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MATHEMATICS AS GUERRILLA WARFARE: THE CASE OF RENE DESCARTES

by Paul Trainor
Providence College

This talk was given in May, 1984, to the Rhode Island Gamma Chapter at Providence College, where Dr. Trainor is a member of the Philosophy Department.

I. More people have read Descartes' Discourse on Method in the last fifty years than in the previous 297 years since its publication in 1637. Yet the economics of the technology which makes Descartes' Discourse so readily available has also distorted our view of exactly what Descartes was doing when he published the work in 1637. Because economics dictates--and academic needs conform--the Discourse is usually published without the other three works to which it was prefaced. Descartes' Discourse on Method was an autobiographical preface to three other works, his Optics, his Geometry and his Meteorology. The mass printing of books has been a great blessing, but it has not been an entirely unmixed blessing because it can lead to a systematic misreading of works. Reading Descartes' Discourse or Preface to his scientific and mathematical treatises without reading the Optics, Geometry and Meteorology themselves--or at least sampling them--is like judging a restaurant by reading its menu but not dining there. As one who has enjoyed a full course Cartesian meal, I would like to tell you what the food is like.

The title of my talk tonight might suggest that I have immature taste buds; it might suggest that Descartes served military K-rations and I liked them. That suggestion may be unavoidable, so I will try to help you digest the entree of thought that I bring from the Cartesian Kitchen by putting on the sauce of a different metaphor. The sauce is not to my taste, but a military metaphor may make Descartes tastier for you.

The times in which Descartes lived (1596-1650) were times of violent religious wars in his native France and in the rest of Europe. Descartes himself witnessed the Thirty Years War as a young military engineer for one of the armies. Indeed, Descartes tells us in his Discourse that it was while returning to the army from the coronation of the Emperor, that, caught by the onslaught of winter, he holed up in a heated room and had the opportunity to consider and develop the philosophical reflections that constitute passages of the Discourse. France itself, as many of you veterans of the Development of Western Civilization may recall, was split religiously between Roman Catholics and Protestant Huguenots, and the religious split was of concern to the monarchy because it reflected and intensified political divisions. The Kings of France became and remained Catholic, in part, because religious unity was seen as essential to political unity. Descartes, in short, lived in a time of civil and religious warfare.

The situation of war in Europe made Descartes' situation as an intellectual very difficult because in times of political and religious strife, orthodoxy is not a social convenience but a necessity. In France, as in most of Europe in the seventeenth century, orthodoxy extended to philosophical and scientific matters as well, and Descartes was acutely sensitive to this fact. In the Fifth of his Six Part Discourse, Descartes tells his readers that he had written a scientific treatise which explained the nature of light, celestial mechanics, myriad phenomena on earth such as earthquakes and the ebb and flow of the tides, as well as the nature of plants, animals and man. That is, Descartes announces that he has developed a new, comprehensive, and powerful scientific theory. But, he tells his reader, he has decided not to publish it, or at least not until after his death.

> It has been three years now, since I reached the end of the treatise containing all these things and began to review it in order to put it in the hands of a publisher, when I learned that people to whom I defer and whose authority over my actions can hardly be less than is that of my own reason over my thoughts, had disapproved of a theory of physics that a certain other person (Galileo) published a little while before. I do not want to say that I agreed with this theory, but before their censure I had noticed nothing in it which I could imagine to be prejudicial to either religion or the state, nor, as a result, anything which should prevent me from writing about it, if reason had persuaded me to do so; and this caused me to fear that in the same way there might be found among my theories some in which I was mistaken, despite the great care I had always taken not to accept into my beliefs any new opinions of which I did not have very certain demonstrations, and not to write of any which turn out to anyone's disadvantage. This was sufficient to oblige me to change my resolution to publish the treatise. [1]

Many interpreters of Descartes--few of whom have read the Optics, Geometry or Meteorology--suggest that this passage and other similar ones show that Descartes was just too cautious a man to stand up for scientific truth against the forces of darkness, that Descartes was too fearful of the Inquisition. In other words, many interpreters think Descartes timid, if not cowardly. We all prefer heroes who fight for their convictions to the frail thinker who retires from the fray. I would like to suggest that this is a serious misreading of Descartes, that in fact Descartes was a cunning guerrilla fighter, and that his Optics, Geometry, and Meteorology were brilliant strategic attacks against an enemy whose conventional forces and firepower were far superior to any he could muster. Descartes himself clues us in to his strategy.

In the Sixth Part of the Discourse, Descartes tells his readers that although he will not publish his works, he will continue to write them and will arrange to have them published posthumously. Nonetheless he will fight for the truth but he will be careful about what battles he gets into because a defeat would be very costly. Better to fight smaller, carefully selected battles than engage directly with larger forces. In his own words, Descartes says that

> trying to conquer all the difficulties and errors that keep us from attaining knowledge of the truth is truly to give battle; and a battle is truly lost when we accept some false opinion concerning a general and important matter. It takes much more skill after such a loss to regain the same state that we had before than to make great progress when we already have principles that are well-founded. As for me, if I have thus far discovered some truths in the sciences. . ., I can say that they are the results and consequences of but five or six principal problems that I have overcome, and I count these as so many battles where I have had luck on my side. I even do not fear to say that I think I need only win two or

three other such battles, in order completely to achieve my goals...[2]

Though he believed he needed only two or three other such battles to achieve his goals, Descartes decided not to publish his treatise because of "the opposition that they would **awaken.**"[3] That is, he refused to engage the opposition directly because he sensed that the conventional army of the scholastics, the troops of the Catholic King, was too powerful. The Galileo case was a sharp reminder of their power. So Descartes took off his military uniform and presented himself as a **gentleman** of leisure.

II. Guerrilla warfare is by definition a war carried on without regard to the Hague conventions of war. By definition, a guerrilla is a fighter who does not wear a uniform and who does not belong to an organization. Without a uniform, a fighter has no rank or military identity; he is outside the order of war. Without an organization a fighter does not have a clearly defined and accepted role to play; nor can he be held accountable for acting in accordance with the rules of war. That is, when a recruit puts on a uniform and is assigned to a unit, the method and objectives of warfare, at least in theory, are defined. **If** captured in uniform, a solider cannot simply be shot; he must be treated as a prisoner of war. A guerrilla need not. In warfare, the guerrilla is not bound by the rules or conventions of war; by the same token, the conventional forces are not obliged to treat a guerrilla by the conventions of war. The guerrilla soldier is neither soldier nor civilian. He is the outsider, and, **if** captured, subject to the caprice of the conventional forces.

Of necessity, the guerrilla fighter is independent, hides his intentions, does not play by the rules, and uses political tactics as much as military to realize his goals. The guerrilla fighter wants to change the status quo which the conventional forces wish to maintain. The guerrilla fighter wants to keep the battle lines fluid and. blurred so he cannot be easily hit by conventional fire. And the guerrilla fighter believes in and lives for the future triumph of his cause even though he knows that at the present time the odds are very much against him. He may not see his cause triumph, but he is confident his children will and so does his bit.

Descartes was a guerrilla fighter fighting the powerful **political-religious** alliance of priests-theologians-philosophers, but he was not a radical. That is, his fight was not with the Catholic Church as such. Descartes was a sincere believer, but he was anti-scholastic and **probably** anti-clerical. The scholastic-clerics controlled the educational **institutions** in France and elsewhere, and given the times, they were not open to new ideas. **It** was a time of intellectual retrenchment, which is a nice way of saying, **it** was a time of dogmatic fixation. Descartes believed in the new science of Copernicus and Galileo and had developed similar theories of his own. But the times were not ripe. How could he advance the cause of the new mathematical sciences in the teeth of militaristic opposition from the scholastic-clerics who controlled the educational institutions in politically volatile France? This is the problem Descartes in his Discourse and his scientific treatises attempted to address.

It is a well-known principle of guerrilla warfare that you don't waste your time talking to the establishment. Talk to the people. Descartes translated this principle into French. That is, he wrote the Discourse not in the de **riguer** academic language of Latin but French. He knew what the statistics of 17th century printing tell us: the buyers of books were increasingly lawyers and merchants, not clerical academics. And in his Discourse, he plays on the prejudices of his prospective readers. Like many of them, Descartes tells us, he had had a scholastic education and he had found **it** a waste of time with very little practical application. Descartes tells his practical middle class readers that he had discovered a new method of thinking which led him to many new and useful discoveries, especially in medicine. The new **method--based** on his work in mathematics--was practical**; it** may some day, Descartes tells his readers, enable us to cure the debilities of old age, and even extend life. Saints might want to rush to God, but the middle class is not known to object to a delay in this life. Descartes chose his target well and knew how to soften **it** up.

Like Galileo, Descartes believed that the Copernican theory was true. He also knew that **it** was not worth dying for. He wanted his readers to accept the theory but he also wanted to avoid drawing the fire of the Aristotelian scholastic army. How could he draw attention to the theory without unacceptable losses?

Disguise, of course, is essential, and so Descartes wore the disguise of the timid harmless intellectual, a gentleman of leisure. He stressed that Copernicus' idea is a hypothesis, a guess, a speculation. Unproven, of course, and so it would seem not true. Now an indirect attack: the Optics.

The Optics, among other things, is a manual on how to manufacture more efficiently cheaper and better telescopes than were available at the time. The opening of the Optics is magnificent subterfuge written to appeal to the money-minded middle class readers Descartes wished to win over.

> All the management of our lives depends on the senses, and since that of sight is the most comprehensive and the noblest of these, there is no doubt that the inventions which serve to augment its power are among the most useful that there can be. And it is difficult to find any of these inventions which augment the power of sight more than that of those marvelous telescopes which, in use for only a short time, have already revealed a greater number of new stars in the sky, and other new objects above the earth, than the sum total of those we have seen there before: so that, carrying our sight much farther than the imagination of our fathers are used to going, they seem to have opened the way for us to obtain a knowledge of nature much greater and more perfect than our fathers had.[4]

Descartes' tactic here is illuminated by the fact that Galileo's most persuasive argument for the Copernican system was the discovery of the four moons rotating around Jupiter--a Copernican system in miniature, Galileo called it--and the discovery of new celestial bodies. Descartes' move here is not to argue directly and on theoretical grounds that the Copernican hypothesis is true but to simply provide an instrument with which the people could see a model of the system in the heavens. Logically, enabling people to look into a telescope proves nothing; psychologically, it wins them over. Psychologically, telescopes played the role computers play today: somehow a computer printout seems much more impressive to us than a typed report, even when they 'say' the same thing. But Descartes' attack in the Optics on the entrenched conventional forces of Aristotelian scholasticism was not simply a matter of psychological pyrotechnics. He appreciated and respected the intellectual capacities of nonacademics. Thus, in explaining the

principles involved in the telescope, Descartes attempted to undercut the Aristotelian epistemology that supported the old science. He did this by undercutting the role that perception plays in cognition.

The old science is based on observed perceptual qualities such as color and taste while the new science is based on mathematical, especially geometrical, reasoning. In support of the old science, the scholastics had argued that the perception of distance, or what we today would call the perception of metrical properties, depended on the perception of sensory qualities such as color. In the course of explaining the principles of how a telescope works, Descartes showed how the perception of metrical properties (like distance) is fundamentally different from the perception of non-metrical properties like color. He showed further that the perception of non-metrical properties like color is a purely subjective response to stimuli. That is to say, Descartes showed that knowledge of the world does not depend on the perception of sensory qualities. Descartes also showed in his Optics that the judgment of distance presupposes geometrical reasoning, and therefore depends upon mathematical thinking. All these points reinforced and made persuasive the claim made by Descartes in his Discourse that knowledge of the world is not through the senses--that is, the old science--but is attained by mathematical thinking.

Descartes followed up his attack on the old science in the Optics by attacking the old mathematics in his Geometry.

Most, if not all of you know, that Descartes is the founder of analytical geometry, but you may not have had the opportunity to appreciate the fact that his discovery of analytical geometry was an attack on the Aristotelians. Although Aristotle himself may have been somewhat unclear about the matter, his followers believed that algebra and geometry could not be combined. Descartes merely showed that they could--and he showed how powerful the combination was by solving problems that for centuries had been considered insoluble. Success is impossible to refute, and Descartes knew it. So he took advantage of his success in combininq algebra and geometry to write a 'hands on' training manual for future guerrilla fighters. Practice may or may not make perfect, but it makes for success. Descartes' audience, as I indicated earlier, is not just contemporary lawyers and merchants and those few intellectuals open to the new, but posterity. Descartes' hope was that the next generation "will.. .be all the more capable of discovery for themselves all that I think I have discovered."[5]

Descartes' attack on the conventional forces of the religious-political establishment is reflected in the style of his Geometry. Descartes' Geometry contrasts sharply with the standard text, Euclid's Elements. Euclid's Elements is polished, as polished as military brass. It is a closed work, a finished work. The reader merely follows and assents to the proofs, once she accepts the axioms and definitions. One follows' the proofs of Euclid almost as though one is following orders. The conclusions are as inescapable as orders once one has put on the uniform of axioms. The 'chain' of reasoning in Euclid--we might accept, we might even get used to it, as one accepts and learns to function in a chain of command,' but by ourselves, individually, we do nothing on our own. We go only where the orders and proofs direct us. That is the strength and the limitation of conventional armies.

With the Geometry of Descartes the guerrilla soldier does not begin with axioms, the order of commands for the subordinate propositions. The guerrilla soldier is first instructed about strategy. Descartes tells his reader that with very limited resources he can reduce all the problems of geometry to manageable arithmetical operations.

> All the problems of geometry can easily be reduced to such terms that thereafter we need to know only the length of certain straight lines in order to construct them.
>
> And just as all of arithmetic is composed of but four or five operations--namely, addition, subtraction, multiplication, division, and the extraction of roots, which may be considered a species of division--so in geometry, in order to find the lines for which we are looking, we need only add to them, or subtract from them, other lines; or else, by taking one line which I shall call unity, in order to relate it as closely as possible to numbers, and which usually can be chosen arbitrarily, and then by taking two others, (we may) find a fourth line which is to one of these two lines as the other is to unity--which is the same as multiplication; or else (we may) find a fourth line which is to one of the two as the unity is to the other--which is the same as division; or finally, (we may) find one, or two, or several mean proportionals between the unity and some other lines--which is the same as extracting the square root, or cube root, etc. And I shall not hesitate to introduce these arithmetical terms into geometry, in order to make myself more intelligible.[6]

Throughout the Geometry, Descartes stresses that his new techniques can solve many problems that the ancient mathematicians could not. Not only could the new techniques solve many problems the ancients could not, it enabled Descartes to recast the proofs of the ancients in a more perspicacious and economical form. The ancients wrote big fat books, says Descartes; I write thin little books which are more intelligible and mathematically more powerful. Moreover, says Descartes, my mathematical discoveries open up an infinite mathematical space, as the new science opened up an infinite physical space. Euclid's Elements is like a closed universe; Descartes' Geometry discloses an open mathematical universe. The parallels with the old and new science are striking.

A distinctive feature of Descartes' Geometry that sharply sets it off from Euclid's Elements is that many of the problems posed in the work are left unsolved. A guerrilla fighter cannot do everything; he must husband his resources; he must get others to join in the work. So Descartes leaves a number of problems unsolved in order that the reader might be engaged in their solution. His style is the mathematical equivalent of what in the U.S. Army today is called 'hands on' training. Descartes wants his readers not to learn about the new way of thinking; he wants them to learn to think in the new way; and for this, there is only one approach: practice, practice, practice. In this vein, Descartes writes:

> And we can always thus reduce all the unknown quantities to a single one, so long as the problem can be constructed by circles and straight lines, or by conic sections, or even by some other line which is only one or two degrees greater. But I shall not pause here to explain this in greater detail, because I should be depriving you the pleasure of learning it for yourself, which is, in my opinion, the principal advantage we can derive from this science. Moreover, I do not observe here anything so difficult that it cannot be discovered by those who are slightly versed in common geometry and in algebra, and who pay close attention to everything in this treatise
>
> This is why I shall content myself here with advising you that in solving these equations, provided that we do not fail to use division whenever possible, we will infallibly reach the simplest term to which the problem can be reduced.[7]

Descartes' advice and invitation to his readers probably attracted his better motivated and mathematically inclined readers, but it is doubtful that Descartes was so naive as to think that all his readers would undertake to solve all the problems he left to their pleasure.

But those who are not highly motivated or have little mathematical aptitude might still be won over. How might one rope the marginal reader in?

Well, what is one of the most popular if boring topics of conversation? What is an oft-used conversation opener? We all know: the weather--rain, snow, earthquakes, rainbows. And so Descartes writes about the science of the weather, a Meteorology.

The beauty of a well-directed guerrilla operation is that it is not where its forces are firing. That is, guerrilla attacks are most successful to the extent that the objectives of the attack are not grasped, or are grasped too late, by the conventional forces. A good guerrilla leader always has his opposite looking the other way. He attacks in the north to capture a city in the southern delta. This, I submit, is what Descartes does when he talks about the weather.

