PI MU EPSILON JOURNAL

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PI MU EPSILON JOURNAL
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MOSES IS YOUR GRANDFATHER AND OTHER APPLICATIONS OF THE GEOMETRIC SERIES

by Mark J. Dugopolski
Southern Louisiana University

This talk was given last spring at a meeting of, the National Council of, Teachers of Mathematics.

Recently, a student of mine showed me a proof, using the geometric series, that Moses is everyone's grandfather. A certain religious group was actually using a mathematical argument to prove that we are all descendants of Moses. Now if it had been Noah, it probably wouldn't have bothered me at all. Finite and infinite geometric series occur in a variety of interesting applications, but this was the first time I had ever seen a biblical application.

A finite geometric series is any sum of the form

$$a + ar + ar^2 + \dots + ar^{n-1},$$

where a and r are real numbers and n is a positive integer. If we let S_n stand for the sum and perform the multiply-and-add trick, we get

$$\begin{aligned} S_n &= a + ar + ar^2 + \dots + ar^{n-1} \\ -rS_n &= - ar - ar^2 - \dots - ar^{n-1} - ar^n \\ \hline (1 - r)S_n &= a - ar^n \\ S_n &= \frac{a(1 - r^n)}{1 - r}. \end{aligned}$$

This gives us a nice closed formula for the sum of the n terms. Now, with just a touch of calculus, we see that $\lim_{n\to\infty} r^n = 0$, provided $|r| < 1$. In this case, we let $S = \lim_{n\to\infty} S_n$ and we get $S = \dfrac{a}{1 - r}$, the sum of an infinite geometric series.

Everyone should plan for retirement, so let's consider the case of a worker who deposits \$100 every month for 40 years into an account paying 12% compounded monthly. This is called an annuity. How much does the worker have at the time of the last deposit? The last deposit receives no interest, the second to the last receives 1% for one month, the third to last receives 1% for two months, and so on. The worth of all these deposits is the following sum, which we easily recognize as a geometric series

$$\$100 + \$100(1.01)^1 + \$100(1.01)^2 + \ldots + \$100(1.01)^{479}$$
$$= \$100 \frac{1-(1.01)^{480}}{1-1.01} = \$1{,}176{,}477.25.$$

Quite a nice nest egg!

Not many of us plan for retirement 40 years in advance, but look what happens to the procrastinators. Saving \$100 per month for 20 years amounts to

$$\$100 \frac{1-(1.01)^{240}}{1-1.01} = \$98{,}925.54$$

which is less than one-tenth as much money.

For the person who would rather spend his \$100 per month on car payments, we can use the geometric series to see the size of loan \$100 per month would pay off in 48 months. To find the present value of these payments, we divide \$100 by the appropriate powers of 1.01

$$\$100(1.01)^{-1} + \$100(1.01)^{-2} + \ldots + \$100(1.01)^{-48}$$
$$= \$100(1.01)^{-1} \frac{1-((1.01)^{-1})^{48}}{1-(1.01)^{-1}} = \$3{,}797.40.$$

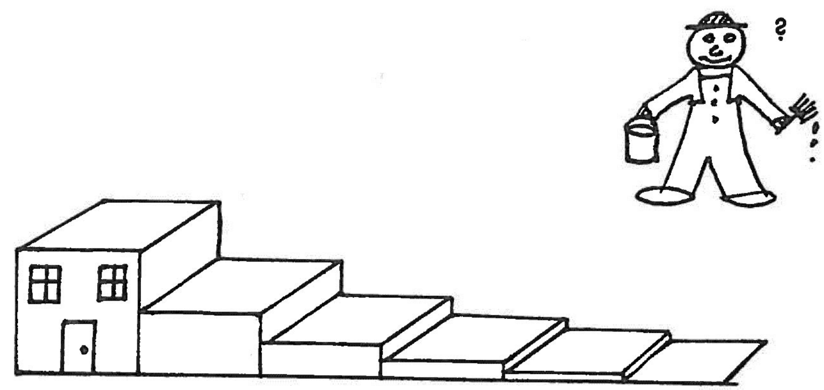
So you can spend your \$100 per month on a used car every four years, or be a millionaire in 40 years!

Probably the most basic application of geometric series occurs in the concept of rational numbers. The number 0.232323... is an infinitely repeating decimal. We can view this decimal as an infinite geometric series and find its sum to be the rational number 23/99.

$$0.232323\ldots = \frac{23}{100} + \frac{23}{100}\left(\frac{1}{100}\right) + \frac{23}{100}\left(\frac{1}{100}\right)^2 + \ldots$$

$$= \frac{23/100}{1-1/100} = \frac{23/100}{99/100} = \frac{23}{99}.$$

So far, we have seen that the geometric series is indispensable in financial planning and even in ordinary arithmetic. Now we will see how the infinite geometric series can be used to construct a maintenance-free house. This is a house that only a mathematician would build. The first room is one foot wide, one foot deep, and one foot high. As the family grows, rooms are added. The second room is the same as the first, but only one-half as high. This mathematician multiplies so well that rooms are added infinitely often, each room being one-half as high as the last.

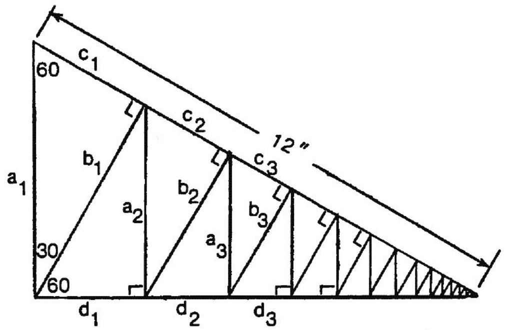


Now that the house is complete, we calculate the volume of the house by totalling the volumes of the rooms:

$$V = 1 + 1/2 + 1/4 + 1/8 + \ldots$$
$$= \frac{1}{1-1/2} = 2 \text{ cubic feet.}$$

This house has finite volume, but since it has infinitely many ceilings of one square foot each, it has infinite surface area. Thus we could fill the house with just 2 cubic feet of paint, but it would take infinitely many gallons of paint to paint the ceilings. Since there is no hope of painting it, we have a maintenance-free house.

Any discussion of the geometric series ought to have at least one purely geometric example. So, consider the 30-60-90 right triangle shown in the diagram which follows.



We leave it to the reader to show that the a's are terms of the geometric series, that the b's are terms of a geometric series, that the c's are terms of a geometric series and that the d's are terms of a geometric series.

So far, we have discussed geometry, arithmetic, financial planning and construction. What could be next? Gambling? Suppose we toss a coin until the first head appears. The probability that the first head appears on the first toss is 1/2, on the second toss 1/4, on the third toss 1/8, and so on. The sum of these probabilities is a geometric series:

$$1/2 + 1/4 + 1/8 + 1/16 + \ldots = \frac{1/2}{1 - 1/2} = 1.$$

The total of one shows us that this is a legitimate probability distribution, the geometric distribution.

Suppose we have a chance to play a simple game where the probability is 1/3 that we win \$6 and 2/3 that we win \$18. On the average, we will win

$$E = \$6(1/3) + \$18(2/3) = \$14.$$

This is the expected value of the game. How much would you be willing to pay in order to play this game? Paying anything less than \$14 would certainly be a bargain.

Now let's go back to tossing a coin until the first head appears. Suppose you win 2^n dollars if the first head appears on the nth toss. How much would you be willing to pay for the privilege of playing this game? You should certainly be willing to put up your maintenance-free house, your used car, and your 40-year annuity, since your expected winnings are infinite!

$$E = \$2(1/2) + \$4(1/4) + \$8(1/8) + \ldots$$

$$= \$1 + \$1 + \$1 + \ldots .$$

This is known as the Petersburg Paradox.

We finally get to the mathematical proof that Moses is your grandfather. Start counting your ancestors. To make things simple, don't count them all, just count yourself, your parents, your grandparents, your great grandparents, your great great grandparents, and so on. This looks like another geometric series

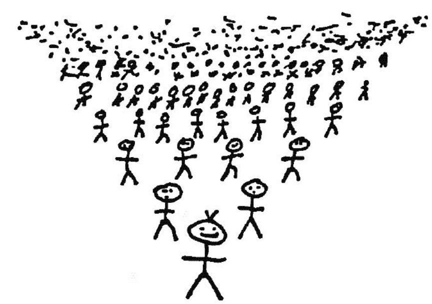
$$1 + 2 + 4 + 8 + 16 + 32 + 64 + \ldots .$$

Now suppose that a new generation occurs every 40 years, then in the last 2,000 years your family tree would have 50 generations, a conservative estimate. Since

$$1 + 2 + 2^2 + 2^3 + 2^4 + \ldots + 2^{50} = \frac{1(1 - 2^{51})}{1 - 2} =$$

$$2,251,799,813,000,000$$

you have over 2 quadrillion people in your family tree and we didn't count aunts and uncles!



Now today the earth has roughly 5 billion people and this is probably a much larger population than at any time in the past. Suppose the population of the earth is renewed each 40 years, then during the last 2000 years there would have been 50 times 5 billion or 250 billion inhabitants of the earth, a very generous estimate. But, you alone have over 2 quadrillion people in your family tree, during the same time. This is certainly a contradiction. Therefore, Moses is your grandfather.

AVERAGE LENGTH OF CHORDS DRAWN FROM A POINT TO A CIRCLE

by **Hung** *C. Li*
University of Southern Colorado

The average (mean) value of a population or a sample plays an important role in statistics. The process for finding the average value of a function in statistics uses an approach which is different from that used in calculus. In calculus, if g is an integrable function on a closed interval [a, b], then the average value of g on [a, b] is defined to be $\int_a^b g(x)dx/(b - a)$. The method depends upon the notion of the limit of an arithmetic mean. In statistics, we utilize the density function $f(x)$ of a random variable X. The average value of g on [a, bl is defined by $\int_{\infty}^{\infty} f(x)g(x)dx$, which is broader. Also, the technique allows us to easily find the standard deviation which describes the dispersion of the data in a population or a sample (see the remark at the end of the paper).

The problem we consider in this paper concerns the intersections of a pencil of straight lines with a circle. The segments of the lines within the circle form chords. We wish to find the average length of the chords using the statistical technique. Since the situation depends on the relative positions of a fixed point and a circle, the discussion falls into three cases.

Case I. The fixed point P is inside the circle with center C and radius r. Without loss of generality, we can take the fixed point P as the pole and the fixed ray PC as the polar axis. Let $PC = c < r$, where c and r are constants, then the polar coordinates of C are $(c,0)$ and the equation of the circle is

$$(1)\qquad \rho^2 - 2c\rho\cos\theta - (r^2-c^2) = 0$$

(see Figure 1.)

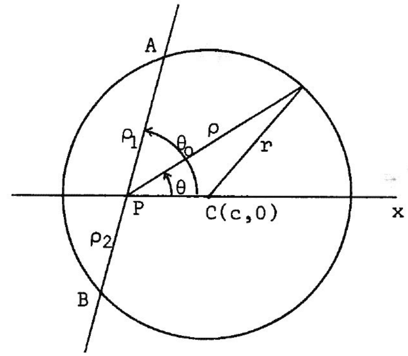


Figure 1

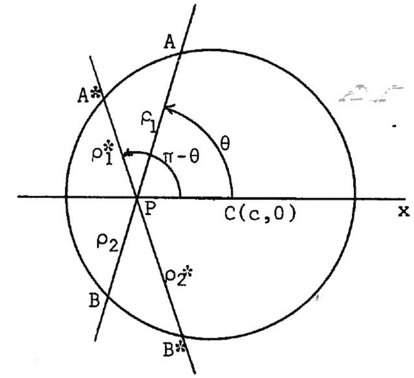


Figure 2

For any fixed θ, say $6 = \theta_0$, the locus of the point (ρ, θ_0) is the straight line AB which intersects the circle at A and B. Using (1), the directed distances from P to A and P to B are $\rho = c\cos 0_o \pm \sqrt{r^2 - c^2\sin^2 0_o}$. Let ρ_1 be the radius vector of $\vec{PA}$ and ρ_2 that of $\vec{PB}$ (see Figure 1).

