A Message from the President ........................................ 633
The C.C. MacDuffee Award for Distinguished Service .......... 634
Pi Mu Epsilon Celebrates 75 Years of Mathematical Activity 636
J. Sutherland Frame .................................................. 636
A Ubiquitous Partition of Subsets of $\mathbb{R}^n$ - Reprint    663
Donald John Nicholson ............................................. 663
X-Matrices: Generalized Complex Numbers 669
Samuel Councilman ................................................... 669
Note on a Well-Known Limit 672
W. Vance Underhill .................................................. 672
$p^q > q^p$ Via Coordinate Geometry 673
Norman Schaumberger ............................................... 673
Proofs Without Words 674
Norman Schaumberger .............................................. 674
Comment on "Counting Bit Strings with a Single Occurrence of 00" 675
J. Sutherland Frame ............................................... 675
In Praise of Calculus and A Calculus Lament - Poetry 676
Hubert Walczak ....................................................... 676
Puzzle Section 677
Joseph D.E. Konhauser .............................................. 677
Problem Department 684
Clayton W. Dodge .................................................... 684
Index to Volume 8 699
In Appreciation ....................................................... 704
A Message from the President ........................................... 633
The C.C. MacDuffee Award for Distinguished Service .............. 634
Pi Mu Epsilon Celebrates 75 Years of Mathematical Activity
   J. Sutherland Frame ........................................... 636
A Ubiquitous Partition of Subsets of $\mathbb{R}^n$ - Reprint
   Donald John Nicholson ........................................ 663
X-Matrices: Generalized Complex Numbers
   Samuel Councilman ........................................... 669
Note on a Well-Known Limit
   W. Vance Underhill ........................................... 672
$p^d > q^q$ Via Coordinate Geometry
   Norman Schaumberger ......................................... 673
Proofs Without Words
   Norman Schaemberger ......................................... 674
Comment on “Counting Bit Strings with a Single Occurrence of 00”
   J. Sutherland Frame ......................................... 675
In Praise of Calculus and A Calculus Lament - Poetry
   Hubert Walczak .............................................. 676
Puzzle Section
   Joseph D.E. Konhauser ....................................... 677
Problem Department
   Clayton W. Dodge ........................................... 684
Index to Volume 8 .................................................. 699
In Appreciation ..................................................... 704
A MESSAGE FROM THE PRESIDENT
COMMEMORATING THE SEVENTY-FIFTH ANNIVERSARY
OF PI MU EPSILON, INC.

The year, 1989, marks the 75th Anniversary of the founding of Pi Mu Epsilon, Inc. and the 40th Anniversary of the establishment of the Pi Mu Epsilon Journal. Chartered in the State of New York on May 25, 1914, Pi Mu Epsilon Fraternity, Inc. began as a non-secret organization "to promote scholarship and mathematics" on college and university campuses and at qualified non-academic institutions. The name of this national honorary mathematics society was changed to Pi Mu Epsilon, Inc. in 1988 to reflect the coeducational composition of the organization since it originated.

Like the Panama Canal which officially opened in 1914, Pi Mu Epsilon unlocked the gates to recognizing student talent in the mathematical sciences that same year. Its membership now boasts over 90,000 individuals and 257 chapters across the United States.

Each year since 1952, Pi Mu Epsilon has been holding its National Meeting in conjunction with the Summer Meetings of the American Mathematical Society and the Mathematical Association of America. The Diamond Jubilee celebration will take place from August 7 to August 10, 1989, on the campus of the University of Colorado at Boulder.

The National Honorary Mathematics Society proudly dedicates this commemorative issue of the Pi Mu Epsilon Journal to the achievements and aspirations of all the outstanding students of mathematics who have been selected to membership in Pi Mu Epsilon and who will join its ranks in the future, as well as to the loyal Faculty Advisors, Faculty Correspondents, and National Officers who are the backbone of the organization. As the 21st century approaches, Pi Mu Epsilon remains committed to its tradition of serving the nation by recognizing promise in the mathematical sciences and by promoting scholarly activities in mathematics.

Eileen L. Poiani
President
Pi Mu Epsilon, Inc.

COVERS

The front and back covers commemorating the 75th Anniversary of the founding of Pi Mu Epsilon were designed and prepared by Professor E. P. Miles, Jr., Florida State University, Tallahassee, Florida, at the FSU Muench Center for Color Graphics, on an INTERCOLOR 2427, DATAVUE, and PRINTACOLOR GP 1024.

Professor Miles presented the J. Sutherland Frame Lecture at the Summer Meeting of Pi Mu Epsilon in Pittsburgh, PA in 1981.
On August 10, 1988, at the Pi Mu Epsilon National Meeting in Providence, Rhode Island, Dr. Milton D. Cox, was honored by Pi Mu Epsilon with the C. C. MacDuffee Award for Distinguished Service. This prestigious recognition, first bestowed in 1965 upon J. Sutherland Frame, is awarded “often enough to be recognized and seldom enough to be meaningful,” according to the Pi Mu Epsilon Governing Council.

As Faculty Advisor of the Ohio Delta chapter at Miami University in Oxford, Ohio, for at least eighteen years, as Councilor of Pi Mu Epsilon from 1975 to 1981, and National President from 1984-87, Dr. Cox has continually contributed to the achievement of Pi Mu Epsilon’s goal “to promote scholarship in mathematics.” A sort of mathematical “pied piper,” he journeyed over the years with many Miami University students - by van, auto, and train, to the annual Pi Mu Epsilon National Meetings at college and university campuses across the United States. His untiring efforts to stimulate student interest in mathematics have been rewarded both by an active Pi Mu Epsilon chapter on his home campus and by a substantial increase in the number of student papers presented at national meetings during his presidency of Pi Mu Epsilon.

Not content to wait for just one summer opportunity for Pi Mu Epsilon students to present papers, Dr. Cox has organized and directed fifteen consecutive Pi Mu Epsilon Student Conferences at Miami University. The sixteenth is scheduled for Fall 1989. These successful conferences continue to serve as models which have been successfully replicated on other campuses.

A graduate of DePauw University, the honoree received his master's and doctoral degrees from Indiana University, specializing in abstract algebra and field theory. At Miami University, he currently divides his time teaching as an Associate Professor of Mathematics and Statistics and serving as the Associate Provost for Teaching Effectiveness Programs. In addition, Dr. Cox coordinates the departmental honors program and chairs the Committee on Student Sections for the Mathematical Association of America’s Ohio Section of which he is a past-president.

A member of Phi Beta Kappa, Dr. Cox has lectured on topics ranging from the theory of tiling to the ingredients of an undergraduate mathematics curriculum. He has brought to Pi Mu Epsilon and to its students a contagious enthusiasm for mathematics.

Pi Mu Epsilon is proud to add the name of Dr. Milton D. Cox to the list of recipients of the C. C. MacDuffee Distinguished Service Award.

AMS ESTABLISHES ANNUAL PRIZE

President Eileen Poiani has announced that the American Mathematical Society has established a prize of up to $1000 per year to promote undergraduate scholarship in mathematics. The prize money will be administered by Pi Mu Epsilon. Officers and councilors of Pi Mu Epsilon will determine how the prize money will be distributed.


See The Rockies!
Share Your Ideas!
Develop New Friendships!
See your PI MU EPSILON advisor for details.
PI MU EPSILON CELEBRATES 75 YEARS OF MATHEMATICAL ACTIVITY

by J. Sutherland Frame
Michigan State University

1. Incorporation and Early Days.

The Pi Mu Epsilon Fraternity, incorporated on May 25, 1914, under the laws of the State of New York, is celebrating its 75th anniversary as a national mathematics honor society, with over 250 chapters in 45 states and the District of Columbia. It is a non-secret organization whose purpose is the promotion among students and faculty of scholarly activity in mathematics.

The Syracuse University Mathematics Club, which parented Pi Mu Epsilon, first met on November 30, 1903, at the home of Dr. W. H. Metzler, who was named Director of the club. Dr. Edward Drake Roe, Jr. was elected Vice-Director, Mary B. Quinlan Secretary and Mr. Carpenter Treasurer, as recounted in the archives at Syracuse University. During the next ten years the club carried on an active program with many student papers (preserved in the archives), directed in turn by Metzler, Roe, Warren G. Bullard, Floyd Fiske Decker, Daniel Pratt and Lapine Hall Price. At the tenth anniversary meeting on November 17, 1913, a committee chaired by Professor Roe was established to consider a possible revision of the club’s constitution. Its report on December 7 presented four options, included in Professor Decker’s historical article about Pi Mu Epsilon (Pi Mu Epsilon Journal, Vol. 1, No. 1 (1949), pp. 8-12). The second option “to reorganize the club as a professional fraternity” was approved. Thus Dr. Roe’s influential ideas led to the establishment of Pi Mu Epsilon.

On March 2, 1914, a convention was held and a constitution was adopted. On March 23, the choice E n M was made among five sets of Greek letters proposed for the fraternity name, subject to changing the order of the letters to Π M E. Eight faculty members and 42 students then took the following pledge and signed their names as charter members:

PLEDGE: I solemnly promise to give my best efforts to the improvement of my scholarship in all my studies and researches and especially in Mathematics; I will maintain a discreet silence concerning all the aims and obligations of this fraternity; I will cheerfully accept advice and admonition as long as I am a member of the Fraternity.

Officers elected at the convention on April 27, 1914, were as follows: Professors E. D. Roe, Jr. and F. F. Decker Director and Vice-Director, undergraduates Helen L. Applebee and Purley J. Bentley Secretary and Treasurer, and Olive Evelyn Jones Historian. Incorporation followed on May 25, 1914.

Additional chapters were chartered at the Ohio State University in 1919, at the University of Pennsylvania in 1921, and at the Universities of Missouri and Alabama in 1922. The officers of the Alpha Chapter at Syracuse University served as national officers until December 1922, when a national office was established. Additional information about the early days is included in Dr. Decker’s article referenced above, and in J. Sutherland Frame’s article “Fifty years in the Pi Mu Epsilon Fraternity” (Pi Mu Epsilon Journal, Vol. 3, No. 10 (1964), pp. 511-5).

2. Pi Mu Epsilon National Reports and Meetings.

Pi Mu Epsilon has always considered its primary aim to be the encouragement of scholarly interest and activity in mathematics among undergraduate and beginning graduate students. This aim was fostered at first mainly by individual chapters, but partly by informal meetings of interested faculty and students, usually held once in three years at meetings of the Mathematical Association of America (MAA). The American Mathematical Monthly (the Monthly) recorded national meetings of Pi Mu Epsilon in 1938 and on January 1, 1942. Meetings without formal talks were held in 1945, 1948, 1953, and 1959. Pi Mu Epsilon activities were reported in the Clubs and Allied Activities section of the Monthly, a section edited from 1931 to June 1935 by F. M. Weida, from December 1935 to July 1938 by F. W. and Helen B. Owens, from August 1938 to August 1942 by E. H. C. Hildebrant, from January 1942 to December 1946 by J. Sutherland Frame, and from 1947 to 1951 by L. F. Ollmann.

In 1927 a vote of nine chapters on the question “Shall we have a National Publication for Π M E?” was 5 to 4 negative. Years later, in 1949, Pi Mu Epsilon launched its own journal, with Ruth Stokes of Syracuse as Editor, Howard C. Bennett as Business Manager, and J. Sutherland Frame, H. T. Kames, N. H. McCoy, and R. J. Walker as Associate Editors. The Journal contains papers written by undergraduates and others deemed to be of interest to them, as well as a puzzle section, a problem department,
chapter reports, invited addresses, and other items of interest to Pi Mu Epsilon members. New member lists were included until 1972.

Pi Mu Epsilon has been especially fortunate in having had a succession of outstanding problem proposers/solvers serve as Editor of the Journal's Problem Department. The first so designated was the late Leo Moser, who edited the department beginning with Vol. 1, No. 4 through Vol. 2, No. 7. Together, C. W. Trigg and Leon Bankoff edited Vol. 2, No. 8. Murray S. Klamkin assumed the editorship with Vol. 2, No. 9 and continued for ten years. Leon Bankoff took over with Vol. 4, No. 9 and served through Vol. 7, No. 2. The next three issues were jointly edited by Bankoff and Clayton W. Dodge. Beginning with Vol. 7, No. 6, Dodge had been Editor.

Presentation of papers by students at Pi Mu Epsilon national meetings began at the national meetings of the M M and American Mathematical Society (AMS) at Michigan State University in 1952, continued at meetings at the University of Michigan in 1955 and at Pennsylvania State University in 1957, and has continued since 1960 at most subsequent Pi Mu Epsilon national summer meetings, held annually with few exceptions (1962, 1974). Student speakers are supported for their travel expenses, encouraged to submit their papers to the Journal for publication, and lured by prizes for the best papers published by students below or just at the master's level. To facilitate interaction between participants, Pi Mu Epsilon holds an informal get-acquainted reception the evening before student papers are delivered. A complete list of the papers presented appears as Appendix A to this article.

Most Pi Mu Epsilon national meetings have included a one-hour address by an invited speaker, notably that in 1972 at Dartmouth by President John Kemeny, entitled "Mathematical Models and the Computer" (Pi Mu Epsilon Journal, Vol. 5, No. 8 (1973), pp. 373-86). At the 1973 meeting of the Council, Past President J. C. Eaves proposed and the Council approved a motion that this activity be formalized in a series of annual addresses to be known as the J. Sutherland Frame Lectures. Past President Frame was asked to inaugurate the series in 1975, and deeply appreciates the honor. Appendix B is a complete list of speakers and topics.

In 1964 a 50-year history of Pi Mu Epsilon by Frame was printed in the Journal. We look forward to our 75th anniversary meeting at the University of Colorado in Boulder in August 1989.


In 1936 when national Secretary John S. Gold became national Secretary-Treasurer for Pi Mu Epsilon his office issued charters for $10, and registration fees and certificates of membership for $0.25 each. The latter two charges were combined into a $1 initiation fee in November 1950, raised to $2 in 1957, to $4 in 1971, and to $10 in 1986, due to increasing expenses. Pi Mu Epsilon has never charged its members annual dues, but does encourage subscriptions to the Journal, which are included for one year in the initiation fee.

In 1937, the L. G. Balfour Company became the official jeweler for Pi Mu Epsilon, and in 1938 began remitting royalties on jewelry which totaled $3384.42 from 1951 to 1954, and varying amounts in subsequent years. At first, 10-karat gold keys or key-pins were sold for $3.75, but when the price of gold skyrocketed in the 1970's and the key price rose above $12, Pi Mu Epsilon began ordering gold-plated pins in quantity from Balfour, and sold them to initiates for $5 in 1972 and for $8 beginning in 1979.

The first 4000-copy issue of the Pi Mu Epsilon Journal in 1949 cost $50. Between 1950 and 1954 the Journal received $0.75 of each $1 initiation fee, which paid for a one-year subscription. Printing costs continued to rise. In 1973 the two-year subscription price was increased from $1.50 to $4 for members, and from $2 to $6 for non-members and libraries; it was raised again in 1980 to $8 and $12 respectively for two years. In 1975-1976 the Journal income of $6108.90 included $4000 from the Pi Mu Epsilon treasury, and expenses were $5426.26. In 1980-1981 Journal costs of $10,850.12 were subsidized by $5000 from the Pi Mu Epsilon treasury. Cash assets of Pi Mu Epsilon on June 30, 1988, totalled $27,185.79.

In 1966 the Council approved a prize of $100, to be awarded each year, for the best paper by an undergraduate that was published in the Journal. Beginning in 1971, first, second, and third prizes of $200, $100, and $50 were awarded, and eligibility was extended to graduate students working on or just completing a master's degree. These awards were named the Richard V. Andree Awards by the Council in 1987. Of all the student-written papers, heretofore receiving first prizes, that of Donald John Nicholson on "A Ubiquitous Partition of Sets of $\mathbb{R}^n$," which appeared in Vol. 8, No. 1, pp. 2-7, has been selected as "best" and is reprinted on pages 663-668.
Pi Mu Epsilon also encouraged local chapters by matching prize money up to $50 per chapter and assisting financially with student participation in sectional meetings. But the largest financial support to students was transportation expense to national meetings. Travel subsidies totaling $486.21 to the 1952 summer meeting in East Lansing, Michigan, with student speakers, and $793.24 for the December 1953 meeting in Baltimore for officers and delegates (no student speakers), was first based on full travel expense up to $60 for speakers, and one-half rail fare plus lower berth up to $35 for delegates. In later years travel subsidies were based on minimum airfare for a speaker, with half these amounts for a chapter delegate, subject, however, to maximums which rose from $60 in 1952 to $150 in 1964, $300 in 1973, $400 in 1977, $500 in 1981, and $600 in 1985.

In 1957 Pi Mu Epsilon contributed $300 to help launch the Mu Alpha Theta high school math honorary sponsored by Dick and Josephine Andree. It contributed $500 in March 1973 to the National Council of Teachers of Mathematics (NCTM) for its building fund, $500 in 1981 to the NCTM toward a memorial to our past president Houston Karnes, $3000 in 1978 to the MAA for its building fund, and $500 in 1987 toward the endowed scholarship in mathematics or computer science at the University of Oklahoma in memory of Richard V. Andree.