Fundamental to the Aristotelian-Ptolemaic view of the universe is that the laws which explain the motion of celestial bodies and the physical constitution of the celestial bodies are fundamentally different from the laws of motion which hold on the earth and the physical constitution of the earth. The New Science rejects this belief. The principle of the uniformity of nature which is essential to accepting the modern scientific way of understanding the universe depends on the belief that the laws which explain the behavior of things on earth obtain throughout the universe and that the planet earth is made of the same chemical elements as other items in the universe. Descartes insinuates this idea in the opening paragraph of the Meteorology:

> It is our nature to have more admiration for the things above us than for those that are on our level, or below. And although the clouds are hardly any higher than the summits of some mountains, and often we even see some that are lower than the pinnacles of our steeples, nevertheless, because we must turn our eyes toward the sky to look at them, we fancy them to be so high that poets and painters even fashion them into God's throne, and picture Him there, using His own hands to open and close the doors of the winds, to sprinkle the dew upon the flowers, and to hurl the lightning against the rocks. This leads me to hope that if I here explain the nature of clouds, in such a way that will no longer have occasion to. wonder at anything that can be seen of them, or anything that descends from them, we will easily believe that it is singularly possible to find the causes of everything that is most admirable above the earth.[8]

Descartes did not end his fight here. After the Discourse, he engaged in other strategic attacks on the conventional armies of the intellectual establishment. But I will save those items on the Cartesian menu for another course. Otherwise, I fear Dr. DeMayo will wish that he had ordered not a Cartesian entree, but rather had ordered his Descartes a la carte.

Footnotes

[1] Rene Descartes, Discourse on Method, Optics, Geometry and Meteorology, tr. Paul J. Olscamp (New York: The Bobbs-Merrill Company, Inc., The Library of Liberal Arts, 1965), p. 49. All citations are from this work.

[2] Discourse, p. 54.

[3] Discourse, p. 55.

[4] Optics, p. 65.

[5] Discourse, p. 57.

[6] Geometry, p. 177.

[7] Geometry, p. 180.

[8] Meteorology, p. 263.

GRAFFITO

Good sense is, of all things among men, the most equally distributed; for everyone thinks himself so abundantly provided with it, that those even who are the most difficult to satisfy in everything else, do not usually desire a larger measure of this quality than they already possess.

Rene Descartes

A MATHEMATICAL MODEL OF VOTER PARTICIPATION

by Mary Beth Dever
Northern Illinois University

The data utilized in this project were made available by the Inter-University Consortium for Political Research. The data for the SRC American National Election Study were originally collected by the Political Behavior Program of the Survey Research Center, Institute for Social Research, The University of Michigan. Neither the original collectors of the data nor the consortium bears any responsibility for the analyses or interpretations presented here.

We shall discuss a voter participation model which was developed by political scientists William Riker and Peter Ordeshook in 1968 [2]. The model is used to predict citizen participation in an election and not the outcome of the election. We assume that each citizen has chosen a preferred candidate and is deciding whether or not to vote.

Our model is described by the relation

$$R = PB - C + D, \quad \text{where}$$

R represents the reward an individual receives from voting;
P represents the probability that by voting the individual affects the outcome of the election;
B represents the differential benefit received by the individual from the success of the preferred candidate over the less preferred;
C represents the costs involved in voting, (e.g., the time taken to go to the polls); and

D represents the sense of citizen duty (the satisfaction some people receive from participating in the political process).

To arrive at this model, Riker and Ordeshook modified an existing model which was not satisfactorily describing behavior. First, they introduced the sense of duty term which was not present in the existing model. They did this because they believed that some people received some satisfaction from participating without regard to the particular candidates or choices in an election, and that this would play a role in the citizen's participation. Secondly, they made a change in the way the probability was calculated, which we shall consider in more detail later.

The model is based on the idea that if an individual's reward from participating in an election is positive, then the individual will vote. We assume that for an individual the costs and sense of duty are constant over several elections, whereas the probability of affecting the outcome and the benefit depend on the particular election at hand. There are three cases to consider.

1) If for an individual $D > C$, (i.e., sense of duty outweighs costs involved) then $R > 0$ and the individual will always vote.
2) If for an individual $C > D$ and $PB > C - D$, then $R > 0$ and the individual will vote.
3) If for an individual $C > D$ and $PB \leq C - D$, then $R \leq 0$ and the individual will not vote.

Notice that in the last two cases the actual election is important in determining participation.

In the testing of the model we do not get precise numerical values. Instead people are placed into categories based on their responses to survey questions. Then given this information about the individual an estimate of the probability he/she will vote in an election can be given. It is assumed that the cost is constant within a category of sense of duty, since those with a high sense of duty would tend to minimize the costs involved in voting and those with a low sense of duty would tend to maximize the costs. So the cost is not used directly in the testing of the model, thus simplifying the testing of the model.

The model testing used responses from the American National Election Study (A.N.E.S.). The A.N.E.S., conducted by the Survey Research Center at the University of Michigan, has been carried out each election year since 1948, and contains questions pertaining to elections and social issues. The results from the A.N.E.S. are used in many branches of the social sciences. More information about the A.N.E.S. can be found in Campbell [1]. Responses to seven of the A.N.E.S. survey questions were used to put citizens into categories based on benefit, probability and sense of duty. The A.N.E.S. includes a post-election survey which contacts the same citizens and inquires, among other things, whether the citizen voted. Within a fixed category of benefit, probability of affecting the outcome, and sense of duty, the level of participation, (i.e., the ratio of the number of people who voted to the number of people in the category), will be used as an estimate of the probability a person in that category will vote.

Let us now consider the components of the model. We will begin by considering the benefit. Let $\{O_1, O_2, \ldots, O_n\}$ be the exhaustive set of outcomes of the election: e.g., candidate A wins, candidate B wins, and so on. Let the outcomes be listed in decreasing order of preference. Define a utility function U satisfying

1) $U(O_1) \geq U(O_2) \geq \ldots \geq U(O_n) \geq 0$; and

2) $\sum_{i=1}^{n} U(O_i) = 1.$

The utility function is intended to be a measure of the relative utility or value each of the outcomes has for the individual. The assignment of actual values to the utility function is not of importance to our analysis but those interested may wish to consult Riker [3]. Let the differential benefit $B = U(O_1) - U(O_2)$. Notice that if the outcomes O_1 and O_2 had the same utility for an individual, then the outcome would not matter to the individual and B would equal zero.

B is a way of kncwing whether the outcome of the election is important to the citizen. To determine benefit the survey question asked was: "How much do you care about the outcome of this election?" Respondents were given several choices and the responses were used to put respondents into one of two categories: high benefit or low benefit.

We next consider P. As mentioned previously, Riker and Ordeshook made a change in the way the probability of affecting the outcome was calculated. The original idea might seem the most natural so we will begin by looking at what was the bad consequence of the old way of calculating the probability, and then see how the new way was developed. The old way of calculating P was simply to let V be the set of voters and v be the number of elements in V, then set $P = 1/v$.

Riker and Ordeshook gave the following argument to show it was not reasonable to proceed in that manner. Consider a one-party state, (for example the South from the Civil War to the early 1950's, where more people vote in the primary than in the general election. So $v_P > v_G$ if v_P = the number of voters in the primary and v_G = the number of voters in the general election. We will now subscript the notation introduced above by P or G to indicate whether we mean the primary or the general election. For an individual in V_P and not in V_G, we have $R_P > R_G$. We also consider cost and sense of duty to be constant for the individual over the primary and general election and will therefore leave them out of the argument. Assume our voter is a loyal Democrat, then $B_P < B_G$, since for a loyal Democrat the difference in benefit between two Democrats would be less than the difference in benefit between a Democrat and a Republican. If $v_P > v_G$, then $1/v_P < 1/v_G$, so $B_P(1/v_P) < B_G(1/v_G)$ which implies $R_P < R_G$ thus arriving at a contradiction. Therefore P should not be calculated as $1/v$.

The new way to look at P is to note that P depends on how close the race is expected to be. Consider an individual who is deciding whether to vote and who prefers candidate A. The citizen makes estimates of the perceived probabilities candidate A will win and lose if the citizen votes and does not vote. Naturally the estimates of these probabilities by some people will be better than that of others. But if each citizen makes his or her own estimate on which to base his or her own decision of whether to vote then that will not matter. It is important to make a distinction between the perceived probabilities of the candidates' winning and the probability of affecting the outcome which we are trying to derive.

Let q be the perceived probability candidate A wins if the citizen votes. Let 1-q be the perceived probability candidate A loses if the citizen votes. Let q' be the perceived probability candidate A wins if the citizen does not vote and $1-q'$ be the perceived probability candidate A loses if the citizen does not vote. If the citizen votes and candidate A wins, then the citizens' utility will be $U(O_1) - C + D$, which is expected with probability q, so the expected utility is $q(U(O_1) - C + D)$. If the citizen does not vote, then the costs and satisfaction from the sense of duty are not experienced by the individual. The expected utilities are summarized below.

	candidate A wins	candidate B wins
citizen votes	$q(U(O_1) - C + D)$	$(1-q)(U(O_2) - C + D)$
citizen does not vote	$q'(U(O_1))$	$(1-q')(U(O_2))$

To compare the expected utility of voting with that of not voting, the expected utility of not voting is subtracted from the expected utility of voting:

$$[q(U(O_1)-C+D) + (1-q)(U(O_2)-C+D)] \quad - [q'U(O_1) + (1-q')U(O_2)]$$

When simplified this yields

$$(q-q')(U(O_1) - U(O_2)) - C + D.$$

This looks amazingly like the original model with $U(O_1) - U(O_2)$ being the differential benefit as defined earlier and $q - q'$ playing the role of P. Riker and Ordeshook assumed the following axiom. The addition of one more member to the set V will not change the preference between candidates for the individual members of V. While this may not be true on small committees, it certainly is reasonable for electorates the size of those in our national elections. This can be used to show that $q \geq q'$, so $q - q'$ is not negative.

Next, we consider the number of votes a candidate needs to win. If there are v voters in V and w is the minimum number of votes needed to win, then

$$w = (v+1)/2, \text{ if } v \text{ is odd; and}$$
$$w = (v/2) + 1, \text{ if } v \text{ is even.}$$

Try this with small numbers and notice that the addition of one more

voter to the set V increases the number of votes needed to win by one if v is odd, but it does not change the number needed to win if v is even. This evenness and oddness will play an important role in the development of P.

Let $pr_v[A,x]$ be the probability candidate A receives exactly x votes if v votes are cast. Candidate A will win if he receives the minimum number of votes needed to win, or any number of votes greater than the minimum up to the total number of votes possible. So if v is odd,

$$q = \sum_{x=\frac{v+1}{2}+1}^{v+1} pr_{v+1}[A,x] = pr_{v+1}[A,(v+1)/2+1] + pr_{v+1}[A,(v+1)/2+2] + \dots + pr_{v+1}[A,v+1];$$

and

$$q' = \sum_{x=\frac{v+1}{2}}^{v} pr_v[A,x] = pr_v[A,(v+1)/2] + pr_v[A,(v+1)/2+1] + \dots + pr_v[A,v].$$

Notice that in this case q and q' have the same number of terms.

If v is even,

$$q = \sum_{x=\frac{v}{2}+1}^{v+1} pr_{v+1}[A,x] = pr_{v+1}[A,(v/2) + 1] + pr_{v+1}[A,(v/2) + 2] + \dots + pr_{v+1}[A,v+1];$$

and

$$q' = \sum_{x=\frac{v}{2}+1}^{v} pr_v[A,x] = pr_v[A,(v/2) + 1] + pr_v[A,(v/2) + 2] + \dots + pr_v[A,v].$$

Notice that in this case q has one more term than q'.

Riker and Ordeshook proved the following lemma.

Lemma. If voter i intends to vote for candidate A and $p_1 = pr_v[A,x]$ and $p_2 = pr_{v+1}[A,x+1]$, then $p_1 = p_2$.

The idea is "one more voter, one more vote." In the case v is odd, applying the lemma we have the 1st tern in the q' summation equals the 1st term in the q summation, 2nd term equals the 2nd term, etc. Since q and q' have the same number of terms $q - q' = 0$. In the case v is even, the lemma gives the 1st tern in q' equals the 2nd term in q, the 2nd term in q' equals the 3rd term in q, etc. Since q has one more term than q', $q - q' = pr_{v+1}[A,(v/2) + 1]$, the unmatched term, which by the lemma equals $pr_v[A,v/2]$.

No one knows whether there will be an even or odd number of people voting in an election, but it can be expected to be even or odd with equal probability. So $P = q-q' = (\frac{1}{2})pr_v[A,v/2]$. Since $pr_v[A,v/2]$ is the probability of a tie, we see P depends on the citizen's estimate of how close the race will be. To determine P, the survey question asked was: "How close do you think the race will be?" Respondents were given several choices which were broken down into two categories, high and low.

Finally, to get an estimate of the sense of citizen duty Riker and Ordeshook constructed a sense of citizen duty scale, which consisted of four statements in the A.N.E.S. survey. Respondents were asked to agree or disagree with the following statements:

1) It isn't so important to vote when you know your party doesn't have a chance to win.
2) A good many local elections aren't important enough to bother with.
3) So many other people vote in the national elections that it doesn't matter much to me whether I vote or not.
4) If a person doesn't care how an election comes out, he or she shouldn't vote in it.

In order to display a high sense of citizen duty the respondent had to disagree with the statements. Disagreeing with all four corresponded to the high category of sense of duty. Disagreeing with three of them corresponded to the medium category. Disagreeing with fewer than three corresponded to the low category of sense of duty.

Thus we have two categories each for benefit and probability of affecting the outcome, and three categories for sense of citizen duty, for a total of twelve potential categories in which citizens can be classified. In each of these categories the level of participation was calculated. I tested the model on data from the 1956 presidential election, which was one of the years Riker and Ordeshook originally used for their testing. My results are included below. Zeros in a category mean I did not have any respondents in that category, so a

level of participation could not be calculated.

1956 Election Results

High D

	High B	Low B
High P	424/848 = 0.88	152/193 = 0.79
Low P	6/7 = 0.86	2/3 = 0.67

Medium D

	High B	Low B
High P	277/356 = 0.78	95/137 = 0.69
Low P	3/4 = 0.75	0

Low D

	High B	Low B
High P	40/78 = 0.51	29/95 = 0.31
Low P	1/1 = 1.00	0

The results did not have a sufficient number of respondents in the low categories of probability to use the estimates of the levels of participation. However in the high categories of probability there are sufficient numbers of respondents.

From these results we can make a few observations. First, keeping the categories of B and P constant, e.g., high B and high P, and comparing along the levels of sense of duty we see that those in high D were more likely to vote than those in medium D, who in turn were more likely to vote than those in low D. These results support the addition of the sense of duty term to the model.

Secondly, within a category of sense of duty, e.g., high D, keeping one of the variables P or B constant and comparing levels of participation between high and low categories of the other variable, one finds the participation level of the former is higher than that of the later, as expected. This shows that both the probability of affecting the outcome and the benefit are important in determining a citizen's participation in an election.

The results support the validity of the model as one that can be used to predict participation in an election. The model could be used to test other hypotheses about voter participation. It would be interesting to see how well the model predicts participation in more current elections as compared to that of elections in the 1950's, and whether there has been a shift in the number of citizens in the various categories such as sense of duty.

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About the author -

Mary Beth Dever is a second-year graduate student at Northern Illinois University.

About the paper -

Mary Beth's paper resulted from a project she did during her senior year and is a summary of a talk which the gave at the National Pi Mu Epsilon Meetings at the University of Oregon in Eugene last August and at the Eleventh Annual Pi Mu Epsilon Student Conference at Miami University in September.

1985 NATIONAL PI MU EPSILON MEETING

It is time to be making plans to send an undergraduate delegate on speaker from your Chapter to the Annual Meeting of Pi Mu Epsilon in Laramie, Wyoming on August 13 and 14. Each student who presents a paper will receive travel support up to $500. Each delegate, up to $250. Only one speaker or delegate can be funded from a single chapter, but others are encouraged to attend. For details, contact Dr. Richard A Good, Secretary-Treasurer, Department of, Mathematics, University of Maryland, Cortege. Park, MU 20742

TAXICAB TRIGONOMETRY

by Ruth Brisbin
Plant H. S., Tampa, FL

and Paul Artola
University of South Florida, Tampa, FL

Introduction: In plane trigonometry, the points of the unit circle correspond to the functions sine and cosine of an induced angle. In this paper, we investigate the equivalent functions defined via the unit circle of the so-called *Taxicab Norm*, the norm function which measures distance in the same fashion that one would measure the distance travelled by a taxicab in going from point A to point B in a city with only north-south and east-west streets. Recent articles in the Journal by Reynolds [1], Moser and Kramer [2], and Iny [3] have investigated the unusual geometry induced by this metric function, but none has tackled the formulation of trigonometric functions.