Since $|c\cos\theta_o| < \sqrt{r^2 - c^2\sin^2 0_o}$, we have

$$\rho_1 = c\cos\theta_o + \sqrt{r^2 - c^2\sin\theta} > 0,$$

and

$$\rho_2 = c\cos\theta_o - \sqrt{r^2 - c^2\sin^2\theta_o} < 0.$$

If I is the length of the chord $\overleftrightarrow{AB}$, then

$$\ell = \rho_1 + |\rho_2| = 2\sqrt{r^2 - c^2\sin^2\theta_o}.$$

Since θ is arbitrary, we can drop the subscript and write

$$(2)\qquad I = \ell(\theta) = 2\sqrt{r^2 - c^2\sin^2 6}.$$

Since the circle is symmetric with respect to the polar axis, it is sufficient to consider 6 from 0 to π. Furthermore, if $0 \le \theta \le \pi/2$ then $\pi/2 \le \pi - 6 \le \pi$, $\overline{PA^*} = \rho_1^* = |\rho_2|$, $\overline{PB^*} = |\rho_2^*| = \rho_1$, and

$\ell^* = \rho_1^* + |\rho_2^*| = |\rho_2| + \rho_1 = \ell$ (see Figure 2). Thus we need only consider θ from 0 to $\pi/2$. Regard the pencil of straight lines produced by a line rotating counter-clockwise uniformly about the fixed point P and starting from the polar axis, then the polar angle θ is a random variable possessing the probability density function

$$f(\theta) = \begin{cases} 2/\pi, & 0 \le \theta \le \pi/2 \\ 0, & \text{otherwise.} \end{cases}$$

The average value μ of ℓ is defined by

$$(3) \qquad \mu = E[\ell(\theta)] = \int_0^{\pi/2} \ell f(\theta)d\theta = \frac{4r}{\pi}\int_0^{\pi/2} \sqrt{1-k^2\sin^2\theta}\,d\theta,$$

where

$$k^2 = (c/r)^2 < 1.$$

The integral in (3) is a complete elliptic integral of the second kind [1], and we can find the approximations of μ.

For instance, if $c = r/2$, then from (3) and tables of elliptic integrals of the second kind, (see [1], p. 533) we have

$$\mu = \frac{4r}{\pi}(1.4675) = 1.868r.$$

One special case of interest is the case when P is the center of the circle. Then $c = 0$ and $\mu = 2r$.

Case II. The fixed point P is on the circumference of the circle; that is, $c = r$. Then (1) and (2) reduce to $\rho = 2r\cos\theta$, $\ell = 2r\cos\theta$, and

$$(4) \qquad \mu = \frac{4r}{\pi}.$$

Case III. The fixed point P is outside the circle. In this case $c > r$. We need only investigate the lines which intersect the circle. Since the figure is symmetric about the polar axis, we need only consider θ from 0 to $\alpha = \sin^{-1}\frac{r}{c} < \pi/2$. For any θ, call the larger of two directed distances ρ_1 and the shorter ρ_2. Since $c > r$ and $c\cos\theta > \sqrt{r^2 - c^2\sin^2\theta} > 0$,

$$\rho_1 = c\cos\theta + \sqrt{r^2 - c^2\sin^2\theta},$$

and

$$\rho_2 = c\cos\theta - \sqrt{r^2 - c^2\sin^2\theta}.$$

Now

$$\ell = \ell(\theta) = \rho_1 - \rho_2 = 2\sqrt{r^2 - c^2\sin^2\theta},$$

which is the same form as (2), but now $c > r$ and $f(\theta) = 1/\alpha$, if $0 \le \theta \le \alpha$; otherwise, $f(\theta) = 0$. Therefore, the average value of ℓ, with respect to θ, is

$$(5) \qquad \mu = E[\ell(\theta)] = \int_0^{\alpha} \ell f(\theta)d\theta = \frac{2r}{\alpha}\int_0^{\alpha}\sqrt{1 - \left(\frac{c}{r}\right)^2\sin^2\theta}\,d\theta,$$

where $\alpha = \sin^{-1}(r/c)$.

The integral in (5) is the same as that in (3), except $(c/r)^2 > 1$. We need to change the variable, before we can use the tables of elliptic integrals.

Transform $\frac{c}{r}\sin\theta = \sin\phi$, then (5) takes the form

$$(6) \qquad \mu = \left[\frac{2r^2}{c\alpha} - \frac{2c}{\alpha}\right]\int_0^{\pi/2}\frac{d\phi}{\sqrt{1-k_1^2\sin^2\phi}} + \frac{2c}{\alpha}\int_0^{\pi/2}\sqrt{1-k_1^2\sin^2\phi}\,d\phi,$$

where $k_1^2 = (r/c)^2 < 1$.

The first integral in (6) is a complete elliptic integral of the first kind.

Example, if $c = 2r$, then $\alpha = \pi/6$. From (6) and tables of elliptic integrals (see [1], pp. 529 and 533), we obtain

$$\mu = \left\{\frac{6r}{\pi} - \frac{24r}{\pi}\right\}(1.6858) + \frac{24r}{\pi}(1.4675) = 1.552r.$$

If we regard c as parameter and allow it to vary, we have two special cases of interest. In one case, if $c \to r^+$ (or $c \to r$ in Case I), then $\alpha \to \pi/2$, $c/r \to 1$, and from (5) (or (3)), we obtain $\mu \to 4r/\pi$ which agrees with (4). In the second case, if $c \to +\infty$, then $\alpha \to 0$, $c\alpha = c \times \sin^{-1}\frac{r}{c} \to r$, and $\mu \to \pi r/2$ ([2]. In a circle, the average length of chords parallel to a given diameter is the same as the average length of chords and

Remark: The standard deviation σ in statistics is defined by

$$\sigma = \sqrt{E[(\ell(\theta))^2] - (E[\ell(\theta)])^2} = \sqrt{E(\ell^2) - \mu^2}.$$

For Case I.

$$E(\ell^2) = \int_0^{\pi/2} \ell^2 f(\theta)d\theta$$

$$= \frac{2}{\pi}\int_0^{\pi/2} 4(r^2 - c^2\sin^2\theta)d\theta = 4r^2 - 2c^2,$$

and hence $\sigma = \sqrt{E(\ell)^2 - \mu^2} = \sqrt{4r^2 - 2c^2 - p^2}$, where μ is defined by (3).

If $c = r/2$, we have $\sigma = \sqrt{3.5r^2 - (1.868r)^2} = 0.1028r$.

For Case II.

$$E(\ell^2) = \int_0^{\pi/2} \ell^2 f(\theta)d\theta = \frac{2}{\pi}\int_0^{\pi/2} 4r^2\cos^2\theta d\theta = 2r^2,$$

and

$$\sigma = \sqrt{2r^2 - (\tfrac{4r}{\pi})^2} = 0.6155r.$$

For Case III.

$$E(\ell^2) = \int_0^{\alpha} \ell^2 f(\theta)d\theta = \frac{1}{\alpha}\int_0^{\alpha} 4(r^2 - c^2\sin^2\theta)d\theta$$

$$= 4r^2 - 2c^2 + \frac{1}{\alpha}c^2\sin 2\alpha$$

and

$$\sigma = \{4r^2 - 2c^2 + \frac{1}{\alpha}c^2\sin 2\alpha - \mu^2\}^{\frac{1}{2}}$$ where μ is defined by (5).

If $c = 2r$, then $\alpha = \frac{\pi}{6}$ and $\sigma = \{(\frac{12\sqrt{3}}{\pi} - 4)r^2 - (1.552r)^2\}^{\frac{1}{2}} = 0.4552r$.

Why is the standard deviation in Case I much smaller than in Cases II and III? Because, in Case I, ℓ spreads only between $\sqrt{3}r$ and $2r$; but in Cases II and III, ℓ is spread widely between 0 and $2r$.

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2. Thomas, George B. and Finney, Ross L., *Calculus and Analytic Geometry, 5th ed.,* Addison-Wesley Publishing Co., 1980, p. 264.

THE QUASI-ISOSCELES QUADRILATERAL

by Jack Garfunkel
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and Clayton W. Dodge
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In the Fall 1981 issue of this Journal, we published an article entitled "The Equilic Quadrilateral" in which we studied a quadrilateral ABCD with AB = CD and angle B + angle C = 120 degrees. Following this theme, in the Fall 1984 issue, we considered "The Squarilic Quadrilateral", defined as a quadrilateral ABCD with AB = CD and angle B + angle C = 90 degrees. Although other special quadrilaterals could be considered, such as the "89° quadrilateral", we realized that many of the results of our previous papers can be derived as special cases, if we consider quadrilaterals in which only AB = CD is required. This is, therefore, the purpose of this paper.

Definition. A quadrilateral ABCD will be called *quasi-isosceles* if AB = CD and angle B + angle C is less than 180 degrees.

Theorem 1. If a quasi-isosceles quadrilateral is cyclic, then it is an isosceles trapezoid.

Proof. The proof is left to the reader. See Figure 1.

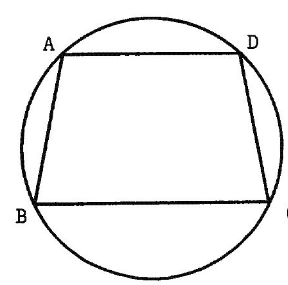

Figure 1

We introduce the following terminology in order to simplify the statements of our results.

Definition. a. For a given quadrilateral ABCD, a <u>special triangle</u> will be any triangle having its base angles add up to angle B + angle C. That is, its apex angle will be equal to angle A + angle D - 180°.

b. An <u>isosceles special triangle</u> is a special triangle whose base angles are equal, that is, each base angle is equal to (1/2)(angle B + angle C).

Theorem 2. Quadrilateral ABCD is quasi-isosceles if and only if special isosceles triangles erected upwardly on sides BC and AD have the same vertex.

Proof. Refer to Figure 2. Let ABCD be a quasi-isosceles quadrilateral, and let EBC be a special isosceles triangle erected upwardly. A rotation about point E through angle BEC carries triangle EBA into a congruent triangle ECD'. Because angle ABC + angle BCD = angle EBC + angle BCE, then angle EBA = angle ECD by subtraction. Also, CD = BA so that CD = CD'. Hence, points D and D' coincide and triangle EAD is special isosceles.

Let EBC and EAD be special isosceles triangles for quadrilateral ABCD. Then a rotation about E carries triangle EBA to triangle ECD, so BA = CD, and quadrilateral ABCD is quasi-isosceles. We leave it to the reader to prove that angle B + angle C < 180°.

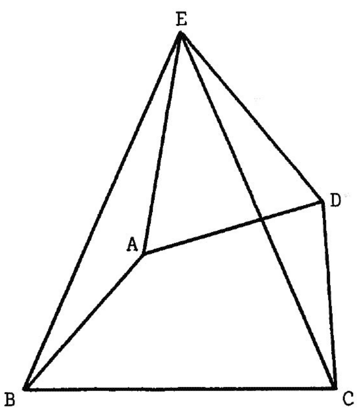

Figure 2

Theorem 3. In a quasi-isosceles quadrilateral ABCD, the midpoints M_3 and M_1 of sides AD and BC, respectively, and the midpoints M_2 and M_4 of the diagonals AC and BD, respectively, are vertices of a rhombus.

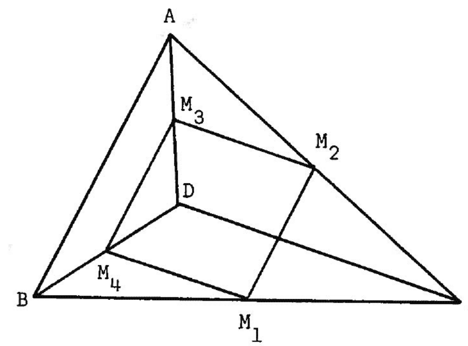

Figure 3

Proof. In Figure 3, from triangles BCD and ACD, we have

$$\overrightarrow{M_1M_4} = \overrightarrow{M_2M_3} = (1/2)\overrightarrow{CD}.$$

From triangles CAB and DAB, we get

$$\overrightarrow{M_1M_2} = \overrightarrow{M_3M_4} = (1/2)\overrightarrow{AB}.$$

Since we are given that AB = CD, $M_1M_2M_3M_4$ is a rhombus.

Theorem 4. Quadrilateral ABCD is quasi-isosceles if and only if the apexes of special isosceles triangles constructed outwardly on sides AB and CD and the midpoint of side AD are collinear.