4. Pi Mu Epsilon growth in chapters and membership.

When national officers were first elected in 1922, there were five chapters of Pi Mu Epsilon (listed in section 1). When Dr. E. D. Roe, Jr., the first Director General, died in 1929 and was succeeded by Dr. Louis Ingold, there were 18 chapters. There were 31 chapters on April 1, 1936, when a new constitution was adopted and the offices of Secretary, Treasurer, and Librarian were combined in the office of Secretary-Treasurer, held by the former Secretary, John S. Gold of Bucknell. Griffith C. Evans succeeded F. W. Owens as Director General, and was followed by W. E. Milne in 1939 and Tomlinson Fort from 1942 to 1948. By fall 1949, when the Pi Mu Epsilon Journal was started, and E. H. C. Hildebrandt was Secretary-Treasurer, there were 48 chapters. Membership totals on April 1, 1951, and July 1, 1954, were 18,857 and 22,897. At that time Richard V. Andree replaced J. Sutherland Frame as Secretary-Treasurer and three-year terms for National Officers began July 1 instead of April 1, to facilitate exchanges of files and information.

During C. C. MacDuffee's term as Director-General, 1948-1954, he installed almost all the new chapters. His successor, S. S. Cairns, had a sabbatical leave in 1954-1955, and asked Vice-Director General Frame to act for him that year. By the end of Frame's three terms as Director General in 1966 there were 120 chapters, of which Frame had installed about 50. Membership in Pi Mu Epsilon increased by 25,196 from 1975 to 1986 when Richard Good was Secretary-Treasurer, and should exceed 90,000 by 1989. On April 1, 1989, there were 257 chapters.

The Constitution and By Laws of 1936 were revised in 1958, 1965, and 1972. A 1958 revision provided that the nominating committee appointed by the Director General should nominate three qualified candidates for the office of Director General, among whom the candidate with the second highest vote of the chapters should become Vice-Director General. Another significant change provided for the chartering of affiliate chapters at Pi Mu Epsilon at non-academic institutions. Only one such chapter was installed, at the urging of George Marks, at the General Electric plant in Evendale, Ohio, in 1958. It became inactive when Marks moved away. Constitutional revisions in 1965 changed the titles of Director General and Vice Director General to President and Vice President, shortened the pledge for initiates, and provided that charter petitions approved by unanimous vote of the Council did not require chapter approval. A 1972 revision changed the office of Vice President to President-Elect, at least one to be nominated, and provided for the automatic succession after a three-year term to a term as President, followed by a term as Past President and Council member.

5. Who were our National Council Members?

Lists of names and dates do not tell the whole story. So what do we know about the men and women who have served as Officers and Councilors of Pi Mu Epsilon? Back numbers of the Pi Mu Epsilon Journal provided added information. The reference (1/1/1-9/F49), meaning Vol. 1, No. 1, pp. 1-9, Fall 1949, refers to a photo and biography of the first Director General E. D. Roe, Jr., who was an astronomer as well as a mathematician. Director-General Griffith C. Evans served as Vice President of both the AMS and M. M., and was AMS President in 1938-1940. Tomlinson Fort served as AMS Associate Secretary and later as Bulletin Editor. C. C. MacDuffee was M. M. President in 1945 and Frame served two terms (1950-1952 and 1958-1960) on the M. M. Board of Governors. R. H. Bing was M. M. President in 1961. Photos and biographies of C. C. MacDuffee, W. M. Whyburn, J. S. Frame, S. S. Cairns and Sophia MacDonald appear in (1/6/229-31/S52),
In 1957 Josephine and Richard Andree launched the high school mathematics honorary society Mu Alpha Theta (2/8/394/S58). Henry Alder and later J. C. Eaves (among others) served as Presidents. Cosponsored now by the NCTM and MAA, Mu Alpha Theta now has over 1200 chapters and 40,000 members in 42 states, the District of Columbia, Puerto Rico, Canada, Germany, Japan, and Switzerland.


After 21 years as Secretary-Treasurer, 1954-1975, Richard Andree served as President-Elect and then President, but died May 8, 1987 after a long illness. Richard Good was his able successor as Secretary-Treasurer 1975-1987 and now serves as a Councilor. In 1978 E. Allan Davis followed Houston Karnes as President. The four Councilors of 1975-1978 have all become national officers: E. Maurice Beasley, Milton Cox and Eileen Poiani as Presidents 1981-1984, 1984-1987, and 1987-1990, respectively, and Robert Woodside as Secretary-Treasurer 1987-1990. David Ballew, our Editor 1978-1984, is now President-Elect. Joseph Konhauser followed Ballew as Editor, Milton Cox is now Past-President, and Richard Posse, Robert Eslinger, and J. Douglas Faires are now Councilors. Under Eileen Poiani and her colleagues Pi Mu Epsilon should have a bright future.

### STUDENT PAPERS

**APPENDIX A**

1952 -- Michigan State College Meeting

- "modification of Infinite Series", Bevan K. House, University of Georgia
- "Rapid Square Roots", Charles D. Parker, Michigan State University
- "almost Periodic Functions", John E. Hoffman, University of Oklahoma
- "matrix Inversion", Verna L. A., University of Delaware
- "extension", Alon J. Goldman, University of Kentucky
- "The G, C. D. Algorithm", Wilson M. Zaring, University of Kentucky

There were no meetings in 1953 or 1954.

1955 -- University of Michigan Meeting

- "magic Squares", Walter W. Turner, Michigan State College
- "Semigroups", Thomas Head, University of Oklahoma
- "Elliptical Wheels on an Inclined Plane", James H. T. Jr., University of Georgia
- "modern developments in Additive Number Theory", Roy Lisker, University of Illinois
- "Reorderings of Sets", Ronald Cleary, Syracuse University
- "An Equation Concerning the Functional Exponent", John Stallings, University of Arkansas
- "Some Cayley Color Groups of Order Less Than Fifteen", Chih Han Sah, University of Illinois
- "On Extensions of Kasner's Circle", Manning L. Rose, University of Kentucky

There was no meeting in 1956.

1957 -- Pennsylvania State University Meeting

- "Envelopes of Certain Families of Conics", Katharine Lipps, St. Louis University
- "Mathematics in Turkish High Schools", Ali Tangren, Louisiana State University
- "Symmetry Groups and Molecular Structure", J. S. Frame, Michigan State University
- "On the Convergence of the Fourier Series", Tom Boehme, University of Oklahoma
- "The Perpetual Calendar", Francis Felix, Pennsylvania State University

There were no meetings in 1958 or 1959.

1960 -- Michigan State University Meeting

- "Königsberg Bridge Problem", Fred Howlett, University of Nebraska
- "Filters and Ultrafilters", John Allen, Oklahoma State University
"A Geometric Interpretation of the Solution of Some 3x3 Games", Virginia Thrasher, University of Buffalo

"On the number of Representations by a Cubic Polynomial modulo p", Stanley Tamanapakis, Hunter College

"Bounds for Waring's Problem, modulo p", Samuel Lomonaco, St. Louis University

"The Construction of the Rfim Plane and its Associated Group in Terms of the Barycentric Calculus", James Herod, University of Alabama

"Characterizations of Certain Lattices", Geraldine Jensen, University of Oregon

"Some Properties of Prime Numbers", Andrew Soma, Michigan State University

1961 -- Oklahoma State University Meeting

"Conformally Elementary Points", Louis DeNoya, Oklahoma State University

"Transcendence of e", John Wells, Ohio University of Dayton

"A Topic in modern Algebra", Charles Mullins, Oklahoma State University

"General Solutions to the Linear Diophantine Equation in the Two Variables", Thurston Shook, Ohio State University

The AMS/AMM Summer Meeting was in Canada and there was no Pi Mu Epsilon Meeting in 1962.

1963 -- Boulder, Colorado Meeting

"Rao's Test", Charles B. Hudman, Miami University

"Some Recent number-Theoretic Research Using a High-speed Calculator", Kenneth E. Kloss, Carnegie Tech

"Definition of Vector Product and Topological Considerations", Robert Jay Buck, University of Buffalo

"A Characteristic Property of Closed Subgroups of E^n", Stephen E. Crick, Jr., Michigan State University


"Commutative Alternative Rings", William A. Thedford, Oklahoma State University

"Catenary Addition -- A method of Sequence Generation", Thomas William Cusick, University of Illinois

"Tauberian Theorems on the Fractional Function", M. Alan Feldstein, UCLA

"A Generalization of the Riemann-Lebesgue Lemma", William E. Heintze, Georgia Institute of Technology

1964 -- University of Massachusetts Meeting

"Existence Theorems for Hyperbolic Partial Differential Equations", R. Brewster Knight, Georgetown University

"Theorem Proving on the Computer... the State of the Art", Carl Douglas Jensen, Florida State University

"Weierstrass Approximation Theorem", James H. Opeda, DePaul University

"Applications of Contraction mappings", Richard H. Bouldin, University of Alabama

"A Study of the Order Properties of Non-Desargussian Planes", Beverly Welles, Denison University

"Generation of Content and measure Spaces", Howard Edward Evans, Franklin & Marshall College

"Some of the Plane Figures Under the Direct motions", Henry Lee Phillips, Southern University

"Determinants in a Division Ring", William A. Bridges, Jr., University of Connecticut

"Mappings of Semigroups Associated with Ordered Pairs", Margaret Viana, SUNY at Buffalo, New York

"Sums of Rational numbers", William A. Webb, Michigan State University

"Subgroups of Rational Numbers", Alan Schwartz, University of Wisconsin

"Transverse Stability of Vessels", Joseph C. Zuker, Marquette University

"Zeros, Identities, and Associativity in Groupoids", Johnny Lundgren, Sacramento State College

1965 -- Cornell University Meeting

"The Solution of a Simple Differential Equation", Ralph Edwin Showalter, North Carolina State at Raleigh

"Applications of the Arithmetics of Simple Quadratic Fields to Diophantine Analysis", Hugh L. Montgomery, University of Illinois

"The Numerical Study of a Boundary Value Problem with One Given Condition at Infinity", Carol Knudson, State University of South Dakota

"An Introduction to Topological Rings", David M. Arnold, Western Washington State College

"The Jordan-Schoenberg Theorem", Richard C. Vile, Jr., Michigan State University

"Solutions of Equations in Finite Groups", Mark Benid, Tulane University

"The Schroedinger Wave Equation", Glenn Thompson, University of Toledo

"Riddt Program for Turing machine", James Woelp, SUNY at Buffalo, New York

"Surveying on the Level: The Schroder-Bernstein Theorem", Frank B. Bernhart, University of Oklahoma

"Random Variables", Louis F. Kemp, Polytechnic Institute of Brooklyn

"An Extension of Liouville's Theorem", Kerb Silverman, Syracuse University

1966 -- Rutgers - The State University Meeting

"Formal Power Series Over Commutative Rings with Identity", James William Brewer, Florida State University

"Extensions of mean Value Theorem", Preston Dinkins, Southern University

"The Fundamental Theorem of Algebra", Paul J. Campbell, University of Dayton

"Pascal's Triangle Generalized and Polytope Geometry", Eric Richard Halsey, Pomona College

"Reciprocal Addition", James Williams, Carleton College

"The Study of a Recursive Sequence", James A. Wingert, John Carroll University

"Continued Fractions and the Geometry of moment Spaces", Thomas McIntosh, Denison University

"Construction of a Banach Space Using the Laplace Transform", Pedro J. Saavedra, Georgetown University
"A. L. P.: Algebraic Programming System", RalphHowenstine, University of Oklahoma
"Indices of Primitivity and Imprimitivity of a Cyclically Vertex-Symmetric Graph", Alexander F. Hunter, Florida Presbyterian College
"Further Varieties of Faces of the Hexahexaflexagon", Robert L. Munson, Oregon State University
1967 -- University of Toronto Meeting
"Some Useful Results in Applications of Power Series Transformations to the Solution of Difference Equations", Ray A. Goskine, Virginia Polytechnic Institute
"Generalizations of Sequences", William L. Reynolds, Florida State University
"Survey of Student Activities to Increase Interest in Mathematics and Mathematical Sciences", Rout A. Deju, New Mexico Institute of Mining and Technology
"A Preliminary Examination of Round-Robin Tournament Theory", Charles A. Bryan, John Carroll University
"A Functional Approach to the Singularities of Plane Curves", Richard J. Bonneau, College of the Holy Cross
1968 -- University of Wisconsin Meeting
"The Fibonacci Sequence: An Introduction", Donald F. Reynolds, Texas Christian University
"A Representation Theorem for Bounded Finite Generation Processes", James Rosensburg, Pomona College
"Revision of a monitor Using Assembly Language", Arthur P. Staddon, Denison University
"Twin 'n-Primes", George F. Grob, Georgetown University
"Series of Polygonal Numbers", Michael C. Kopkas, John Carroll University
1969 -- University of Oregon Meeting
"Rational Points on Elliptic Curves", Bruce Rienzo, Rutgers University
"The Cantor Set", Jerry Lynn West, Southern University
"A Semi-Number System", Ann K. Miller, Saint Louis University
"Determination of Polynomial Roots: A Construction", John Wallis, Texas Christian University
"Simplicial Decompositions of Convex Polytopes", Allan L. Edmonds, Oklahoma State University
"A Decimal Approximation to Pi. Utilizing a Power Series", T. Golian and J. Hennicken, Ohio University
"Number Theory, Density of Quadratic [Ion-Residues in \( \{p\}\)]", Richard W. Johnson, East Carolina University
"Tauberian Theorems", Addison M. Fischer, West Virginia University
"Topological Rings and Topologies on Rings", Gary Sargent, New Mexico Institute of Technology
"An Important Counterexample in the Field of Round Robin Tournament Theory", Anthony J. Kenzie, John Carroll University
1970 -- University of Wyoming Meeting
"Linear Programming", Margaret M. Lynch, St. Louis University
"Program for Finding Endomorphisms and Associated Near Rings of a Finite Group", Russell Schexnayder, University of Southwestern Louisiana
"Positive Definite matrices", Stephen M. Zemany, University of Delaware
"Newton and the Development of Calculus", Thomas R. Bingham, State University College of New York
"Wronskian Identities", Martin J. Swatkowski, John Carroll University
"Phi Curious Yellow: The Golden Ratio", Linda Riede, University of Denver
"Game Theory", Elliot Wiesner, Polytechnic Institute of Brooklyn
"Zero Divisors in an Enveloping Ring", Robert Usery, East Carolina University
"On Almost Unitary Perfect numbers", Sidney West Graham, University of Oklahoma
1971 -- Pennsylvania State University Meeting
"Lonesome Points in a Topological Space", Bobby Beckom, East Carolina University
"Dieting: An Application of Decision Theory", Rebecca Jane Klemm, Miami University
"The Legendre Polynomials", Barbara F. Beier, Fontbonne College
"S-Spaces Generalization of the Pythagorean Theorem", Sister Carol Ann Dodd, Western Washington State
"Finite Near Domains", Russell Schexnayder, University of Southern Louisiana
"An Example of Ion-numerical Analysis", Frederick C. Druseifcis, University of Arizona
"The Unique Equilateral Triangle", Janet B. McDonald, State University of New York at Albany
1972 -- Dartmouth College Meeting
"K-Automorphisms of K((1/(x))", Hugh Barnett, East Carolina University
"Integers That Are a Function of Their Digits", Victor G. Feser, St. Louis University
"A General Test for Divisibility", Robb T. Koither, University of Richmond
"Eulerian numbers", Angela Kenison, Brigham Young University
"Characterization of the Differentiable Functions of a Quaternion Variable", Joseph J. Buff, New York University
"Distributively Generated Near Rings of Groups of Order 16", Kenneth R. Woodland, Worcester Polytechnic Institute
"Happy numbers", Daniel Wensing, John Carroll University
"Intuitionism (School of Intuition)", Beverly Waiters, Gonzaga University
"Goldbach's Conjecture", Chris Scosed, Michigan State University
"A Program to Determine the Number of Topologies on a Finite Set", Roger W. Foster, Miami University
"Computer Generated Random Vectors with Specified Density", Richard M. White, Miami University

"Complex Analysis", Ronald Umble, Temple University

1973 -- University of Montana Meeting

"Some Slick Tricks with Fourier Series", Joseph Jonathan Biff, New York University

"Necessity of the Secs and a Characterization of Groups of Bounded Order", San W. Talley, Western Kentucky University

"Applications of Finite Differences to the Summation of Series", H. Joseph Straight, State University of New York

"Partial Differentiation in metric Spaces", Roseann Mordillo, Seton Hall University

"Properties of CT-Topological Spaces", Daniel Campbell, Samford University

"Book Thickness and Graphs", David Keys, Louisiana State University

"A Look at Predator-Prey mathematical models", Raymond J. Marszalowicz, St. Peter's College

"Configurations of the Platonic Solids of Positive Genus", Kurt J. Schmucker, Michigan State University

"Some Properties of the Beltrami Differential Equation", Shirley M. foreman, Prairie View A&M College

"Mathematics and Language", Vincen Fon, University of Kansas

"The ilgebras of Floating Point number Systems", Paul Vitasek May, University of Oklahoma

"A Formal System for Quantification Theory", Mitchell Spector, John Carroll University

"The Theory of Queues", Kristine R. Hueseman and Cynthia M. Warren, St. Louis University

"Graphical approach to Game Theory", Evelyn Wallace Jenkins, University of East Carolina

"An All-Rule Axiomatization of the Propositional Calculus", William David Hook, Occidental College

"Constructing Polyhedron (Regular)", Karin Ann Statler, St. Louis University

"Soap Films", Jack Eckhardt, St. Louis University

"Metric Topological Invariants and Some Elementary Consequences", Gerald Hyde Jaimovich, U.C.L.A.

The AMS/AMM Summer Meeting was not held and thus there was no Pi Mu Epsilon Meeting in 1974.