Metric Functions and Unit Circles: We recall that if $x = (x_1, x_2, \ldots, x_n)$ is a point in R^n, then the function

$$(1)\qquad ||x||_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$$

is a metric function for the vector space R^n. In particular, for $p = 2$, (1) becomes the Euclidean metric function, a familiar one to all of us. When $p = 1$, however, we have the *Taxicab* metric function

$$(2)\qquad ||x||_1 = \sum_{i=1}^{n} |x_i| \ .$$

Once we have a metric function, we can define the distance between two points by

$$(3)\qquad d_p(x,y) = ||x - y||_p$$

so that, in R^2, for points $A = (x_1, y_1)$ and $B = (x_2, y_2)$, we have the familiar distance formula

(4) $\qquad d_2(A,B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2},$

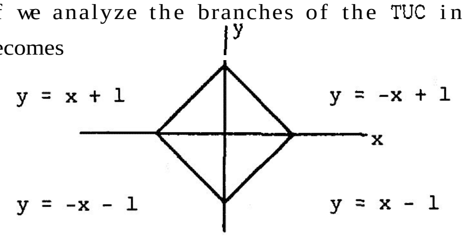
while in the taxicab metric, we have

(5) $\qquad d1(A,B) = |x_2 - x_1| + |y_2 - y_1| .$

With the concept of distance between two points in mind, we recall that the *Unit Circle* is $\{A\varepsilon R^2 : d_p(A,0) = 1\}$, where $0 = (0,0)$. Of course, for $p = 2$, we have the Euclidean Unit Circle, while for $p = 1$, the *Taxicab Unit Circle* (TUC) takes the form

(6) $\qquad d_1(A,0) = |x_1| + |y_1| = 1.$

If we analyze the branches of the TUC in the various quadrants, the graph becomes



Quadrant	Equation
I	y = - x + 1
II	y = x + 1
III	y = - x - 1
IV	y = x - 1

Diamond Sine and Diamond Cosine: We shall now define the trigonometric functions *Diamond Sine* (sind) and *Diamond Cosine* (cosd) of some angle 6 in the same way one determines their Euclidean analogues sine and cosine. Recall that if (x,y) is some point on the Euclidean unit circle (so that $x^2 + y^2 = 1$) then $x = \cos\theta$ and $y = \sin\theta$, where θ is the angle with initial side the positive x-axis and terminal side the radial line passing through point (x,y). Now if we use this idea for finding $\text{cosd}\theta$ and $\text{sind}\theta$ by (x,y) be a point on the TUC, then $|x| + |y| = 1$. At the same time, the equation of the line joining (x,y) and (0,0) is

(7) $\qquad y = (\tan\theta)x.$

Solving the system [for (x,y) in the first quadrant]:

$$\begin{cases} y = (\tan\theta)x \\ y = -x + 1 \end{cases}$$

we obtain

$$(\tan\theta)x = -x + 1$$

so

$$(\tan\theta + 1)x = 1$$

$$\frac{\sin\theta + \cos\theta}{\cos\theta}\, x = 1$$

therefore

$$x = \text{cosd}\theta = \frac{\cos\theta}{\sin\theta + \cos\theta}$$

Since $y = \text{sind}\theta = 1 - x$, then we see that for $0 \le \theta \le 90^\circ$:

(8) $\qquad \text{cosd}\theta = \dfrac{\cos\theta}{\sin 6 + \cos\theta}$ and $\text{sind}\theta = \dfrac{\sin 9}{\sin\theta + \cos\theta}$

Moving into the second quadrant ($90^\circ \le 6 \le 180^\circ$), we must solve:

$$\begin{cases} y = (\tan\theta)x \\ y = x + 1 \end{cases},$$

Thus, we find:

(9) $\qquad \text{cosd}\theta = \dfrac{\cos\theta}{\sin\theta - \cos\theta}$ and $\text{sind}\theta = \dfrac{\sin 8}{\sin\theta - \cos\theta}$

Continuing in this fashion for the third and fourth quadrants:

Quadrant	cosdθ	sindθ
I	$\dfrac{\cos\theta}{\sin\theta + \cos\theta}$	$\dfrac{\sin\theta}{\sin\theta + \cos\theta}$
II	$\dfrac{\cos\theta}{\sin\theta - \cos\theta}$	$\dfrac{\sin\theta}{\sin\theta - \cos\theta}$
III	$\dfrac{-\cos\theta}{\sin\theta + \cos\theta}$	$\dfrac{-\sin\theta}{\sin\theta + \cos\theta}$
IV	$\dfrac{-\cos\theta}{\sin\theta - \cos\theta}$	$\dfrac{-\sin\theta}{\sin 6 - \cos\theta}$

Identities: In Euclidean trigonometry, several identities are derived from the relationship between the coordinates of the points on the unit circle and the trigonometric functions. Of course, the fundamental identity is the Pythagorean Identity:

$$\cos^2 6 + \sin^2 6 = 1.$$

Clearly, this relation comes from the fact that for any point (x,y) on the unit circle, $x^2 + y^2 = 1$. Turning to the TUC, we recall that $|x| + |y| = 1$, so it follows that analogous to the Pythagorean Identity, we have the *Taxicab Identity*:

(10) $\qquad |\text{cosd}\theta| + |\text{sind}\theta| = 1$

Moving on, we ask what identities exist for angles -6. To motivate this, let us look at Figure 6.

Quadrant I: Quadrant II:

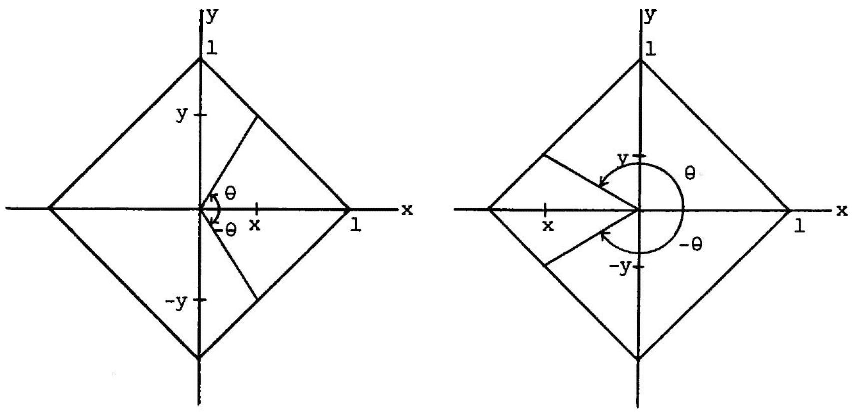


Figure 6.

From these graphs, it appears that

(11) $\quad \mathrm{cosd}(-\theta) = \mathrm{cosd}\theta$ and $\mathrm{sind}(-\theta) = -\mathrm{sind}\theta$.

To prove this, we shall take -9 to be a third quadrant angle so that 6 is a second quadrant angle. Thus:

$$\mathrm{cosd}(-\theta) = \frac{-\cos(-\theta)}{\sin(-0) + \cos(-\theta)} = \frac{-\cos\theta}{-\sin\theta + \cos\theta} = \frac{\cos\theta}{\sin 0 - \cos\theta} = \mathrm{cosd}\theta,$$

and

$$\mathrm{sind}(-\theta) = \frac{-\sin(-6)}{\sin(-\theta) + \cos(-\theta)} = \frac{\sin 0}{-\sin\theta + \cos\theta} = \frac{-\sin\theta}{\mathrm{sine} - \cos\theta} = -\mathrm{sind}\theta.$$

Continuing in this fashion for the other quadrants, we establish (11).

The last class of identities we develop involves working with the reference angle, θ', of a given angle, 6. Recall that for the ease of using trigonometric tables, the knowledge of reference angle identities enables us to only have a minimal table handy. By associating any given angle with a first quadrant angle, we need only use a table of values for first quadrant angles. The reference angles for all quadrants are as follows:

Quadrant	Angle	Reference Angle
I	θ	$\theta' = 8$
II	θ	$\theta' = 180^\circ - \theta$
III	θ	$\theta' = \theta - 180^\circ$
IV	θ	$\theta' = 360^\circ - \theta$

Also, recall that

$$\cos\theta = -\cos(180^\circ - \theta) = -\cos(\theta - 180^\circ) = \cos(360^\circ - \theta), \text{ and}$$

$$\sin\theta = \sin(180^\circ - \theta) = -\sin(\theta - 180^\circ) = -\sin(360^\circ - \theta).$$

Let θ_i be a given angle in Quadrant i and let θ_i' be its associated reference angle, then between them we have the following relationships for the diamond trig functions.

$$\mathrm{cosd}\theta_2 = \frac{\cos(180^\circ - \theta_2')}{\sin(180^\circ - \theta_2') - \cos(180^\circ - \theta_2')} = \frac{-\cos\theta_2'}{\sin\theta_2' + \cos\theta_2'} = -\mathrm{cosd}\theta_2' ;$$

$$\mathrm{sind}\theta_2 = \frac{\sin(180^\circ - \theta_2')}{\sin(180^\circ - \theta_2') - \cos(180^\circ - \theta_2')} = \frac{\sin\theta_2'}{\sin\theta_2' + \cos\theta_2'} = \mathrm{sind}\theta_2' ;$$

$$\mathrm{cosd}\theta_3 = \frac{-\cos(\theta_3' + 180^\circ)}{\sin(\theta_3' + 180^\circ) + \cos(\theta_3' + 180^\circ)} = \frac{-\cos\theta_3'}{\sin\theta_3' + \cos\theta_3'} = -\mathrm{cosd}\theta_3' ;$$

$$\mathrm{sind}\theta_3 = \frac{-\sin(\theta_3' + 180^\circ)}{\sin(\theta_3' + 180^\circ) + \cos(\theta_3' + 180^\circ)} = \frac{-\sin\theta_3'}{\sin\theta_3' + \cos\theta_3'} = -\mathrm{sind}\theta_3' ;$$

$$\mathrm{cosd}\theta_4 = \frac{-\cos(360^\circ - \theta_4')}{\sin(360^\circ - \theta_4') - \cos(360^\circ - \theta_4')} = \frac{\cos\theta_4'}{\sin\theta_4' + \cos\theta_4'} = \mathrm{cosd}\theta_4' ;$$

$$\mathrm{sind}\theta_4 = \frac{-\sin(360^\circ - \theta_4')}{\sin(360^\circ - \theta_4') - \cos(360^\circ - \theta_4')} = \frac{-\sin\theta_4'}{\sin\theta_4' + \cos\theta_4'} = -\mathrm{sind}\theta_4' .$$

THE TAXI CAB UNIT CIRCLE

θ	COS θ	SIN θ	COSM	SIN Dθ
0	1.0000	0.0000	1.0000	0.0100
1	.9998	.0175	.9828	.0172
2	.9994	.0349	.9663	.0337
3	.9986	.0523	.9502	.0498
4	.9976	.0698	.9346	.0654
5	.9962	.0872	.9195	.0805
6	.9945	.1045	.9049	.0951
7	.9925	,1219	.8906	.1094
8	.9903	.1392	.8768	.1232
9	.9877	.1564	.8633	.1367
10	,9848	.1736	.8501	.1499
11	.9816	,1908	.8373	.1627
12	.9781	,2079	.8247	.1753
13	,9744	.2250	.8124	.1876
14	,9703	.2419	.8004	.1996
IS	.9659	.2588	.7887	.2113
16	.9613	.2756	,7772	.2228
17	.9563	.2924	.7659	.2341
18	.9511	,3090	,7548	.2452
19	.9455	.3256	.7439	.2561
20	.9397	.3420	,7332	.2668
21	.9336	.3584	.7226	.2774
22	.9272	.3746	.7122	.2878
23	.9205	.3907	.7020	.2980
24	.9135	,4067	.6919	.3081
25	.9063	.4226	.6820	.3180
24	.8988	.4384	.6722	.3278
27	.8910	.4540	.6625	.3375
28	.8829	.4695	.6529	.3471
29	.8746	.4848	.6434	.3566
30	.8660	.5000	.6340	.3660
31	.8572	.5150	.6247	.3753
32	.8480	.5299	.6154	.3846
33	.8387	.5446	.6063	.3937
34	.8290	.5592	.5972	.4028
35	.8192	.5736	.5882	.4118
36	.8090	,5878	.5792	.4208
37	.7986	.6018	.5703	.4297
38	.7880	.6157	.5614	.4386
39	.7771	.6293	.5526	.4474
40	.7660	.6428	.5437	.4563
41	.7547	.6561	.5350	.4650
42	.7431	.6691	.5262	.4738
43	.7314	.6820	.5175	.4825
44	,7193	,4947	.5087	.4913
45	.7071	,7071	.5000	.5000

θ	COS θ	SIN θ	COSM	SMM
46	.6947	.7193	.4913	.5087
47	.6820	.7314	.4825	.5175
48	.6691	.7431	.4738	,5262
49	.6561	.7547	,4650	.5350
50	.6428	,7660	.4563	.5437
51	.6293	.7771	.4474	.5526
52	.6157	.7880	.4386	.5614
53	,6018	.7986	.4297	.5703
54	.5878	.8090	.4208	.5792
55	.5736	.8192	.4118	.5882
56	.5592	.8290	.4028	.5972
17	.5446	.8387	.3937	.6063
58	.5299	.8480	.3846	.6154
59	.5150	,8572	.3753	,6247
60	.5000	.8660	.3660	.6340
61	.4848	,8746	.3566	,6434
42	.4695	.8829	.3471	.6529
63	.4540	.8910	.3375	.6625
64	,4384	.8988	.3278	.6722
65	.4226	.9063	.3180	.6820
U	,4047	,9135	.3081	.6919
67	.3907	.9205	.2980	.7020
68	,3746	,9272	.2878	.7122
49	.3584	,9334	.2774	.7226
70	,3420	.9397	,2668	.7332
71	.3256	.9455	.2561	.7439
72	.3090	.9511	.2452	.7548
73	.2924	.9563	.2341	.7659
74	.2756	.9613	.2228	.7772
75	.2588	,9659	.2113	.7887
76	.2419	.9703	.1996	.8004
77	.2250	.9744	.1876	.8124
78	.2079	.9781	.1753	.8247
79	.1908	,9814	.1627	.8373
81	.1736	.9848	.1499	.8501
81	.1564	.9877	,1367	.8633
82	.1392	.9903	.1232	.8768
83	,1219	.9925	,1094	.8906
84	.1045	.9945	.0951	.9049
85	.0872	,9962	.0805	.9195
86	.0698	.9976	.0654	.9346
87	.0523	.9986	.0498	.9502
88	.0349	.9994	,0337	.9663
89	.0175	.9998	.0172	.9828

In closing, we note the non-uniform change in arc length as we increment the angle by a fixed amount. This "irregularity" does not occur on the Euclidean unit circle and double-angle formulas are not difficult to formulate. We leave the derivation of analogous formulas in the taxicab trigonometry as a challenge to the reader.

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About the authors -

Ruth Brisbin will be graduating from H. 8. Plant High School this spring and hopes to attend college in the Northeast. She is active in the Spanish Honor and Thespian Societies at school, although she maintains her interest in geometry. She gratefully acknowledges the help of her father, Mr. George Brisbin, who wrote software to generate the table of values at the end of this paper.

Pout Artola is a mathematician with the Department of Defense at Ft. Meade, Maryland. He attended the University of South Florida in Tampa and received his B.A. in Mathematics in 7980 and his M.A. in 7985. He has been a member of Pi Mu Epsilon since 7979 and was Student Correspondent of the Florida Epsilon Chapter in the 1979-1980 school year.

About the paper -

This paper was written under the Science Mentor Program grant from the Center for Excellence at the University of South Florida in conjunction with the Center for Mathematical Services. The Science Mentor Program, under the supervision of Dr. Fredric Zerla, provides gifted high school students a chance to pursue an independent study project under the tutelage of a graduate student attending the University of South Florida.

EDGE-LABELLED TREES

by Julie Yancey
Durango, Colorado

The problem discussed in this paper was presented in a beginning graph theory course taught by Dr. Richard Gibbs at Fort Lewis College in Fall, 1982. The problem appears as an exercise in Introduction to Graph Theory, 2nd edition, by *Robin Wilson*, Academic Press.

In graph theory, a tree is a graph drawn on n vertices with n-1 edges. Each two vertices of the graph are connected by a unique path. In a tree, there are no cycles, isolated points, or multiple edges.

One way which graph theorists use to describe a graph is by labelling. Figure 1 is a vertex-labelled tree and Figure 2 is an edge-labelled tree. Notice that Figure 3 is also an edge-labelled tree, but the edge-labelling is different (non-isomorphic) from that in Figure 2.