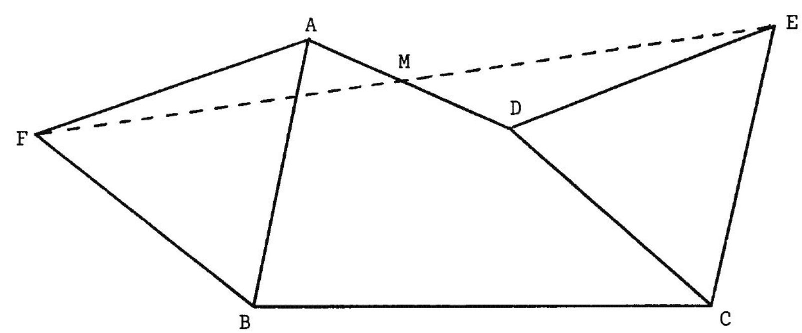

Figure 4

Proof. Refer to Figure 4. Let FBA and EDC be special isosceles triangles constructed outwardly on sides AB and CD of quasi-isosceles quadrilateral ABCD, and let M be the midpoint of AD. Recall that

$$A + B + C + D = 360^\circ,$$

thus

$$\begin{aligned} \text{angle } MDE &= 360^\circ - D - (1/2)(B + C) \\ &= A + B + C - (1/2)(B + C) \\ &= A + (1/2)(B + C) = \text{angle } FAM. \end{aligned}$$

Now, triangles FAM and EDM are congruent by S. A. S., so M is the midpoint of EF. Conversely, let M be the midpoint of EF and also of AD where FAB and ECD are special isosceles triangles for quadrilateral ABCD. Then triangles AFM and DEM are congruent. Then AB = CD and quadrilateral ABCD is quasi-isosceles. (We have angle B + angle C $< 180^\circ$, since FAB is a triangle and special for ABCD.)

Theorem 5. If special isosceles triangles are constructed on sides AB and DA (outwardly) and on DC (downwardly) of quasi-isosceles quadrilateral ABCD, then the join of their vertices determines a special isosceles triangle.

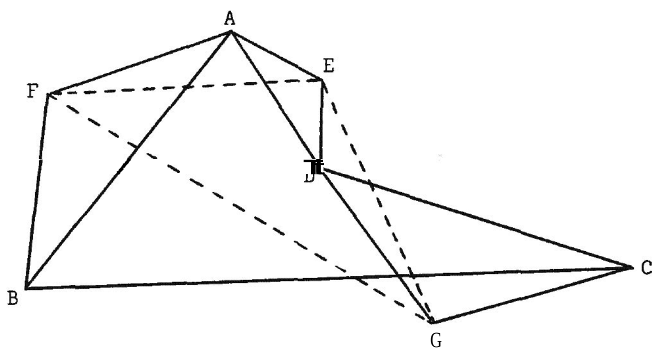

Figure 5

Proof. Refer to Figure 5. We have

$$\begin{aligned} \text{angle } FAE &= \text{angle } FAB + \text{angle } BAD + \text{angle } DAE \\ &= (1/2)(B + C) + A + (1/2)(B + C) \end{aligned}$$

$$\begin{aligned} &= A + B + C \\ &= 360^\circ - D \\ &= \text{angle } ADC = \text{angle } EDG \end{aligned}$$

since angle EDA = angle GDC. Because EA = ED and AF = DG, triangles EAF and EDG are congruent by S. A. S. Thus EF = EG and angle FEG = angle AED since a rotation about E through angle AED carries triangle EAF to triangle EDG.

Theorem 6. If special isosceles triangles QAD, RAC and PBD for quasi-isosceles quadrilateral ABCD are constructed upwardly on side AD and on diagonals AC and BD, then the apex vertices are collinear and Q is the midpoint of PR.

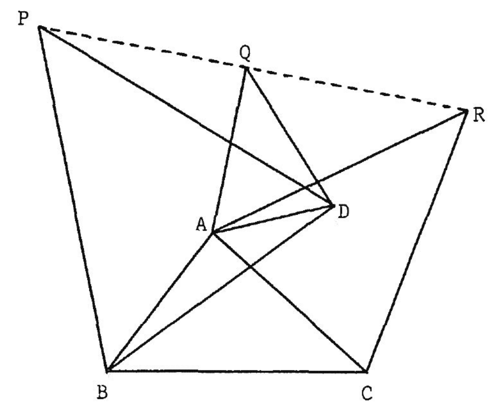

Figure 6

Proof. Refer to Figure 6. A rotation-stretch with center D, through angle BDP = (1/2)(B + C) and with ratio PD/BD carries segment BA to segment PQ, so the angle between BA and PQ is (1/2)(B + C). Hence, the angle between BC and PQ is

$$(1/2)(B + C) - B = (1/2)(C - B).$$

A similar rotation-stretch with center A, through angle CAR = (1/2)(B + C) and with ratio RA/RC (= PD/BD) maps CD to RQ. Hence, the angle between CD and RQ is

$$C - (1/2)(B + C) = (1/2)(C - B).$$

Thus, P, Q and R are collinear. Furthermore, PQ = QR since PQ/AB = PD/BD = RA/AC = QR/CD and AB = CD.

Theorem 7. If PAC and P'BD are isosceles triangles whose bases are the diagonals of quasi-isosceles quadrilateral ABCD and whose apex angles are each equal to angle B + angle C and which are oriented downward, then P and P' coincide.

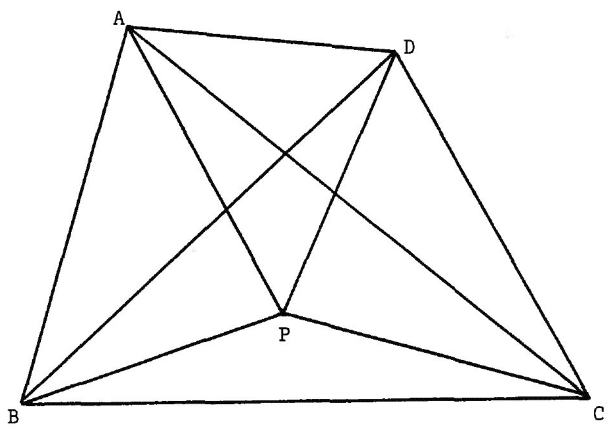

Figure 7

Proof. Refer to Figure 7. Let the perpendicular bisectors of AC and of BD meet at point P for given quasi-isosceles quadrilateral ABCD. Then PC = PA, PD = PB, and, since AB = CD, triangles PCD and PAB are congruent. Hence, angle ABP = angle CDP. Let x = angle PBD = angle PDB, then in triangle BDC we have

$$(\text{angle } B - \text{angle } ABP + x) + (x + \text{angle } PDC) + \text{angle } C = 180^\circ$$

so that $$2x + B + C = 180^\circ$$

and $$\text{angle } BPD = 180^\circ - 2x = B + C.$$

Similarly, from triangle ABC, we angle APC = B + C. The theorem follows.

Theorem 8. If special isosceles triangles EAB, GCB, ICD, GDA, FDB and HCA are constructed, all upwardly, on each side and on each diagonal of quasi-isosceles quadrilateral ABCD, then the vertices of these triangles determine a quasi-isosceles quadrilateral.

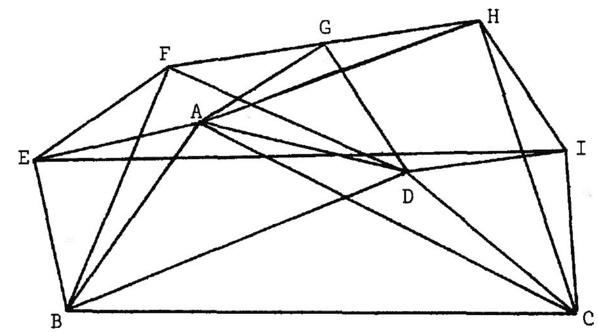

Figure 8

Proof. Refer to Figure 8. A rotation-stretch about point B, through angle DBF and with ratio EA/BA, carries AD to EF. A rotation-stretch about point C, through angle ACH and with ratio IC/DC, carries AD to HI. Since EA/BA = IC/DC, ye have FE = HI, so FEIH is a quasi-isosceles quadrilateral. Furthermore, the angle between EF and IH equals angle B + angle C since AD is rotated (1/2)(B + C) in either direction to get EF and IH.

Surely, more results can be discovered about the quasi-isosceles quadrilateral. However, it may be more instructive to examine the consequences that follow when we let this quadrilateral degenerate in various ways, and transfer the proved properties to these degenerations. As is to be expected, we obtain theorems about triangles.

Degenerate Case 1. Given a triangle ABC with AC > AB and D, a point on AC such that AB = CD. If isosceles triangles with base angles equal to (1/2)(B + C) are erected upwardly on AD and on BC, they have' the same third vertex. See Figure 9. The proof follows from Theorem 2.

LIMITS OF MEANS FOR LARGE VALUES OF THE VARIABLES

by J. L. Brenner
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Polo Alto, CA 95053

1. Introduction. This article concerns the limit of the difference between two power means when all the variables become large. The definition of power mean follows and a precise statement of the theorem we prove appears in Section 3.

Let $a_1, a_2, \ldots, a_n$ be n positive numbers, not all equal, with n at least two. Let $(\underline{a})$ represent the n-tuple $(a_1, a_2, \ldots, a_n)$.

Definition. The $r\underline{th}$ power mean of $(\underline{a})$, written $M_r(\underline{a})$, is given by

$$(1) \qquad M_r(\underline{a}) = \left[\frac{1}{n}(a_1^r + \cdots + a_n^r)\right]^{1/r} = \left[\frac{1}{n}\sum a_i^r\right]^{1/r}.$$

In this paper, all summations will be from $i = 1$ to $i = n$ and the limits on the summation symbol will be omitted.

Examples of power means are the arithmetic mean $(r = 1)$, the harmonic mean $(r = -1)$, and the root-mean-square $(r = 2)$. When $r = 0$, (1) has no meaning.

By the following argument, L'Hôpital's Rule can be used to prove that

$$\lim_{r\to 0} M_r(\underline{a}) = (a_1a_2 \cdots a_n)^{1/n},$$

which is the geometric mean of the n positive numbers $a_1, a_2, \ldots, a_n$. Taking logarithms to base e in (1), we have

$$\log M_r(\underline{a}) = \left[\log \frac{1}{n}\sum a_i^r\right]\Big/ r.$$

If $r \to 0$, then $a_i^r \to 1$, and

$$\frac{1}{n}\sum a_i^r \to 1, \quad \log \frac{1}{n}\sum a_i^r \to 0.$$

So we differentiate numerator and denominator with respect to r to get

$$\lim_{r\to 0} \log M_r(\underline{a}) = \lim_{r\to 0} \frac{\left[\frac{1}{n}\sum a_i^r \log a_i\right]\Big/\frac{1}{n}\sum a_i^r}{1}$$

$$= \lim_{r\to 0} \frac{\sum a_i^r \log a_i}{\sum a_i^r}$$

$$= \frac{\sum \log a_i}{n} = \log(a_1a_2 \cdots a_n)^{1/n}.$$

But $\lim_{r\to 0} \log M_r(\underline{a}) = \log \lim_{r\to 0} M_r(\underline{a})$, and hence $\lim_{r\to 0} M_r(\underline{a}) = (a_1a_2 \cdots a_n)^{1/n}$, as we claimed. An alternative proof of this result appears in [1], page 15, no. 3.

In this article, $M_0(\underline{a})$ will be defined as $G(\underline{a})$, the geometric mean.

A fundamental theorem on power means (see reference [1], page 26, no. 16) in the case of n positive numbers a_i is

Theorem of the (Power) Means. If $r > s$, then $M_r(\underline{a}) > M_s(\underline{a})$, unless all the a_i are equal.

2. The Theorem of Hoehn-Niven. In a recent article in *Mathematics Magazine* [2], the following result is proved.

Theorem. If $r = -1, 0, 1,$ or 2, the value of $M_r(\underline{a} + \underline{x}) - M_1(\underline{a} + \underline{x})$ approaches 0 as $x \to \infty$, where $\underline{a} + \underline{x}$ denotes the n-tuple $(a_1 + x, a_2 + x, \ldots, a_n + x)$.

For orientation, note that if the numbers $b_1, b_2, \ldots, b_n$ all equal b, $b > 0$, then $M_r(b_1, b_2, \ldots, b_n) = b$ for each value of r. In the theorem of Hoehn-Niven, if x is a large positive quantity, it could be argued that the numbers $a_1 + x, a_2 + x, \ldots, a_n + x$ are "nearly equal," since any differences are swamped by the (large) value of x. This remark does not prove the theorem, or even give any clue as to how it might be established.