1975 -- Western Michigan University Meeting

"Sequential Convergence", Lynn L. Mineo, Eastern Carolina University

"The Philosophy and Proof of Godel's Incompleteness Theorem", Charles M. 'Robins, Brooklyn College

"Continuous Non-Differentiable Functions", Brent Haaslern, University of Denver

"A Proof of the Theorem that Any Two Tape Turing machines may be Simulated by a One Tape Turing Machine", Zack D. Cox, Jr., University of North Carolina at Wilmington

1976 -- University of Toronto Meeting

"Chainable Continuum not homeomorphic to an Inverse Limit on [0,1] with Only One Bonding map", Dorothy Marsh, University of Houston

"An Informal math Lab", Kevin Bucal, Creighton University

"[0,1] Is not Compact: A Discussion of the Hyper-Real", Thomas Sweeney, St. Louis University

"On the Problem of the Lion and the man", Mark Showers, Southern Illinois University

"Ridge Regression", Dete Boroviak, University of Akron

"Fugue in Z# major", William Stone, University of Utah

"Fixed Point Theorems in metric Spaces", Carol Collins, East Carolina University

"Identities and Con-actions to Elementary number Theory in Certain Subsets of the Integers I and II", William Lenhart and Karen McColloch, St. Joseph's University

"Topology on Geometrics with Betweenness", Arne Rozsakow, Andrews University

"High School mathematical models", Ellen Hearns, St. Peter's College

"The Whitney Theory for Maps Between 2-Manifolds", Jane Hawkins, College of the Holy Cross

"On Diffusional and Circular Relations", Alma E. Posey, Hendrick College


1977 -- University of Washington Meeting

"Paradox in the Development of mathematics", Wayne Heym, Miami University

"The Value of mathematics in Pro-Legal Education", Bruce Fox, University of Illinois

"Continuous rings and Fields", Kathryn Dowell, St. Joseph's University

"Continuous Convergence of Functions", Robert Childs, East Carolina University

"Continuous Convergence in C(X)", James Lewis, East Carolina University

"A Characterization of G-Sets in metric Spaces", Jeff Thompson, Cornell University

"A Problem in Geometry and Probability", John Q. Walker, UI, Southern Illinois University at Edwardsville

"A Growth model for the Plastics Industry", Lawrence R. Rogers, Miami University

"The Busy Beaver Problem for Turing machines", Kathleen Marra, Manhattan College

"Coherent States in mathematical Physics", Thomas Ronayne, University of South Florida

"Mathematical models in nuclear Fuel management", Michael P. Manahan, Michigan State University

"Calculus of Variations", John R. Andreozzi, Rhode Island College

"Distance and Interiority in a Boolean metric Space", H. Elizabeth Newton, St. Louis University

"Numerical Solution of a Non-Linear Electron Conduction Equation with Boundary Values", James Delucia, St. Joseph's College

"Snakes on the 7-Dimensional Cube", Robert Myers, Hope College
1978 -- Providence Meeting

"Why I like Graph Theory", Nancy L. Burger, State University of New York at Potsdam
"Calculation of the Period of the Lotka-Volterra Predator Prey Model", Kathy Stuewe, University of Tennessee
"The Unique Number 15", J. B. Zipperer, Jr., Armstrong State College
"A Paradox in Quantum Theory", Linda Whiteyman, Sam Houston State University
"Is it Possible to Lose the only Magic?", Douglas W. Boone, Miami University
"The Double Ferris Wheel Problem", Albert E. Parish, College of Charleston
"The Use of Fractional Calculus in Solving Certain Difference Equations", David Challoner, Iowa State University
"Causes of math Anxiety at the University", Kathleen V. Walker, Southern Illinois University at Edwardsville
"Linear-Linear Additive Functions", Julie R. Anderson, Hendrix College
"The Converse of Lagrange's Theorem and Finite Nilpotent Groups", Marsha James, East Carolina University
"Matrix models in Biology", Michael Young, Portland State University
"Multivariate Discriminant Analysis and the Prediction of Loan Defaults", Nick Belloit, University of Northern Florida
"Scaling in mammals", Timothy O'Shea, St. Peter's College
"Concerning Irreducible Compact Continua", W. Dwayne Collins, University of Houston
"Exam Scheduling: An Example of math modeling", Coralee H. Cook, Miami University
"Generalized Lipschitz Criteria for First Order Differential Equations", Mark L. Burton, Hendrix College
"A method of Finding the Complement of a Sequence", Susan McIntosh, East Carolina University

1979 -- Duluth Meeting

"A Pulse-Time model for mathematics Class Enrollments", Julie Montgomery, Sam Houston State University
"Division Algorithms for Prime Factorization", John Anderson, Rose-Hulman Institute of Technology
"Problems: Stimulation to Research and Application", Steven From, Rose-Hulman Institute of Technology
"Numerical Treatment of meteorological Data", Gregory Battle, Washington University
"Magic Card Squares, Cubes, and Hypercubes", Bernard P. Smith, St. Louis University

1980 -- Providence Meeting

"On Continued Fraction Representations of Liouville numbers", Michael Filaseta, University of Arizona
"Behavior of the Permanent of a Special Class of Doubly Stochastic Matrices", Phuong Anh Vu, University of Houston
"Physical models for Applications of Rational numbers in Reduces Farm", JoAnn Fiene, Southern Illinois University-Carbondale
"Enzyme Kinetics of Phenylketonuria", C. P. Kuchinad, Marquette University
"A Hard (But Interesting) Problem", Michael L. Call, Rose-Hulman Institute of Technology
"Time as a Dynamical Variable in Quantum Physics", Stephen L. Ken, University of Tennessee-Knoxville
"Love Sonnet for a mathematician", Mark walker, South Dakota School of Mines and Technology
"Infinitesimals: Where They Come From and What They Can Do", Professor H. Jerome Keisler and J. Sutherland Frame, University of Wisconsin-Madison
"The Algebra of Choice", John B. Vaughn, St. Louis University
"The Kernel of the Laplace Transformation", David C. Sutherland, Hendrix College
"The Knight's Tour Problem", Gary Ricard, South Dakota School of Mines and Technology
"A Simple Proof of a Theorem by H. Schell", Peter Westfall, University of California-Davis
"Mathematics and the Boiling Points of Alkanes", Michael J. Schell, University of North Carolina at Charlotte
"Two Problems in number Theory", Janet Reid, University of North Florida
"Uniform Algebras and Scattered Spaces", Robert C. Smith, University of Arkansas
"A Case for against Computer Crime", Lonnie Eward, Northwest Missouri State University
"Measure Construction Using Cauchy Sequences", Stephen W. Semmes, Armstrong State College
"Cyclic numbers", Richard O. Griffin, University of Lowell
"The Consequences of Cauchy's Integral Theorem", Alfred Earl Byrum, East Carolina University
"An Infinite number of magic Squares", Stephen J. Hubert, Miami University

1980 -- University of Michigan Meeting

"Some Sums of Sums", Ann Zabiniski, College of St. Benedict
"Simulating Transformations in the Planes Symmetry Patterns Using the Tektronix 4051", James McKelvey, Hope College
"A New Algorithm To Solve Convolution Kernel Volterra Integral Equations", Aly Graham, University of Arizona
"Characterization of Primes", Kathryn Nolan, University of Arkansas at Pine Bluff
"Resolution of Russell's Paradox", Christopher J. Roosmer, University of Dayton
"Equivalent Statements", Wood-Wai Lee, Lamar University
"Applications of the Programmable Calculator in mathematics and Statistics", Daniel Pollak, Miami University

1982 -- University of Michigan Meeting

"The Gamma Function and Extensions", Bro. Longinus Anyanwu, Morgan State University
"Dirichlet Integrals and Their Applications", Ravi Salgia, Loyola University
"Transformation of Computer Programs into Functions", Brian Summer, University of Denver

1981 -- Pittsburgh Meeting

"Introduction to Box-Jenkins Time Series", Beth Snyder, Miami University
"Stokes' Theorem for Quaternion Integral Operators", Daniel Maguck, South Dakota School of Mines and Technology
"A Look at Formal Theory", Dean Shea, St. John's University
"An Approximation to the normal Distribution", Edward P. Cowry, Washington University
"The Centralizer of a Linear Transformation", Christopher J. Roosmer, University of Dayton
"Discrete Versus Continuous: Or, What is the Calculus of Finite Differences?", Beverly J. Skeans, Marshall University
"The Centralizer of a Linear Transformation", Douglas Eugene Jewett, Texas A&M University
"Finite State Machines", Martha Blackwelder, Appalachian State University
"A Nested Interval Theorem With Two Applications", Kurt L. Wiese, University of Arkansas
"Relativity in Perspective", William Terkurst, Hope College
"Problems With Infinity", Michael K. May, S.J., St. Louis University

1981 -- University of Michigan Meeting

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" Transformation of Computer Programs into Functions", Brian Summer, University of Denver
"A Complex Parabola in Four Dimensions", Robert Xear, West Carolina University
"Using the method of maximum Likelihood Estimation in Genetics", Margaret H. Wallace, Miami University
"Rubik's magic Cubes", Kevin Sayfors, Pomona College
"Descartes: Philosopher or mathematician?", James F. Goekke, S.J., St. Louis University
"A Non-Parametric multiple Comparison Test for Differences in Variances", Dean Foelmann, Northern Illinois University

1982 -- University of Michigan Meeting

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"Descartes: Philosopher or mathematician?", James F. Goekke, S.J., St. Louis University
"A Non-Parametric multiple Comparison Test for Differences in Variances", Dean Foelmann, Northern Illinois University
"Computer Graphics in numerical Analysis", Daniel John Pierce, Northern Illinois University

"Peppermint Patty's Dilemma --- Math Anxiety", Cathie Spino, Miami University

"Love and Psyche-Mathematics", Richard R. Porter, Douglass College

"Investigations of Maxfield's Theorem", Laura Southard, University of Oklahoma

1983 -- Albany Meeting

"A BASIC Program for the Schredering Equation", Mary Anne Bromdemier, Miami University

"Subset Selection", David Van Brockie, University of Central Florida

"Two Ancient Greek Construction Problems in Euclidean and Hyperbolic Geometry", Jack M. Rau, Oklahoma State University

"What Difference Does it Make?", Dariusch Saghafi, John Carroll University

"Ubiquitous Partition of Subsets of R^n", Donald John Nicholson, Iowa State University

"Leo Moser's Theorem", Susan McDonald Brits, East Carolina University

"Computability and Analogy Using Microcomputers", Thomas TenHove VII, Hope College

"Cryptology --- The Science of Secret Writing", Denise Vining, Miami University

1984 -- University of Oregon Meeting

"Complements --- mathematically Speaking", Leslie Youngdahl, Miami University

"Finding the Center of Complaints", Michael H. Cox, Marshall University

"Solving Linear-Linear Systems of Equations", Deborah Whitfield, Youngstown State University

"Money and Math: An Investigation of Linear Economic Models", Renee L. Larson, South Dakota State University

"Dealings in n-Dimensional Geometry", Karin Remington, College of St. Benedict

"Wedging Those Vector Integral Theorems", Calvin Johnson, University of California-Davis

"A Mathematical Model of Voter Participation", Mary Beth Dever, Northern Illinois University

"Number Nina", Patrick Tamer, Appalachian State University

"Graph measure in Euclidean n-Space", Jody Trout, The College of Charleston

"Finite Laplace Transforms", David W. Barnette, East Carolina University

"A Way to Generate Arbitrarily High-Order Root-Finding methods", Jeffrey Michael Kubina, Youngstown State University

"Samuelson's Interaction Between the Incidence and the Multiplier", Suguna Pappu, Miami University

"Commutativity and Distributivity: Different Perspectives", Benjamin L. Marshall, Hendrix College

1985 -- University of Wyoming Meeting

"Box-Jenkins Autoregressive Integrated Moving Average Forecasting of Common Stock Prices", Milam W. Aiken, University of Oklahoma

"An Historical Look at Some Interesting Functions", George M. Alexander, St. John's University

"A Two-Dimensional Ion-Lineal Population Model", Kenneth Mark Alot, University of Houston-Downtown

"A Shortcut to Solving Polynomials and Rational Inequalities", Christa Blackwell, University of Oklahoma

"Create Your Own Geometry", David Cameron, Miami University

"Metagame Theory and Political Behavior", Andrew Chin, University of Texas Which Came First, the Explicit Formula for the Determinant or its Properties?", Kevin T. Christian, University of California-Davis

"Historic Geometric Problems Involving Compass and Straightedge Construction", Anthony Clark, Youngstown State University

"Iont-negative Integer Solutions of \( \sum_{i=1}^{n} x_i = k \), Marie Coffin, South Dakota State University

"Some Applications of the Gray Code", Henry L. Culver, Youngstown State University

"Using Extrapolation to Obtain Approximations to 'Pi'", Raymond E. Flannery, Jr., Youngstown State University

"A Report on the Precision of Floating Point Math Functions on Selected Computers", John C. Fosholker, Georgia Institute of Technology

"A Recursive Desert Scanner in LCOQ", Mark E. Frydengen, University of Hartford

"Monte Carlo Studies in Statistical Research", Kathleen A. Steiglmann, Northern Illinois University

"Fooling Around with Mother Nature --- An Application in Genetics", John Gneskovitch, Miami University

"Mystery's Example of a Regular Topological Space That is Not Completely Regular", Elizabeth Stratton, Miami University

Some Results in the Theory of Intangibles Numbers", James L. Hart, Hendrix College

The Shifted QR Algorithm in Real Matrix Computations", Karalee Howell, Occidental College

"Sabermetrics", James D. Johnston, St. Bonaventure University

"Randomness and Complexity", Jorge O. Ochoa-Lions, University of Arizona

"Expanding a System of Propositional Logic", Mitchell Pollack, Bucknell University

Valuations on Monoids", Terry Reilly, University of Montana

"Mathematical Models in International Relations", Sandra R. Rogers, Auburn University

"Hidden Lines, The Unseen Graphic", Henry Rosche III, Southeastern Louisiana University

"A Microcomputer Application of Karush's Polynomial-Time Algorithm for Linear Programming", Annuar Mohd Saffar, University of Missouri-Columbia

"From Lead Pipes to Telestar --- History and Applications of Fourier Analysis", James J. Shea, Worcester Polytechnic Institute
network modeling in a Transportation Environment". Cynthia M. Stuber, St. Norbert College

"A General Solution Procedure for the Steady-State Probability of State-Homogeneous Production Line models". Karim Tofang-Sazi and Sencer Yordan, University of Missouri-Columbia

"A Phase-Plane Analysis of Coulomb Damping". John Tokar, Rose-Hulman Institute of Technology

"Dale's Core of Experience and Improved Math Teaching". Rob Walking, Miami University

"Lattices of Periodic Functions". Kirk Wetter, Hope College

"Boolean Hemispherical matrices and I applictions". Sherry Wheeler, Alabama State University

"Figurate numbers". Terry Wiencek, Youngstown State University

"From Pac-Man to the Dihedral Group, D4: An I applictation of Group Theory to Video Games". Erik L. Winters, University of New Hampshire

1986 - University of California-Berkeley Meeting

"Generation of One million Prime numbers". William O. Haas, Southeastern Louisiana University

"A Proof of Primality Utilizing Fermat's Theorem". Sara S. Fagan, Marquette University

"Solving Linear Diophantine Equations Using Euclid's Algorithms". Kathy Kowell, Miami University

"Lame's Theorem and the Euclidean Algorithm". Frederick Taverner, University of California-Davis

"A New Proof of a Lemma to the Quadratic Reciprocity Law". David Claussen, Miami University

"A Representation of Squares in Generalized Fibonacci Sequences". Russ Shuttleworth, Wichita State University

"Class numbers of Cyclotomic Fields". John E. Fischer, Jr., University of Pennsylvania

"Starting with Pascal's Triangle". Erlan Wheeler 11, Virginia Polytechnic Institute

"$1000 Howard: Sam Loyd's 14-15 Puzzle". Judith Barciewicz-Symula, St. John's University

"mathematical models in Population Genetics". Richard Tabbaa, University of Houston-Downtown

"Entropy of the M/B/1 Queuing System". Milam W. Aiken, University of Oklahoma

"Estimating Age-Specific Fecundity of Soft Shell Clams". Anne Kochendorfer, Fairfield University

"An Objective Analysis of Rainfall Data". Thomas A. Kreitzberg, Drexel University

"The Absorbing Markov Process as Applied to a Random Behavior Model". Bridget Moore, Miami University

"Linear-Time Three-Dimensional Graphics with Hidden Line Elimination". Jeffrey S. Bonwick, University of Delaware

"A Look at the DoD's Trusted Computer System Evaluation Criteria". John Masplholer, Georgia Institute of Technology

"The mathematical World of Cryptology". Barry Schoch, Fairfield University

"Symbolic Computation". Erri1 J. Vochek, University of Delaware

"Does 4^k+1/k Converge or Diverge?". Dawn Alisha Lott, Bucknell University

"Infinite investments". Hunter Marshall, Texas A&M University

"A Method of Defining Infiniteinvestments and Extending Functions". Karen Sue Billings, Hendrix College

"Beauty from Boredom - A Visa of Fractal Geometry". Fun Shea, Worcester Polytechnic Institute

"Fractal Curves". Michael J. Cullen, Marquette University

"The Dynamics of F(z) = z^2 - 1". Connie Lou Overzet, Boston University

"The Equilateral Quadrilateral". Rob Walking, Miami University

"Counting Rectangles in a Multiregional Region". Steven D. Van Lieshout, St. Norbert College

"Graphs Uniquely Hamiltonian-Connected from a Vertex". Carolyn R. Thomas, St. Lawrence University

"Approximating the Solution to Ordinary Differential Equations Using Taylor Polynomial Expansions". Anthony Clarko, Youngstown State University