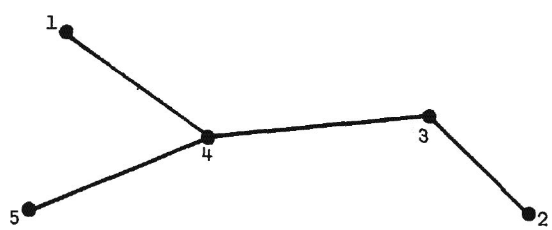

Figure 1
Vertex-Labelled Tree

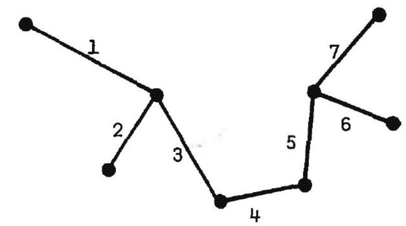

Figure 2
Edge-Labelled Tree

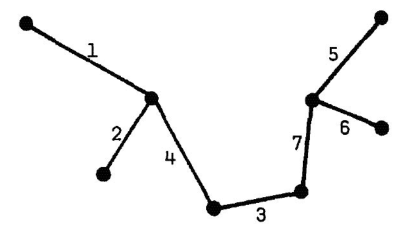

Figure 3
Edge-Labelled Tree

As part-time combinatorialists, graph theorists spend much time counting graphs. Cayley's theorem, which is proven in many graph theory texts, states that there are n^{n-2} non-isomorphic vertex-labelled trees on n vertices. My proof is of the assertion that there are n^{n-3} non-isomorphic edge-labelled trees on n vertices.

I chose an algorithmic approach to the problem, and found that two algorithms were necessary to complete the proof. The first algorithm is a method for proceeding from a given edge-labelled tree to a family of n non-isomorphic vertex-labelled (*n.i.v.l.*) trees on n vertices and the second algorithm shows that this process is reversible. We show that one edge-labelled tree corresponds to n *n.i.v.l.* trees. Thus n^{n-2} *n.i.v.l.* trees (from Cayley's theorem) divide into unique families of n members each, and this yields the required result of n^{n-3} non-isomorphic edge-labelled trees (*n.i.e.l.*) trees on n vertices.

The first algorithm begins by constructing any unlabelled tree on n vertices. The edges are then labelled from 1 to (n-1), as in Figure 4-A. From this edge-labelled tree, T, we are going to construct

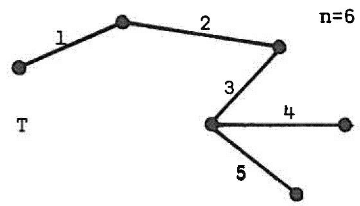

Figure 4-A

a family of *n.i.v.l.* trees. For the first tree in the family, T-1, label the edges the same as those of T, and label a random vertex with n. The rest of the vertices of the tree are then labelled by proceeding outward from the vertex labelled n, with each vertex bearing the label of the preceding edge. See Figure 4-B.

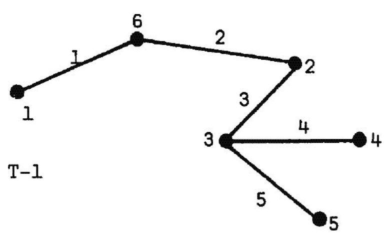

Figure 4-B

The unique path property assures us that each vertex of T-1 will be joined to any other vertex in the tree by a unique path, so each vertex label is defined and no two vertices will share the same label.

The second tree in the family, T-2, has the same edge-labelling as T-1, but a different vertex is labelled n. The rest of the vertices of T-2 are labelled as in T-1. See Figure 4-C.

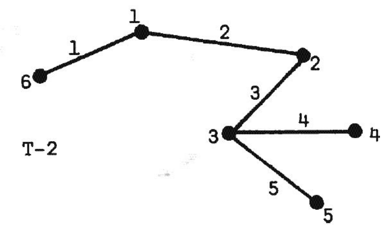

Figure 4-C

Continuing in this way, the family of n trees (Figures 4B-4G) will be complete when all of the n possibilities for a vertex labelled n have been exhausted.

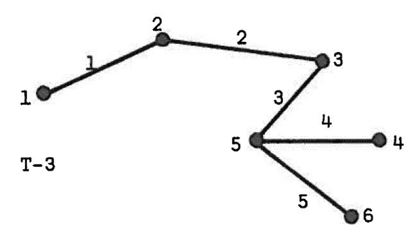

Figure 4-D

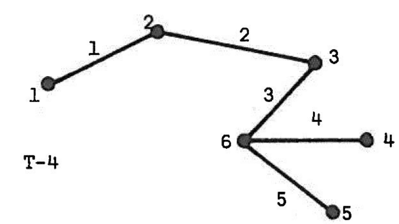

Figure 4-E

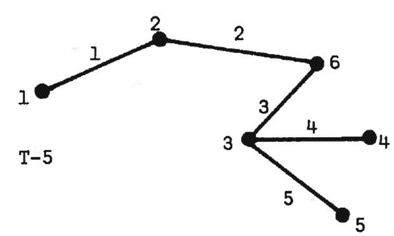

Figure 4-F

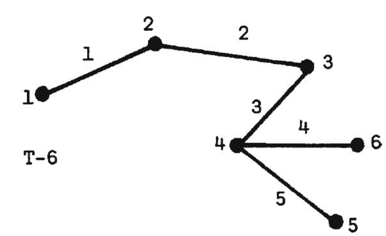

Figure 4-G

Notice (Figures 4B–4G) that each tree in a particular family is non-isomorphically vertex-labelled. Each time a different vertex of T is labelled n, n will be adjacent to at least one different labelled edge, so the method of vertex labelling from this first algorithm will produce a non-isomorphic vertex-labelling each time n is moved. Therefore, we have shown that one *n.i.e.l.* tree corresponds to *n.i.v.l.* trees, where n is the number of vertices.

The second algorithm will show that there is a family of n non-isomorphic vertex-labelled trees which correspond to one edge-labelled tree. Beginning with any unlabelled tree, we randomly label the vertices from 1 to n. See Figure 5–A. We edge-label this tree by labelling each edge of the tree with the label of the following vertex, proceeding outward from the vertex labelled n. Denote the tree by t-1. Since there are n-1 edges in the tree, and no edge will carry the label n (Figure 5–A), the unique path property assures us that no two edges will share the same label.

To obtain t-2, the second tree in the family, begin with the same isomorphic unlabelled tree. Exchange the label n from t-1 with the label of an adjacent vertex, and that will be the vertex labelled n in the second tree (Figure 5-B). The rest of the vertices and edges are labelled in the same way as in the first tree of the family (Figure 5-C).

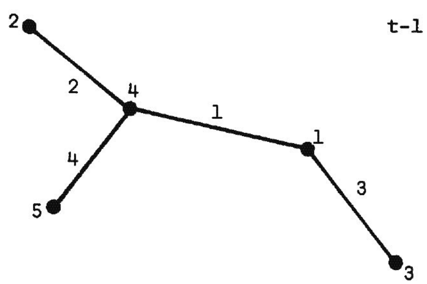

Figure 5–A

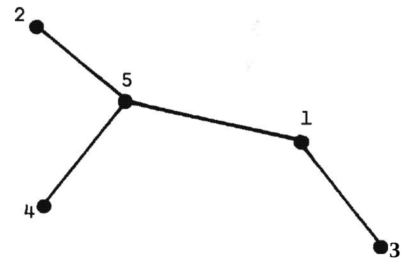

Figure 5–B

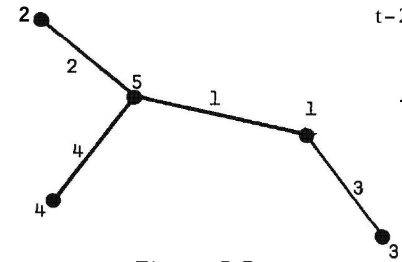

Figure 5–C

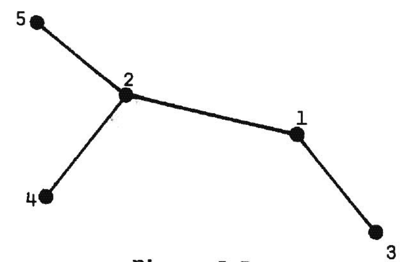

Figure 5-D

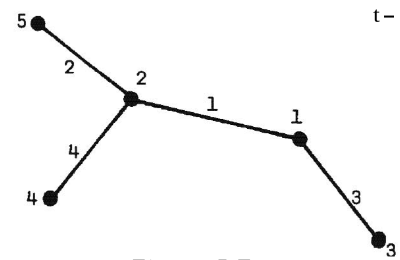

Figure 5–E

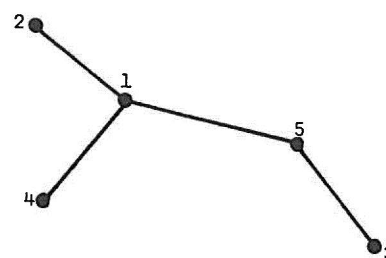

Figure 5–F

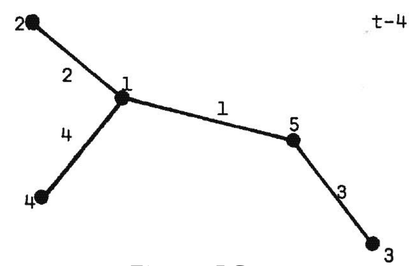

Figure 5–G

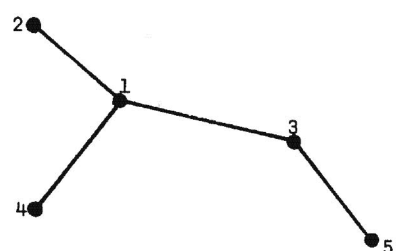

Figure 5–H

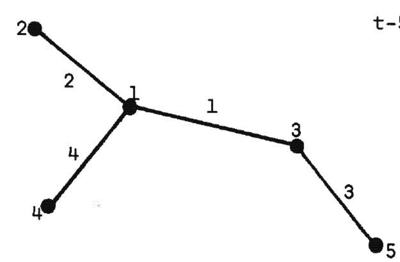

Figure 5–1

We find that the same, unique edge-labelling is produced. This is because the label of any edge is the same as one of the vertices adjacent to that edge, and the algorithm is designed to preserve the vertex-labelling as much as possible within the family of trees.

All of the *n.i.v.l.* trees in the family will be labelled by the same method of exchanging n and the label of the adjacent vertex for each new tree (Figures 5-B, D, F, H). n will eventually appear at each vertex and this will complete the family of n *n.i.v.l.* trees corresponding to one edge-labelled tree (Figures 5-A, C, E, G, I).

Observe that the family of n *n.i.v.l.* trees corresponding to a unique edge-labelled tree as defined by the second algorithm is the same as the collection of n *n.i.v.l.* trees which are produced from one edge-labelled tree in the first algorithm. Both algorithms produce the same family of trees.

The reversibility of the algorithm assures us that the family of n *n.i.v.l.* trees corresponding to one edge-labelled tree is complete. A 1 : n correspondence has been defined between edge-labelled and vertex-labelled trees and an n : 1 correspondence has been established between vertex-labelled and edge-labelled trees. Since it has already been proven that there are n^{n-2} *n.i.v.l.* trees on n vertices, there must be n^{n-3} *n.i.e.l.* trees on n vertices.

About the author. -

Julie Yancey graduated from Fort Lewis College at Durango, Colorado with a B.A. in Mathematics in 1984.

About the paper -

The problem discussed in the paper came up in a beginning graph theory course which Julie was taking in the Fall of 1982.

GRAFFITO

The theory of graphs is one of the few fields of mathematics with a definite birth date. The first paper on graphs was written by the Swiss mathematician Leonhard Euler (1707 - 1783) and it appeared in the 1736 volume of the publications of the Academy of Science in St. Petersburg (Leningrad).

Oystein Ore
Graphs and Their Uses

GENERATING ARBITRARILY HIGH-ORDER ROOT-FINDING METHODS

by Jeffrey M. Kubina
Youngstown State University

A common problem in applied mathematics is approximating the zeros of a continuous function. To be more precise, consider the following problem:

> Suppose f is a continuous function on [a, b], and suppose there exists a point p in [a, b] such that $f(p) = 0$. Given tolerance $\varepsilon > 0$, find an approximation q to p such that $|p - q| < \varepsilon$.

One way to solve this problem is to use a fixed-point method, which involves changing the equation $f(x) = 0$ into the equation $g(x) = x$, so that $g(p) = p$ implies $f(p) = 0$.

Given some initial approximation $p(0)$ to p, the sequence $\{p(n)\}$, defined by $p(n+1) = g(p(n))$, for $n \geq 0$, converges to p. If $f^{(3)}$ is continuous on [a, b] and $f'(p) \neq 0$, then there exists $d > 0$ such that, for every $p(0)$ in $(p - d, p + d) \subseteq [a, b]$, the fixed-point method $g(x) = x - f(x)/f'(x)$ converges quickly to p. This is a famous fixed-point procedure called Newton's Method. If f satisfies certain conditions on [a, b] it is possible to define functions g, from f, so that for any $p(0)$ in some neighborhood of p, the fixed-point method $g(x) = x$ will converge to p as rapidly as one wishes. In this article we will derive such fixed-point methods.

To begin, we first need to define the rate of convergence of a convergent sequence, and that of a fixed-point method. The following definitions do this.

<u>*Definition 1.*</u> If the sequence $\{p(n)\}$ converges to p, the sequence has mth order convergence if and only if $m \geq 1$ and the limit, as $n \to \infty$, of $|p(n+1) - p|/|p(n) - p|^m$ exists and is positive. If m = 1, 2, or 3, then the convergence is called linear, quadratic, or

cubic, respectively.

Definition 2. A fixed-point method $g(x) = x$ has mth order convergence if and only if every sequence generated by g and converging to p has mth order convergence.

An important characteristic of quadratically convergent fixed-point methods, which is proven in Faires and Burden (1985), is that the number of correct decimal digits of the fixed point is approximately doubled at each iteration. For example, if $|p(n) - p| \approx 0.001$, then $|p(n+1) - p| \approx 0.00001$. A similar statement may be made about higher order methods. For example, cubic convergent fixed-point methods approximately triple the number of correct decimal digits at each iteration.

The following theorem, which is a generalization of a theorem proven in Faires and Burden (1985), states the conditions necessary for a fixed-point method to have mth order convergence.

Theorem 1. Suppose the following is true of the function g:

1) for some integer $m \geq 2$, $g^{(m)}$ is continuous on $[a, b]$;
2) there exists a point p in $[a, b]$ such that $g(p) = p$;
3) for $k = 1, \dots, m-1$: $g^{(k)}(p) = 0$.

Then there exists $d > 0$ such that for every $p(0)$ in $(p - d, p + d) \subseteq [a, b]$ the fixed-point method $g(x) = x$ has mth order convergence.

Our objective is to define a fixed-point method which converges to the zero p, of f, with order m. One technique for deriving such a method is to assume that $g(x) = x + c_1(x)f(x) + c_2(x)[f(x)]^2 + \dots + c_{m-1}(x)[f(x)]^{(m-1)}$, and then determine the functions $c_1, c_2, \dots, c_{m-1}$ by forcing g to satisfy the criteria of Theorem 1. Notice that we do not know the restrictions necessary on f to ensure such a method exists. However, as the derivation of the functions $c_1, c_2, \dots, c_{m-1}$ proceeds, the restrictions necessary on f can be determined. As an example, we will derive a quadratically converging fixed-point method.

Derivation 1. Assume $g(x) = x + c(x)f(x)$. In this derivation we will make g satisfy the third, second, and then the first criterion of Theorem 1.

The third criterion of Theorem 1 requires that $g'(p) = 0$.

Assuming c' and f are continuous on $[a, b]$, $g'(p) = 1 + c(p)f'(p)$ is zero if and only if $c(p) = -1/f'(p)$. Hence, we must require $f'(p) \neq 0$ and we define $c = -1/f'$. It is possible that f may have zeros on $[a, b]$. Since $f'(p) \neq 0$, by the continuity of f', there exists an $r > 0$ such that, for every x in $(p - r, p + r) \subseteq [a, b]$, $f'(x) \neq 0$. Let $[c, d] = [p - r/2, p + r/2]$, then g' is continuous on $[c, d]$.

The second criterion of Theorem 1 requires that $g(p) = p$. Since g is continuous on $[c, d]$, $g(p) = p + f(p)/f'(p) = p$.

The first criterion of Theorem 1 requires $g^{(2)}$ to be continuous on $[c, d]$. Since $g(x) = x - f(x)/f'(x)$, $f^{(3)}$ must be continuous on $[c, d]$.

By Theorem 1, if $f^{(3)}$ is continuous on $[c, d]$, and $f'(p) \neq 0$, then there exists a $d > 0$ such that, for every $p(0)$ in $(p - d, p + d) \subseteq [c, d]$, the fixed-point method $g(x) = x - f(x)/f'(x)$ converges quadratically to p. This is Newton's Method. Hence, provided f satisfies certain conditions, Newton's Method will yield quadratic convergence.