It is known (and easily proved) that the $r\underline{th}$ mean lies between the smallest value, say ℓ, and the largest value, say L, of the numbers $a_1, a_2, \ldots, a_n$; that is,

$$\ell = \min a_j < M_r(\underline{a}) < \max a_j = L.$$

Thus $(x + \ell) - (x + L) \le M_r(\underline{a} + \underline{x}) - M_1(\underline{a} + \underline{x}) \le (x + L) - (x + \ell)$, so that the Hoehn-Niven difference lies between $-(L - \ell)$ and $(L - \ell)$. This fact is consistent with the theorem, but it is still a long way from proving it (or even from proving that the difference approaches a limit).

Finally, it can be proved that, as $r \to \infty$, $M_r(\underline{a})$ approaches L, and as $r \to -\infty$, $M_r(\underline{a})$ approaches ℓ, but this, even together with the theorem of the means, does not prove anything about the Hoehn-Niven limit.

3. *A generalization.* As far as I know, nothing like the Hoehn-Niven theorem has previously appeared in the literature. The theorem is the special case (for $r = -1, 0, 1$, or 2) of the following

Theorem 1. If r is any integer, then the value of $M_r(\underline{a} + \underline{x}) - M_1(\underline{a} + \underline{x})$ approaches zero as x becomes infinite.

This generalization will be proved only for $r > 0$. The proof for $r < 0$ is more difficult, except that if $n = 2$ and $r < 0$, it is a reasonable exercise for readers of this Journal. If r is not an integer, the generalization is also true. See Section 4. If $r > 0$ and r is an integer, the proof goes this way. Write $M = M_r(\underline{a} + \underline{x})$, and compute

$$M - x = (M^r - x^r)/(M^{r-1} + M^{r-2}x + \ldots + Mx^{r-2} + x^{r-1}). \tag{2}$$

Note $M^r = (a_1 + x)^r/n + \ldots + (a_n + x)^r/n$. Use the binomial theorem on each of the n terms: $(a_1 + x)^r = x^r + ra_1x^{r-1} + \ldots$, and so on. The terms in x^r in the above numerator cancel. Thus

$$M^r - x^r = rx^{r-1}(a_1 + a_2 + \ldots + a_n)/n + T,$$

where T is the sum of terms of degree r-2 or less in x. As to the denominator, there are r terms of the form $M^{r-1-j}x^j$. It will be proved first that each of these r terms has the property that the limit of $M^{r-1-j}x^j/x^{r-1}$ is 1 for $x \to \infty$.

For instance, the first term is M^{r-1}, and

$$M^{r-1}/x^{r-1} = \left[\frac{1}{n}\sum (a_i + x)^r\right]^{(r-1)/r} / x^{r-1}$$

$$= \left[\frac{1}{n}\sum (a_i/x + 1)^r\right]^{(r-1)/r} \to 1^{(r-1)/r}$$

The general term in the denominator of (2) is $M^{r-1-j}x^j$. If this term is divided by x^{r-1}, the result is M^{r-1-j}/x^{r-1-j}, which also has limit 1, since M/x has limit 1, as has just been proved.

In all, there are r terms in the denominator of (2), the fraction that represents $M - x$. Dividing numerator and denominator by x^{r-1} and letting $x \to \infty$ gives the result

$$\lim_{x\to\infty} (M - x) = (a_1 + a_2 + \ldots + a_n)/n$$
$$= M_1(\underline{a}),$$

since $\lim_{x\to\infty} (T/x^{r-1}) = 0$.

Only one more step is needed. Write $M - M_1 = (M - x) - (M_1 - x)$, and note that it has just been proved that $(M - x)$ and $(M_1 - x)$ have the same limit. This completes the proof of Theorem 1 if $r > 0$ and r is an integer.

4. *The case r not integral.* To lift the restriction that r is an integer, use the theorem of the (power) means stated in the introduction. Let s be a (positive) integer greater than the (positive) number r. First, take the case $s > r \geq 1$. Then, by the theorem of the (power) means,

$$0 \leq M_r(\underline{a} + \underline{x}) - M_1(\underline{a} + \underline{x}) \leq M_s(\underline{a} + \underline{x}) - M_1(\underline{a} + \underline{x}),$$

and Theorem 1 (with r nonintegral) follows at once from Section 3 (with s integral). Next, take the case $0 < r < 1$. By the theorem of the (power) means,

$$0 \leq M_1(\underline{a} + \underline{x}) - M_r(\underline{a} + \underline{x}) \leq M_1(\underline{a} + \underline{x}) - M_0(\underline{a} + \underline{x}),$$

and Theorem 1 (with r nonintegral) is proved in this case also, according to the second assertion in the Hoehn-Niven theorem. Thus Theorem 1 is proved for every positive value of r.

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2. Hoehn, L. and Niven, I., Averages on the Move, Mathematics Magazine, Vol. 58, No. 3, May 1985, pp. 151-156.

GRAFFITO

Learning without thought is labour lost;
thought without learning is perilous.

Confucius

A NEW PROOF OF A FAMILIAR RESULT

by Sylvan Burgstahler
University of Minnesota-Duluth

It is well known that if sides **AB** and **AC** of triangle **ABC** are of unequal length, then the ray that bisects angle **A** will meet the perpendicular bisector of side **BC** in a point D that is <u>outside</u> the triangle, as in Figure 1. Of the variety of known proofs of this fact perhaps the easiest involves showing that D lies on the circle that circumscribes triangle **ABC**. Here, we present a new proof of this familiar result that relies on a continuity argument.

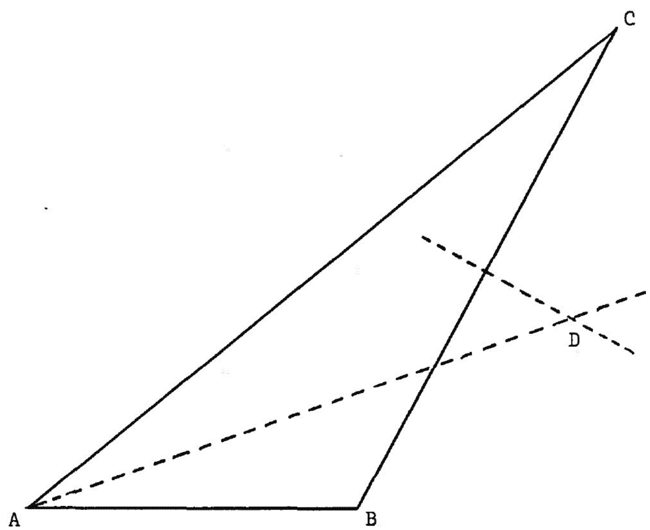

Figure 1

Without loss of generality, suppose that side **AB** is shorter than side **AC**. It is convenient to consider three cases depending upon whether angle **B** is acute, is a right angle, or is obtuse, although in almost all respects the proof is identical in all three cases. In each of the three cases, shown in Figure 2, M_1 is the midpoint of side **BC**.

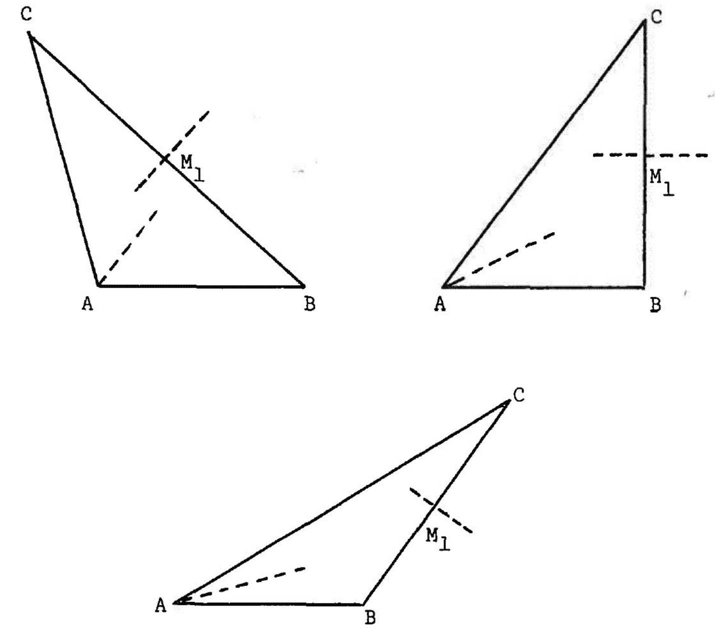

Figure 2

Observe, first, that point D falls <u>inside</u> triangle **ABC** only if the ray that bisects angle **A** cuts **BC** <u>above</u> its midpoint. (In truth, this is not entirely "obvious" when angle **B** is acute but we will return to this detail at the conclusion of the main argument.) If this is accepted, it follows that the theorem will be established if we can show that the point S in which the angle bisector intersects **BC** is closer to **B** than to **C**. We are thus led to consider the ratio r

$$r = \frac{BS}{SC} .$$

Next, extend (short) side **AB** to point **C'** such that **AC** = **AC'** and locate M_2 at the midpoint of side **CC'**, as shown in Figure 3. (Only the case for B obtuse is shown; the other cases proceed in exactly the same way.) Then, since triangle **AC'C** is isosceles, we know that AM_2 bisects angle **A**.

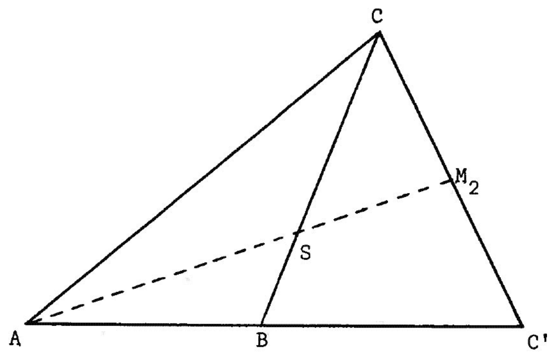

Figure 3

While B was originally a "fixed" point on the line segment AC', we now ask what happens if B is allowed to vary back and forth on AC'. To study this question, we introduce a variable x to measure the location of B during these wanderings. In particular, we choose x to be a linear scaling of the interval AC' such that if B is at A, the leftmost 'extremity" of AC', then x = 0, whereas, if B is at C' (the other extremity), then x = 1. In effect, we make B a "function" of x, say B = B(x). If S continues to designate the point in which the (fixed) segment A M intersects the (newly varying) side BC, S also becomes a function of x and, more importantly, so does the ratio r introduced above. It is the behavior of the continuous function r(x) that we now investigate.

Clearly, r(0) = 0 since S = A when x = 0. Likewise, r(1) = 1 since S = M_2 when x = 1 and M_2 is the midpoint of CC'. As noted earlier, our proof will be finished if we can show that r(x) stays <u>below</u> 1 for $0 < x < 1$.

Suppose it does not, that is, suppose there exists some point x such that $r(x\ ) \geq 1$. (The case $r(x\ ) > 1$ is shown in Figure 4; the other case is easier.) By continuity of r(x) there must then exist a point u between 0 and x such that r(u) = 1. Because of the way r was defined, this means that the corresponding point S(u) is the midpoint of side BC. But we have now reached a contradiction because if S is the midpoint of side BC and M is the midpoint of side CC' in triangle BC'C, then segment SM_2 must be parallel to side BC' and we know it is

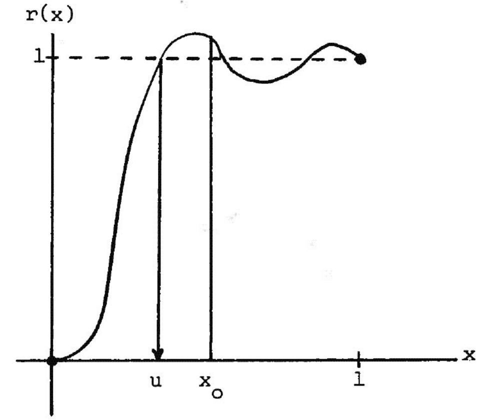

Figure 4

not because A is a common point on the lines containing these supposedly "parallel" segments.