"Stability on a Finite Interval of Time-Average Differential-equations". Michael P. Perrone, Worcester Polytechnic Institute

"Pressure Analysis in a Biomedical Device". Paula A. Michaels, Miami University

"The Iraq-Iran War". Conchita Minor, Alabama State University

"Locating Emergency Facilities in Order to minimize Response Time". Craig J. Cole, Miami University

"The Effect of Einstein's Theory of Relativity on Interstellar navigation". James A. Kirklin, Mount Union College

"modeling a magnetic Oscillator". Bradley O. Pad, Miami University

"Applications of Bally's Theorem to the approximation of Functions by Polynomials". Donna Vigant, University of Lowell

"Using Residues to Evaluate Certain Infinite Series". Mark Russell Smith, East Carolina University

"The Development of Outstanding Secondary mathematics Students". Brian A. Twitchell, University of Maine

1987 - University of Utah Meeting

"Deterministic and Probabilistic Fire Models". Warren E. Maisel, Worcester Polytechnic Institute

"The Epidemiology of the RDS Virus". Aaron Klebanoff, University of California-Davis

"The Strangely Attracted Bouncing Ball". Stephanie Ruth Land, University of Texas

"Applications of Signal Processing". Debra Shure, University of Lowell

"Self-Calibration of Complex Visibility Data from a Very Large Array of Antennas". Ali Saffari-nejad, Iowa State University

"Representations and Characters of Groups". Ken Kopp, Miami University

"An Algebraic Construction of a Projective Geometry". Stephanie Dumoski, Occidental College

"The RSA Public Key Cryptosystem: An Application for Modern Algebra". Stephen Fieo, West Virginia University
16/64 = 1/4 and Other Digit Cancellations", David A. Messineo, University of Hartford

"An Algebraist's View of Competitive Games", Erlan Wheeler UI, Virginia Polytechnic Institute

"Perpendicular Least Square Estimators", Brian Anderson, Western Kentucky University

"The Isoperimetric Inequality", Jeff Diller, University of Dayton

"Inverting a Pin in $\mathbb{R}^2$", Russell Goldwin, University of Arkansas

"Fourier Series and the 'Best' Mean Square Approximation", Margaret M. Lineberger, East Carolina University

"Dynamic Programming Applied to Computer Voice Recognition", Thomas Eugene Glabson, St. John's University

"A Formal Sum Method to the Traveling Salesman Problem", Jeffrey Horn, Marquette University

"A Bit of Checking and Correcting", Summer Quimby, St. Norbert College

"A Graphical Illustration of the Convergence of Karmarkar's Linear Programming Algorithm", Joseph S. Crepeau, University of Montana

The Existence of Eulerian and Hamiltonian Circuits in Graphs and their Line Graphs", Carol Parker, Hendrix College

"Circuit Spaces and Cut-Spaces of a Connected Graph", Michael Tackett, Miami University

"Yet Another Discussion of Graceful Graphs", Andrew P. Ferreira, Miami University

"Sub-Families of Venn Diagrams", Philip Beymer, University of Oregon

"Soap Films as Minimal Surfaces", George Matter, St. John's University

"Theology, mathematics and meaning", Mary Ehl, St. Norbert College

"Curves Length Minimizing Module in $\mathbb{R}^2$", Jeff Abramson, Massachusetts Institute of Technology

"NP-Completeness and the Traveling Salesman Problem", Melanie X. Braker, Northeast Missouri State University

"The Classical Problems of Antiquity in the Hyperbolic Plane", Robert Curtis, University of California-Santa Cruz

"A Physical Derivation of the Well-Tempered Musical Scale", Timothy Koponen, Aquinas College

"Games of Timing with Two or Three Players", Timothy P. Ronana, Moravian College

"A Glimpse at the Theory of Restricted Choice", Scott Krutsch, Rose-Hulman Institute of Technology

1988 -- Providence Meeting

"Maximal Polygons for Convex Periodic Tilings", Annette M. Matthews, Portland State University

"On Introduction to Equitransitive Tilings", Gerry Wachter, Miami University

"Finding a Generator of a Finitely Generated Abelian Group", David L. Fakes, Miami University

"The Cross Product in n-Space", Joel Atkins, Rose-Hulman Institute of Technology

"A Continued Fraction Approach for Factoring Large Numbers", Robert Coury, University of Washington

Perfect numbers, Abundant numbers, and Deficient numbers. Sarah Taylor, East Carolina University

The Problem of the Traveling Salesman - A study in Computational Complexity and Heuristic Algorithms", Douglas Galarus, The University of Montana

"Using Portable Intermediate Code in Compiler Construction", Paul Stodgill, Dickinson College

"Numerical Solutions for the Three-Dimensional Heat Conduction Equation", Charles Jobour, University of Houston-Downtown

"The Effect of Intra-Cluster Correlations on the Regression Estimation in the Finite Population Inference", Devi J. Sales, Wichita State University

"QR Revisited - A Parallel approach to the QR Decomposition", Lora Aist, University of Maryland-Baltimore County

"A Brute Force approach to Solving a Putnam Problem", Summer Quimby, St. Norbert College

The Sturm-Liouville mathematical System", Lorie Ceremuga, Youngstown State University

"The Annihilation Operator", Marc Ahrens, Elmhurst College

"Mathematical measures in the Analysis of Image", Clifford D. Krumvieda, Texas A&M University

"Squaring a Square", Terry Hendrickson, Miami University

"The Area of Wasted Space in a Rosette Design", George Anderson, Miami University

"Some Results for Chromatic and Isospectral Polynomials of Graphs", Laura Cuttersberth, Hendrix College

"The Greedy Spy", Janelle Becke, St. John's University/College of St. Benedict

"Heuristic Arguments in Mathematics", David. Petry, University of Oregon

"Removing the Fibrosilicate from Relativity", Katie Conen, St. Norbert College

"Dynamics of PDP models", George Ashline, St. Lawrence University

"A Venture into Chaos", Erlan Wheeler, Virginia Tech

"Fractals - A mathematical Chaos", Michael F. Marcon, McNeese State University

Editor's Note: The author and the Editor are indebted to David Ballew for compiling Appendices A and C.
APPENDIX B

J. SUTHERLAND FRAME LECTURES

The J. Sutherland Frame Lecture is named in honor of Pi Mu Epsilon’s ninth president who served from 1957-1966. In 1952, Sud Frame initiated the concept and made possible the Pi Mu Epsilon student participation and student paper sessions at the Joint Summer Meetings of the AMS and MAA. Since then he has offered insights and inspiration to our student mathematicians at the summer meetings.

1988 Boris W. Schuttessnider
“You, Too, Can Tile the Conway Waga”

1987 Clayton W. Dodge
“Reflections of a Problems Editor”

1986 Paul Halmos
‘Problems I Cannot Solve”

1985 Ernst Snapper
“The Philosophy of mathematics”

1984 John L. Kelley
“The Concept of Plane Area”

1983 Henry Alder
“How to Discover and Prove Theorems: A Demonstration with Partitions”

1982 Israel Halperin
“The Changing Face of mathematics”

1981 E. P. Mueh, Jr.
“The Beauties of mathematics Revealed in Color Block Graphs”

1980 Richard A. Askey
“Ramanujan and Some Extensions of the Gamma and Beta Functions”

1979 H. Jerome Keisler
“Infinities: Where They Come From and What They Can Do”

1978 Herbert E. Robbins
“The Statistics of Incidents and Incidents”

1977 Ivan Niven
“Techniques of Solving Extremal Problems”

1976 H. S. MacDonald Coxeter
“The Pappus Configuration and Its Groups”

1975 J. Sutherland Frame
“matrix Functions: A Powerful Tool”

APPENDIX C

NATIONAL OFFICERS FOR PI MU EPSILON

<table>
<thead>
<tr>
<th>PERIOD</th>
<th>DIRECTOR</th>
<th>VICE-DIRECTOR</th>
<th>SECRETARY</th>
<th>TREASURER</th>
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<tbody>
<tr>
<td>1914-1915</td>
<td>E. D. Roe, Jr.</td>
<td>F. F. Decker</td>
<td>Helen Applebee</td>
<td>P. J. Bentley Olive Jones</td>
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</table>

Local Officers of the Syracuse Chapter served as General Officers until 1922.

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<thead>
<tr>
<th>PERIOD</th>
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<th>VICE-DIRECTOR</th>
<th>SECRETARY</th>
<th>TREASURER</th>
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<td>1922-1927</td>
<td>E. D. Roe, Jr.</td>
<td>W. G. Bullard</td>
<td>H. S. Everett</td>
<td>Louisa Lotz Mabel Kessler</td>
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<tr>
<td>1928-1929</td>
<td>E. D. Roe, Jr.</td>
<td>T. Fort</td>
<td>John S. Gold</td>
<td>Louisa Lotz Mabel Kessler</td>
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<tr>
<td>1929-1933</td>
<td>Louis Ingold</td>
<td>T. Fort</td>
<td>John S. Gold</td>
<td>Louisa Lotz Mabel Kessler</td>
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The Offices of Secretary and Treasurer were combined and the Office of Librarian was discontinued.

EDITOR OF THE
PI MU EPSILON JOURNAL

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<th>PERIOD</th>
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<tr>
<td>1951-1954</td>
<td>C. C. MacDuffee</td>
<td>W. M. Whyburn</td>
<td>J. S. Frame</td>
<td>Ruth Stokes</td>
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<tr>
<td>1955-1957</td>
<td>S. S. Calms</td>
<td>J. S. Frame</td>
<td>R. V. Andree</td>
<td>Francs Regan</td>
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<tr>
<td>1972-1976</td>
<td>H. T. Kames</td>
<td>E. Allan Davis</td>
<td>Richard Good</td>
<td>David Kay</td>
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<tr>
<td>1975-1978</td>
<td>E. Allan Davis</td>
<td>R. V. Andree</td>
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<td>David Kay</td>
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<td>1978-1981</td>
<td>E. Allan Davis</td>
<td>E. Maurice Beasley</td>
<td>Richard Good</td>
<td>David Ballew</td>
</tr>
<tr>
<td>1981-1984</td>
<td>E. Maurice Beasley</td>
<td>Milton Cox</td>
<td>Richard Good</td>
<td>David Ballew</td>
</tr>
</tbody>
</table>
COUNCILORS OF PI MU EPSILON

1914-1922 Florence A. Lane
1922-1926 E. R. Hedrick
1926-1929 J. C. Eaves
1929-1933 E. D. Roe, Jr.
1933-1936 E. R. Hedrick
1936-1939 W. C. Breske
1939-1942 H. H. Downing
1942-1945 W. C. Breske
1945-1948 George Williams
1948-1951 S. S. Calms
1951-1954 S. S. Calms
1954-1957 H. C. Hightower
1957-1960 E. H. C. Hildebrandt
1960-1963 J. C. Eaves
1963-1966 Josephine Chanter
1966-1969 L. E. Bush
1969-1972 E. R. Hedrick
1972-1975 Gloria Hewitt
1975-1978 Milton Cox
1978-1981 Milton Cox
1981-1984 David Barnelle
1984-1987 David Baliew
1987-1990 Richard Good

Helen Mary Barnard
W. H. Roeber
R. D. Carmichael
Tomlinson Fort
Alain Campbell
W. W. Elliot
P. J. Daus
C. A. Hutchinson
Tomlinson Fort
Tomlinson Fort

Edward Cottrell
S. E. Rasor
E. R. Hedrick
C. S. Latham
D. Lehmer
G. C. Evans
E. H. C. Hildebrandt
C. H. Richardson
J. S. Gold
S. L. McDonald
Adolph Sussman
U. G. Mitchell
Louis Ingold
F. W. Owens
R. A. Johnson
W. P. Orby
E. R. Smith
A. J. Kempner
Ruth Stokles

FROM THE CONSTITUTION OF PI MU EPSILON

ARTICLE VII.
FORMS AND CEREMONIES

Section 1. Motto. The Pi Mu Epsilon motto is the following:

"To Promote Scholarship and Mathematics"

Section 2. Insignia. The insignia of the fraternity are the following:

Colors—Violet, gold, and lavender.
Seal—Round, design of pin raised in center. The words, 'Pi Mu Epsilon, Incorporated' around the edge.

FLOWER—Violet, Shield—Divided into four parts. In the upper right hand corner, the design of a conventionalized violet; in the lower right, a summation sign; in the upper left, an integral sign; and in the lower left, three stars representing fraternity, morality and scholarship. Just above the shield proper the pin is represented, and below, on a streamer, is the Greek motto.

A UBQUITOUS PARTITION OF SUBSETS OF R^n

by Donald John Nicholson
Iowa State University

We are about to embark on an adventure into the beauty of mathematical inquiry. The journey will begin with a simple yet wonderful property of the real numbers, and in the end we will arrive at an elegant result: a ubiquitous partition of subsets of R^n, i.e., a partition in which every element of the partition has a non-empty intersection with each neighborhood (abbreviated nbd) in the space.

The path will lead only through Euclidean spaces, so each space must be understood to be a subspace of R^n. When nbds are mentioned, they refer to nonempty intersections of open subsets of R^n with the space. We will use the symbol \( \bar{X} \) to represent the closure in R^n of a set X, X' to denote the set of limit points in R^n of X, and then define \( X'' = X' \setminus \bar{X} \). Finally, \( U(x) \) is the set of all points whose distance from x is less than r, and \( B_r(x_0) \) is its closure. Now let us begin.

Let Q and P be the rational and irrational numbers respectively, and let U be a nonempty interval in R. Both \( Q \cup Q \) and \( P \cup P \) are infinite no matter how small U is. Consequently, each point of \( Q \cup Q \) is a limit point of the set, but there must always be irrational limit points as well. It is this property which we shall generalize in the following definition.

Definition 1. A space X is a Gurioxiw space* if \( \forall \text{nbds } U \text{ in } X \), \( \bar{U} \neq X \).

Proposition 1. If X is a Gurioxiw space, then X is dense in itself.

Proof. Let x be an isolated point in X. Then \( \{x\} \) is a nbd of \( x_0 \) in X. But \( \{x_0\} \cap X = \emptyset \), thus X is not a Gurioxiw space.

* see excerpt from a letter by the author at end of the paper
If $X$ is a Grunowix space, then $X \times Y'$, but $X \not\# Y'$. This raises a question: what kind of space is $X''$? Before answering, let us define a new concept.

**Definition 2.** Two spaces $X$ and $Y$ are mutually dense iff $X = Y$. An arbitrary collection of spaces is mutually dense iff they are mutually dense in pairs.

**Lemma 1.** Let $X$ and $Y$ be disjoint spaces. Then the following are equivalent: (a) $X \subset Y''$, (b) $X \subset Y'$, and (c) $X \sqcup Y$ and $Y \sqcup X'$.

**Proof.** Suppose $X = Y$. Then $X \subset Y''$ and $Y \subset X'$, and since $X$ and $Y$ are disjoint, $X \subset Y''$ and $Y \subset X'$, hence $X \subset Y'$ and $Y \subset X'$. This implies that $X \subset Y''$ and $Y \subset X'$, so $X \subset Y''$ and $Y \subset X'$. This returns us to our original supposition that $X = Y$.

**Lemma 2.** If $X$ is a Grunowix space, then $X = X''$.

**Proof.** By the definition of $X''$, $X$ and $X''$ are disjoint, and $X'' \times X'$; therefore, to satisfy Lemma 1 we need only show that $X \subset (X'')'$. Let $x \in X$ and let $U$ be a nbhd of $x$ in $R^n$. Then $3 > 0 : U(3x) \subset U$. Now choose $r' > 3 \in X' r'$; this will give us $U(3x) \subset U \circ (3x) \subset U$. Observe that $\bigcup_{n=1}^{\infty} U(3x) \subset U$, and $U(3x) \subset U$ is a nbhd of $x$ in $X$, so that, since $X$ is a Grunowix space, $U(3x) \subset U - x \not\# y$. This implies that $\bigcup_{n=1}^{\infty} U(x) \subset U$ and thus $U$ contain points in $X''$; it follows that $x$ is a limit point of $X''$. Since $x$ is arbitrary, we infer that $X \subset (X'')'$.

**Theorem 1.** Let $X$ and $Y$ be mutually dense, disjoint spaces. Then $X$ and $Y$ are Grunowix spaces.

**Proof.** Let $x \in X$ and let $U$ be a nbhd of $x$ in $R^n$. By Lemma 1 $X \subset Y'$ and $Y \subset X'$, so $y \in U \cap Y$ implies $y \not\in X$. This implies that $y \in U \cap Y \subset X' \times X'$. Since $X' \times X'$ is an arbitrary nbhd of $x$ in $X$, $X$ is a Grunowix space. By a similar argument $Y$ is a Grunowix space.

**Lemma 2 and Theorem 1** answer our question: if $X$ is a Grunowix space, then so is $X''$; e.g., $\emptyset$ and $P$ are Grunowix spaces. But Theorem 1 has a much more interesting consequence: every collection of spaces which is mutually dense and pairwise disjoint is a collection of Grunowix spaces, and we shall see that such a collection is a ubiquitous partition of its union.