The following corollary gives a more direct procedure for determining the functions $c_1, c_2, \dots, c_{m-1}$, and states the restrictions on f necessary to ensure the fixed-point method $g(x) = x$ has mth order convergence.

Corollary 1. Let the function f have a zero p in $[a, b]$. Suppose further that for some integer $m \geq 2$, $f^{(2m-1)}$ is continuous on $[a, b]$, and for every x in $[a, b]$, $f'(x) \neq 0$. Define the functions $c_0, c_1, \dots, c_{m-1}$ and g as follows:

1) $c_0(x) = x$;
2) for $k = 1, \dots, m-1$: $c_k = -c'_{k-1}/(k f')$;
3) $g = c_0 + c_1 f + c_2[f]^2 + \dots + c_{m-1}[f]^{m-1}$.

Then there exists a $d > 0$ such that, for every $p(0)$ in $(p - d, p + d) \subseteq [a, b]$, the fixed-point method $g(x) = x$ has mth order convergence.

Proof. To prove the corollary we will show that the function g, defined in statement (3) above, satisfies the three criteria of Theorem 1.

The first criterion of Theorem 1 is that $g^{(m)}$ be continuous on $[a, b]$. To prove this we first show that $c_0^{(m)}, c_1^{(m)}, \dots, c_{m-1}^{(m)}$ are all continuous on $[a, b]$. Obviously, $c_0^{(m)} = 0$ is continuous on $[a, b]$. Since the highest derivative of f in c_1 is f', the highest derivative of f in c_2 is f''. Inductively, the highest derivative of f in c_k is $f^{(k)}$.

Hence, the highest derivative of f in $c_k^{(m)}$ is $f^{(m+k)}$. Because k is at most $m-1$ and because $f^{(2m-1)}$ is continuous on $[a, b]$, each $c_k^{(m)}$ is continuous on $[a, b]$. Since $g = c_0 + c_1 f + c_2[f]^2 + \dots + c_{m-1}[f]^{m-1}$, $g^{(m)}$ is continuous on $[a, b]$.

The second criterion of Theorem 1 requires that $g(p) = p$. Since $g^{(m)}$ is continuous on $[a, b]$, g is continuous on $[a, b]$. Hence, $g(p) = p$ since $c_0(p) = p$ and $f(p) = 0$.

The last criterion of Theorem 1 requires that for $k = 1, 2, \dots, m-1$, $g^{(k)}(p) = 0$. Since

$$g(x) = \sum_{k=0}^{m-1} c_k(x)[f(x)]^k,\quad g'(x) = 1 + \sum_{k=1}^{m-1} \{c_k'(x)[f(x)]^k + kc_k(x)f'(x)[f(x)]^{k-1}\}.$$

From statement (2) of the corollary

$$g'(x) = 1 + \sum_{k=1}^{m-1} \{c_k'(x)[f(x)]^k - c_{k-1}'(x)[f(x)]^{k-1}\}$$

$$= c_{m-1}'(x)[f(x)]^{m-1}.$$

Since $g'(x) = c_{m-1}'(x)[f(x)]^{m-1}$, $g'(p) = c_{m-1}'(p)[f(p)]^{m-1} = 0$. Because g' has a zero of multiplicity $m-1$ at p (that is, there exists a function h such that $g'(x) = h(x)(x-p)^{m-2}$), the next $m-2$ derivatives of g' will also have zeros at p. Hence, for $k = 1, 2, \dots, m-1$, $g^{(k)}(p) = 0$.

By Theorem 1, there exists some $d > 0$ such that, for every $p(0)$ in $(p-d, p+d) \subseteq [a, b]$, the fixed-point method $g(x) = x$ has mth order convergence.

Notice that higher order methods increase the number of functional evaluations, multiplications, and divisions of the fixed-point function. Hence, generally, it is better to use Newton's Method, which compared to higher order methods generally requires the least amount of computational effort to achieve any desired degree of accuracy.

REFERENCES

1. Burden, R. and J. Faires, *Numerical Analysis, 3rd Edition*, PWS Publishers, Boston, 1985.

About the author -

Jeff Kubina is a senior at Youngstown State University and is majoring in mathematics. He plans to continue his study of mathematics in graduate school.

About the paper -

Jeff presented the contents of the paper at the National Pi Mu Epsilon Meetings at the University of Oregon in Eugene last August. For his encouragement in the preparation of the paper for publication, Jeff offers "special thanks to Dr. J. Douglas Faires of Youngstown State University for his time and inspiration."

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GRAFFITO

A teacher affects eternity; he can never tell where his influence stops.

Henry Adams

NON-NEGATIVE INTEGER SOLUTIONS OF $\sum_{i=1}^{n} X_i = k$

by Marie Coffin
South Dakota State University

In several contexts, (for example in number theory and Bose-Einstein equations), it is necessary to find the number of non-negative integer solutions of the equation $\sum_{i=1}^{n} X_i = k$, where $k \geq 0$. The usual proof offered involves a combinatorial scheme of arranging k objects in n boxes. We present a straightforward proof that uses induction on the number of variables.

Theorem. The equation $X_1 + X_2 + X_3 + \ldots + X_n = k$, where k is a on-negative integer, has $\binom{k+n-1}{n-1}$ non-negative integer solutions.

We will use the following result

(i) $\sum_{i=0}^{n} \qquad = \qquad)$, where r is any non-negative integer.

First, we introduce some notation. Let $T_{n,k}$ be the number of non-negative integer solutions to $X_1 + X_2 + \ldots + X_n = k$. To prove the theorem, we will show that $T_{n,k} = \binom{k+n-1}{n-1}$. We proceed by induction on n.

Let n = 1. The equation $X_1 = k$ has exactly one solution; that is, $T_{1,k} = 1 = \binom{k}{0} = \binom{k+1-1}{1-1}$, so the theorem is true for n = 1. Now assume that for $n \geq 1$, $T_{n,k} = \binom{k+n-1}{n-1}$, where k is a non-negative

integer. Consider the equation $X_1 + X_2 + \ldots + X_{n+1} = k$. Choose X_1 to be any integer between 0 and k inclusive, say $X_1 = a$, $0 \leq a \leq k$, then $X1 + X2 + \ldots + X_{n+1} = k$ if and only if $a + X_2 + \ldots + X_{n+1} = k$; or

(ii) $X2 + X_3 + \ldots + X_{n+1} = k - a$

Since equation (ii) has n variables, the induction hypothesis applies, and (ii) has $T_{n,k-a} = \binom{k-a+n-1}{n-1}$ solutions. Then

(iii) $T_{n+1,k} = \sum_{\text{all } a} T_{n,k-a}$ (see below)

$= \sum_{a=0}^{k} \qquad)$ by the induction hypothesis

$= \sum_{i=0}^{k} \binom{i+n-1}{n-1}$ changing the notation,

or

$T_{n+1,k} = \binom{k+n-1+1}{n+1+1}$ by (i)

$= \binom{k+(n+1) \quad 1}{(n+1)-}$, and the induction is complete.

Therefore, $X_1 + X_2 + \ldots + X_n = k$ has $\binom{k}{\ }$ non-negative integer solutions, where k is a non-negative integer.

To justify the summation process used in (iii), let S be the set of non-negative integer solutions to $X_1 + X_2 + \ldots + X_n = k$. S is a finite set of ordered n-tuples. Let $(b_1, b_2, \ldots, b_n)$ denote an arbitrary element of S, and

$$Sa = \{(b_1, b_2, \ldots, b_n) : b_1 = a\}.$$

$\{Sa \mid 0 \leq a \leq k\}$ forms a partition of S. Because {Sa} is a partition of S, the number of elements in S is the sum of the number of elements in each Sa over all a.

This summation process may be further clarified by considering the case n = 3. Choose $X_1 = a$, $0 \leq a \leq k$, then $X_1 + X_2 + X_3 = k$ if and only if $a + X_2 + X \ = k$, or $X_2 + X_3 = k - a$, which by the induction hypothesis, has $\binom{k-a+1}{1}$ solutions. Therefore,

$$T_{3,k} = \sum_{\text{all } a} \binom{k-a+1}{1} = \sum_{a=0}^{k} (k-a+1) = \sum_{i=1}^{k+1} (i)$$

$$= \frac{(k+2)(k+1)}{2} = \binom{k+2}{2} = \binom{k+n-1}{n-1}, \text{ with } n = 3.$$

About the author -

Marie Coffin is a senior mathematics major at South Dakota State University.

About the paper -

In this paper, Marie offers a proof by mathematical induction of a well-known result which is usually established by a combinatorial argument. The paper was submitted for publication in the Journal - in Marie's behalf - by one of her teachers, Dr. Robert Lacher, Professor of Mathematics and Statistics.

NATIONAL PAPER COMPETITION

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GRAFFITO

Clearly, the Hungarian educational system has been the most successful for pure mathematics; it's a model that ought to be studied very carefully because it works.

Donald Knuth
Two Year College Mathematics Journal
Volume 13, January 1982

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OXFORD, OHIO

SEPTEMBER 27-28, 1985

WE INVITE YOU TO JOIN US! THERE WILL BE SESSIONS OF THE STUDENT CONFERENCE ON FRIDAY EVENING AND SATURDAY AFTERNOON. FREE OVERNIGHT LODGING FOR ALL STUDENTS WILL BE ARRANGED WITH MIAMI STUDENTS. EACH STUDENT SHOULD BRING A SLEEPING BAG. ALL STUDENT GUESTS ARE INVITED TO A FREE FRIDAY EVENING PIZZA PARTY SUPPER AND SPEAKERS WILL BE TREATED TO A SATURDAY NOON PICNIC LUNCH. TALKS MAY BE ON ANY TOPIC RELATED TO MATHEMATICS, STATISTICS OR COMPUTING. WE WELCOME ITEMS RANGING FROM EXPOSITORY TO RESEARCH, INTERESTING APPLICATIONS, PROBLEMS, SUMMER EMPLOYMENT, ETC. PRESENTATION TIME SHOULD BE FIFTEEN OR THIRTY MINUTES.

WE NEED YOUR TITLE, PRESENTATION TIME (15 OR 30 MINUTES), PREFERRED DATE (FRIDAY OR SATURDAY) AND A 50 (APPROXIMATELY) WORD ABSTRACT BY SEPTEMBER 20, 1985.

PLEASE SEND TO

PROFESSOR MILTON D. COX
DEPARTMENT OF MATHEMATICS AND STATISTICS
MIAMI UNIVERSITY
OXFORD, OHIO 45056

WE ALSO ENCOURAGE YOU TO ATTEND THE CONFERENCE ON "STATISTICS" WHICH BEGINS FRIDAY AFTERNOON, SEPTEMBER 27. FEATURED SPEAKERS INCLUDE MYLES HOLLANDER, RICHARD L. SCHEAFFER, AND RONALD D. SNEE. CONTACT US FOR MORE DETAILS.

PUZZLE SECTION

Edited by
Joseph D. E. Konhauser

The PUZZLE SECTION is for the enjoyment of those renders who are addicted to working ***doublecrostics*** *or who find an occasional mathematical puzzle attractive. We consider mathematical puzzles to be problems whose solutions consist of answers immediately* ***recognizable*** *as correct by simple observation and requiring little formal proof. Material* ***submitted*** *and not used here will be sent to the Problem Editor if deemed appropriate for the PROBLEM DEPARTMENT.*

Address ail proposed puzzles and puzzle solutions to Prof. Joseph D. E. ***Konhauser,*** *Mathematics and Computer Science Department,* ***Macalester*** *College, St.* ***Paul,*** *Minnesota 55105. Deadlines for puzzles appearing in the Fail Issue* ***will*** *be the next* ***February*** *15, and for puzzles appearing in the Spring Issue will be the next September IS.*

Mathacrostic No. 20

Proposed by Joseph P. E. Konhauser
Macalester College, St. Paul, Minnesota

The word puzzle on pages 116 and 117 is a keyed anagram. The 265 letters to be entered in the diagram in the numbered spaces will be identical with those in the 30 keyed *words* at the matching numbers. The key numbers have been entered in the diagram to assist in constructing the solution. When completed, the initial letters of the *words* will give the name of an author and the title of a book; the completed diagram will be a quotation from that book. For an example, see the solution to the last mathacrostic on page 115.

SOLUTION

Mathacrostic No. 19. (See Fall 1984 Issue) *(Proposed by Joseph D. E. Konhauser, Macalester College, St. Paul, Minnesota).*

Words:

A. H-shaped
B. Ornament
C. Double
D. Graph theory
E. Empathy
F. Shibboleth
G. At the bat
H. Lambda-calculus
I. Apotheosis
J. Nerd
K. The Met
L. Uvarovite
M. Rheostat
N. Indeed
O. Nephelometer
P. Golden
Q. Tower of Hanoi
R. Hexenbesen
S. Event
T. Eldest hand
U. Nuthatch
V. Ineffable twaddle
W. Growth Form
X. Method
Y. Adiabolist

First Letters: HODGES, ALAN TURING: THE ENIGMA

Quotation: ***... he had found an unsolvable phobtem, and ... had dealt the death-blow to the, Hilbert programme. He. had shown that mathematics could never be. exhausted by any finite set of procedures. He had gone to the. heart of the phobtem, and settled it with one simple, elegant observation.***

Solved by: Jeanette Bickley, Webster Groves High School, MO; Betsy Curtis, Meadville, PA; Charles R. Diminnie, St. Bonaventure University, NY; Victor G. Feser, Mary College, Bismarck, ND; Robert Forsberg, Lexington, MA; Robert C. Gebhardt, County College of Morris, Randolph, NJ; Allan Gilbertson, University of Maryland, MD; Dr. Theodor Kaufman, Nassau Hospital, Mineola, NY; Lt. Timothy B. Killam, USAF, Offutt Air Force Base, NE; Henry S. Lieberman, John Hancock Mutual Life Insurance Co., Boston MA; Robert Prielipp, The University of Wisconsin-Oshkosh, WI; Stephanie Sloyan, Georgian Court College, Lakewood, NJ.

1 d	2 V		3 N	4 W	5 X	6 b	7 R		8 J	9 E	10 a	11 F	12 L	13 P
14 G	15 D	16 b	17 Z	18 T		19 E	20 c	21 M	22 d	23 a	24 Q	25 S		26 Z
27 O		28 A	29 B	30 Z	31 Y	32 F	33 S	34 c	35 b	36 U		37 D	38 P	
39 E	40 M	41 N		42 G	43 H	44 D	45 J	46 O		47 Q	48 b	49 T		50 B
51 c	52 A	53 H	54 F	55 L		56 b	57 T	58 I		59 Q	60 R	61 a	62 A	
63 O	64 M		65 d	66 L	67 Y	68 Q	69 R	70 F		71 I	72 P		73 H	74 W
75 R	76 F	77 c		78 I	79 N	80 d	81 E	82 S		83 W	84 b	85 c	86 X	87 L
88 I	89 D	90 J	91 U	92 Y	93 N	94 d		95 T	96 R		97 N	98 C		99 K
100 P	101 V		102 X	103 R	104 d	105 J	106 E	107 U	108 Q	109 V		110 P	111 A	
112 L	113 K	114 N	115 F	116 X	117 Z	118 D		119 W	120 K		121 I		122 U	123 c
124 X	125 d	126 R	127 B	128 K		129 W	130 V		131 A	132 N	133 P	134 U	135 L	
136 O	137 F		138 K	139 J	140 b	141 D	142 G		143 I	144 Z		145 R	146 X	147 F
148 L	149 Y	150 P	151 W	152 K	153 A		154 D	155 L	156 R	157 E	158 d	159 a		160 Q
161 M	162 N	163 K	164 b	165 R	166 L	167 D		168 C	169 N		170 V	171 b	172 S	
173 U	174 N	175 O	176 W	177 L		178 X	179 J	180 b	181 H	182 A		183 G	184 E	185 J
186 W	187 N	188 Y	189 B		190 a	191 V		192 T	193 R	194 I		195 R	196 Z	197 F
198 U	199 L	200 S	201 H	202 Q	203 O	204 E	205 J		206 a	207 I		208 b	209 c	
210 F	211 U	212 X	213 N	214 d		215 T	216 K	217 R		218 W	219 d	220 c	221 A	222 G
	223 F	224 T	225 C		226 D	227 K	228 L	229 E	230 S	231 A	232 X	233 R	234 V	
235 a	236 Z		237 b	238 L	239 M		240 Y	241 F	242 P		243 R	244 N	245 b	246 L
	247 I	248 C		249 G	250 a	251 K		252 H	253 B	254 O		255 Z	256 T	
257 L	258 D	259 R	260 B	261 b	262 J	263 U	264 E	265 P						