We turn, finally, to the "detail" mentioned at the outset of the proof. It might be argued that our decision that D is outside triangle ABC is unassailable if angle B $\geq 90^\circ$ but is faulty for acute angles B if the perpendicular bisector of side BC in such triangles actually cut AB, as in the (distorted) Figure 5. In short, our "proof" for the case

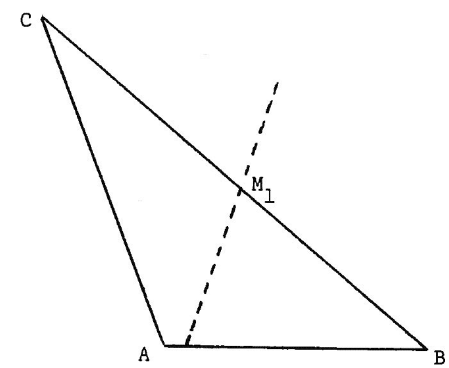

Figure 5

of acute angles B would rest upon the state of affairs indicated by a figure rather than upon logic. This objection disappears if we consider the location of P in Figure 6, where P is the foot of the perpendicular drawn from A.

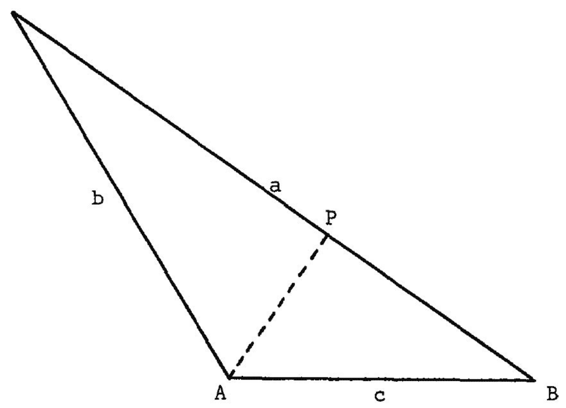

Figure 6

To show that P lies between B and the midpoint of BC, we apply the Law of Cosines to triangle ABC using cosB. From $b^2 = a^2 + c^2 - 2ac\cos B$, we obtain $b^2 - c^2 = a(a - 2c\cos B)$. But $b^2 - c^2 > 0$, so $a > 2c\cos B$, and $a/2 > c\cos B$. Therefore, our proof is valid after all.

SIMPLIFIED PROOFS FOR SOME MATRIX THEOREMS

by John R. Schue
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The idea for this note grew out of a mention in [1], page 88, that for two matrices A and B, $R_i(AB) = R_i(A)B$, where $R_i(X)$ represents the ith row of matrix X. This relation turns out to be quite useful in simplifying proofs for a number of the basic algebraic relations in matrix multiplication. Two examples are given below.

For the proofs we need two additional relations, both of which are obvious and easily proved. They are

(1) $$R_i(A)B = a_{i1}R_1(B) + \dots + a_{in}R_n(B)$$ and

(2) $$c(AB) = (cA)B$$

with both holding for A mxn, B nxp, and c a scalar.

Our first result is a proof of the associative law for matrix multiplication which avoids any use of double sums.

Theorem 1. If (AB)C is defined then it is equal to A(BC)

Proof. For any i,

$$\begin{aligned} R_i((AB)C) &= R_i(AB)C = (R_i(A)B)C \\ &= (a_{i1}R_1(B)C + \dots + (a_{in}R_n(B))C \\ &= a_{i1}(R_1(B)C) + \dots + a_{in}(R_n(B)C) \\ &= a_{i1}R_1(BC) + \dots + a_{in}(R_n(BC)) \\ &= R_i(A)BC \\ &= R_i(A(BC)). \end{aligned}$$

Thus, (AB)C = A(BC).

A second example is given by the familiar result that an elementary row operation can be effected by a pre-multiplication with an elementary matrix.

Theorem 2. Suppose A and B are $m \times n$ and B is obtained from A by one elementary row operation. Let E be the matrix obtained from the $m \times m$ identity matrix I by using the same row operation. Then, $B = EA$.

Proof. The proof will be given only for an operation of the third kind. The other two are quite similar. Thus, suppose $R_i(B) = R_i(A) + cR_j(A)$ for some $j \neq i$. Then $R_i(E) = R_i(I) + cR_j(I)$. For $k \neq i$, $R_k(EA) = R_k(E)A = R_k(I)A = R_k(IA) = R_k(A) = R_k(B)$ and $R_i(EA) = R_i(E)A = (R_i(I) + cR_j(I))A = R_i(A) + cR_j(A) = R_i(B)$ so that $EA = B$.

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1. Murdock, David C., *Linear Algebra,* John Wiley and Sons, 1970

THE LEAST MEMBER METHOD, AN ALTERNATIVE TO INDUCTION

by Robert Dinion
Virginia Polytechnic Institute
and State. University

The axiom of induction is often used when proving theorems about the set of natural numbers. Many times, however, an equivalent principle, which says that every non-empty set of natural numbers has a least member, is overlooked. This principle can often make a proof shorter and easier. As an example, we will prove the theorem that the product of any three consecutive natural numbers is divisible by six. The ground rules are that we are not allowed to notice that every other natural number is even nor that every third number is a multiple of three.

First Proof. (By induction) Let $S = \{n \in N: n(n + 1)(n + 2)$ is divisible by $6\}$. Clearly, 1 is a member of S. Suppose that $k \in S$. Then $(k + 1)(k + 2)(k + 3) = k(k + 1)(k + 2) + 3(k + 1)(k + 2)$. By the inductive hypothesis, $k(k + 1)(k + 2)$ is divisible by 6, and so it suffices to show that $3(k + 1)(k + 2)$ is a multiple of 6.

Let $T = \{n \in N: 3(n + 1)(n + 2)$ is a multiple of $6\}$. Clearly, $1 \in T$. Suppose that $n \in T$. Then $3(n + 1 + 1)(n + 1 + 2) = 3(n + 2) \times (n + 1 + 2) = 3(n + 1)(n + 2) + 6(n + 2)$. By the inductive hypothesis, $3(n + 1)(n + 2)$ is divisible by 6, and, since $6(n + 2)$ is certainly a multiple of 6, it follows that $n + 1 \in T$. We have, by the axiom of induction, that T is the set of natural numbers. In particular, $3(k + 1)(k + 2) \in T$ and so $3(k + 1)(k + 2)$ is a multiple of 6, as is required.

Second Proof. The proof is by contradiction. Let $Q = \{n \in N: n(n + 1)(n + 2)$ is not a multiple of $6\}$ and suppose that Q is non-empty. Then Q has a least member q. Since $1 \times 2 \times 3 = 6$, $2 \times 3 \times 4 = 24$, and $3 \times 4 \times 5 = 60$, we see that q does not equal 1, 2, or 3. Therefore, there is a natural number k such that $q = k + 3$. But $q(q + 1)(q + 2) = (k + 3)(k^2 + 9k + 20) = (k + 3)(k^2 + 3k + 2 + 6k + 18) = (k + 1)(k + 2) \times (k + 3) + 6(k + 3)$. Since $k + 1 < q$, $(k + 1)(k + 2)(k + 3)$ is a multiple of 6, as is $6(k + 3)$. We have reached a contradiction.

About the author -

Robert Dinion is an undergraduate at Virginia Polytechnic Institute and State University.

About the. paper -

Robert submitted this note. for publication at the. urging of his teacher, Professor Peter Fletcher, in a course called Methods of, Phoof,.

WINNERS - NATIONAL PAPER COMPETITION

Pi Mu Epsilon encourages student research and the. presentation of that research in this, Journal. The. National Paper competition awards prizes of $ 200, $ 100 and $ 50 each year in which at least five student papers have. been submitted to the. Editor. All students who have. not yet received a Master's Degree, oh higher, are eligible for these awards.

First prize winner for 1984-1985 is Donald John Nicholson for his paper "A Ubiquitous Partition of Subsets of, R^n," which appeared in the. Fall 1984 issue of, the. Journal.

Second prize winners am Paul Artola and Ruth Brisbin for their joint paper "Taxicab Trigonometry" in the. Spring 7985 issue.

Third prize winner is Julie Yancey for her paper "Edge-labelled Trees" in the. Spring 1985 issue.

PUZZLE SECTION

Edited by

Joseph P. E. Konhauser

The PUZZLE SECTION is for the enjoyment of those readers who are addicted to working doublecrostics or who find an occasional mathematical puzzle attractive. We consider mathematical puzzles to be problems whose solutions consist of answers immediately recognizable as correct by simple observation and requiring little formal proof. Material submitted and not used here will be sent to the 'Problem Editor if deemed appropriate for the PROBLEM DEPARTMENT.

Address all proposed puzzles and puzzle solutions to Professor Joseph D. E. Konhauser, Mathematics and Computer Science Department, Macalester College, St. Paul, Minnesota 55105. Deadlines for puzzles appearing in the Fall Issue will be the next February 15, and for puzzles appearing in the Spring Issue will be the next September 15.

Mathacrostic No. 21

Proposed by Joseph P. E. Konhauser
Macalester College, St. Paul, Minnesota

The word puzzle on pages 174 and 175 is a keyed anagram. The 264 letters to be entered in the diagram in the numbered spaces will be identical with those in the 26 keyed *words* at the matching numbers. The key numbers have been entered in the diagram to assist in constructing the solution. When completed, the initial letters of the *words* will give the name of an author and the title of a book; the completed diagram will be a quotation from that book. For an example, see the solution to the last mathacrostic on page 173.

SOLUTION

Mathacrostic No. 20. (See Spring 1985 Issue) *(proposed by Joseph V. E. Konhauser, Macalester College, St. Paul, Minnesota).*

Words:

A. Hands-down
B. Argosy
C. Node
D. Konrad Zuse
E. Inosculate
F. Napier's bones
G. Sashay
H. Spline
I. Ipso Facto
J. Rafflesia
K. Waterscrew
L. Irascible Genius
M. Lunes
N. Law of the lever
O. Isogony
P. Atanasoff
Q. Mancala
R. Rules of inference
S. Orrery
T. Whydunit
U. Astragali
V. Nest egg
W. Hodograph
X. Apodictic
Y. Misere
Z. Illation
a. Logistic
b. Think thinkable
c. One-horse
d. Naprapathy

First Letters: HANKINS, SIR WILLIAM ROWAN HAMILTON

Quotation: ***He wrote incessantly, usually in notebooks of, all sizes and and shapes, but also on pieces of, loose pap&, particularly if he was drafting an article oh a lecture. He wrote on walks, in carriages, during meetings of, the. Royal Irish Academy, on his fingernails if no paper was handy, and, according to his Aon, even on his egg at breakfast.***

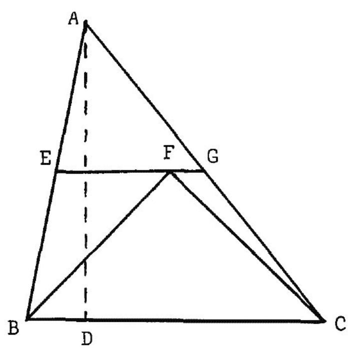
Solved by: Jeanette Bickley, Webster Groves High School, MO; Charles R. Diminnie, St. Bonaventure University, NY; Victor G. Feser, Mary College, Bismarck, ND; Robert Forsberg, Lexington, MA; Dr. Theodor Kaufman, Winthrop-University Hospital, Mineola, NY; Henry S. Lieberman, John Hancock Mutual Life Insurance Co., Boston, MA; Robert Prielipp, The University of Wisconsin-Oshkosh, WI; Stephanie Sloyan, Georgian Court College, Lakewood, NJ; Barbara Zeeburg, Denver, CO.

