, quoted from Casey's *Euclid*, is not true.

1984 NATIONAL PI MU EPSILON MEETING

The National Meeting of the PI MU EPSILON FRATERNITY was held at the University of Oregon in Eugene on August 16 through August 18. Highlights were a reception for members and guests, a Dutch Treat Breakfast and the Annual Banquet, at which Past-President E. Maurice Beesley was honored. The J. Sutherland Frame Lecture was given by Professor John L. Kelley, University of California, Berkeley and entitled "The Concept of Plane Area."

Professor Frame has served Pi Mu Epsilon as Associate Editor of the Journal, as Secretary-Treasurer General and as Vice-Director General. He was Director General during the period 1957-1966. During one of the paper sessions he gave a brief history of the fraternity. He is the author of "Fifty Years in the Pi Mu Epsilon Fraternity," Pi Mu Epsilon Journal, Vol. 3, No. 10, 1964.

The program of student papers included:

Complements -- Mathematically Speaking	Leslie Youngdahl Ohio Delta Miami University
Finding the Center of Complaints	Michael H. Cox West Virginia Beta Marshall University
Solving Non-Linear Systems of Equations	Deborah Whitfield Ohio Xi Youngstown State University
Money and Math: An Investigation of Linear Economic Models	Renee L. Larson South Dakota Gamma South Dakota State University
Dea Zings in n-Dimensional Geometry	Karin Remington Minnesota Delta College of St. Benedict
Wedging Those Vector Integral Theorems	Calvin Johnson California Lambda University of California, Davis
A Mathematical Model of Voter Participation	Mary Beth Dever Illinois Epsilon Northern Illinois University

Number Nine	Patrick Tamer North Carolina Eta Appalachian State University
Graph Measure in Euclidean n-Space	Jodg Trout South Carolina Gamma The College of Charleston
Finite Laplace Transforms	David W. Barnette North Carolina Delta East Carolina University
A Way to Generate Arbitrarily High-Order Root-Finding Methods	Jeffrey Michael Kubina Ohio Xi Youngstown State University
Samuelson's Interaction Between the Accelerator and the Multiplier	Suguna Pappu Ohio Delta Miami University
Commutativity and Distributivity: Different Perspectives	Benjamin L. Marshall Arkansas Beta Hendrix College

1985 NATIONAL PI MU EPSILON MEETING

It is time to be making plans to send an undergraduate delegate or speaker from your Chapter to the Annual Meeting of Pi Mu Epsilon in Laramie, Wyoming in August 1985. Each student who presents a paper will receive travel support up to $500. Each delegate, up to $250. Only one speaker or delegate can be funded from a single chapter, but others are encouraged to attend. For details, contact Dr. Richard A. Good, Secretary-Treasurer, Department of Mathematics, University of Maryland, College Park, MD 20742.

REGIONAL MEETINGS

Many regional meetings of the Mathematical Association of America regularly have sessions for the presentation of student papers. If two or more colleges and at least one local chapter of Pi Mu Epsilon help sponsor, or participate in, such sessions, financial help up to $50 is available. Write to Dr. Richard A. Good, Secretary-Treasurer, Department of Mathematics, University of Maryland, College Park, MD 20742.

GLEANINGS FROM CHAPTER REPORTS

ARKANSAS BETA (Hendrix College). The Undergraduate Research Program continued to be very active. Seniors *Karen Anderson, Diane Crockett, Ben Marshall* and *Grey Williams* presented papers at the Oklahoma-Arkansas MAA Meeting at Arkansas Tech University in Russellville in March. *Karen* and *Diane* gave talks at North Texas State University in April. *Diane, Ben* and *Grey* presented their papers again in April at the Annual Hendrix-Sewanee-Southwestern Math Symposium in Memphis. Guest speakers during the school year included *Walter Smiley* from Systematics, Inc., *Jerry Mauldin,* president of Arkansas Power and Light, Hendrix graduate, *David Sutherland,* now studying at North Texas State University in Stillwater, *Dr. Jim Choike* from Oklahoma State University in Stillwater and *Dr. Tim Wright* of the University of Missouri at Rolla. The McHenry-Lane Freshman Math Award was given to *Gary Thacker.* The Hogan Senior Math Award was shared by *Karen Anderson* and *Grey Williams.* The Phillip Parker Undergraduate Research Award was given to *Ben Marshall.*

DISTRICT OF COLUMBIA ALPHA (Howard University). *Mr. Hari Thariani* was presented with an award for his performance in the Elbert F. Cox Undergraduate Mathematics Competition. The Competition perpetuates the memory of Dr. Elbert F. Cox, the first black American to receive the Ph. D. degree in mathematics.

MASSACHUSETTS GAMMA (Bridgewater State College). During the winter of 1983-1984 the membership met weekly with the faculty advisor, *Thomas E. Moore,* for problem solving sessions. In the spring, the chapter sponsored a mathematics week, "Women in Mathematics and Computer Science." The speakers included men and women from both the business and academic worlds.

MINNESOTA GAMMA (Macalester College). Along with the annual game night, film showing and spring and fall picnics, the chapter sponsored invited talks by *Professor Frank Harary* on "Graph Theoretic Models in the Physical and Social Sciences," *Professor Gerald Bergum* on "Interesting but Unsolved Problems in Number Theory," and *Professor Michael Tangredi* on "Volterra's Population Models." A T-shirt sale was a very successful fund-raiser.