Definitions	Words
A. Unquestionable (comp.)	182 52 28 221 153 231 62 131 111
8. A rich supply	260 127 253 29 50 189
C. Double point of a curve	248 168 225 98
D. German relay computer designer (b. 1910) (full name)	141 37 15 258 226 154 44 89 167 118
E. To unite by apposition or contact	157 9 229 264 184 19 204 39 106 81
F. Forerunner of the modern slide rule (2 wds.)	197 223 210 115 54 147 76 32 241 137 11 70
G. To proceed or move in a diagonal or sideways manner	142 183 42 249 14 222
H. A fitting curve named after a draftsman's aid	181 53 73 43 201 252
I. By the very nature of the case (2 wds.)	143 78 194 71 207 121 88 58 247
J. Tropical parasite which is a competitor for the title "world's largest infloresence"	179 139 262 105 90 45 205 8 185
K. Archimedean "pump"	99 216 163 152 120 251 227 113 128 138
L. Title of Naboth Moseley's 1970 full-scale biography of Charles Babbage (2 wds.)	66 148 112 12 228 238 257 177 199 166 135 246 87 155 55
M. Figures that have been squared by using the give-and-take principle	40 21 64 161 239
N. Basis of Archimedes' mechanical method (4 wds.)	93 79 3 174 169 114 97 162 41 213 244 187 132
0. Equivalent relative growth of parts such that size relations remain constant	203 46 136 254 63 27 175
P. Court-declared (Honeywell vs Sperry-Rand, October 19, 1973) inventor of the electronic computer (b. 1904) (last name only)	150 265 100 242 110 13 133 72 38
Q. Played by primitive people, it might be the world's oldest game	160 202 108 68 47 24 59
R. Directions as to how sentences already known as true may be transformed so as to yield new true sentences (3 wds.)	156 126 60 7 217 75 195 193 233 96 69 103 243 165 145 259
S. Ball and wheel apparatus for showing the relative positions and motions of the bodies in the solar system	33 230 200 172 82 25
T. Psychological thriller	215 192 18 49 57 224 95 256
U. Early randomizers, used in the Middle East around 3600 B.C.	263 36 134 173 91 198 211 122 107
V. Reserve fund (2 wds.)	191 130 101 170 2 234 109
W. For an orbit, curve defined by the velocity vectors when drawn from one point	218 74 186 119 151 4 176 83 129
X. Clearly demonstrable	146 212 5 102 178 124 86 232 116
Y. The version of any game in which the game objective is reversed	188 149 240 67 92 31
Z. A conclusion inferred	196 117 17 255 30 26 236 144
a. ____/Russell = formalistic/Hilbert = intuistic/Brouwer	23 190 159 206 61 235 250 10
b. "To ____ the ____ -- that is the mathematician's am" J. Keyser (2 wds.)	16 171 180 208 261 6 237 164 48 35 84 56 140 245
c. Inferior; of low quality (coup.)	209 220 77 51 34 85 20 123
d. A therapeutic system of treatment by manipulation and without the use of drugs	158 219 80 214 104 65 22 125 1 94

COMMENTS ON PUZZLES 1 - 7, FALL 1984

For *Puzzle # 1*, only the proposer, **John M. Howell,** and **Victor G. Feser** submitted solutions. The number of distinguishable squares is 70. **Feser** and **Howell** provided arrangements of the 70 squares into rectangles 7x10. Both provided arrangements of the 64 squares, none containing four different colors, in 8x8 arrays. **Feser** also provided a 5x14 rectangle and said that 1x70 and 2x35 rectangles were easy to obtain. The Editor will send photocopies of the arrangements upon request. For *Puzzle. # 2*, **Leonard Volovets** and **David Ehren** submitted $39_{57} = 264 = 180$. **Ehren** also submitted $75_{39} = 426_8 = 278$. **Marc Farley** contributed $9_{3_{5_7}} =$

4 = 84. The interpretation of **Farley's** solution will be left to the reader. *Puzzle # 3* drew the largest number of responses. These varied from particular solutions to formulas for generating families of solutions, The quartet (1, 22, 41, 58), including 1, was among the solutions obtained by most of the respondents. Only three readers responded to *Puzzle # 4*. A composite of solutions from **John H Scott, Robert Sartini** and **Stephen W. Nelson** goes as follows: cb = cbab = cabb = acbb = abacbb = baacbb = babaacbb = bbaaacbb = bbaacabb = bbacaabb = bbcaaabb = bbcaabab = bbcaabba = bbcababa = bbcaba = bbca = bbac = babc = bc. Three readers responded to *Puzzle # 5*. **Victor G. Feser** referred to Problem 566 in the Journal of Recreational Mathematics on "prime chains." **Feser's** contribution toward the solution of Problem 566 appeared on pages 311-312 of Volume 10, No. 4, 1977-78, where the extension of "prime chains" to "prime rings" is discussed. **David Ehren** showed that if N - 1, N + 1 and 2N - 1 are primes, the arrangement

$$N,\ 1,\ N-2,\ 3,\ N-4,\ 5,\ \ldots\ ,\ 2,\ N-1,\ N$$

is a "prime ring." One sum of adjacent members of the ring is 2N - 1. The rest are either N - 1 or N + 1. *Puzzle. # 6* drew responses from **Robert Prielipp** and **Robert Sartini.** **Sartini** submitted the set (11, 17, 20, 22, 23, 24). All 63 non-empty subsets have different sums of elements. **Prielipp** reported that the problem is not new. In the course of "searching for something else, he found the Sartini-submitted set as part of the solution to problem E1062 (proposed by **Leo Moser).** The solution appears on pages 713-714 of the December 1953 issue. The other part of the problem was to show that no seven positive integers, strictly less than 25, can have sums of all subsets different. *Puzzle # 7* drew responses from six readers. All agreed, but not all proved, that 14 is the largest <u>balanced</u> number. **Victor Feser's** argument follows: "The largest balanced number is 14. The example 10 shows that 'between 1 and N' is to be taken strictly: 1 and N are not included. Therefore, a balanced number must be even. Now 16 is unbalanced, in favor of composites. For any larger N, the new integers to be considered must be at least half composite (the even ones !), so the imbalance remains."

<u>List of solvers</u>: **Edward Aboufadel** (*3*), **Patrick Costello** (*3*), **David Ehren** (*3,5*), **Mark Evans** (*6,7*), **Marc Farley** (*2*), **Victor G. Feser** (*1,5,7*), **Allan Gilbertson** (*3*), **John M Howell** (*1,3,7*), **Thomas M Mitchell** (*3*), **Stephen W. Nelson** (*3,4,7*), **Robert Prielipp** (*3,6,7*), **Robert Sartini** (*4,6*), **John Howe Scott** (*2,3,4,5,7*), **Brian Varney** (3) **and Leonard Volovets** (2).

PUZZLES FOR SOLUTION

1. *Proposed by John M. Howell, Littlerock, California.*

Which of the following alphametics can be solved with all one-digit integers?

Base 5	Base 6	Base 7	Base 8	Base 9	Base 10
$\begin{array}{r} AB \\ +\ C \\ \hline DE \end{array}$	$\begin{array}{r} AB \\ +\ CD \\ \hline EF \end{array}$	$\begin{array}{r} AB \\ +\ CD \\ \hline EFG \end{array}$	$\begin{array}{r} ABC \\ +\ DE \\ \hline FGH \end{array}$	$\begin{array}{r} ABC \\ +\ DEF \\ \hline GHI \end{array}$	$\begin{array}{r} ABC \\ +\ DEF \\ \hline GHIJ \end{array}$

2. *Proposed by Victor G. Feser, Mary College, Bismarck, North Dakota.*

What is the next integer in the sequence: 1, 3, 5, 7, 9, 15, 17, 21, 27, 31, 33, ... ?

3. *Proposed by J. O. E. Konhauser, Macalester College, St. Pout, Minnesota.*

Is there a five-digit number abcde such that 9 x abcde = edcba?

4. *Proposed by J. D. E. Konhauser, Macalester Cortege, St. Paul, Minnesota.*

In acute-angled triangle ABC the point D is the foot of the altitude from A to BC. If AD = BC, dissect the triangle into four pieces which can be reassembled to form a square.

5. *Proposed by J. D. E. Konhauser, Macalester College, St. Paul, Minnesota.*

Are there three positive integers such that the sum of the squares of any two of them is a perfect square?

6. *Proposed by J. D. E. Konhauser, Macalester College, St. Paul, Minnesota.*

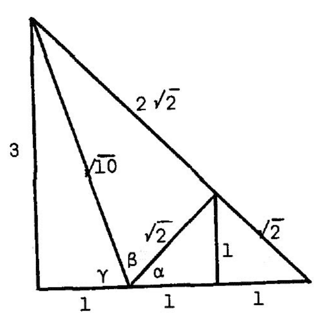
Here is a first-class puzzler. Posthaste, give the next two terms in the sequence: 2, 3, 4, 5, 6, 8, 10, 13, 15, 18, ...

7. *Proposed by J. D. E. Konhauser, Macalester College, St. Paul, Minnesota.*

With a pair of compasses draw a circle on a plane. Then, without changing the opening of the compasses, draw a circle on a sufficiently large sphere. Which circle encloses the larger area?

LETTER TO THE EDITOR:

Dear Editor:



$\alpha + \beta + \gamma = \pi$ (straight angle)

$\arctan 1 + \arctan 2 + \arctan 3 = \pi$

John M. Howell
Littlerock, California

PROBLEM DEPARTMENT

Edited by Clayton W. Dodge
University of Maine

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

All communications should be addressed to C. W. Dodge, Math. Dept., University of Maine, Orono, ME 04469. Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly identified with name and address. Solutions to problems in this issue should be mailed by December 15, 1985.

Problems for Solution

587. *Proposed by Morris Katz, Macwahoc, Maine.*

As a tribute to an Editor Emeritus of this department, find positive integers x and y, with $y > 2$, such that $x^y = BANKOFF$.

588. *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, Pennsylvania.*

Find all solutions to the quadratic congruence

$$x^2 \equiv -1 \pmod{m}$$

where m is of the form $m = (rn \pm 1)^2 + r^2$.

589. *Proposed by Joyce W. Williams, University of Lowell, Massachusetts.*

The integers 7, 3, and 10 are related by

$$7^3 = 3^5 + 10^2.$$

Is this the only set of positive integers that satisfies the relation

$$a^3 = b^5 + c^2?$$

Find all solutions.

590. *Proposed by Emmanuel O. C. Imonitie, Northwest Missouri State University, Maryville.*

Find all solutions to the simultaneous equations

$$2^{x+y} = 6^y \quad \text{and} \quad 3^{x-1} = 2^{y+1}.$$

591. *Proposed by Charles W. Trigg, San Diego, California.*

Find all three-term arithmetic progressions of three-digit primes in the decimal system with first and last terms that are permutations of the same digit set and with only four consecutive digits involved in the three terms of each progression.

592. *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.*

Find all 2 by 2 matrices A whose entries are distinct non-zero integers such that for all positive integers n the absolute value of the entries of A^n are all less than some finite bound M.

593. *Proposed by Joe Van Austin, Emory University, Atlanta, Georgia.*

Russian roulette is played with a gun having n chambers, in which k bullets are placed at random $(0 < k < n)$. Find the expected number of tries until the first bullet is fired if the chambers are spun

(i) before each shot.

(ii) only before the first shot.

594. *Proposed by R. S. Luthar, University of Wisconsin Center, Janesville.*

Prove that

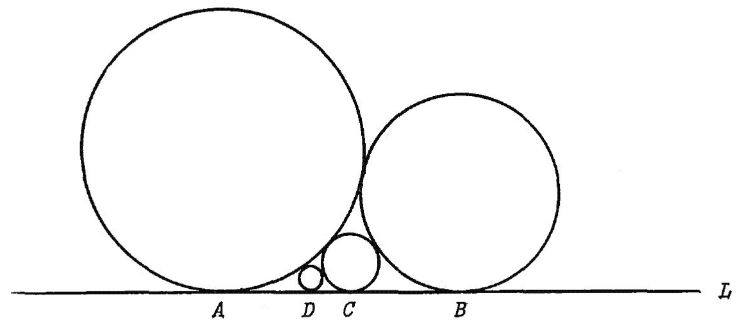
$$\int_0^1 x^x \ln x \, dx = -\int_0^1 x^x \, dx.$$

595. *Proposed by Harry Nelson, Livermore, California.*

If the integers from 1 to 5000 are listed in equivalence classes according to the number of written characters (including blanks and hyphens) needed to write them out in full in correct English, there are exactly forty such non-empty classes. For example, class "4" contains 4, 5, and 9, since FOUR, FIVE, and NINE are the only such numbers that can be written out with exactly four characters. Similarly, class "42" contains 3373, 3377, 3378, 3773, 3777, 3778, 3873, 3877, and 3878. Find the unique class "n" that contains just one number.

596. *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.*

Two circles are externally tangent and tangent to a line L at points A and B. A third circle is inscribed in the curvilinear triangle bounded by these two circles and L and it touches L at point C. A fourth circle is inscribed in the curvilinear triangle bounded by line L and the circles at A and C and it touches the line at D. Find the relationship between the lengths AD, DC, and CB.



597. *Proposed by Stanley Rabinowitz, Digital Equipment: Cow., Nashua, New Hampshire.*

Find the smallest n such that there exists a polyhedron of non-zero volume and with n edges of lengths $1, 2, 3, \ldots, n$.

598. *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, Pennsylvania.*

Establish the formula

$$D^n(e^{rx}\cos ax) = e^{rx}\cos ax\,[r^n - \binom{n}{2} r^{n-2} a^2 + \binom{n}{4} r^{n-4} a^4 - \ldots]$$
$$+ e^{rx}\sin ax\,[-\binom{n}{1} r^{n-1} a + \binom{n}{3} r^{n-3} a^3 - \binom{n}{5} r^{n-5} a^5 + \ldots]$$

and find the corresponding formula for

$$D^n\ (e^{rx}\ \sin ax).$$

599. *Proposed jointly by Gregg Patruno, Princeton University, New Jersey, and Murray S. Klamkin, University of Alberta, Edmonton, Canada.*

Prove that

$$\frac{\cos^2 x \cos y}{\cot^2 x \cot^2 y} = \frac{\cos^2 x - \cos^2 y}{\cot^2 x - \cot^2 y}$$

and generalize this result by finding under what conditions on functions f and g it is true that

$$\frac{f(x)\cdot f(y)}{g(x)\cdot g(y)} = \frac{f(x) - f(y)}{g(x) - g(y)}.$$

Solutions

544. [Fall 1983, Spring 1984] *Proposed by Charles W. Trigg, San Diego, California.*

The S.P.F.A. (Society for Persecution of Feline Animals) established a

P U R *R*
F R E E
A R E A at its headquarters.

In the word square each letter uniquely represents a decimal digit, and each word and acronym represents a square integer. What are these squares?

Solution by Vicki Schell, Pensacola Junior College, Florida.

The only four-digit squares with the same last two digits end in *00* or *44*. The only possibility for *44* without repeating a *4* or another digit is *3844*. Therefore **PURR** and *FREE* must be *3844* and *6400*. Now *AREA* must be *9409* and *SPFA* is *1369*. The square is

1 3 6 9
3 8 4 4
6 4 0 0
9 4 0 9 .

Also solved by FRANK P. BATTLES, *Massachusetts Maritime Academy, Buzzards Bay,* MARK EVANS, *Lousiville, KY,* VICTOR G. FESER, *Mary College, Bismark, ND,* ROBERT C. GEBHARDT, *Hopatcong, NJ,* RICHARD I. HESS, *Rancho Palos Verdes, CA,* EDWIN M. KLEIN, *University of, Wisconsin, Whitewater,* GLEN E. MILLS, *Pensacola Junior College, FL,* BOB PRIELIPP, *University of, Wisconsin-Oshkosh,* JOHN RUEBUSH, *St. Xavier High School., Cincinnati, OH,* STEPHANIE SLOYAN, *Georgian Court College, Lakewood, NJ,* W. R. UTZ, *Columbia, MO,* KENNETH M. WILKE, *Topeka, KS,* THEODORE G. ZAVALES, *Bergenfield, NJ, and the* PROPOSER.

*561. [Spring 19841 *Proposed by I. Don, Guiva Dam, California.*

For what values of n does $n!$ have *6* for its last nonzero digit?