1 H	2 A	3 I	4 E	5 K	6 C		7 J	8 M	9 N	10 B		11 L	12 S	13 W
14 R	15 U		16 V	17 Q	18 Y	19 O	20 T	21 D		22 F	23 X		24 H	25 V
26 Z	27 C	28 E	29 K		30 G	31 S	32 Y	33 P	34 F		35 J	36 N	37 R	38 L
	39 B	40 S	41 H		42 M	43 D	44 X	45 W	46 Z	47 C	48 E	49 N	50 B	51 A
52 H		53 S	54 Z	55 F		56 R	57 T	58 U	59 B	60 N		61 S	62 X	63 Y
	64 O	65 T	66 W		67 C	68 F	69 D	70 I	71 A	72 S		73 V	74 O	
75 X	76 Q	77 Z	78 B	79 J		80 I	81 D	82 C	83 V		84 P	85 X	86 K	
87 H	88 C	89 Q	90 U	91 L	92 G	93 J	94 I	95 D	96 V	97 W	98 R		99 C	100 E
101 X	102 B	103 I	104 P	105 Y	106 O		107 B	108 H	109 G	110 L	111 K		112 J	113 A
114 N	115 Q	116 B		117 H	118 C	119 M		120 U	121 Q	122 Z	123 W	124 R	125 O	126 L
127 V		128 S	129 F		130 A	131 G	132 K	133 I	134 D	135 R	136 U	137 N	138 J	139 T
140 Q	141 X		142 E	143 D	144 P		145 O	146 F	147 P	148 V	149 W	150 Y		151 Z
152 K	153 J	154 V	155 I	156 L		157 M	158 Q		159 U	160 X	161 B	162 F	163 D	
164 R	165 I	166 N		167 H	168 M	169 Q	170 J	171 P	172 L	173 Z	174 Y	175 O		176 Q
177 N	178 F	179 W	180 G	181 L	182 D	183 O		184 C	185 F		186 I	187 V	188 P	189 S
190 K	191 B	192 U	193 Q	194 Y	195 T		196 H	197 N	198 K		199 F	200 S	201 R	
202 T	203 C	204 Z	205 Q	206 K	207 J	208 O	209 M	210 U		211 P	212 K	213 Q	214 N	
215 V	216 N	217 I	218 X	219 H		220 L	221 Q	222 P		223 T	224 P	225 Z	226 Y	227 V
228 L	229 O	230 U	231 S	232 C		233 R	234 X	235 C	236 Q	237 H	238 F	239 K	240 O	
241 G	242 I		243 P	244 N	245 F	246 C	247 L		248 S	249 M	250 V	251 E	252 Q	
253 M		254 P	255 B	256 V	257 U	258 O		259 L	260 S	261 Q	262 I	263 T	264 C	

Definitions	Words
A. A square with an arm adjustable to any angle	113 130 71 2 51
B. Inconsequential	59 78 116 102 39 107 50 255 161 10 191
C. Vocal cords (2 wds.)	99 118 27 246 47 264 184 82 203 67 88 235 6 232
D. Swiss inkblotter	43 81 134 21 163 143 69 182 95
E. The last syllable of a word	251 28 142 4 48 100
F. A product of finitely many shears	68 173 146 162 22 129 185 238 55 245 199 34
G. Cover completely	92 131 109 180 30 241
H. In the hyperbolic plane every line intersects it in two points (2 wds.)	117 219 167 196 1 52 108 87 24 237 41
I. The cry of an owl (2 wds., both comp.)	155 70 80 3 217 94 133 262 103 165 186 242
J. The original fauna and flora of a geographical area	112 170 35 79 138 207 153 93 7
K. Device for measuring wind speeds aloft	206 190 152 212 111 29 132 239 5 198 86
L. Used to coordinatize projective and affine planes (2 wds.)	126 110 259 228 220 91 38 172 11 181 247 156
M. Blindness	8 249 168 119 42 209 157 253
N. A full period of a night and a day	36 60 9 214 137 216 177 49 166 244 114 197
O. Said of a set whose every point is a limit point of the set (comp.)	19 229 240 106 183 125 158 208 64 145 175 258 74
P. Landmark of linguistic confusion (3 wds.)	84 224 211 144 33 171 254 243 104 147 188 222
Q. Ratio of galaxy velocity to distance (2 wds.)	76 121 17 89 221 205 261 236 193 115 176 252 169 140 213
R. Brilliant	98 56 233 135 37 14 201 124 164
S. A volume of selections from an author	53 40 189 260 61 231 128 12 248 31 200 72
T. Said of a precious stone without a setting	139 57 223 65 263 20 202 195
U. A collection without a natural ordering relation	210 192 120 136 15 58 159 257 230 90
V. Driving force that pushes successful scientists beyond the frontiers of human knowledge (3 wds.)	187 16 127 154 96 73 25 148 250 83 256 227 215
W. Obliquely truncated cones or cylinders	123 97 45 179 149 13 66
X. Standard	101 44 234 218 85 141 75 160 23 62
Y. Center of perspectivity	63 105 150 226 32 18 194 174
Z. Process which converts a conical projection into a plane perspectivity	46 173 122 54 151 225 77 204 26

COMMENTS ON PUZZLES 1 - 7, SPRING 1985

Puzzle # 1 attracted eight responses. There was near unanimous agreement that solutions (not necessarily unique) exist for the sums set in bases 5, 6, 8, 9 and 10 and that no solution exists for the sum set in base 7. Only two readers, John H. Scott and John M. Howell, responded to ***Puzzle.* #** 2. Both produced correct, but somewhat cumbersome formulas/algorithms for generating the given terms in the sequence. Neither discerned the scheme that the proposer, Victor G. Feser, had in mind. When written in base 2 notation, the numbers 1, 3, 5, 7, 9, 15, 17, 21, 27, 31, 33 are palindromic, namely, 1, 11, 101, 111, 1001, 1111, 10001, 10101, 11011, 11111, 100001, so the next number in the sequence is 45 (45_{10} = 101101). Eleven readers responded to *Puzzle # 3*. Some produced the solution 10989 x 9 = 98901. Others, interpreting the problem statement to mean that a, b, c, d and e are different, showed that there is no solution. Stephen Bloch proved that in base n, n > 2, the product of n-1 and the five-digit number 1 0 n-2 n-1 n-2 is the five-digit number n-2 n-1 n-2 0 1. Moreover, the solution is unique. In base 2, there are four solutions. For a similar problem see ***Puzzle.* # 1** in this issue. John H. Scott, John M. Howell and Jeffery Cook submitted essentially equivalent solutions to the dissection posed in *Puzzle* **# 4.** Their solution follows.



Triangle ABC is acute-angled and AD = BC. Points E and G are midpoints of sides AB and AC, respectively. Point F on EG is such that the triangle BFG is an isosceles right triangle with right angle at F. Cuts along EG, BF and FC produce four pieces which are easily reassembled to form a square. Howell remarked that three-piece solutions exist if if AD = 2 x BC or if 2 x AD = BC. About ***Puzzle.* #** 5, Laurent Hodges, Professor of Physics at Iowa State University, wrote "There is nothing new under the sun" and referred to Volume II, Chapter XIV, section 238, of Euler's ***Elements of Algebra,*** where Question 15 is "Required three square numbers such that the sum of every two of them may be a square." Euler gives three solutions (117, 240, 44), (429, 2340, 880) and (6325, 5796, 528) and shows how others may be obtained. In particular, if (x, y, z) is a solution, then so is (xy, yz, zx). Robert Prielipp, University of Wisconsin - Oshkosh, provided photocopies of pages 61-63 of Sierpinski's ***Elementary Theory of Numbers.*** Sierpinski remarks that the solution (117, 240, 44) was obtained by P. Halcke in 1719 (before Euler). Léo Sauvé, Algonquin College, Editor of *Crux Mathematicorum*, referred to Dickson's ***History of Number Theory,*** Vol. II, page 497. Six readers wrote regarding *Puzzle* # 6. Henry J. Osner said "20, 22 - posthumorous answer." John H. Scott said "Thanks for the posthaste hint. I can remember 2 and 3." Jeanette Bickley thanked the proposer for the "first-class" hint. Victor G. Feser wrote that he "... hoped there isn't another number in the sequence by the time the answer is published." The puzzle was suggested by a news release which appeared in the Minneapolis Star and Tribune in 1981 when the minimum first-class mail rate jumped from 18 to 20 cents. Seven responses were received for ***Puzzle.*** # **7.** All solutions involved calculus - directly or indirectly. The answer, surprising to many readers, is that the areas of the spherical cap and the planar disk are equal.

<u>List of Responders</u>: Jeanette Bickley (*6*), Stephen Bloch (*3,7*), James Campbell (*1,3*), Jeffery Cook (*4*), Mark Evans (*1,3,7*), Victor G. Feser (*1,3,6,7*), Ruben Gamboa (*1,3*), Laurent Hodges (*5*), John M. Howell (*2,3,4,5,6,7*), Ralph King (*7*), Bob LaBarre (*3*), Glen E. Mills (*1,3*), Henry J. Osner (*1,3,6,7*), Robert Prielipp (*1,3,5*), Léo Sauvé (*5*), John H. Scott (*1,2,3,4,5,6,7*) and Stephanie Sloyan (6).

GRAFFITO

In five minutes you will bag that it is all so absurdly simple.

Sherlock Holmes
The. Adventures of the. Dancing Men
Sir Arthur Conan Doyle

PUZZLES FOR SOLUTION

1. *Proposed by Bob LaBarre, United Technologies Research Center, East Hartford, Connecticut.*

Are there any single-digit positive integers k, other than 1 and 9, such that kxabcde = edcba?

2. *Proposed by Joseph V. E. Konhauser, Macalester College, St. Paul, Minnesota.*

Using the usual arithmetic symbols and the digits 1, 2, 3, 4 and 5, in that order from left to right, are you able to form 22/7?

3. *Suggested by results in a 1966 paper by S. J. Einhorn and I. J. Schoenbag.*

a. Arrange four points in the plane so that the six distances between pairs of points fall into just two classes. For example, if the four points are vertices of a unit square then the distances are 1, 1, 1, 1, $\sqrt{2}$ and $\sqrt{2}$.

b. In three-space, in how many ways can five points be arranged so that the distances between pair's of points fall into just two classes?

4. *Proposed by Joseph D. E. Konhauser, Macalester College, St. Paul, Minnesota.*

With disjoint line segments (endpoints included and different), is it possible to cover

a. a triangle plus its interior,

b. a circle plus its interior?

5. *Suggested by a hemahk in a paper on covering problems by L. M. Kelly.*

Arrange five points in a plane so that each subset of four can be covered by a unit square tile (a square plus its interior) but such that the unit square tile cannot simultaneously cover all five points.

6. *An oldie.*

Find three different numbers x, y and z such that x, y, z are in arithmetic progression; y, z, x are in geometric progression; and z, x, y are in harmonic progression.

PROBLEM DEPARTMENT

Edited by Clayton W. Dodge
University of Maine

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

All communications should be addressed to C. W. Dodge, Math. Dept., University of Maine, Orono, ME 04469. Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly identified with name and address. Solutions to problems in this issue should be mailed by July 1, 1986.

Problems for Solution

600. *Proposed by John M. Howell, Littlerock, California.*

$$\begin{array}{r} I \\ AM \\ \underline{HOT} \\ SURE \end{array}$$

SURE, but if $I < M < T$ and $A < O$, I think there are only five solutions to this alphametric.

601. *Proposed by Charles W. Trigg, San Diego, California.*

Without table searching, identify the three consecutive integers in the decimal system whose squares have the form *abcdef* with distinct digits and whose reverses have squares with the same digits in the order *efcdab*.

***602.** *Proposed by Jack Garfunkel, Flushing, New York.*

Given isosceles triangle ABC and a point O in the plane of the triangle, erect directly similar isosceles triangles *POA*, *QOB*, *ROC* (but not necessarily similar to triangle *ABC*). Prove that the apexes *P*, *Q*, *R* of these triangles determine a triangle similar to triangle ABC.

603. *Proposed by Russell Euler, Northwest Missouri State University, Maryville.*

Evaluate

$$\lim_{x\to\infty} \sum_{n=0}^{\infty} (-1)^n\left(\frac{x^n}{n!}\right)^2.$$

604. *Proposed by David Iny, Renssalaer Polytechnic Institute, Troy, New York.*

A unit square is covered by n congruent equilateral triangles of side s with or without the triangles overlapping each other. Find the minimum values for s for $n = 1, 2,$ and 3.

605. *Proposed by Jack Garfunkel, Flushing, New York.*

Given that x is an acute angle, find the value of x if

$$\frac{\sin 4x}{2\cos 3x} = \frac{\sin x}{\cos 2x} + 2\sin x.$$

606. *Proposed by Russell Euler, Northwest Missouri State University, Maryville.*

Prove that

$$\prod_{k=0}^{p-1} \left\{r^2 - 2r\cos\left[x - \frac{(2k+1)\pi}{p}\right] + 1\right\} = r^{2p} + 2r^p\cos px + 1.$$

607. *Proposed by Jack Garfunkel, Flushing, New York.*

Triangles ABC and $A'B'C'$ are right triangles with right angles at C and C'. Prove that if $s/r > s'/r'$, then $s/R < s'/R'$, where s, s', r, r', R, R' are respectively the semiperimeters, inradii, and circumradii of ABC, $A'B'C'$.