Let us construct such a collection. Let $r, sc(0,1)$ and define $r-s$ iff $\log r \in \mathbb{Q}$. It should be a simple exercise for the reader to show that $r$ is an equivalence relation and hence partitions $(0,1)$. Let $[r]$ and $[s]$ be distinct equivalence classes in the partition of $(0,1)$, then $[r] \cap [s]$ is disjoint. Furthermore, $\log s \in \mathbb{Q}$, i.e., $\exists p \in \mathbb{Z}$ such that $r^{1/p} = s$. Since $p \in \mathbb{Z}$, $\exists n \in \mathbb{Z}$ such that $n^{1/p} = s$. Each $r \in [r]$, thus $s$ is a limit point of $[r]$ as is every element of $[s]$. Similarly, every element of $[r]$ is a limit point of $[s]$. Let us define $a$ equivalence classes in the partition of $(0,1)$, and $[r]$ and $[s]$ are Grunowix spaces by Theorem 1. Note that each equivalence class is countable, whereas $(0,1)$ is uncountable, thus $([r], m(0,1))$ is uncountable.

The above partition is interesting in that the elements are mutually dense; they are like chemically inert gases in a closed container at thermal equilibrium. Just as the molecules of each gas distribute themselves throughout the container, every nbhd in $(0,1)$ contains points from every equivalence class; i.e., the partition is ubiquitous.

Let us define a **Grunowix partition** as a partition consisting of mutually dense spaces. By Lemma 1 and the definition of a limit point, this is equivalent to the definition of a ubiquitous partition. Thus every Grunowix partition is ubiquitous, but no other partition is; this is the beauty of the Grunowix space.

Do NOT BLINK! We are about to show how to construct Grunowix partition's of an infinite number of spaces by using only three theorems and our partition of $(0,1)$!

**Theorem 2.** Let $(X_a : a \in A)$ where $A$ is an indexing set be a Grunowix partition of $X$, and let $f : X \rightarrow Y$ be a continuous bijection. Then $(f(X_a) : a \in A)$ is a Grunowix partition of $Y$.

**Proof.** The bijectivity condition insures that $(f(X_a) : a \in A)$ will be a partition of $Y$. If $x \in X_a$ and $y \in Y$, $\exists x \in X_a$ such that $f(x_a) = y$. Since $f$ is continuous, $f(X_a) = f(X)$, thus $f(X_a)$ and $f(X)$ are mutually dense. It follows that $(f(X_a) : a \in A)$ is a Grunowix partition of $Y$.

**Theorem 3.** Let $(X_i : a \in A)$ be a Grunowix partition of the $i$th factor space. Then $(X_i : a \in A)$ is a Grunowix partition of its union.
Proof. Let \( a, bcA \) and \( a \neq b \). Then \( x_1 x_2 \ldots x_i x_{i+1} \ldots x_n = x_1 x_2 \ldots x_i x_{i+1} \ldots x_n \). Now \( x_1 x_2 \ldots x_i x_{i+1} \ldots x_n \) thus \( \{ x_1 x_2 \ldots x_i x_{i+1} \ldots x_n : acA \} \) is a mutually dense collection of spaces. Since they are disjoint and their union is \( X \), they form a Grunowix partition of \( X \) (we will call this a Grunowix partition of \( X \) in the \( i \)'th factor space or \( i \)'th coordinate).

**Theorem 4.** Let \( \{ x : acA \} \) be a Grunowix partition of \( X \), and \( B \) an open subset of \( X \). Then \( \{ X \cap B : acA \} \) and \( \{ X \cap B : acA \} \) are Grunowix partitions of \( B \) and \( B \) respectively.

**Proof.** Here we will use the equivalence of the definitions of Grunowix and ubiquitous partitions. Now \( \forall x \in X, \forall \text{nbd } U \text{ of } x \text{ in } X, \text{ and } \forall acA, \cup \exists x \notin B \). Since \( B \) is open in \( X \), each \( \text{nbd} \) in \( B \) is a \( \text{nbd} \) in \( X \), thus \( \forall x \in B, \forall \text{nbd } U \text{ of } x \text{ in } B, \text{ and } \forall acA, \cup \exists x \notin B \). But \( \cup \exists x = \cup \exists \forall x \notin B \), thus \( \{ x \land B : acA \} \) is a Grunowix partition of \( B \).

Now let \( x \in B \) and \( U \) a \( \text{nbd} \) of \( x \) in \( X \). Then \( \cup \exists x \notin B \), and since \( B \) is open in \( X \), \( \cup \exists x \notin B \) is open in \( X \), so that \( \cup \exists x \notin B \forall acA \). Since

\[
\cup \exists x \notin B \forall \cup \exists x \notin B \forall acA
\]

is a Grunowix partition of \( B \).

These three theorems are our tools for constructing Grunowix partitions. We may construct a Grunowix partition of any open subset of \( R^n \) or its closure by collecting the intersections of the set with each element of a Grunowix partition of \( R^n \). A Grunowix partition of \( R^n \) may be easily constructed by mapping \( (0,1) \) homeomorphically onto one or more factor spaces of \( R^n \) and using Theorem 3. Look how many spaces we can ubiquitously partition with a simple equivalence relation on \( (0,1) \)!

We would like to make one final observation. Constructing a Grunowix partition of \( R^n \) by partitioning one or more of its factor spaces imposes a variety of geometric structures on the connected components of the elements of the partition. Since the elements of a Grunowix partition of \( R \) are totally disconnected, a Grunowix partition of \( R^n \) in one coordinate will consist of elements whose components are parallel planes, in two coordinates the components will be parallel lines, and in three coordinates the components will be points. In general the components of the elements of a Grunowix partition of \( R^n \) in \( m \) of its coordinates will be \( n-m \) dimensional hyperplanes.

There is more. We may use any curvilinear coordinate system in \( R^n \) to construct a Grunowix partition so long as each point has unique coordinates. By using spherical or cylindrical coordinates we may have a Grunowix partition of \( R^3 \) whose elements consist of concentric spheres or coaxial cones or cylinders. The mysteries of the Grunowix space know no bounds!

The purpose of Figure 1 is to aid in visualizing the Grunowix partition \( \{ Q^2, R^3-Q^3 \} \) of \( R^3 \). The element \( Q^2 \) is totally disconnected, and since it is countable, \( R^3-Q^3 \) is connected. A very rough description of the partition is a countable collection of infinitesimal "boxes" enclosing points. How exquisite! And it all began with the property of the real numbers that every interval contains infinitely many rationals and irrationals.
REFERENCES


EXCERPT FROM A LETTER

... The idea of a Grunowix space arose from an undergraduate topology homework problem in reference 3. The problem is Exercise 1-28 on page 25: prove that the set of rationals as a subspace of the reals is not locally compact.

When I finally decided what property of the set of rationals prevented it from being locally compact, it struck me as being very fascinating. I asked the instructor if this property had a name. He knew of none, so I generalized the property and tried to come up with a name myself. My wife suggested inventing a word, so we tried constructing words with a scrabble board and letter tiles, and 'Grunowix' was the first construction we could pionounce.

My original definition for the Grunowix space was formulated differently, but it was equivalent to the present one. If my paper is devoid of references, it is only because none of the math professors I was acquainted with could steer me toward any sources that helped me with this interesting space, nor did any searching on my part uncover any useful references. I was forced to develop the definitions and propositions on my own; it required walking many months and wandering down home blind alleys, but I am pleased with the results and what I learned from the experience. It has made me love mathematics.

I would like to thank Dr. Donald Sanderson of Iowa State University for the multitude of occasions on which he listened to me sort out ideas to clarify my ideas and for being kind and patient enough to read and constructively criticize five versions of this paper ...

Sincerely,  
Donald John Nicholson

Editor's note: This paper was prepared while the author was a senior undergraduate majoring in mathematics and physics at Iowa State University. Donald presented the paper at the National Meeting of Pi Mu Epsilon in Albany, NY in August 1983 and has entered the paper in the Journal's National Paper Competition. The competition is open to students who have not received their master's degree at the time of submission. Papers may be submitted to the Editor at any time.

X-MATRICES: GENERALIZED COMPLEX NUMBERS

by Samuel Councilman  
California State University, Long Beach

In introductory Abstract and Linear Algebra courses one encounters the familiar isomorphism between complex numbers and $2 \times 2$ matrices:

$$a_1 + a_2 i \leftrightarrow \begin{pmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{pmatrix}$$

Thus all matrices of the form

$$\begin{pmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{pmatrix} = a_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + a_2 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

yield a commutative subalgebra of $M_{2,2}(\mathbb{R})$, the algebra of $2 \times 2$ real matrices. A natural question is "can this be generalized to higher dimensions so as to obtain commutative subalgebras of $M_{n,n}(\mathbb{R})$ for $n \geq 2$?"

In even dimensions $n = 2k$ the matrix

$$J = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & -1 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & 0 & -1 \\ 1 & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

behaves just like an "imaginary unit." $J^2 = -I$. Thus one may be inclined to consider matrices of the form $a_1 I + a_2 J$. But you can do better! The scalars $a_1$ and $a_2$ can be replaced by biisymmetric diagonal matrices that is, diagonal matrices which are symmetric with respect to the diagonal from the lower left to the upper right corner.
Definition. \( D = (d_{ij}) \) is a bi-symmetric diagonal matrix if and only if \( d_{ij} = 0 \) for \( i \neq j \) and \( d_{ii} = d_{(n+1-i)(n+1-i)} \).

In \( M_{2k,2k}(R) \), consider the set of matrices of the form
\[
D_1 I + D_2 J = D_1 + D_2 J
\]
where \( D_1 \) and \( D_2 \) are bi-symmetric diagonal matrices and \( J \) is the imaginary unit matrix defined above. Because of the pattern formed by the (possibly) non-zero elements, these \( D_1 + D_2 J \) matrices may be called "\( X \)-matrices."

The set of \( X \)-matrices is obviously closed under scalar multiplication, and the commutative and distributive properties of matrix addition show that \( X \)-matrices add just like complex numbers:
\[
(D_1 + D_2 J) + (D_3 + D_4 J) = (D_1 + D_3) + (D_2 + D_4) J.
\]

To verify that \( X \)-matrices multiply like complex numbers, a preliminary observation is required: since premultiplication by a diagonal matrix multiplies rows and postmultiplication multiplies columns, it is easily seen that
\[
D J = JD
\]
for any bi-symmetric diagonal matrix \( D \). With this result, a straightforward application of commutativity of matrix addition and the distributive law shows that \( X \)-matrices multiply exactly like complex numbers:
\[
(D_1 + D_2 J)(D_3 + D_4 J) = D_1 D_3 + D_1 D_4 J + D_2 D_3 J + D_2 D_4 J.
\]
Since multiplication of diagonal matrices is commutative, this multiplication of \( X \)-matrices is also commutative, and in even dimensions \( 2k \) the set of \( X \)-matrices \( D_1 + D_2 J \) is a commutative subalgebra of \( M_{2k,2k}(R) \).

To complete the analogy with complex numbers, one should note that the \( X \)-matrix \( D_1 + D_2 J \) will have inverse \( (D_1^2 + D_2^2)^{-1}(D_1 - D_2 J) \) just in case the bi-symmetric diagonal matrix \( D_1^2 + D_2^2 \) is nonsingular (has no 0 elements on its main diagonal).

Since the imaginary unit matrix \( J \) fails to exist in odd dimensions, one may wonder whether analogous commutative algebras exist in \( M_{n,n}(R) \).

Struggling through an \( \epsilon-\delta \) proof of the continuity of a simple function, the instructor stopped, faced the students, and asked "Are there any questions?"

One student asked "How much do we have to make to get an A?"

The instructor answered "Somewhere in the neighborhood of 85."

Another student, who seemed to have been asleep, woke up and asked "How large is \( \varepsilon \)??"
NOTE ON A WELL-KNOWN LIMIT

by W. Vance Underhill
East Texas State University

In the Spring 1986 issue of this journal Norman Schaumberger obtained the limit

\[ \lim_{n \to \infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e} \]

by use of the Mean Value Theorem. In that derivation, however, it was necessary to first establish an intermediate double inequality. Here is a shorter and quite different elementary approach to the same limit.

The radius of convergence \( R \) of a power series \( \sum_{n=0}^{\infty} a_n x^n \) is given by either of the formulas

\[ \frac{1}{R} = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| \quad (1) \]

\[ \frac{1}{R} = \lim_{n \to \infty} \sqrt[n]{|a_n|} \quad (2) \]

provided the limit in (1) exists. Consider the series \( \sum_{n=0}^{\infty} \frac{n!}{n^n} x^n \).

Using (1), a routine calculation shows that

\[ \frac{1}{R} = \frac{1}{e} \]

Using (2), we find that

\[ \frac{1}{R} = \lim_{n \to \infty} \sqrt[n]{\frac{n!}{n^n}} = \lim_{n \to \infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e} \]

Therefore,

\[ \lim_{n \to \infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e} \]

\[ p^q > q^p \] VIA COORDINATE GEOMETRY

by Norman Schaumberger
Bronx Community College, Bronx, NY

In this note we offer a geometric proof of the familiar theorem that states that if (a) \( q > p \geq e \), or if (b) \( 0 < q < p \leq e \), then \( p^q > q^p \).

To prove case (a), we observe that the graph of \( y = \ln x \) is concave downward and that the tangent line, \( y = \frac{x}{e} \), at \((e,1)\) lies above the curve. Hence \( x/e > \ln x \) for all positive \( x \), with equality only in case \( x = e \).

Putting \( x = eq/p \) in this inequality gives \( q/p > \ln(eq/p) \), or

\[ (1) \quad q - p > \ln(q/p)^e \]

with strict inequality because \( q/p \neq 1 \).

Since \( p/e > 1 \) and \( q > p > 0 \), it follows that \( (q - p)\ln(p/e) > 0 \), or

\[ (2) \quad \ln(p/e)^{q-p} > q - p. \]

Combining (1) and (2) gives the desired result.

A special case of (a) is \( e^x > x^e \).

In case (b), if \( p > q > 0 \) and \( e/p > 1 \). Hence \( p - q \ln(e/p) > 0 \) which gives the inequality in (2).
Dear Professor Moore:

I enjoyed reading your Pi Mu Epsilon Journal article (Vol. 8, No. 9, pp. 572-5) on "Counting Bit Strings with a Single Occurrence of 00," and was especially interested in the role played by the Fibonacci numbers.

Perhaps I overlooked it, or perhaps you did not notice that the sum $a_n$ in (1), p. 573, can be expressed as follows for $n > 0$.

$$a_n = \frac{(n+1)F_{n-1}}{5} + \frac{(n-1)F_{n+1}}{5}$$

Proof: Let $r$ and $r'$ denote the positive and negative roots of the equation $r^2 = r + 1$. Then for $n > 0$ we have

$$r^n + r'^n = \frac{1}{5^{n-1}}(r_{n-1}F_n + F_{n+1})$$

Thus

$$a_n = \sum_{k=1}^{n-1}(r_k + r_{n-k})(r_{n-k} - r_{n-k})$$

$$5a_n = (n-1)(F_{n-1} + F_{n+1}) - 2\sum_{k=1}^{n-1}r_k(n-k)$$

$$= (n-1)(F_{n-1} + F_{n+1}) - 2r(r^{n-1} - r^n)/(r - r')$$

$$= (n+1)F_{n-1} + (n)F_{n+1}$$

Verification:

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<td>$F_n$</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>8</td>
<td>13</td>
<td>21</td>
<td>34</td>
<td>55</td>
<td>89</td>
</tr>
<tr>
<td>$(n+1)F_{n-1}$</td>
<td>0</td>
<td>3</td>
<td>4</td>
<td>10</td>
<td>18</td>
<td>35</td>
<td>64</td>
<td>117</td>
<td>210</td>
<td>374</td>
<td></td>
</tr>
<tr>
<td>$(n-1)F_{n+1}$</td>
<td>0</td>
<td>2</td>
<td>6</td>
<td>15</td>
<td>32</td>
<td>65</td>
<td>126</td>
<td>238</td>
<td>440</td>
<td>801</td>
<td></td>
</tr>
<tr>
<td>$a_n$</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>10</td>
<td>20</td>
<td>38</td>
<td>71</td>
<td>130</td>
<td>235</td>
<td></td>
</tr>
</tbody>
</table>

I thought that this might interest you.

Sincerely yours,

J. Sutherland Frame
IN PRAISE OF CALCULUS AND A CALCULUS LAMENT • POETRY

by Hubert Walczak
College of St. Thomas

In Praise of Calculus

Oh Calculus!
You free me from the false and inexact
You elevate my mind to thoughts abstract

Oh Calculus!
My fountain of delight
You please me - when my answers come out right!

A Calculus Lament

Oh Calculus!
You fill me with frustrations
Plagues of limits, weird manipulations
Derivatives, those ghosts of vanishing fractions
Defy my most determined actions
Related rates, a mystery profound
Who cares how shadows move around?
Square roots, sines and secants, sources of my sorrow
I hope they all clear up - before that test tomorrow!

PUZZLE SECTION

Edited by Joseph D. E. Konhauser
Macalester College

The PUZZLE SECTION is for the enjoyment of those readers who are addicted to working doublecrosstics or who find an occasional mathematical puzzle or word puzzle attractive. We consider mathematical puzzles to be problems whose solutions consist of answers immediately recognizable as correct by simple observation and requiring little formal proof. Material submitted and not used here will be sent to the Problem Editor if deemed suitable for the PROBLEM DEPARTMENT.

Address all proposed puzzles and puzzle solutions to Professor Joseph D. E. Konhauser, Mathematics and Computer Science Department, Macalester College, St. Paul, MN 55105. Deadlines for puzzles appearing in the Fall Issue will be the next February 15, and for the puzzles in the Spring Issue will be the next September 15.

PUZZLES FOR SOLUTION


   Restore the digits:

   $$* \star \star \star \star \star \star\star \star$$


   Devise a rule of formulation for the square array

   $$\begin{array}{ccc}
   9 & 3 & 2 \\
   4 & 8 & 1 \\
   5 & 6 & 7 \\
   \end{array}$$

3. *Proposed by the Editor.*

   Characterize the members of the set \( \{1, 3, 4, 9, 10, 12, 13, 27, 28, 30, 31, 36, 37, 39, 40, 81, 82.84, 85, 90, 91, 93, 94, 108, 109, 111, 112, 117, \ldots \} \).