I. *Solution by Harry Sedinger and Chuck Diminnie, St. Bonaventure University, New Yohk.*

More generally let $L(x)$ denote the last nonzero digit of x. Let the base *5* expansion of n be $a_k a_{k-1}\cdots a_1 a_0$. Let p be the number of twos in this expansion and q the number of fours. Let

$$t = (p + 2q + (a_1 + a_5 + \ldots) + 2(a_2 + a_6 + \ldots)$$
$$+ 3(a_3 + a_7 + \ldots)) \pmod 4.$$

We then have $L(0!) = L(1!) = 1$ and for $n > 1$,

$$L(n!) \equiv (6)(2^t) \pmod{10}.$$

Let $n > 1$ and take

$$n = a_0 + 5b_0,$$

$$b_0 = a_1 + 5b_1,$$

$$\ldots$$

$$b_{k-2} = a_{k-1} + 5b_{k-1},$$

$$b_{k-1} = a_k.$$

If $x = a + 5b$, then

$$(1)\qquad x! = f(a,b)\cdot g(b)\cdot 10^b\cdot b!$$

where

$$f(a,b) = \frac{(a+5b)!}{(5b)!} \quad\text{and}\quad g(b) = \frac{(5b)!}{(5)(10)\cdots(5b)\cdot 2^t}$$

Repeated applications of (1) give

$$n! = f(a_0,b_0)\cdot g(b_0)\cdot 10^{b_0}\cdot f(a_1,b_1)\cdot g(b_1)\cdot 10^{b_1}\cdots$$
$$\cdot f(a_{k-1},b_{k-1})\cdot g(b_{k-1})\cdot 10^{b_{k-1}}\cdot b_{k-1}!$$
$$= f(a_0,b_0)\cdot f(a_1,b_1)\cdots f(a_{k-1},b_{k-1})\cdot g(b_0)\cdot g(b_1)\cdots$$

$$\cdot g(b_{k-1})\cdot 10^{b_0+b_1+\cdots+b_{k-1}}\cdot a_k!.$$

Since $L(10x) = L(x)$, we have

$$L(n!) = L(f(a_0,b_0)\cdot f(a_1,b_1)\cdots f(a_{k-1},b_{k-1})\cdot g(b_0)\cdot g(b_1)\cdots \cdot g(b_{k-1})\cdot a_k!).$$

It is easily seen that if x is even, $z \equiv y \pmod 5$, and $y \not\equiv 0 \pmod 5$, then $L(xy) = L(xz)$. Since $f(a_i,b_i) \equiv a_i! \pmod 5$ and

$$g(b) \equiv (1\cdot 2\cdot 3\cdot 4)^b/2^b \equiv 12^b \equiv 2^b \pmod 5,$$

it follows that

$$L(n!) = L(a_0!\cdot a_1!\cdots a_k!\cdot 2^{b_0+b_1\cdots+b_{k-1}})$$

Since $0! \equiv 1! \equiv 3! \equiv 1 \pmod 5$, $2! \equiv 3 \pmod 5$, and $4! \equiv 2^2 \pmod 5$, then

$$L(n!) = L(2^{(p+2q+b_0+b_1+\cdots+b_{k-1}) \pmod 4}).$$

Now we also have that

$$b_0 + b_1 + \cdots + b_{k-1} = c_1a_1 + c_2a_2 + \cdots + c_ka_k$$

where $c_1 = 1$ and $c_{i+1} = 5c_i + 1$, so that $c_{i+1} \equiv c_i + 1 \pmod 4$.

Thus

$$p + 2q + b_0 + b_1 + \cdots + b_k$$
$$(p + 2q + a_1 + 2a_2 + \cdots + ka_k) \pmod 4$$
$$\equiv (p + 2q + (a_1 + a_5 + \cdots) + 2(a_2 + a_6 + \cdots) + (a_3 + a_7 + \cdots)) \pmod 4.$$

Clearly $L(n!) = 2, 4, 6,$ or 8, and the desired result follows. Finally, $L(n!) = 6$ whenever $n > 1$ and $t = 0$.

11. *Comment by John C. Mairhuber, University of, Maine, Ohono.*

Starting with any multiple of 5, the next four values of $L(n!)$ are obtained successively by multiplying by 1, 2, 3, and 4. This is evident for n an even multiple of 5; for odd multiples of 5 it follows from the fact that $L(n!)$ is even and that $6 \equiv 1$, $7 \equiv 2$, $8 \equiv 3$, and $9 \equiv 4 \pmod 5$.

It is of interest to note that the formula for $L(n!)$ does not have periodicity, though subcycles recur. Thus the values of $L(n!)$ for $n=2$ to 624 are repeated from $n = 627$ to 1249. In general, the

values from $n = 2$ to $5^{4k} - 1$ recur from $5^{4k} + 2$ to $2\cdot 5^{4k} - 1$.

Also wived by RICHARD I. HESS, Rancho Palos Verdes, CA, and JOHN C. MAIRHUBER, University of, Maine, Orono. One incorrect solution was d o received.

562. [Spring 1984] *Proposed by Walter Blumberg, Coral Springs, Florida.*

Prove than $\tan 1^o \tan 61^o = \tan 3^o \tan 31^o$.

Amalgam of, solutions submitted independently by HAO-NHIEN QUI VU, Purdue University, Lafayette, Indiana, and KENNETH M. WILKE, Topeka, Kansas.

We get that

$$\tan 3x = \frac{\tan x\,(3 - \tan^2 x)}{1 - 3\tan^2 x} \text{ from } \tan(a + b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}.$$

Also we have that

$$\tan(30^o + x) = \frac{\tan 30^o + \tan x}{1 - \tan 30^o \tan x} = \frac{1/\sqrt{3} + \tan x}{1 - (1/\sqrt{3})\tan x}$$

$$= \frac{1 + \sqrt{3}\tan x}{\sqrt{3} - \tan x}$$

and

$$\tan(60^o + x) = \frac{\sqrt{3} + \tan x}{1 - \sqrt{3}\tan x}$$

Now we combine these results to get the more general statement

$$\tan 3x \tan(30^o + x) = \tan x\,\frac{3 - \tan^2 x}{1 - 3\tan^2 x}\,\frac{1 + \sqrt{3}\tan x}{\sqrt{3} - \tan x}$$

$$= \tan x\,\frac{\sqrt{3} + \tan x}{1 - \sqrt{3}\tan x} = \tan x \tan(60^o + x),$$

which holds for all x for which none of x, $3x$, $30^o + x$, and $60^o + x$ is equal to an odd multiple of 90^o. The desired identity follows when $x = 1^o$.

Also solved by GEORGE W. BARRATT, Maryville, MO, FRANK P. BATTLES, Massachusetts Maritime Academy, Buzzards Bay, ANTONE R. COSTA, Old Saybhook, CT, RUSSELL EULER, Northwest Missouri State University, Maryville, JACK GARFUNKEL, Flushing, NY, ROBERT C. GEBHARDT, Hopatcong, *NJ*, RICHARD I. HESS, Rancho *Palos Verdes*, CA, JOHN M. HOWELL, *Littlerock*, CA, JAMES JOHNSTON, *St. Bonaventure University*, NY, RALPH KING, *St. Bonaventure University*, NY, EDWIN M. KLEIN, *University of Wisconsin, Whitewater*, HENRY S. LIEBERMAN, Waban, MA, GLEN E. MILLS, *Pensacola Junior College*, FL, BILL OLK, *Clintonville, WI*, PI MU EPSILON PROBLEM SOLVING TEAM, *Louisiana State University*, Baton Rouge, BOB PRIELIPP, *University of Wisconsin-Oshkosh*, HARRY SEDINGER, *St. Bonaventure University*, NY, WADE W. SHERARD, *Furman University*, *Greenville*, SC, and the. PROPOSER.

563. [Spring 1984] *Proposed by Morris Katz, Macwahoc, Maine.*

There is a unique solution to this odd alphametric when TEN is divisible by 9 and when *TEN* is taken either odd or even (I've forgotten which).

$$\begin{array}{r} TWELVE \\ TEN \\ \underline{TEN} \\ THIRTY \end{array}$$

Solution by Glen E. Mills, Pensacola Junior College, Florida.

Let a, b, and c be the carries into the tens', hundreds', and thousands' columns respectively. Also let *XX* denote that we have arrived at a contradiction. Since $c = 1$ or 2, then $E = 8$ or 9 and $I = 0$ or 1.

Let $E = 8$, so $c = 2$ and $I = 0$. Then $(N,Y) = (3,4)$, $(4,6)$, $(7,2)$, or $(9,6)$:

If $(N,Y) = (3,4)$, then $T = 7$ and $V = 0$. *XX*

If $(N,Y) = (4,6)$, then $T = 6$. *XX*

If $(N,Y) = (7,2)$, then $T = 3$, but then $c \neq 2$. *XX*

If $(N,Y) = (9,6)$, then $T = 1$, but then c # 2. *XX*

Therefore $E = 9$ and $I = 0$ or 1.

If $I = 0$, then $(N,Y) = (2,3)$, $(3,5)$, $(4,7)$, $(6,1)$, $(7,3)$, or $(8,5)$. The latter three cases all lead to $T = V$, a contradiction. Similarly $(2,3)$ and $(4,7)$ lead to contradictions. Only $(3,5)$ yields a solution,

$$\begin{array}{r} 619479 \\ 693 \\ \underline{693} \\ 620865 \end{array} \quad \text{in which } TEN \text{ is odd.}$$

Now let $I = 1$, which implies $c = 2$ and $(N,Y) = (2,3)$, $(3,5)$, $(4,7)$, $(7,3)$, or $(8,5)$. Here the latter two cases yield the contradiction $T = V$. Also $(3,5)$ produces a contradiction. The remaining two cases lead to the solutions

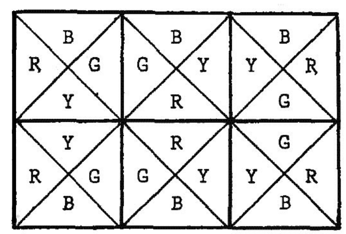
$$\begin{array}{r} 759489 \\ 792 \\ \underline{792} \\ 761073 \end{array} \quad \text{and-} \quad \begin{array}{r} 529869 \\ 594 \\ \underline{594} \\ 531057. \end{array}$$

Since *TEN* is even in these last two solutions, we have the unique solution given above when *TEN* is odd.

Also solved by MARK EVANS, *Louisville*, KY, VICTOR G. FESER, *Mary College, Bismark, ND*, ROBERT C. GEBHARDT, *Hopatcong, NJ*, RICHARD I. HESS, Rancho *Palos Verdes*, CA, PI MU EPSILON PROBLEM SOLVING TEAM, *Louisiana* State. *University*, Baton Rouge., CHARLES W. TRIGG, San *Diego*, CA, KENNETH M. WILKE, Topeka, KS, and the. PROPOSER.

564. [Spring 19841 *Proposed by Charles W. Trigg, San Diego, California.*

A tetrachromatic square is a square in which each of the four triangles formed by drawing the diagonals has a different color. With four specific different colors, six distinct tetrachromatic squares can be formed, not counting rotations. The six distinct tetrachromatic unit squares can be assembled into a2-by-3rectangle with matching colors on the edges that come into contact. The rectangle then contains seven solidly colored squares. This may be done in a variety of ways, one of which is shown in the figure.



Show that in any matched-edge assembly:

a) There are never only two colors of solidly colored squares;

b) The assembly can never have central symmetry; and

c) The perimeter of the rectangle can never consist of unit segments of just two alternating colors.

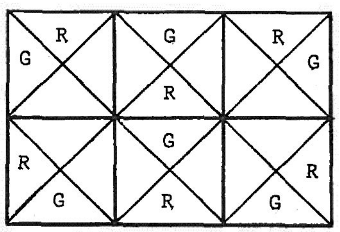
(For a related problem, see problem 282 [Fall 1973, pp. 480-1].)

Solution by Richard I. Hess, Rancho Palos Verdes, California.

a) There can never be only two colors in the seven solid squares because they include 14 triangles and there are only 6 triangles of each color.

b) Central symmetry would require three pairs of congruent squares, contradicting the requirement that the six squares must be distinct.

c) If the perimeter has just two alternating colors, say red and green, then the four tiles with those colors adjacent must occupy the four comer squares, leaving the two squares with red and green opposite to occupy the middle positions. But then we do not have a solidly colored square in the center, as shown in the accompanying figure.

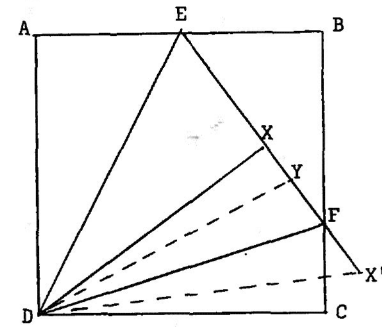


Also solved by the PROPOSER.

565. *[Spring* 1984] *Proposed by Walter Blumberg, Coral Springs, Florida.*

Let $ABCD$ be a square and choose point E on segment AB and point F on segment BC such that angles AED and DEF are equal. Prove that $EF = AE + FC$.

Solution by Henry S. Lieberman, Waban, Massachusetts.



Locate the point X on the ray EF such that $AE = EX$ and draw DX. Now $\angle AED = \angle DEX$ and $DE = DE$ so that triangles ADE and XDE are congruent by SAS. Then $AD = DX$ and DXE is a right triangle. If X does not lie between E and F, as depicted by X' in the figure, then $DX' > DF > DC = AD$, a contradiction. Hence X lies between E and F. Since $DX = DC$ and $DF = DF$, then right triangles DXF and DCF are congruent by HL. Now

$$EF = EX + XF = AE + FC.$$

The converse is also true. That is, if E lies on side AB and F on side BC of square $ABCD$ such that $EF = AE + FC$, then angles AED and DEF are equal.

There is a point X between E and F such that $AE = EX$. We wish to prove that $\angle DXE$ is a right angle, since then right triangles DAE and DXE will be congruent by HL and $\angle AED = \angle DEF$. So suppose that DXE is not a right angle, in particular suppose it is obtuse. Then $DX < AD$ from triangles DAE and DXE. Now locate point Y on EF so that DY is perpendicular to EF. Then $DY < DX < AD$. If Y lies outside the square (on EF extended), then $DY > DC = AD$, a contradiction. Thus Y lies on segment EF. Since $EY >$ EX, then $YF < XF = FC$. Therefore, from right triangles DYF and DCF, since leg $FC >$ leg FY, then leg $DC <$ leg DY, another contradiction. Hence angle DXE cannot be obtuse. A similar argument shows it cannot be acute either. Hence it is a right angle and the converse is established.

Also solved by ANTONE R. COSTA, *Old Saybrook, CT,* MARK EVANS, *Louisville, KY,* JACK GARFUNKEL, *Flushing, NY,* RICHARD I. HESS, *Rancho Palos Verdes, CA,* CAROLYN KAY, *Bridgewater State College, MA,* RALPH KING, *St. Bonaventure University, NY,* G. MAVRIGIAN, *Youngstown State University, OH,* BILL OLK, *Clintonville, WI,* PI MU EPSILON PROBLEM SOLVING TEAM, *Louisiana State University, Baton Rouge.,* BOB PRIELIPP, *University of Wisconsin-Oshkosh,* HARRY SEDINGER, *St. Bonaventure University, NY,* WADE H. SHERARD, *Furman University, Greenville, SC,* CHARLES W. TRIGG, *San Diego, CA,* KENNETH M. WILKE, *Topeka, KS, and the* PROPOSER. *The figure for this problem was drawn by* CHARLES W. TRIGG.

566. [Spring 1984] *Proposed by N. J. Kuenzi, University of Wisconsin-Oshkosh.*

If $\{p_n\}$ is a sequence of probabilities generated by the recurrence relation

$$p_{n+1} = p_n - \frac{1}{2} p_n^2 \qquad n \geq 0,$$

for which initial probabilities p_0 does $\lim_{n\to\infty} p_n$ exist?

I. Amalgam of solutions submitted independently by SYLVAIN BOIVIN, *Université du Quebec a Chicoutimi, Canada,* RUSSELL EULER, *Northwest Missouri State University, Maryville,* ROBERT C. GEBHARDT, *Hopatcong, New Jersey,* RICHARD I. HESS, *Rancho Palos Verdes, California,* EDWIN M. KLEIN, *University of Wisconsin, Whitewater,* HENRY S. LIEBERMAN, *Waban, Massachusetts,* PI MU EPSILON PROBLEM SOLVING TEAM, *Louisiana State University, Baton Rouge,* RICHARD QUINDLEY *and* I. PHILIP SCALISI, *Bridgewater State College, Massachusetts,* HARRY SEDINGER, *St. Bonaventure University, New York, and the* PROPOSER.

If $p_0 = 0$, then all $p_n = 0$. (If $0 < p_n \leq 1$, then

$$\frac{1}{2} \leq 1 - \frac{1}{2} p_n < 1, \qquad \text{so} \qquad 0 < p_{n+1} < p_n.$$

Hence $\{p_n\}$ is a monotone nonincreasing sequence bounded below by zero. Thus it has a limit L that must satisfy the relation

$$L = L - \frac{1}{2} L^2,$$

from which it follows that $L = 0$. That is, the sequence converges to 0 for all initial probabilities p_0.

II. Additional comment by Morris Katz, Macwahoc, Maine.

If $x > -1$, then

$$1 - x^2 < 1, \qquad \text{so} \qquad 1 - x < \frac{1}{1 + x}.$$

If $p > 0$, then there is a positive number m such that $p = 2/m$. Then we have

$$p(1 - \frac{p}{2}) < \frac{p}{1 + p/2} = \frac{2/m}{1 + 1/m} = \frac{2/m}{(m + 1)/m} = \frac{2}{m + 1}$$

It follows that for $n > 0$, we have $p_n < 1/n$. Since also $p_n \geq 0$, then $0 \leq \lim p_n \leq \lim (1/n) = 0$.