MINNESOTA ZETA (Saint Mary's College). Featured colloquium speakers were *Dr. Dick Jarvinen* on "Vasectomy and the Death of Euler," *Sister Kathleen Sullivan* on "Infinitesimals Revisited," *Judith Knapp* on "Trigonometric Substitutions as an Integration Technique," and *Barbara J. Carlson* on "Mathematical-Musical Applications for the Classroom."

MISSISSIPPI ALPHA (The University of Mississippi). After a year of inactivity, Mississippi Alpha has regrouped. During the spring 1984 semester, monthly meetings were held. Twenty new members were initiated. Highlights were a talk on job prospects in mathematics by *Dr. William R. Trott* and a spring finale cook-out.

NEW JERSEY BETA (Douglass College). Math Career Day on November 19 featured talks by math majors now holding positions as computer programmer, teacher, financial analyst and actuary. In invited talks, *Richard Porter* discussed a possible model for love (and hate) and *Dr. Jean Taylor* discussed the mathematics of soap films. The chapter continued its tradition of offering free tutoring in mathematics courses. A new event was a prize exam for area high school mathematics teams held jointly with the New Jersey Alpha (Rutgers University) chapter.

NEW YORK ALPHA BETA (LeMoyne College). *Eileen Poiani,* Pi Mu Epsilon Councilor (and now President-Elect), represented the fraternity at the chapter installation ceremonies in October. Thirty-six students, graduates and faculty were inducted. In March, *Professor Steven Brams* of New York University spoke on "Biblical Games."

NEW YORK PHI (State University of New York at Potsdam). The invited speaker at October ceremonies inducting 40 new members was *Dr. Harris Schlessinger,* a chapter alumnus, who spoke on "What is Mathematics?" In April, 36 new inductees were welcomed to membership. *Dr. Richard DelGuidice,* Dean of the School of Liberal Studies, was principal speaker. The school year ended with the annual picnic.

NEW YORK OMEGA (Saint Bonaventure University). Activities included the showing of the films "Symmetries of the Cube" and "Space Filling Curves." Invited lecturers were *Professor L. F. Lardy* on "Computing the Zeroes of a Polynomial," *Professor David Hanson* on "Some Unexpected Results in Coin Tossing," and *Gary Myers* on "Uniquely Intersectable Graphs - an Open Problem." Students *James Brahaney* and *Janet McMahon* shared the Pi Mu Epsilon Award at the university's annual Honors Banquet. *Joan Cugell* was recognized for honorable mention.

NORTH CAROLINA KAPPA (North Carolina Agricultural and Technical State University). *Dr. John Tolle,* University of North Carolina at Chapel Hill, spoke on "Optimization." One hundred twenty-five students representing 40 schools participated in the chapter-sponsored State Regional High School Geometry Contest which is to become an annual event.

PENNSYLVANIA BETA (Bucknell University). For the 12th consecutive year the chapter sponsored the John Steiner Gold Mathematical Competition for students from area high schools. Seventy-five students representing 26 schools participated. The competition was established to discover and encourage mathematical talent. The competition honors Professor Gold who served the fraternity as Secretary and/or Secretary-Treasurer from 1927-1947 and later as Councilor. At the annual initiation banquet, *Professor James Dudziak* spoke on the "Kakeya-Besicovitch Needle Problem." The Pi Mu Epsilon Fraternity Prize, awarded to the member of the graduating class whose work in mathematics has been outstanding, was presented to *John L. Allen.*

SOUTH CAROLINA GAMMA (The College of Charleston). Activities included helping to conduct the annual Math Meet at The College of Charleston for 1200 high school students from South Carolina and neighboring states. Students, faculty at the college and at nearby institutions gave talks on a variety of mathematical topics. Speakers included *Dr. Rose Hamm, Dr. Al Parrish, Arthur Squillante, John Trout* and *Dr. Hurshell Hunt.*

TEXAS IOTA (University of Texas at Arlington). The chapter had a very successful year with twelve lectures on mathematics outside the classroom. Speakers were from E-Systems, General Dynamics, Rockwell International, Mobil Oil and Texas Instruments.

VIRGINIA GAMMA (James Madison University). The chapter recognized *Pamela Ficalora* with its Outstanding Senior Award for scholastic achievement in mathematics and service to Pi Mu Epsilon.

WEST VIRGINIA BETA (Marshall University). Guest speakers were *Dr. Melton,* a former faculty member, and *Reginald Spencer* from the Placement Center. A tutoring file was established. The annual book sale was a successful fund-raiser. Four students attended the 67th Annual Meeting of the MAA in Louisville. Six students attended the spring meeting of the Ohio Section of the MAA with the support of Pi Mu Epsilon Marshall University. *Mike Cox* presented a paper at the student paper session. The chapter's Annual Job Fair with a panel of representatives from local businesses and industries was a big success. The annual Math Competition, supported by Pi Mu Epsilon and the Marshall University Foundation, awarded $500 in cash prizes to highest scorers. The competition is open to outstanding high school students in West Virginia and the Tri-State Area who have been chosen to participate by their teachers.

ATTENTION — FACULTY ADVISORS

Is your chapter's report here? If not, please consider sharing a summary of your chapter's activities with other members of the fraternity. Accounts of programs which have been successful at your institution are especially welcome. Send copies of your report to Dr. Richard A. Good, Secretary-Treasurer, Department of Mathematics, University of Maryland, College Park, MD 20742 and to Dr. Joseph P. E. Konhauser, Editor, Mathematics and Computer Science Department, Macalester College, St. Paul, MN 55105.

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