608. *Proposed by R. S. Luthar, University of Wisconsin, Waukesha.*

Evaluate the following determinant:

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \binom{n}{1} & \binom{n+1}{1} & \cdots & \binom{n+k}{1} \\ \binom{n+1}{2} & \binom{n+2}{2} & \cdots & \binom{n+k+1}{2} \\ & & \cdots & \\ \binom{n+k-1}{k} & \binom{n+k}{k} & \cdots & \binom{n+2k-1}{k} \end{vmatrix}.$$

609. *Proposed by R. C. Gebhardt, Parsippany, New Jersey.*

Determine whether there exist nonzero integers a, b, c, and d such that $a^2 + b^2 = c^2$ and $a^2 - b^2 = d^2$.

610. *Proposed by Russell Euler, Northwest Missouri State University, Maryville.*

Find all twice-differentiable functions f such that the average value of f on each closed subinterval of $[a,b]$, $a < b$, is the mean of f at the endpoints of the subinterval.

611. *Proposed by Hao-Nhien Qui Vu, Purdue University, West Lafayette, Indiana.*

Calculate the following integrals:

$$a)\quad \int_0^{\ln 2} \frac{x\,dx}{e^x - 1}$$

$$b)\quad \int_0^{\infty} \frac{x\,dx}{e^x - 1}.$$

612. *Proposed by David Iny, Renssalaer Polytechnic Institute, Troy, New York.*

A friend writes the letters A, B, C, D in some order unknown to you. You may ask a fixed number of yes-no questions about the permutation.

a) If they are answered truthfully, show that less than half a dozen questions will suffice to determine the permutation.

b) If there is at most one lie, then not over 10 questions are needed.

c) If there are at most two lies, show that not more than 15 questions are required.

*d) Are these limits the best possible?

Solutions

407. **[Fall 1977, Fall 1978]** *Proposed by Ben Gold, John H. Howell and Vance Stine, Los Angeles City College, California.*

Two sets of n dice are rolled ($n = 1, 2, 3, 4, 5, 6$). What is the probability of k matches ($k = 0, 1, \ldots, n$)?

II. Comment by John Howell and Ben Gold.

I think that the published solution is wrong. It looks to me as if the solution would apply only if the dice were ordered.

503. [Fall 1981, Fall 1982] *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, Pennsylvania.*

Find the equation of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ with minimum volume which shall pass through the point $P(r,s,t)$, $0 < r < a$, $0 < s < b$, $0 < t < c$.

II. Solution by M. S. Klamkin, University of Alberta, Canada.

The published solutions to this problem and to Problem 527 use multi-variate calculus which is overkill. Also the first solution is incomplete in that there was no verification of the minimum.

Both of these problems and many other extremum problems in multi-variate calculus books can be done completely, more elementarily, and more simply and quickly by using the arithmetic-geometric mean inequality.

Here we wish to minimize $(4/3)\pi abc$, given that

$$\left(\frac{r}{a}\right)^2 + \left(\frac{s}{b}\right)^2 + \left(\frac{t}{a}\right)^2 = 1, \text{ where } r,\ s,\ t \text{ are given.}$$

By the AM-GM inequality we have

$$1 = \left(\frac{r}{a}\right)^2 + \left(\frac{s}{b}\right)^2 + \left(\frac{t}{c}\right)^2 \geq 3\left(\frac{rst}{abc}\right)^{2/3}.$$

Thus $min(abc)^2 = 3^3(rst)^2$ and is taken for

$$\left(\frac{r}{a}\right)^2 = \left(\frac{s}{b}\right)^2 = \left(\frac{t}{c}\right)^2 = \frac{1}{3}.$$

527. [Fall 1982, Fall 1983] *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, Pennsylvania.*

Find the volume of the largest rectangular parallelopiped with upper vertices on the surface and lower vertices on the xy-plane that can be inscribed in the elliptic paraboloid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2h - 2z.$$

II. Solution by M. S. Klamkin, University of Alberta, Canada..

Here we wish to maximize

$$V = 4xy\left(h - \frac{x^2}{2a^2} - \frac{y^2}{2b^2}\right), \text{ where } a,\ b,\ h \text{ are given.}$$

By the AM-GM inequality,

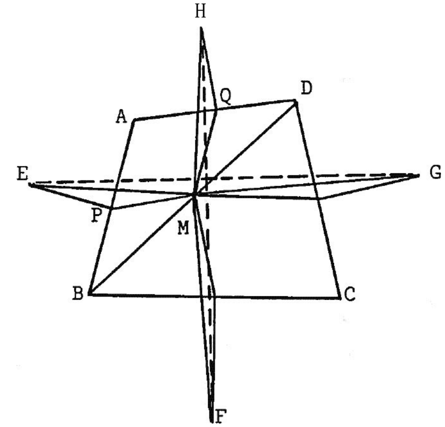
$$h = \frac{x^2}{2a^2} + \frac{y^2}{2b^2} + \frac{1}{2}\left(h - \frac{x^2}{2a^2} - \frac{y^2}{2b^2}\right) + \frac{1}{2}\left(h - \frac{x^2}{2a^2} - \frac{y^2}{2b^2}\right)$$

$$\geq 4\left\{\frac{x^2y^2}{16a^2b^2}\left(h - \frac{x^2}{2a^2} - \frac{y^2}{2b^2}\right)^2\right\}^{1/4} = (V/ab)^{1/2}.$$

Thus $V_{max} = abh^2$ and is taken for $\frac{x^2}{2a^2} = \frac{y^2}{2b^2} = \frac{h}{4}$.

558. [Fall 1983, Fall 19841 *Proposed by Richard 1. Hess, Palos Verdes, California.*

Let $ABCD$ be a quadrilateral. Let each of the sides AB, BC, CD, DA be the diagonal of a square. Let E, F, G, H be those vertices of the squares that lie outside the quadrilateral. That is, EAB, FBC, GCD, and HRA are directly similar isosceles right triangles with apexes E, F, G, H. Prove that EG and FH are perpendicular. See the figure below.



Solution by Leon Bankoff, Los Angeles, California.

If Euclid knew anything about transformations, vectors, or complex numbers, he kept it a closely guarded secret. That is why, when confronted with Van Aubel's quadrilateral theorem, of which this problem is a partial statement, he pulled a rabbit out of his hat to boost his stock of old-fashioned, high school Greek geometry, using nothing but theorems out of his *Elements*.

You must remember that Euclid had a great track record in motivating and inspiring mathematical neophytes and that his *Elements* hit the best-seller lists long before it was challenged by the Holy Bible. Descending from the heights of Mount Olympus, he found Van Aubel trying to persuade his incredulous students that the two lines joining the centers of opposite squares described externally on the sides of a quadrilateral are equal and mutually perpendicular. Euclid drew Van Aubel aside and whispered a self-contained, synthetic proof that went like this.

Label the vertices of the quadrilateral A, B, C, D and let M denote the midpoint of diagonal BD. Call the centers of the squares on AB, BC, CD, and DA, E, F, G, and H. Let P, Q be the midpoints of AB, AD. Then $MQ = PA = PE$ and $MP = AQ = HQ$. Now MP is parallel to AQ and is therefore perpendicular to HQ, while MQ is parallel to AP and is perpendicular to EP. It follows that triangles HMQ and MEP are congruent and that the sides HM and EM are mutually perpendicular and equal. In a similar manner we can show that the lines MF and MG are equal and perpendicular.

Now draw EG and FH and consider the triangles EMG and HMF. We already know that EM and HM are equal and perpendicular, as are also FM and GM. Hence HF is equal and perpendicular to EG.

Van Aubel escourted Euclid back to the foot of Mount Olympus, patted the old boy on the back and said, "If I ever go in for that New Math they're talking about, fear not. Remember the old saying-- I love my wife but Oh, Euclid!"

Also solved by **RALPH KING**, *St. Bonaventure*, **NY**, **HARRY SEDINGER**, *St. Bonaventure University*, **NY**, **KENNETH M. WILKE**, **Topeka, KS**, *and the* **PROPOSER**.

Editorial note. Wilke pointed out that this well known theorem appeared as Problem 308 in *The Pentagon*, Fall 1979, where it was solved by vectors, and in Garfunkel, "Solving Problems in Geometry by using Complex Numbers," *Mathematics Teacher*, Nov. 1967, pp. 731-734, where it was solved by complex numbers. It also appears as Exercise 20.22d in Dodge, *Euclidean Geometry and Transformations*, Addison-Wesley, 1972, p. 93, which was stolen from R. L. Finney, "Dynamic proofs of Euclidean Theorems," *Mathematics Magazine* 43 (1970) pp. 177-185, where it is solved by elementary isometries. Your Problems Editor certainly should have spotted this last reference!

574. *Proposed by* **S. E. Uuceh, Rogue Bluffs, Maine.**

Although there are many solutions to this unfortunate base 8 alphametric, there is only one prime $MOOD$. Find that $MOOD$.

$$\begin{array}{r} SOT \\ IS \\ \underline{THE} \\ MOOD \end{array}$$

Solution by **Charles W. Trigg, San Diego, California.**

Immediately $M = 1$. Since $MOOD$ is prime in an even base, then $D = 3$, 5, or 7. Now the columns of the alphametric, reading from the right, determine the following equations:

(1) $T + N + E = D + 8$,

(2) $I + H + 1 = 8$

(3) $N + T + 1 = O + 8$.

From (3), $7 + 6 + 1 = 6 + 8$ involves a duplicated digit, so $O < 6$. Neither N, I, nor T can be zero.

Method 1. When the ten eligible values of $MOOD$ from 1223 to 1557 (base 8), inclusively, are converted to base ten and checked against a list of primes, only three prove to be prime, namely

$$1223_8 = 659_{10},\quad 1225_8 = 661_{10},\ \text{and}\ 1335_8 = 733_{10}.$$

In each case there is only one pair of non-duplicating digits N, T that satisfies (3) and then only one digit E that satisfies (1). In two cases there is a duplicate digit. Thus

M	O	D	N	T	E	I	H
1	2	3	4	5	2	duplicate	
1	3	5	4	6	3	duplicate	
1	2	5	3	6	4	7	0.

In the last case the two unused digits satisfy *(2)*, affording the reconstruction

$$\begin{array}{r} 623 \\ 76 \\ \underline{304} \\ 1225 \end{array}$$

where the 3 and the 6 are interchangeable.

Method 2. There are only seven solutions to *(1)* devoid of duplicated digits. These are shown below together with the solutions of *(2)* using available digits and the residual digit assigned to O.

T	N	E	D	I	H	O
7	6	0	5	3	4	2
7	4	2	5	none		
7	4	0	3	2	5	6
6	5	4	7	none		
6	5	0	3	none		
5	4	2	3	7	0	6
6	4	3	5	7	0	2

The first set of values does not satisfy *(3)*, and since $0 < 6$, the last set provides the only solution, with E = 4, I = 7, and H = 0, as shown in Method 1 above.

Also solved by EDWARD ABOUFADEL, Ft. Wayne, IN, FRANK P. BATTLES, *Massachusetts Maritime* Academy, *Buzzards* Bay, MARK EVANS, *Louisville*, KY, VICTOR G. FESER, *Mary College, Bismarck, ND*, RICHARD I. HESS, Rancho *Palos Verdes*, CA, JOHN M. HOWELL, *Littlerock, CA*, GLEN E. MILLS, *Valencia Community College, Orlando*, FL, SOUTH DAKOTA STATE UNIVERSITY PROBLEM SOLVING GROUP, *Brookings*, KENNETH M. WILKE, Topeka, *KS*, and the PROPOSER.

575. *Proposed* by *Charles* W. *Trigg*, San *Diego, California.*

The sum of the digits of a two-digit integer *N* is S and the product of the digits is P. One of the differences *N* - S and N - P is a square and the other is a cube. Find *N* and show it to be unique.

Solution by *Frank* P. *Battles, Massachusetts Maritime* Academy, *Buzzards Bay*.

Let a and b be the digits of *N*, so N = 10a + b, S = a + b, and P = ab. If N - S = *9a* is a square, then a = 1, 4 or 9. But then N - P = *10a* + b - *ab* is 10, 40 - *3b*, or 90 - *8b*, none of which can be a cube. Hence we must have *N* - S = *9a* equal to a cube. so a = 3. Now N - P = 30 - *2b* must be a square, so b = 7. Hence *N* = *37* is the unique solution.