567. [Spring 1984] *Proposed by R. S. Luthar, University of Wisconsin-Janesville.*

Find the exact value of $\sin 20^\circ \sin 40^\circ \sin 80^\circ$.

I. Solution by Russell Euler, Northwest Missouri State University, Maryville.

Substituting $k = 9$ into the well-known identity

$$\prod_{s=1}^{k-1} \sin \frac{\pi s}{k} = \frac{k}{2^{k-1}} \qquad (k > 2)$$

yields

$$(1) \qquad \prod_{s=1}^{8} \sin \frac{\pi s}{9} = \frac{9}{256}$$

Using the identity $\sin x = \sin (180^\circ - x)$, equation (1) becomes

$$\frac{3}{4} \sin^2 20^\circ \sin^2 40^\circ \sin^2 80^\circ = \frac{9}{256}$$

and so

$$\sin 20^\circ \sin 40^\circ \sin 80^\circ = \frac{\sqrt{3}}{8}.$$

II. Solution by Leon Bankoff, Los Angeles, California.

In *Solutions of the Examples in Hall and Knight's Elementary Trigonometry* by H. S. Hall (London: MacMillan, 1953), answer 29 on page 47 states:

$$\sin 20^\circ \sin 40^\circ \sin 80^\circ = \frac{1}{2} \sin 20^\circ (\cos 40^\circ - \cos 120^\circ)$$

$$= \frac{1}{2} \sin 20^\circ (\frac{3}{2} - 2 \sin^2 20^\circ)$$

$$= \frac{1}{4} \sin 60^o = \frac{\sqrt{3}}{8}.$$

This last equality follows from the identity

$$\frac{1}{4} \sin 3x = \frac{1}{4} (3 \sin x - 4 \sin^3 x) = \frac{1}{2} \sin x \left(\frac{3}{2} - 2 \sin^2 x\right).$$

Also solved by FRANK P. BATTLES, *Massachusetts Maritime Academy, Buzzards Bay,* ANTONE R. COSTA, *Old Saybrook, CT,* JACK GARFUNKEL, *Flushing, NY,* ROBERT C. GEBHARDT, *Hopatcong, NJ,* RICHARD I. HESS, *Rancho Palos Verdes, CA,* JOHN M. HOWELL, *Littlerock, CA,* JAMES JOHNSTON, *St. Bonaventure University, NY,* RALPH KING, *St. Bonaventure University, NY,* EDWIN M. KLEIN, *University of Wisconsin, Whitewater,* HENRY S. LIEBERMAN, *Waban, MA,* GLEN E. MILLS, *Pensacola Junior College, FL,* BILL OLK, *Clintonville, WI,* PI MU EPSILON PROBLEM SOLVING TEAM, *Louisiana State University, Baton Rouge,* BOB PRIELIPP, *University of, Wisconsin-Oshkosh,* JOHN RUEBUSH, *Cincinnati, OH,* I. PHILIP SCALISI, *Bridgewater State College, MA,* HARRY SEDINGER, *St. Bonaventure University, NY,* WADE H. SHERARD, *Furman University, Greenville, SC,* W. R. UTZ, *Columbia, MO,* and *the* PROPOSER.

568. [Spring 1984] *Proposed by Robert C. Gebhardt, Hopatcong, New Jersey.*

Find a **simple expression** for the power series

$$1 + \frac{x}{2!} - \frac{x^2}{3!} - \frac{x^3}{4!} + \frac{x^4}{5!} + \frac{x^5}{6!} - \frac{x^6}{7!} - \frac{x^7}{8!} + \cdots .$$

Solution by Edwin M. Klein, University of, Wisconsin-Whitewater.

Let $f(x)$ denote the given series. From the expansions

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

and

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots ,$$

we see that $x \cdot f(x) = \sin x + 1 - \cos x$. Thus

$$f(x) = \frac{\sin x + 1 - \cos x}{x} \quad \text{for } x \neq 0 \text{ and } f(0) = 1.$$

Also solved by FRANK P. BATTLES, *Massachusetts Maritime Academy, Buzzards Bay,* SYLVAIN BOIVIN, *Universite du Quebec a Chicoutimi, Canada,* ANTONE R. COSTA, *Old Saybrook, CT,* RUSSELL EULER, *Northwest Missouri State University, Maryville,* MARK EVANS, *Louisville, KY,* RICHARD I. HESS, *Rancho Palos Verdes, CA,* HENRY S. LIEBERMAN, *Waban, MA,* G. MAVRIGIAN, *Youngstown State University, OH,* BILL OLK, *Clintonville, WI,* PI MU EPSILON PROBLEM SOLVING TEAM, *Louisiana State University, Baton Rouge,* BOB PRIELIPP, *University of Wisconsin-Oshkosh,* I. PHILIP SCALISI, *Bridgewater State College, MA,* HARRY SEDINGER, *St. Bonaventure University, NY,* W. R. UTZ, *Columbia, MO,* HAO-NHIEN QUI VU, *Lafayette, IN,* KENNETH M. WILKE, *Topeka, KS,* and *the* PROPOSER. *Not all solvers recognized the necessity to define $f(0)$.*

569. [Spring 1984] *Proposed by Robert C. Gebhardt, Hopatcong, Nw Jersey.*

a) Find the largest regular tetrahedron that can be folded from a square piece of paper (without cutting).

b) Prove whether it is possible to fold a regular tetrahedron from a square piece of paper without overlapping or cutting.

Solution by Charles W. Trigg, San Diego, California.

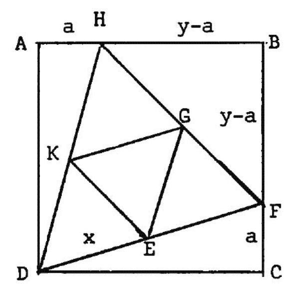


Figure 1

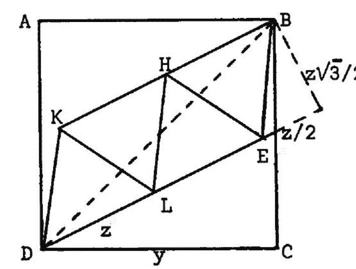
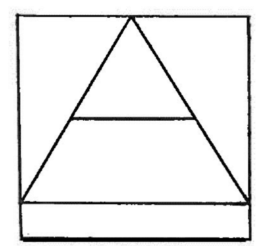


Figure 2



a) The regular tetrahedron has two nets: four equilateral triangles assembled into a larger equilateral triangle as in Figure 1, and four equilateral triangles constituting a parallelogram as in Figure 2.

Triangle *DHF* in Figure 1 is the largest equilateral triangle that can be inscribed in the square *ABCD*. Triangles *DAH* and *DCF* are congruent, so $AH = CF = a$. Then $HB = BF = y - a$, where y is the side of the square. Now $HF = DF$, so from right triangles *HBF* and *DCF*,

$$2(y-a)^2 = a^2 + y^2,$$

$$y^2 - 4ay + a^2 = 0,$$

$$a = (2 - \sqrt{3})y.$$

Now $DE = EF = x$, an edge of the tetrahedron. Then from triangle *DCF*,

$$(2x)^2 = y^2 + a^2 = y^2 + (2-\sqrt{3})^2y^2,$$

$$4x^2 = (8 - 4\sqrt{3})y^2,$$

$$x = \sqrt{2-\sqrt{3}}\ y = (\sqrt{6}-\sqrt{2})y/2 = 0.5176\ y.$$

In Figure 2, *DKBE* is the largest 4-triangle strip that can be inscribed in square *ABCD*. Then with y equal to the side of the square and z the edge of the tetrahedron, from the right triangle *DPF* we have

$$(y\sqrt{2})^2 = (z\sqrt{3}/2)^2 + (5z/2)^2 = 7z^2$$

so

$$z = \sqrt{2/7}\ y = 0.5346\ y,$$

the edge of the largest tetrahedron that can be folded from a square of side y.

b) Clearly, when the squares and the nets they contain that we have seen in Figures 1 and 2 are folded into tetrahedrons, overlapping occurs.

The sum of the face angles at a vertex of a regular tetrahedron is $3(60^\circ) = 180^\circ$. If no overlapping is to occur, one vertex must be at the midpoint of a side of the square, whereupon the edge of the tetrahedron is equal to one-half the side of the square. It is evident that to complete the tetrahedron overlapping must occur.

Also solved by the PROPOSER.

570. [Spring 19841 *Proposed by Richard I. Hub, Rancho Palos Verdes, California.*

The natural logarithm of a complex number $z = re^{i\theta}$ is defined by

$$\ln z = se^{i\lambda}$$

where

$$s = ((\ln r)^2 + \theta^2)^{\frac{1}{2}}, \qquad \lambda = \tan^{-1}(\theta/\ln r),$$

and

$$0 \le \lambda \le \pi/2 \text{ for } r \ge 1 \qquad \text{or} \qquad \pi/2 < \lambda < \pi \text{ for } 0 < r < 1$$

Find a number z_0 such that $\ln z_0 = z_0$.

Solution by Bill Olk, Clintonville, WI.

If $z = \ln z$, then

$$r = ((\ln r)^2 + \theta^2)^{\frac{1}{2}} \quad \text{and} \quad \theta = \tan^{-1}(\theta/\ln r),$$

from which it follows that

$$\ln r = \theta/\tan\theta \text{ and } r = e^{\theta/\tan\theta}$$

Now we have that

$$e^{\theta/\tan\theta} = ((\theta/\tan\theta)^2 + \theta^2)^{\frac{1}{2}}$$

which simplifies to

$$e^{\theta/\tan\theta}\sin\theta - \theta = 0.$$

Using Newton's method and a calculator we find that

$$\theta = 1.3372357 \quad \text{and hence} \quad r = 1.374557.$$

Also solved by ROBERT C. GEBHARDT, *Hopatcong, NJ, and the* PROPOSER.

571. [Spring 19841 *Proposed by Chuck Allison, Huntington Beach, California.*

Assume a pegboard with one line of holes numbered 1 through n. Find the probability of picking correspondingly numbered pegs one at a time at random and placing them in their corresponding holes contiguously. That is, if peg k is chosen first, then the second peg must be next to it, either number $k-1$ or $k+1$. If pegs $p, p+1, p+2, \ldots, q$ have already been chosen, the next peg must be either

$p - 1$ or $q + 1$, so that no gaps ever appear between pegs.

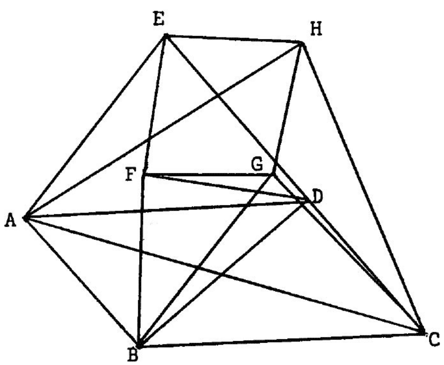
Solution by the Proposer.

There are as many ways of removing the pegs contiguously from a filled board as there are of filling it, since any successful sequence can be reversed. Then the first peg removed will be an end peg, which can be done in two ways. We now have a "reduced board" of $n - 1$ contiguous pegs, so again we must remove an end peg. This process continues until one peg remains, which can be removed in just 1 way. The board can be thus undone in 2^{n-1} ways, so the probability of undoing it as well as filling it contiguously is $2^{n-1}/n!$.

Also solved by RICHARD I. HESS, *Rancho Palos Verdes, CA,* EDWIN M. KLEIN, *University of Wisconsin, Whitewater,* HENRY S. LIEBERMAN (2 *solutions*), *Waban, MA, and* HARRY SEDINGER, *St. Bonaventure University, NY.*

572. [Spring 1984] *Proposed by Jack Garfunkel, Flushing, NY.*

Let $ABCD$ be a parallelogram and construct directly similar triangles on sides AD, BC, and diagonals AC and BD. See the figure, in which triangles ADE, ACH, BDF, and BCG are the directly similar triangles. What restrictions on the appended triangles are necessary for $EFGH$ to be a rhombus?



Solution by Morris Katz, Macwahoc, Maine.

Let $x = \angle DAE$ and $r = AE/AD$. Then a rotation through angle x with ratio r about point A carries DC to EH by considering similar triangles HAC and EAD. The same rotation about point B carries DC to FG by triangles GBC and FBD. Now FG and EH are equal and parallel, so $EFGH$ is a parallelogram. Since a rotation through angle ADE with ratio DE/AD about point D carries AB to FE by triangles FBD and EAD, then

$$EF = (DE/AD)AB = r \cdot AB = r \cdot CD = EH = FG$$

and $EFGH$ is a rhombus provided $DE/AD = r = AE/AD$. Thus the appended triangles must be isosceles.

Also solved by RICHARD I. HESS, *Rancho Palos Verdes, CA, and the* PROPOSER.

573. [Spring 1984] *Proposed by William S. Cariens, Loraine County Community College, Elyria, Ohio.*

Prove that when any parabola of the form

$$(1) \qquad y = x^2 + ax + b$$

is intersected by a straight line

$$(2) \qquad y = px + q,$$

then the sum of the derivatives of equation (1) at the two points of intersection is always twice the slope of the straight line.

Amalgam of essentially identical solutions submitted independently by ANTONE R. COSTA, *Old Saybrook, Connecticut,* JAMES JOHNSTON, *St. Bonaventure University, New York,* BOB PRIELIPP, *University of Wisconsin-Oshkosh,* I. PHILIP SCALISI, *Bridgewater State University, Massachusetts,* W. R. UTZ, *Columbia, Missouri, and* HAO-NHIEN QUI VU, *Purdue University, Lafayette, Indiana.*

To find the points of intersection, we must solve the equation

$$x^2 + ax + b = px + q.$$

The sum of the roots of this equation is $x_1 + x_2 = -(a - p)$.

Since $y' = 2x + a$, then the sum of the derivatives at these points is

$$y'(x_1) + y'(x_2) = 2x_1 + 2x_2 + 2a$$
$$= -2(a - p) + 2a = 2p,$$

which is twice the slope of the line.

Also solved by FRANK P. BATTLES, *Massachusetts Maritime Academy, Buzzards Bay,* MARC COCHRAN, *Rensselaer Polytechnic Institute, Troy, NY,* RUSSELL EULER, *Northwest Missouri State University, Maryville,* MARK EVANS, *Louisville, KY,* VICTOR G. FESER, *Mary College, Bismark, ND,* JACK GARFUNKEL, *Flushing, NY,* ROBERT C. GEBHARDT, *Hopatcong*, NY, RICHARD I. HESS, *Rancho Palos Verdes*, CA, RALPH KING, *St. Bonaventure University*, NY, HENRY S. LIEBERMAN, *Waban*, MA, GLEN E. MILLS, *Pensacola Junior College*, FL, BILL OLK, *Clintonville*, WI, PI MU EPSILON PROBLEM SOLVING TEAM, *Louisiana State University*, Baton Rouge, JOHN RUEBUSCH, *Cincinnati*, OH, WADE H. SHERARD, *Furman University*, *Greenville*, SC, PHILLIP J. SLOAN, *Pembroke State University*, NC, KENNETH M. WILKE, Topeka, KS, ONE UNSIGNED SOLVER, and the PROPOSER.

Late Solutions and Comments

Solutions to problem 552 by FRANK P. BATTLES, *Massachusetts Maritime Academy*, *Buzzards Bay*, and by ROGER PINKHAM, Ifoboken, NJ. *Solution to problem* 555 by CHARLES W. TRIGG, San *Diego*, CA.

556. [Fall 1983, Fall 1984] *Proposed by Richard 1. Hess*, *Palos Verdes*, *California*.

A normal pair of unbiased dice give a total of 2 through 12 according to the distribution 1, 2, 3, 4, 5, 6, 5, 4, 3, 2, 1. How should you change the spots on the dice so that the sums 2 through 12 and only those sums still occur but with as uniform a distribution as possible? (Minimize the sum of the squares of the deviations from completely uniform).

III. Comment by Roger Pinkham, *Hoboken*, NJ.

Two possible distributions were suggested in the published solutions: Howell's 3 3 3 3 3 6 3 3 3 3 3 and Hess' 2 3 4 4 3 4 3 4 4 3 2, and the question was raised as to which was more uniform. One common measure of uniformity is the amount of information $\Sigma p_i \ln p_i$ inherent in the distribution. In Howell's case this amount is 2.369 while that for Hess' solution is 2.370. Thus on these grounds Howell's solution is preferred.

GRAFFITO

We shall have to evolve.
problem solvers galore -
since each problem we *solve*
creates tin. problems mote.

Piet Hein
Short Giookb 1

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