4. *Proposed by the Editor.*

   A laddergram (shades of 1928). By changing just one letter at each step and not changing the order of the letters, one can go from SOUP to NUTS in just four (the minimum possible number) steps:

   $$\text{SOUP} \rightarrow \text{SOUS} \rightarrow \text{SOTS} \rightarrow \text{NOTS} \rightarrow \text{NUTS}$$

   In how few steps can you go from ONE to TWO?
5. Proposed by the Editor.

In a test of short-term memory, four participants \{ A, B, C, D \} were shown cards numbered 1 through 4. Each card bore the picture of one of these common objects - a key, a comb, a book and a pen - each appearing on just one card. After two minutes, the participants were asked to pair card numbers and objects. The results are tabulated below on the left:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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</thead>
<tbody>
<tr>
<td>A</td>
<td>comb</td>
<td>pen</td>
<td>book</td>
<td>key</td>
</tr>
<tr>
<td>B</td>
<td>comb</td>
<td>book</td>
<td>key</td>
<td>pen</td>
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<tr>
<td>C</td>
<td>key</td>
<td>book</td>
<td>comb</td>
<td>pen</td>
</tr>
<tr>
<td>D</td>
<td>pen</td>
<td>comb</td>
<td>key</td>
<td>book</td>
</tr>
</tbody>
</table>

Three of the participants got exactly two correct matches and the fourth did not get any correct. Match the numbers and the objects.

6. Contributed.

Place the integers 1 through 15 in the 3x5 array (above right) so that the five column sums are equal and so that the three row sums are equal.

7. contributed.

The "transform" of an isosceles triangle with base angles \( B \) is an isosceles triangle with vertex angle \( B \). A "chain" is a succession of transforms. For example, the transform of the 8-80-20 triangle is the 50-50-80 triangle, and the transform of the latter is the 65-65-50 triangle. If the first triangle is not equilateral, what is the length of the longest chain that can be formed in which all the angles of all the triangles have an integral number of degrees?

COMMENTS ON PUZZLES 1 - 7, FALL 1988

ALAN WAYNE submitted a program in BASIC for the solution of Puzzle # 1. He determined "the only base ten prime (less than 100) which has the same period length does not continue. " In a further remark, he says "I mentioned this problem on a card to Samuel Yates, Delray, FL (author of Prime Period Lengths, 1975; and Repunits and Repetends, 1982) and he instantly wrote back". WAYNE continued "In base thirty-two, the period length of 71 is 7 so the pattern does not continue." In a further remark, he says "I mentioned this problem on a card to Samuel Yates, Delray, FL (author of Prime Period Lengths, 1975; and Repunits and Repetends, 1982) and he instantly wrote back. ... I went directly to pages 98-99 of Beiler's book [Albert H. Beiler, Recreations in the Theory of Numbers, Second Edition, Dover Publications, Inc., New York, 1966] and looked to see which primes \( p \) had the same exponent to which 2, 4, 8, 10, and 16 belong (mod p) ... . Of course, this gives the instantaneous answer, \( p = 71 \)."

RICHARD I. HESS provided the additional computer-generated solutions 191, 311, 359, 479, 599, 719 and 839, with corresponding \( n \) values 95, 155, 179, 239, 299, 359 and 419. Other responses to Puzzle # 1 were received from BILL BOULGER, WILLIAM CHAU and EMIL SLOWINSKI.

Puzzle # 2 (see Figure 1) drew twelve responses. A composite of the solutions follows. "From the area condition, \((AB)^2 = 2(AM)^2\). From the perimeter condition, \((2AM = 2(AB - AM) + BC\). It follows that \(BC = (2/\sqrt{2})AB\). The angle opposite side \( BC \) has measure \( \text{sin}^{-1}(BC/(2AB)) = \text{sin}^{-1}(1/2 + 1) = 48.94^\circ \)." Solvers were ALI AMIR-MOEZ, JEANETTE BICKLEY, BILL BOULGER, WILLIAM CHAU, RICHARD DUNLAP, GEORGE P. EVANOVICH, MARK EVANS, RICHARD I. HESS, DONALD B. ONNEN, DON PFAFF, EMIL SLOWINSKI and ALAN WAYNE.

Puzzle # 4 drew only two responses and one of these was incorrect. RICHARD I. HESS submitted the colorings in the two arrays on the left, where a "1" denotes one color and a "2" denotes the other. The coloring on the right is from Martin Gardner's New Mathematical Diversions from Scientific American, Simon and Schuster, New York, 1966. According to Gardner, other solutions are possible with different combinations of 1's and 2's in the positions which are underlined in the array on the right.

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</tbody>
</table>

Figure 1

Figure 2

Eleven readers submitted solutions to Puzzle # 3 (see Figure 2). Only WILLIAM CHAU, RICHARD DUNLAP and RICHARD I. HESS avoided differentiation. CHAU's argument is given next. "Let \( x \) and \( z \) denote the lengths of \( AX \), \( AY \) and \( XY \), respectively, where \( A \) is the vertex opposite the side of length 3. Since the two portions separated by the line segment \( XY \) have equal area, it is clear that \( 2\text{sin}A = 3(4) \). But \( \text{sin}A = 315 \), so \( xy = 10 \). By the law of cosines, \( z^2 = x^2 + y^2 - 2xy \cos A \). By \( x = 10 \) and \( \cos A = 415 \), so \( z^2 = x^2 + y^2 - 16 - x^2 + y^2 - 20 + 4 = \text{sin}A + \text{cos}A \). The minimum of \( z^2 \), or \( z \), attains clearly at \( x = y \). Therefore, the line segment \( XY \) is shortest when \( AX = AY = 10 \)." THOMAS MITCHELL solved the same problem for \( XY \) connecting the other two pairs of sides of the original 3-4-5 triangle. The lengths of minimizing segments turned out to be 48 and 243. MITCHELL commented":"... the \( XY \) connecting the sides of lengths 4 and 5 is the shortest of the three. In each case, the minimum-length line segment \( XY \) is the base of an isosceles triangle. This is likely an intuitive clue which some would employ to solve the puzzle." To this point, here is RICHARD I. HESS' argument. "It must happen that \( AX = AY \) to minimize \( XY \). This can be seen by imagining otherwise and pivoting \( XY \) about its midpoint in a small increment. Areas won't change but length will change unless \( AX = AY \)." Comments, anyone? ALI AMIR-MOEZ generalized the problem by considering an arbitrary right triangle. Other contributors of solutions to Puzzle # 3 were JEANETTE BICKLEY, BILL BOULGER, GEORGE P. EVANOVICH, MARK EVANS, RICHARD I. HESS, DONALD B. ONNEN, DON PFAFF, EMIL SLOWINSKI and ALAN WAYNE.
HESS made the following assertions:

1. For a 3x3 square there are 20 ways.
2. For a 4x4 square there are 221 ways.
3. For a 5x5 square there are 338 ways.
4. For squares larger than 6x6 there are no ways.

He then asks "How about results for three colors or k colors or mxn rectangles?" HESS' results came "from a program written in BASIC on a PC."

Puzzle # 5 drew responses from twelve readers. The nine respondents with the correct answer 8/21 were CHARLES ASHBACHER, BILL BOULGER, WILLIAM CHAU, RICHARD DUNLAP, GEORGE P. EVANOVICH, MARK EVANS, RICHARD I. HESS, JOHN M. HOWELL and DONALD B. ONNEN. CHAD's argument was There are eight ways to locate three O's in the same row, column or diagonal. For each, we need six ways to locate the remaining five X's and the one O. These total 48 ways. But overall we have 9 choose 4, or 126 arrangements of the X's and O's. So the required probability is 48/126 = 8/21."

RICHARD I. HESS and JOHN M. HOWELL added that the probability of at least one column, row or diagonal with three X's is 7/9. HESS further stated that the probability of both is 2/7 and of neither is 8/63.

Puzzle # 6 asked for a rule of formulation for the 3x3 array with 1, 6 and 2 in the first row, 5, 4 and 9 in the second, and 8, 7 and 3 in the third. JEANETTE BICKLEY observed "The numbers in row 1 (2, 3) each have three (four, five) letters and are arranged in alphabetical order. The same observation was made by DONALD B. ONNEN (who credited MIKE SANKNER with the solution), BOB PRIELIPP and ALAN WAYNE. In a second solution, WAYNE showed that

\[ a_{ij} = -6i^2j^2 + 21i^2j + 51ij^2/2 - 31j^2 - 183ij/2 - 24j^2 - 143i + 89j - 69, \]

where \( a_{ij} \) is the jth entry in the ith row.

VALERIE ALBANO, JEANETTE BICKLEY, WILLIAM CHAU, RICHARD DUNLAP and RICHARD I. HESS all provided essentially the same dissection of the staircase-like piece in Puzzle # 7 into three pieces which can be reassembled to form a square. Below, their dissection is shown at the left and their reassembly in the middle. A different dissection found by the Editor is shown at the right. The reassembly is left as an exercise.

The staircase-like piece in Puzzle # 7 has eight steps and can be dissected into three pieces which can be reassembled to form a square. Given a staircase-like piece with nine (or some number other than eight) steps, are you able to dissect the piece into three pieces which can be reassembled to form a square?

Editor's Note: Apologies are due RICHARD I. HESS for overlooking his correct solution to Puzzle # 7 in the Spring 1988 issue. In the FALL 1988 issue the Editor said that no correct solutions had been submitted when, in fact, HESS had supplied an infinity of them. In the Editor's solution one need only replace the number 15 by any positive number less than 30, say a, and the number 150 by 180 \( \times 2a \), then one has a set of nine points such that each is one unit away from exactly four others.

Solution to Mathacrostic No. 27 (See FALL 1988 issue.)

**WORDS:**

| A | B | C | D | E | F | G | H | I | J | K | L | M | N | O | P | Q | R | S | T | U | V | W | X | Y | Z |
| endless cycles | implicate | affright | thwart | implicite | wyrm | acutum | 210 | antediluvian | notable | nucleotide | nitrate | nitrate | nitrate | nitrate | nitrate | nitrate | nitrate | nitrate | nitrate | nitrate | nitrate | nitrate | nitrate | nitrate | nitrate | nitrate | nitrate |

**AUTHOR AND TITLE:** ELI MAOR TO INFINITY AND BEYOND

**QUOTATION:** The common theme of all Islamic art is geometric regularity, spatial rhythm, periodic repetition. Islam, with its central creed of an omnipotent God to all humans must humbly defer, found in the infinite pattern a supreme artistic expression of its philosophy.

**SOLVERS:** JEANETTE BICKLEY, Webster Groves High School, MO; CHARLES R. DIMINNIE, St. Bonaventure University, NY; ROBERT FORSBERG, Lexington, MA; META HARRSEN, Neptune High School, NJ; DICK AND JOAN JORDAN, Indianapolis, IN; DR. THEODOR KAUFMAN, Brooklyn, NY; CHARLOTTE MAINES, Rochester, NY; DON PFAB, University of Nevada-Reno, NV; STEPHANIE SLOYAN, Georgian Court College, Lakewood, NJ; MICHAEL J. TAYLOR, Indianapolis Power and Light Co., Indianapolis, IN; and ALAN WAYNE, Holiday, FL.


For clue, see page 97 of the April, 1988, Smithsonian Magazine.

Mathacrostic No. 28

**Proposed by Joseph D. E. Konhauser**

The 277 letters to be entered in the numbered spaces in the grid will be identical to those in the 29 keyed Words at the matching numbers. The key numbers have been entered in the diagram to assist in constructing the solution. When completed, the initial letters of the Words will give the name(s) of the author(s) and the title of a book; the completed grid will be a quotation from that book.
Definitions

A. a counter take-away game played in China under the name "tsian-shih-tzu" (picking stones) (2 wds.)

B. a method of "zoom" and "zoom" these quadrilaterals, properly arranged, provide a non-periodic tesselation of the plane (2 wds.)

C. derived branch

D. inlogent

E. a Greek word for a construction technique using a marked straightedge

F. Mark Katz's expression for a sterile generalization pursued for its own sake (2 wds.)

G. a coupling device that permits relative rotational motion of the connected parts (2 wds.)

H. temporal reflection (2 wds.)

I. a commonly used cipher uncrackable even in principle (2 wds., 1 comp.)

J. an authorized rule or standard

K. a zone of transformation between ecological communities

L. residue

M. first stated by Zermelo in 1904, its acceptance or rejection reflects underlying philosophical conceptions about the nature of mathematics and mathematical existence (3 Ms.)

N. why the easy pages of a table of logarithms become degenerated first (2 wds.)

O. to the physicist what a connection on a fibre bundle is to a topologist (3 Ms., 1 comp.)

P. an Escher lithograph featuring three totally different words built together into a unified whole

Q. "an intellectual is someone whose mind watches ..." Albert Camus

R. axon

S. name for an encoding function that can be described exactly while keeping its inverse secret

T. an imaginary norminghouse with no rooms (2 wds.)

U. its utilization enables the Voyager spacecraft to continue its journey through the solar system (2 wds.)

V. a statistical observation that is not homogeneous with others of a sample

W. William Gospels space-fillingcurve based on an arrangement of seven congruent regular hexagons

X. a track or channel in which something rolls or slides

Y. physicist for whom the annual Hungarian mathematical competition, open to high school graduates of that year, is named

Z. wind-blown
PROBLEM DEPARTMENT

Edited by Clayton W. Dodge
University of Maine

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by a y information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

All communications should be addressed to C. W. Dodge, Math. Dept., University of Maine, Orono, ME 04469. Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly identified with name and address. Solutions to problems in this issue should be mailed by December 15, 1989.

Problems for Solution

691. Proposed by Charles W. Trigg, San Diego, California.
Find the smallest possible FACE on the largest possible CUBE of this addition alphametric:

SIX + FACE = CUBE

Solve the equation

3(30^x) - 6(15^x) - 3(6^x) + 6(3^x) + 2(5^x) - 10^x + 2^x - 2 = 0.

693. Proposed by Barry Brunson, Western Kentucky University, Bowling Green, Kentucky.
Solve the equation

3+2\log x(3x-2) + 2\times3\log x(3x-2) = 5\times6\log y(3x-2),


694. Proposed by Russell Euler, Northwest Missouri State University, Maryville, Missouri.

An ellipse has its foci at the vertices of a hyperbola and its vertices at the hyperbola's foci. Under what conditions, if any, will the ellipse and the hyperbola be orthogonal at their points of intersection.

If ABC is a triangle, prove that

\sqrt{\sin A} + \sqrt{\sin B} + \sqrt{\sin C} \geq 6 \sqrt{\frac{1}{2} \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}}.

Consider \int \int xy \sqrt{x^2 + y^2} \, dA, where the integral is taken over the region bounded by the y-axis and the semicircle \( x = \sqrt{1 - y^2} \) that lies in the first and fourth quadrants.

In rectangular coordinates we have

\int_{-1}^{1} \int_{0}^{\sqrt{1-y^2}} xy \, d\zeta + y^2 \, dx \, dy = \int_{-1}^{1} \frac{1}{3} (x^2 + y^2)^{3/2} \, dy = \frac{1}{15}.

In polar coordinates, however, we have

\int_{-\pi/2}^{\pi/2} \int_{0}^{\pi/2} r^4 \cos \theta \sin \theta \, dr \, d\theta = \frac{1}{5} \int_{-\pi/2}^{\pi/2} \cos \theta \sin \theta \, d\theta = 0.

What is the correct answer? Explain the discrepancy.

697. Proposed by Keith Goggin, Joseph Puthoff and John Riebusch, St. Xavier High School, Cincinnati, Ohio.

Circle (B) is internally tangent to circle (A) at K and to diameter VAW at center A. Circle (C) is internally tangent to circle (A) at K externally tangent to circle (B) at L, and tangent to segment AW at Y, as shown in the figure at right. Find the ratios of the areas of the three circles to one another. (This problem was adapted from an MAA test.)
698. Proposed by John M. Howell, Littlerock, California.
Find all solutions in integers
a) to the equation
\[ 2^n = n^x. \]
*b) to the simultaneous inequalities
\[ 2^n > n^x \quad \text{and} \quad 2^{n-1} < (n-1)^x. \]

699. Proposed by Peter A. Lindstrom, North Lake College, Irving, Texas.
Let \( A(k) \) be the unpaid principal on a loan of \( S A \) after the \( k \)th payment out of \( n \) equal payments has been made, so that \( A(0) = S A \) and \( A(n) = 0 \). Let \( i \) be the interest rate per payment period and \( I \) be the total interest paid over the life of the loan. Show that
\[ \frac{n-1}{k=0} A(k) = \frac{1}{I}. \]

700. Proposed by R. S. Lusher, University Of Wisconsin Center, Janesville, Wisconsin.
Let \( a, b, c, p, q, r \) be any positive numbers satisfying the equation \( \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1 \). Prove that
\[ \frac{ap}{b} + \frac{bq}{c} + \frac{cr}{a} \geq abc. \]

701. Proposed by Dmitry P. Maslo, Moscow, U.S.S.R.
Let \( a, b, c \) be nonnegative numbers such that \( L \sqrt{3} + 9B = 9 \sqrt{3} \). Prove that in any triangle \( ABC \)
\[ \frac{\tan \frac{A}{2} + \tan \frac{B}{2} + \tan \frac{C}{2}}{\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2}} \geq \left( \tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2} \right)^{2} + B \left( \tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2} \right) \]
with equality if and only if the triangle is equilateral.

702. Proposed by Dmitry P. Maslo, Moscow, U.S.S.R.
In right triangle \( ABC \) with right angle at \( C \) the altitude \( CD \) and the median \( CE \) are drawn. Determine the ratio of the sides containing the right angle if \( AB = 3a \) (from the SYMP-86 Entrance Exam).

703. Proposed by Christopher Stuart, New Mexico State University, Las Cruces, New Mexico.
If \( f(x) \neq 0 \) and differentiable, find \( \lim_{h \to 0} \left[ \left( \frac{f(x+h)}{f(x)} \right)^{1/h} \right] \).

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SOLUTIONS

An international committee translated the EINS-ZWEI-DREI problem, Problem 626 [Fall 1986, Fall 1987]. Even though the languages are all mixed up, there is a base ten solution in which \( O, D, T, S \) were the initial letters of ONE, DOS, and TRE, are in increasing arithmetic progression. Find that solution.

\[ O \quad N \quad E \quad D \quad O \quad S \quad T \quad R \quad E \quad S \quad X \]

The only possibilities for the ordered triple \( O, D, T \) are \((1,2,3), (2,3,4), \) and \((3,4,5) \). In any case, \( E \neq 0 \) and \( E \neq 5 \). Let \( (O,D,T) = (1,2,3) \) and suppose \( S = 9 \). Clearly \( X \) is even. If \( X = 4, \) then \( E = 9, \) so \( N + R = 6, \) which is not possible. If \( X = 8, \) then \( E = 1 \) or \( 6, \) another impossibility. If \( X = 0, \) then \( E = 7 \) and \( N + R = 4, \) again unobtainable. Thus we cannot have \( S = 6 \).

If \( (O,D,T,S) = (1,2,3,7), \) then \( X \) is odd. If \( X = 5, \) then either \( E = 4 \) and \( N + R = 12 \) or \( E = 9 \) and \( N + R = 16, \) neither of which is possible. If \( X = 9, \) the only other possibility, then \( E = 6 \) and \( N + R = 14, \) another hopeless equation.

If \( (O,D,T,S) = (1,2,3,8), \) then \( 2 \) must be carried from the ten's column, requiring that \( N + R \) be at least 17, which is not possible. Thus \( O = D = T \) cannot be \((1,2,3) \).

If \( (O,D,T) = (2,3,4), \) then \( S = 9 \) and \( X \) is odd. If \( X = 1, \) then \( E = 6 \) and \( N + R = 2, \) which is possible. If \( X = 5, \) then \( E = 8 \) and \( N + R = 4, \) again not solvable. Finally, if \( X = 9, \) then \( E = 4 \) or \( E = 9, \) neither case being admissible.

Hence \( (O,D,T,S) = (1,3,5,9) \) and \( X \) is odd. Hence \( X = 7 \) and \( E = 4, \) which leads to \( N + R = 2. \) Then \( (N,R) = (2,0) \) or \( (0,2), \) and the problem has the two solutions:

\[ 104 \quad 124 \]
\[ 524 \quad 504 \]
\[ 937 \quad 937 \]

Also solved by STEVE ASHER, McNeil Pharmaceutical, Spring House, PA, CHARLES ASHBACHER, Mount Mercy College, Cedar Rapids, IA, FRANK P. BATTLES, Massachusetts Maritime Academy, Buzzards Bay, ROY BENTON, Columbia Union College, Takoma Park, MD, WILLIAM BOULGER, St. Paul Academy, MN, UNDERWOOD DUDLEY, DePaul University, Greencoast, PA, MARK EVANS, Louisville, KY, VICTOR G. FESER, University of May, Bismarck, ND, RICHARD I. HESS, Rancho Solano, CA, WADE H. SHERARD, Furman University, Greenville, SC, MEDITPA TRIPATHI, SUNY at Buffalo, L. J. UPTON, Mississauga, Ontario, Canada, H. J. MICHEL WIJERS, Almond, The Netherlands, and the PROPOSER.

Five dice are rolled to form a "poker hand." Find the probabilities of the hands: no matches, one pair, two pairs, three of a kind, four of a kind, five of a kind, full house, and straight.
I. Solution by Henry S. Lieberman, West bran, Massachusetts.

We count the number of ways each "poker hand" can be accomplished, then divide by $6^5 = 7776$, the total number of permutations of the five dice, to get the desired probability.

Straight: There are 2 straight combinations and $5!$ permutations of the numbers for $2 \times 5! = 240 \times \text{ways of obtaining a straight}$.

No matches and no straight: There are $\binom{6}{5} = 6$ subsets of five elements taken from the set $\{1, 2, 3, 4, 5, 6\}$ and $5!$ permutations of the five subset elements. Since 240 of these are straights, then there are $6 \times 5! - 240 = 480$ "bust" hands.

One pair: There are $\binom{5}{1} = 5$ choices for the pair candidate and then $\binom{5}{3} = 10$ choices for the rest of the hand. Each hand has $5! / 2! = 60$ permutations, so one pair occurs in $6 \times 5 \times 10 \times 60 = 3600 \times \text{ways}$.

Two pair: There are $\binom{6}{2} = 15$ choices for the pair numbers and then $\binom{4}{1} = 4$ possibilities for the odd card. Since each hand has $5! / (2! \times 2!) = 30$ permutations, there are $15 \times 4 \times 30 = 1800 \times \text{two-pair hands}$.

Three of a kind: Similarly there are $\binom{6}{1} = 6$ and $\binom{5}{2} = 10$ choices for the triplet number and the remaining numbers and the hand has $5! / 3! = 20 \times \text{permutations}$. Yielding 1200 hands.

Four of a kind: Here the numbers are $\binom{6}{1} = 6$, $\binom{5}{1} = 5$, and $5! / 4! = 5$, producing 150 hands.

Five of a kind: There are $\binom{6}{1} = 6$ choices for this quintet.

Full house: There are $\binom{6}{1} = 6$ choices for the triple and then $\binom{5}{2} = 10$ for the pair. Each hand has $5! / (2! \times 3!) = 10 \times \text{permutations}$, so the full house occurs in 300 ways.

We summarize these results in the following table.

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<td>0.0617</td>
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<td>0.0193</td>
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<tr>
<td>Five of a kind</td>
<td>6</td>
<td>0.0008</td>
</tr>
<tr>
<td>Full house</td>
<td>300</td>
<td>0.0386</td>
</tr>
<tr>
<td>Straight</td>
<td>240</td>
<td>0.0309</td>
</tr>
</tbody>
</table>

II. Comment by James E. Campbell, Indiana University at Bloomington, Bloomington, Indiana.

A challenging problem, but I would expect #666 to be a bit more, um, ...

Also solved by CHARLES ASHBACHER, Mount Mercy College, Cedar Rapids, IA, JAMES E. CAMPBELL, Indiana University at Bloomington, UNDERWOOD DUDLEY, DePauw University, Greencastle, IN, RICHARD DUNLAP, Georgia Tech, Atlanta, MARK EVANS, Louisville, KY, RICHARD I. HESS, Rancho Palos Verdes, CA, JEREMY PARTIN, Hendrix College, Conway, AR, and the PROPOSER.


Each special die has one face with 1 spot, two faces with 2 spots each, and three faces with 3 spots each. Find the probability of tossing a sum of 8 with four special dice.

Composite of solutions submitted by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada, and Michael V. Williams, Walsh Forest University, Winston-Salem, North Carolina.

We count the number of ways each "poker hand" can be accomplished, then divide by $6^5 = 7776$, the total number of permutations of the five dice, to get the desired probability.

Straight: There are 2 straight combinations and $5!$ permutations of the numbers for $2 \times 5! = 240 \times \text{ways of obtaining a straight}$.

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Also solved by CHARLES ASHBACHER, Mount Mercy College, Cedar Rapids, IA, STEVE ASHER, McNeil Pharmaceutical, Spring House, PA, WILLIAM BOULGER, St. Paul Academy, MN, JAMES E. CAMPBELL, Indiana University at Bloomington, UNDERWOOD DUDLEY, DePauw University, Greencastle, IN, RICHARD DUNLAP, Georgia Tech, Atlanta, DAVID EHREN, University of Wisconsin, Milwaukee, MARK EVANS, Louisville, KY, ROBERT C. GEBHARDT, Hopatcong, NJ, RICHARD I. HESS, Rancho Palos Verdes, CA, PATRICK P. T. LEONG, John Carroll University, Cleveland, OH, RICHARDS. LIEBERMAN, West bran, MA, RICHARD I. HESS, University of Wisconsin, Milwaukee, MARK EVANS, Louisville, KY, ROBERT C. GEBHARDT, Hopatcong, NJ, RICHARD I. HESS, Rancho Palos Verdes, CA, PATRICK P. T. LEONG, John Carroll University, Cleveland, OH, HENRY SCHAEFER, St. Bonaventure University, NY, and the PROPOSER.

The integral is

$\int_{0}^{\pi/2} \frac{\cos^3 x}{2 - \sin 2x} \, dx$

1. Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.

By making the substitution $x = \pi/2 - t$, we have that

$J = \int_{0}^{\pi/2} \frac{\cos^3 x}{2 - \sin 2x} \, dx = \int_{0}^{\pi/2} \frac{\cos^3 x \, dx}{1 - \sin x \cos x} = \int_{0}^{\pi/2} \frac{\sin^3 t \, dt}{1 - \sin t \cos t}$

Thus we have

$2J = \int_{0}^{\pi/2} \frac{\cos^3 x}{2 - \sin 2x} \, dx = \int_{0}^{\pi/2} \frac{\cos^3 x \, dx}{1 - \sin x \cos x} = \frac{\pi}{2}$

and $J = 1.$
More generally, if \( G(a - x) = G(x) \), then the substitution \( t = a - x \) verifies that

\[
\frac{a}{G(x)} = \int_0^a \frac{F(a - x)}{G(x)} \, dx
\]

so that

\[
J = \frac{1}{2} \int_0^a F(x) + F(a - x) \, dx.
\]

For example, since

\[
\sin^5 x + \cos^5 x = \frac{(\sin x + \cos x)\left(1 - \frac{\sin^2 x}{2}\right)}{4},
\]

we have

\[
\int_0^{\pi/2} \frac{4 \cos^5 x}{4 - 2 \sin 2x - \sin^2 2x} \, dx = 1 \text{ also.}
\]

II. Solution by Amitabha Tripathi, SUNY at Buffalo, Buffalo, New York.

We observe that

\[
\frac{2 \cos^3 x}{2 \sin 2x} - \frac{\cos^3 x}{1 - \sin^2 x \cos x}
\]

\[
\frac{\cos^3 x (1 + \sin x \cos x)}{1 - \sin^2 x \cos^2 x} = \frac{\cos x (1 - \frac{\sin^2 x}{2})}{1 - \sin^2 x \cos x}
\]

so that we have

\[
\frac{\pi}{2} \int_0^{\pi/2} \frac{2 \cos^3 x}{2 - \sin 2x} \, dx
\]

\[
\frac{\pi}{2} \int_0^{\pi/2} \cos x (1 - \frac{\sin^2 x}{2}) \, dx - \frac{\pi}{2} \int_0^{\pi/2} \frac{\cos^4 x \sin x}{1 - \sin^2 x \cos^2 x} \, dx
\]

\[
\frac{\pi}{2} \int_0^{\pi/2} \frac{1 - u^2}{1 - u^2 (1 - u)} \, du + \frac{\pi}{2} \int_0^{\pi/2} \frac{u^4}{1 - u^2 (1 - u^2)} \, du
\]

\[
= \int_0^{\pi/2} \frac{1 - u^2 + u^4}{1 - u^2 + u^4} \, du = 1
\]

III. Solution by Richard I. Hess, Rancho Palos Verdes, California.

We have that

\[
\frac{2 \cos^3 x}{2 \sin 2x} - \frac{\cos^3 x}{1 - \sin x \cos x}
\]

\[
= \frac{\cos x + \cos^3 x - \sin x (1 - \sin x \cos x)}{1 - \sin x \cos x}
\]

\[
= \frac{\cos x + \cos^3 x - \sin x \cos x}{1 - \sin x \cos x}
\]

By letting \( u = \pi/2 - x \) in the following integral \( J \), we find that

\[
\pi/2 \int_0^{\pi/2} \cos x \sin x \, dx = \int_0^{\pi/2} \sin x \cos u \, du - J
\]

so \( J = 0 \). Now we have that

\[
\frac{\pi}{2} \int_0^{\pi/2} \frac{2 \cos^3 x}{2 - \sin 2x} \, dx = \int_0^{\pi/2} \sin x \, dx + J = 1 + 0 = 1.
\]

Also solved by FRANK P. BATTLES, Massachusetts Maritime Academy, Buzzards Bay, ROY BENSON, Columbia Union College, Takoma Park, MD, DAVID G. CARABALLO, Hillside, NJ, CHARLES R. DIMINNIE, St. Bonaventure University, NY, UNDERWOOD DUDLEY, DePauw University, Greencastle, IN, RICHARD DUNLAP, Georgia Tech, Atlanta, ROBERT I. EGBERT, The Wichita State University, KS, EDWIN M. KLEIN, University of Wisconsin, Whitewater, HENRY S. LIEBERMAN, Youngstown State University, OH, BOB PRIELIPP, University of Wisconsin-Oshkosh, G. MAVRIGIAN, Youngstown State University, OH, WADE H. SHERARD, Truman University, Greeniville, SC, SAHIP SINGH, Clarion University of Pennsylvania, and the PROPOSER. One incorrect solution was received.


Let \( n \) be any positive integer. Show that \( \prod_{k=1}^{n} (k!+1) \) has a factor greater than \( \prod_{k=1}^{n} k \).

I. Solution by the Proposer.

For each positive integer \( k \) there is a prime \( p_k \) such that \( p_k \mid (k!+1) \) and \( p_k \geq k \).

Hence

\[
\prod_{k=1}^{n} (k!+1) \]
II. Solution by Underwood Dudley, DePauw University, Greencastle, Indiana.

One answer would be to say that the given product has itself as a factor; another answer is that the last factor (2n) + 1 of the given product also satisfies the stated conditions. Both answers are unsatisfying and lead one to suspect that the problem is not stated correctly. However, the first emendation that springs to mind is to say that the given product has itself as a factor; another answer is that the last factor (2n) + 1 is prime. Clearly, the statement of the problem is inadequate and the editor should have caught the difficulty before it got into print. One rewording would be "has a factor that is a product of not more than n + 1 primes and is greater than 1.

670. [Spring 1988] Proposed by T. A. Lindstrom, North Lake College, Lakeville, Texas. al

If $F_n = 2^{2^n} + 1$ is the nth Fermat number, find all values of n so that $F_n$ and $F_{n+1}$ are twin primes.

I. Solution by Mark Evans, Louisville, Kentucky.

Clearly one integer in any three consecutive integers is divisible by 3. Since $F_n$ is obviously not, then either $F_n$ or $F_{n+1} - 2$ must be divisible by 3. Hence only $F_1 = 2$, $F_2 = 3$ and $F_4 = 5$ are twin primes.

II. Solution by Sahib Singh, Clarion University, Clarion, Pennsylvania.

The only twin primes of the family are 3 and 5 for n = 1. The number $F_n$ is always prime when $n > 1$ because $F_n$ divides $F_{n+1}$ whenever $n > m > 0$. See Burton, Elementary Number Theory, page 237.

Also solved by William Boulger, St. Paul Academy, MN; David G. Caraballo, Hillsdale, NJ; Richard Dunlap, Georgia Tech, Atlanta; David Euler, Northeast Missouri State University, Maryville; Mark Evans, Louisville, KY; Victor G. Feser, University of Maryland, Bismarck, ND; Richard I. Hess, Rancho Palos Verdes, CA; Peter A. Lindstrom, North Lake College, Irving, TX; Henry S. Lieberman, Waban, MA; Don Paff, University of Nevada, Reno, Sahib Singh, Clarion University of Pennsylvania, Tom, Kim, Cammie and Christine, Massachusetts Gamma, Bridgewater State College, and Amitabha Tripathi, SUNY at Buffalo, Michael D. Williams, Wake Forest University, Winston-Salem, NC, and the PROPOSER.


Find all sequences of 2k + 1 consecutive integers $a, a+1,..., a+2k$ such that the sum of the squares of the first $k + 1$ of these integers is equal to the sum of the squares of the last k. That is, find a formula for $a_k$ as a function of k. For example, $a_1 = 3$ since $3^2 + 4^2 = 5^2$.

Solution by Mark Evans, Louisville, Kentucky.

Letting all the summations run from 1 to k, we wish to have

$$a^2 + \sum_{i=1}^{k} (a+i)^2 = \sum_{i=1}^{k} (a+i) = 2a^2 + 2\sum_{i=1}^{k} i + 2\sum_{i=1}^{k} i^2$$

so that

$$a^2 = 2\sum_{i=1}^{k} i + 2\sum_{i=1}^{k} i^2 = 2ak + k(k+1) + k^3 = 0$$

$$\Rightarrow a^2 - 2a^2 - 2k^2 = (a+k)(a - 2k^2 - k),$$

and $a = 2k^2 + k$ (the 2kth triangular number) or $a = -k$.

Also solved by Steve Asher, McKiel Pharmaceutical, Spring House, PA; Frank P. Battles, Massachusetts Maritime Academy, Buzzards Bay; William Boulger, St. Paul Academy, MN; James E. Campbell, Indiana University at Bloomington, Charles R. Diminnie, St. Bonaventure University, NY; Underwood Dudley, DePauw University, Greencastle, IN; David Ehren, University of Wisconsin, Milwaukee; Russell Euler, Northeast Missouri State University, Maryville; Victor G. Feser, University of Maryland, Bismarck, ND; Richard I. Hess, Rancho Palos Verdes, CA; Frank Hubeny, University of Maine, Orono, Henry S. Lieberman, Waban, MA; Don Paff, University of Nevada, Reno, Bob Prielipp, University of Wisconsin-Oshkosh, Sahib Singh, Clarion University of Pennsylvania, Amitabha Tripathi, SUNY at Buffalo, and the PROPOSER.

672. [Spring 1988] Proposed by Barry Brunson, Western Kentucky University, Bowling Green, Kentucky.

Find a series representation for

$$\int_0^1 x^n \, dx$$

Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.

More generally,

Let AB be an edge of a regular tesseract (a four-dimensional cube) and let C be the tesseract's vertex that is furthest from A. Find the measure of angle ACB.

Solution by Richard I. Mess, Rancho Palos Verdes, California.

We take the n-dimensional simplex whose vertices are the points \((x, x, x, \ldots, x)\) where each \(x\) is either 0 or 1. We take \(A = (0, 0, 0, \ldots, 0)\), a nearest neighbor \(B = (1, 0, 0, \ldots, 0)\), and the farthest point \(C = (1, 1, 1, \ldots, 1)\). Then we have

\[
\cos ACB = \frac{(C - B) \cdot (C - A)}{|C - B| |C - A|} = \frac{(0, 1, 1, \ldots, 1) \cdot (1, 1, 1, \ldots, 1)}{\sqrt{n - 1} \sqrt{n}} = \frac{n - 1}{n}.
\]

For \(n = 4\) we have \(\cos ACB = \sqrt{3}/2\), so \(ACB \approx 60^\circ\). Angles for other values of \(n\) are now readily computed. For example, \(ACB = 45^\circ\) for the square \((n = 2)\) and \(ACB = 35.264^\circ\) for the cube \((n = 3)\).

Also solved by James E. Campbell, Indiana University at Bloomington.

674. [Spring 1988] Proposed by Russell Euler, Northwest Missouri State University, Maryville, Missouri.

Find necessary and sufficient conditions for the arithmetic mean of the roots of the polynomial equation

\[a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n = 0\]

to be equal to the geometric mean of the roots.

Solution by Alphonso, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin.

It is known that the sum of the roots of the given polynomial is equal to \(-a_1/a_0\) and their product is \((-1)^n a_n/a_0^n\). Hence the arithmetic mean of the roots equals their geometric mean if and only if

\[(-a_1/a_0)^n = ((-1)^n a_n/a_0)^{1/n},\]

which can be rewritten as

\[a_1^n/n = a_n a_0^{n-1} a_n^{1/n}.
\]


Although this is a relatively simple problem, one must be careful to examine signs when taking roots. Thus, for example, \((-1)^n/n = (-1)^n\) and not \(-1\). Now, if all the roots are positive, then the stated condition implies that the roots must all be equal. If some of the roots are negative, of course their geometric mean must be a real number. Thus, if \(n\) is even, an even number of roots must be negative or a root must be zero. A nontrivial example is the quartic polynomial whose zeros are \(3 \cdot 3 \sqrt{10}, 3 = \sqrt{10}, 3 + \sqrt{10},\) and \(3 + 3 \sqrt{10}\).


Erect a semicircle on segment AB as diameter. From point D on the semicircle drop a perpendicular to point C on AB. Draw a circle tangent to CB at J and tangent to the semicircle and to segment CD. Prove that angles CDJ and JDB have equal measures. See the figure below.

Solution by Henry S. Lieberman, Waian, Massachusetts.

On pages 84 and 85 of Problems and Solutions in Euclidean Geometry by Aref and Wernick (Dover, 1968), it is proven that \(\angle AJD = \angle CJD\). But \(\angle ZADJ = \angle ZDC + \angle ZJD\) and \(\angle ZJD = \angle ZCBS + \angle ZJDB\). Since ZADC and ZCBZ are both the complement of \(\angle BAD\), it follows that they are equal. Therefore \(\angle ZCDJ = \angle ZJDB\).

Also solved by Jack Garfunkel, flushing, NY, Richard I. Hess, Rancho Palos Verdes, CA, King W. Jamison, Middle Tennessee State University, Murfreesboro, TN, Mavrigian, Jounstown State University, Off, Wade H. Sherard, Furman University, Greenville, SC, and the proposer.

Sherard, whose solution closely parallels the featured solution, located the theorem that \(\angle AJD = \angle AD\) as problem 494 in this Journal [Spring 1982], proposed by Zelda Katz and solved by Henry S. Lieberman. Jack Garfunkel was one of the also-solvers. Also, this is not the first time that the editor of this Department has missed (related) problems that have appeared here previously.


Show that, for \(k > 0\),

\[
\sum_{i=1}^{n} \left( \frac{k}{\Pi(i + 1)} \right)^{-1} = \frac{1}{k, k^{1/2}} \frac{1}{k} \left( \Pi(n + 1) \right)^{-1}.
\]

The left side of the given equation can be written as

\[ S = \sum_{r=0}^{n-1} \frac{n!}{(k + r + 1)!} = \frac{1}{k} \sum_{r=0}^{n-1} \frac{n!}{(k + r + 1)!}, \]

which is a collapsing series that reduces to

\[ S = \frac{1}{k} \left( \frac{n!}{(k + n)!} \right), \]

a form equal to the right side of the given equation.

Also solved by UNDERWOOD DUDLEY, DePauw University, Greencastle, Ind.; RICHARD DUNLAP, Georgia Tech, Atlanta; MARK EVANS, Louisville, Ky.; RICHARD HESS, Rancho Palos Verdes, CA; MURRAY S. KLAMKIN, University of Alberta, Canada; HENRY S. LIEBERMAN, Winona, MN, and the PROPOSER. Klamkin pointed out that this result is well known.


If A, B, and C are the angles of a triangle, then show that

\[ \cos A \cos B \cos C = \frac{1}{2} \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} \leq \frac{\sqrt{3}}{9}. \]

Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.

The inequality is obviously true if any angle is greater than or equal to 90°, so we assume that the triangle is acute. Multiplying the numerator and denominator of the left side by 8 sin (A/2) sin (B/2) sin (C/2), we get

\[ 8 \sin (A/2) \sin (B/2) \sin (C/2) \cos A \cos B \cos C \leq \frac{\sqrt{3}}{9}. \]

This result follows from items 2.12 and 2.40 in Geometric Inequalities, Noordhoff, Groningen, 1968:

\[ \sin A \sin B \sin C \leq \frac{1}{4R} \quad \text{and} \quad \cot A \cot B \cot C \leq \frac{\sqrt{3}}{9}, \]

the latter inequality valid only for acute triangles.

Also solved by MARK EVANS, Louisville, KY; J. S. FRAME, Michigan State University, Lansing, Richard H., HESS, Rancho Palos Verdes, CA; EDWIN M. KLEIN, University of Wisconsin, Whitewater, YOSHINOBUMURAYOSHI, Portland, OR; BOB PRIELIPP, University of Wisconsin-Oshkosh, AMITABHA TRIPATHI, SUNY at Buffalo, and the PROPOSER. Klein, Murayoshi, and Tripathi each pointed out that equality holds if and only if the triangle is equilateral.

Edwin M. Klein, University of Wisconsin, Whitewater, and Bob Prielipp, University of Wisconsin-Oshkosh, each pointed out that the stated problem was problem A-4 in the 1969 William Lowell Putnam Mathematical Competition (see the August-September 1970 issue of The American Mathematical Monthly, pages 723 and 725). Prielipp also noted that on page 676 of the June-July 1971 issue of the Monthly C. D. Olds gives a detailed solution to this problem, including its numerical value 0.78343051...
INDEX TO VOLUME 8. Fall 1984-Spring 1989

No. 1 Fall 1984 1-68 No. 6 Spring 1987 357-424
No. 2 Spring 1985 69-140 No. 7 Fall 1987 425-492
No. 3 Fall 1985 141-208 No. 8 Spring 1988 493-556
No. 4 Spring 1986 209-280 No. 9 Fall 1988 557-632
No. 5 Fall 1986 281-356 No. 10 Spring 1989 633-704

AUTHOR INDEX

AMIR-MOIZ, ALI R. The Focal Distance of a Conic Section .......... 246
AMIR-MOIZ, ALI R. On Reduction of Conic Sections ............... 576
ANDERSON, OLIVER D. Multiplier Problem Revisited .......... 590
ARTOLA, PAUL and RUTH BRISBIN. Taxicab Trigonometry ...... 89
AZARIAN, MOHAMMAD K. The Improper Integral \( \int \sin(x) \) as an Alternating Series ........... 521
BAJAJ, PREM N. An Algorithm for Partial Fractions ............... 317
BANKOFF, LEN. Arctan1 + Arctan2 + Arctan3 = \( \pi \) .......... 31
BENANDER, ALAN C. A Graphical Approach to \( e^x > x^n \) ... 325
BENANDER, BARBARA A. A Theorem of Phillip Hall ............. 225
BERTRAM, RICHARD. Maximum Trigonometry ..................... 503
BOAS, R. P. The Sign of \( e^x - x^n \) .......................... 522
BOYD, JAMES N. A Locus of Points from Convex Coordinates ... 508
BRENNER, J. L. Asymptotic Solution of the Problems of Erdos and Sierpiński Concerning N/N .................................. 24
BRENNER, J. L. Limits of Means for Large Values of the Variables . 160
BRENNER, J. L. Marcel Riesz - An Anecdote ....................... 252
BRENNER, J. L. Paul Turán - Some Anecdotes ..................... 390
BRISBIN, RUTH and PAUL ARTOLA. Taxicab Trigonometry .... 89
BRUSCARDI, BARRY J. Lies, Spies, AIDS, and Drugs ............ 559
BURGSTAHLER, SYLVAN. A New Proof of a Familiar Result .... 164
CASHING, DOUGLAS L. and CHARLES R. DIMINNIE. Convergence of Weighted Averages ............................. 377
CHAMBERS, KELLY ANN. The Isomorphism of the Lattice of Congruence Relations on a Group and the Lattice of Normal Subgroups of a Group ................................................... 450
CHAN, WAI KEEUN. On the Largest RAT-free Subset of a Finite Set of Points ............................................. 357

n
COFFIN, MARIE. Non-Negative Integer Solutions of \( \sum_{i=1}^n x_i = k \) ........................................... 108
COUNCILMAN, SAMUEL. X-Matrices: Generalized Complex Numbers ... 669
DEVER, MARY BETH. A Mathematical Model of Voter Participation ... 90
DICK, THOMAS. Of Exotic Integers and Quaternions -- An Introduction to Representation Theory .......................... 209
DIMINNIE, CHARLES R. and DOUGLAS L. CASHING. Convergence of Weighted Averages ............................. 377
DINION, ROBERT. The Least Member Method. An Alternative to Induction ................................................... 170
DODGE, CLAYTON W. and JACK GARFUNKEL. The Squarilic Quadrilateral . 8
DODGE, CLAYTON W. and JACk GARFUNKEL. The Quadillar Quadrilateral ................................................... 151
DUGOPOLSKI, MARK J., Moses is Your Grandfather and Other Applications of the Geometric Series .................. 141
ECKOFF, JANET. Random Cantor Sets .............................. 311
Mathematics as Guerrilla Warfare: The Case of René Descartes. PAUL TRAINOR ........................................ 69
Maximal Polygons for Equi-transitive Periodic Tilings. JAMES E.
GEORGES and ANNETTE M. MATTHEWS ..................... 557
Maximum Trigonometry. RICHARD BERTRAM .................. 503
Moses is Your Grandfather and Other Applications of the Geometric
Series. MARK J. DUGOPOLSKI ............................ 141
A Multiplier Problem. WILLIAM M. PEREL .................. 518
Multiplier Problem Revisited. OLIVER D. ANDERSON .... 590
A New Proof of a Familiar Result. SYLVAN BURGSTHALER .... 164
Non-Negative Integer Solutions of \( \sum_{i=1}^{n} x_i - k \). MARIE COFFIN .............. 108
Note on a Well-Known Limit. W. VANCE UNDERHILL .......... 672
A Note on Evaluating Definite Integrals by Substitution. PETER A.
LINDSTROM ............................................. 516
Of Exotic Integers and Quaternions — An Introduction to
Representation Theory. THOMAS DICK ........................ 209
On Buffon's Needle Problem Using Concentric Circles. H. J. KHANIS . 368
On Reduction of Conic Sections. ALI R. AMIR-MOEZ ........ 576
On the Largest RAT-Free Subset of a Finite Set of Points. WAH
MEUNG CHAN ............................................ 357
Paul Turán - Some Anecdotes. J. L. BRENNER .............. 390
Pi Mu Epsilon Celebrates 75 Years of Mathematical Activity.
J. SUTHERLAND FRAME ................................... 636
p^q > q^p ............................................. 673
Proving Without Words. NORMAN SCHAUERBERGER ........ 674
The Quasi-Isosceles Quadrilateral. JACK GARFUNKEL and CLAYTON W.
DODGE ................................................. 151
Random Cantor Sets. JANET ECKOFF ...................... 311
The Role of Russell's Paradox in the Development of Twentieth
Century Mathematics. KAREN P. MIDDLETON .............. 234
The Russell Paradox. ERNST SNAPPER ..................... 281
The Specter of Non-Measurable Sets. STANLEY RABINDRITZ .... 441
Simplified Proofs for Some Matrix Theorems. JOHN R. SCHUE .... 169
The Sign of \( e^x - x^e \). R. P. BOAS ........................... 522
Some Families of Convergent Series With Sums. H. M. SRIVASTAVA .... 292
The Square-In Quadrilateral. CLAYTON W. DODGE and JACK GARFUNKEL . 8
A Summation Formula. JOHN SCHUE ........................ 391
Taxicab Trigonometry. RUTH BRISBIN and PAUL AULTA ........ 89
A Theorem of Philip Hall. BARBARA A. BENEDNER .......... 225
A Theorem on Homothetic Simplexes. M. S. KLAMIN ........ 29
Three New Convergent Series Via the Mean Value Theorem. NORMAN
SCHAUERBERGER .......................................... 250
Tree. Branch and Root. NORMAN WOO ....................... 380
A Ubiquitous Partition of Subsets of \( \mathbb{R}^n \). DONALD JOHN NICHOLSON . 2 663
Using Areas to Obtain the AM-GM Inequality. NORMAN SCHAUERBERGER .... 461
A Variation of the Standard Proof of the Infinitude of Primes.
NORMAN SCHAUERBERGER ................................ 159
X-Matrices: Generalized Complex Numbers. SAMUEL COUNCILMAN .... 669
You Can't Hurry Love or How Much More Must Art Wait? J. S.
VERDUCCI ............................................. 383

LETTERS TO THE EDITOR

John M. Howell ............................................. 120
R. S. Luthar ............................................. 280
Edmund F. Marks, Jr. .................................... 418
J. Suck .................................................. 419
R. P. Boas ............................................. 321
John M. Howell ............................................ 592
Henry J. Osman .......................................... 593
I. J. Good .............................................. 593

PUZZLE SECTION

Edited by Joseph D. E. Konhauser ....................... 33, 114, 172, 258, 327
 ........................................... 394, 462, 525, 594, 677

PROBLEM DEPARTMENT

Edited by Clayton W. Dodge .......................... 41, 121, 179, 265, 332
 ........................................... 401, 468, 532, 600, 684

NATIONAL PI MU EPSILON MEETINGS

1984 - Eugene ........................................... 64
1985 - Laramie ........................................... 199
1986 - Berkeley .......................................... 353
1987 - Salt Lake City .................................... 489
1988 - Providence ......................................... 623

GLEANINGS FROM CHAPTER REPORTS

Edited by Joseph D. E. Konhauser ............... 66, 203, 356, 420, 492, 552, 626

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Secretary-Treasurer, Robert M. Woodside. East Carolina University.
In Appreciation

The Editor wishes to thank the following for contributing their time and skill in evaluating papers for publication in the Pi Mu Epsilon Journal during the past five years - Volume 8, Nos. 1 - 10. Their comments and their criticisms have greatly assisted the Editor in the publication process.

Richard V. Andree, The University of Oklahoma; David Ballew, Western Illinois University; Dr. Leon Bankoff, Los Angeles, California; Gerald E. Bergum, South Dakota State University; Ralph P. Boas, Northwestern University; Barry W. Brunson, Western Kentucky University; Jane W. DiPaola, Cheyenne, Wyoming; Robert C. Eslinger, Hendrix College; Joseph Gallian, University of Minnesota - Duluth; Allan M. Kirch, Macalester College; R. S. Luther, University of Wisconsin Center; Suzanne Molnar, College of St. Catherine; Ivan Niven, University of Oregon; Richard Poss, St. Norbert College; A. Wayne Roberts, Macalester College; John R. Schue, Macalester College; Dale Varberg, Hamline University and Joseph S. Verducci, Ohio State University.

The Editor also wishes to thank those persons who have anonymously aided in the selection of the prize-winning student-written papers which have appeared in the Journal during this period.

Special thanks are extended to the editor of the Problem Department, Clayton W. Dodge. For each issue he supplied carefully edited copy, well before the deadline, in a form easily converted to camera-ready copy for the printer.

Finally, thanks to Kathy Grundhoefer who typed many of the 704 pages in Volume 8, to Carl Gadow who assisted the Editor in all phases of the preparation of copy and who maintained the subscription lists for four years and to current student assistants Andrea Heiberg and Timothy Kremer.

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