Also solved by EDWARD ABOUFADEL, *Ft. Wayne*, IN, DAVID DEL SESTO, No. *Scituate*, RI, CHARLES R. DIMINNIE, St. *Bonaventure University*, NY, MICHAEL W. ECKER, *Clarks* Summit, PA, MARK EVANS, *Louisville*, KY, VICTOR G. FESER, *Mary College, Bismarck*, NO, ROBERT C. GEBHARDT, *Hopatcong*, NY, RICHARD I. HESS, Rancho *Palos Verdes*, CA, JOHN M. HOWELL, *Littlerock*, CA, EDWIN M. KLEIN, *University of Wisconsin, Whitewater*, HENRY S. LIEBERMAN, John Hancock *Mutual Life Insurance* Company, *Boston*, MA, GLEN E. MILLS, *Valencia Community College, Orlando*, FL, THOMAS M. MITCHELL, *Southern Illinois University, Carbondale*, PI MU EPSILON PROBLEM SOLVING TEAM, *Louisiana* State. *University, Baton* Rouge, BOB PRIELIPP, *University of Wisconsin-Oshkosh*, JOHN PUTZ, *Alma College*, MI, GEORGE W. RAINEY, Los Angeles, *CA*, JOHN RUEBUSCH, St. *Xavier* High School, *Cincinnati*, OH, HARRY SEDINGER, St. *Bonaventure University*, NY, WADE H. SHERARD, *Furman University, Greenville*, SC, SOUTH DAKOTA STATE UNIVERSITY PROBLEM SOLVING GROUP, *Brookings*, VIS UPATISRINGA, *Humboldt* State *University, Arcata*, CA, KENNETH M. WILKE, Topeka, KS, and the PROPOSER.

576. *Proposed* by *David* Iny, *Rensselaer Polytechnic Institute, Troy*, New *York*.

Prove the following for all natural numbers *n*:

(a) $$1\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \ldots + n\binom{n}{n} = 2^{n-1}n,$$

$$1^2\binom{n}{1} + 2^2\binom{n}{2} + 3^2\binom{n}{3} + \ldots + n^2\binom{n}{n} = 2^{n-2}n(n+1),$$

$$1^3\binom{n}{1} + 2^3\binom{n}{2} + 3^3\binom{n}{3} + \ldots + n^3\binom{n}{n} = 2^{n-3}n^2(n+3).$$

*(b) for each positive integer *p* there exists a polynomial *q(n)* of degree *p* such that:

(i) $$1^p\binom{n}{1} + 2^p\binom{n}{2} + 3^p\binom{n}{3} + \ldots + n^p\binom{n}{n} = 2^{n-p}q(n),$$

(ii) *q(n)* has integral coefficients and leading coefficient 1.

(*iii*) when $p > 1$ is odd, then $q(n)$ is divisible by n^2;

(*iv*) when $p > 2$ is even, then $q(n)$ is divisible by $n(n+1)$.

Solution by *Richard A. Gibbs, Fort Lewis College, Durango, Colorado.*

Due to the length of this solution I leave the details of the various proofs by mathematical induction for the pleasure of the reader. Let

$$f(0,n,x) = (1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i,$$

$$f(1,n,x) = x\,f'(0,n,x) = xn(1+x)^{n-1} = \sum_{i=1}^{n} i \binom{n}{i} x^i,$$

where the prime indicates derivative with respect to x,

$$f(2,n,x) = x\,f'(1,n,x) = (x^2n^2 + xn)(1+x)^{n-2} = \sum_{i=1}^{n} i^2\binom{n}{i} x^i,$$

$$f(3,n,x) = x\,f'(2,n,x) = (x^3n^3 + 3x^2n^2 + (-x^2+x)n)(1+x)^{n-3}$$

$$= \sum_{i=1}^{n} i^3 \binom{n}{i} x^i,$$

and it is easy to show by induction that in general we have

$$f(p,n,x) = x\,f'(p-1,n,x) = (1+x)^{n-p}\,Q(p,n,x) = \sum_{i=1}^{n} i^p\binom{n}{i} x^i$$

where $Q(p,n,x)$ is a polynomial in n of degree p with coefficients which are themselves polynomials in x.

Part (a) of this problem now follows by setting $x = 1$ in the equations for $f(1,n,x)$, $f(2,n,x)$, and $f(3,n,x)$.

In fact, $Q(p,n,x)$ satisfies the recursion

(*a*) $Q(p,n,x) = x(n-p-1)\,Q(p-1,n,x) + x(1+x)\,Q'(p-1,n,x)$.

Now set

$$Q(p,n,x) = \sum_{i=1}^{p} g(p,i,x)\,n^i$$

in equation (a) to get

(*b*) $g(0,0,x) = 1$ and, for $p \geq 1$,

(*c*) $g(p,0,x) = (x^2+x)\,g'(p-1,0,x) - (p-1)\,g(p-1,0,x)$,

(*d*) $g(p,i,x) =$

$(x^2+x)\,g'(p-1,i,x) - (p-1)x\,g(p-1,i,x) + x\,g(p-1,i-1,x)$

for $i = 1, 2, \ldots, p-1$, and

(*e*) $g(p,p,x) = x\,g(p-1,p-1,x)$.

From these recursions it is readily shown that

(*1*) $g(p,0,x) = 0$ for all $p \geq 1$ (from (*b*) and (*c*)),

(*2*) $g(p,p,x) = x^p$ (from (*e*)), and

(*3*) $g(p,i,x)$ for $i = 1, 2, \ldots, p-1$ is a polynomial in x of degree $p-1$ with integral coefficients and with a factor of x^i (from (*d*)).

(Note that our $Q(p,n,1)$ is the $q(n)$ given in the proposal.) From equations (*1*), (*2*), and (*3*) we see that (*i*) and (*ii*) of part (*b*) of the proposal now follow.

Since $g(p,0,x) = g(p,0,1) = 0$ by (*1*) for $p \geq 1$, we know that $Q(p,n,1)$ is divisible by n for $p \geq 1$. To establish (*iii*) it suffices to show that the coefficient of n in $Q(2k+1,n,1)$ is also zero. To that end we will show that

(*) $\quad q(2k+1,1,1) = 0$ for all $k \geq 1$.

To establish (*iv*) it suffices to show that $n = -1$ is a zero of $Q(2k,n,1)$. To that end we will show that

(**) $\quad Q(2k,-1,1) = \sum_{i=0}^{2k} (-1)^i\,g(2k,i,1) = 0$ for all $k \geq 1$.

Observe from (*d*) and (*1*) that

(*f*) $g(p,1,x) = (x^2+x)\,g'(p-1,1,x) - x(p-1)\,g(p-1,1,x)$.

If

$$g(p-1,1,x) = a_{p-2}x^{p-2} + a_{p-3}x^{p-3} + \ldots + a_1x,$$

then we see from (*f*) that the coefficient of x^m in $g(p,1,x)$ is

$$a_m + (m-p)a_{m-1}.$$

This observation allows us to establish by induction that

(*4*) if $g(2k+1,1,x) = A_{2k}x^{2k} + A_{2k-1}x^{2k-1} + \ldots + A_1x$ for $k \geq 1$, then $A_i = -A_{2k+1-i}$, and

(*5*) if $g(2k,1,x) = B_{2k-1}x^{2k-1} + B_{2k-2}x^{2k-2} + \ldots + B_1x$ for $k \geq 1$, then $B_i = B_{2k-i}$.

In (*4*) set $x = 1$ to obtain (*) and thus part (*iii*). Finally, from

(f) it can be shown by induction that

(g) $\sum_{i=0}^{j} (-1)^i g(j,i,x) = -g(j+1,1,x)$ for all $j \geq 1$.

Setting $j = 2k$ in (g), we see that (**), and hence part (iv), follows from (*).

Also partially solved by RUSSELL EULER, *Northwest Missouri State University, Maryville,* RICHARD I. HESS, *Rancho Palos Verdes, CA,* JOHN M. HOWELL, *Littlerock, CA,* SOUTH DAKOTA STATE UNIVERSITY PROBLEM SOLVING GROUP, *Brookings,* KENNETH M. WILKE, *Topeka, KS, and the* PROPOSER. WILKE *pointed out that problem 956 in Crux Mathematicorum is closely related to this problem.*

***577.** *Proposed by David E. Penney, The University of Georgia, Athens.*

In the 3 by 3 by 3 cubical array below, the sum of the eight digits in each of the eight 2 by 2 by 2 corner cubes is a fixed rational multiple (100/13) of the integer in the center. Does there exist such an array of the integers from 1 to 27 in which the eight corner sums are the same *integral* multiple of the integer in the center? [See Problem 504 (Fall 1981) for a similar two-dimensional problem.]

Top:			Center:			Bottom:		
24	12	27	2	19	1	25	11	26
10	3	7	17	13	18	9	4	8
20	15	23	6	16	5	21	14	22

Solution by Morris Katz, Macwahoc, Maine.

By a tedious partial computer search we find the solution

Top:			Center:			Bottom:		
20	10	22	1	17	3	24	15	27
14	18	9	4	7	5	12	11	6
23	19	21	2	13	8	26	16	25

where the constant multiple is the integer 13. Thus we have answered the given question: yes, there does exist such an array and one example appears above. A complete solution to the problem, the finding of all such arrays, may well be too tedious to undertake.

578. *Proposed by Emmanuel O. C. Imonitie, Northwest Missouri State University, Maryville.*

Given that x and y have opposite signs, solve the simultaneous equations

$$x + y + xy = -5 \text{ and } x^2 + y^2 + x^2y^2 = 49.$$

Amalgam of solutions submitted independently by CHARLES R. DIMINNIE, *St. Bonaventure University,* New *York,* VICTOR G. FESER, *Mary College, Bismarck, North Dakota,* JACK GARFUNKEL, *Flushing, New York,* EDWIN M. KLEIN, *University of Wisconsin, Whitewater,* HENRY S. LIEBERMAN, *John Hancock Mutual Life Insurance Company, Boston, Massachusetts,* SAM PEARSALL, *Loyola Marymount University, Los Angeles, California,* PI MU EPSILON PROBLEM SOLVING TEAM, *Louisiana State University, Baton Rouge,* JOSEPH PUTHOFF, *St. Xavier High School, Cincinnati, Ohio,* GEORGE W. RAINEY, *Los Angeles, California,* MICHIEL SMID, *Tilburg, The Netherlands,* SOUTH DAKOTA STATE UNIVERSITY PROBLEM SOLVING GROUP, *Brookings,* VIS UPATISRINGA, *Humboldt State University, Arcata, California,* HAO-NHIEN QUI VU, *Purdue University, West Lafayette, Indiana, and* KENNETH M. WILKE, *Topeka, Kansas.*

Rearrange the first equation to get

$$x + y = -5 - xy,$$

which when squared yields

$$x^2 + 2xy + y^2 = 25 + 10xy + x^2y^2.$$

Now subtract the second given equation to obtain

$$2x^2y^2 + 8xy - 24 = 0,$$

which can be factored:

$$2(xy + 6)(xy - 2) = 0.$$

Since x and y have opposite signs, only $xy = -6$ is acceptable, so substitute $y = -6/x$ into the first given equation to get

$$x^2 - x - 6 = 0,$$

whose roots 3 and -2 yield the solutions $(x,y) = (3,-2)$ and $(-2,3)$.

Also solved by EDWARD ABOUFADEL, *Ft. Wayne, IN,* GEORGE W. BARRATT, *Maryville, MO,* FRANK P. BATTLES, *Massachusetts Maritime Academy, Buzzards Bay,* RUSSELL EULER, *Northwest Missouri State University, Maryville,* MARK EVANS, *Louisville, KY,* ROBERT C. GEBHARDT, *Hopatcong*, NY, RICHARD I. HESS, *Rancho Palos Verdes*, CA, JOHN M. HOWELL, *Littlerock* CA, PETER A. LINDSTROM, *North Lake College, Irving*, TX, GLEN E. MILLS, *Valencia Community College, Orlando*, FL, BOB PRIELIPP, *University of Wisconsin-Oshkosh*, WADE H. SHERARD, *Furman University, Greenville*, SC, W. R. UTZ, *Columbia*, MO, and the PROPOSER.

579. *Proposed by R. S. Luthar, University of Wisconsin Center, Janesville.*

Prove that for any positive integer n,

$$4^n(n!)^3 < (n+1)^{3n}.$$

I. Solution by South Dakota State University Problem Solving Group, Brookings.

Replace 4 by 8 and take cube roots to obtain the sharper inequality

$$2^n(n!) \le (n+1)^n,$$

which can be rearranged to get

$$\prod_{k=1}^{n} \frac{2k}{n+1} \le 1.$$

Letting $m = [(n+1)/2]$, we have

$$\prod_{k=1}^{n} \frac{2k}{n+1} = \left(\prod_{k=1}^{m} \frac{2k}{n+1}\right)\left(\prod_{k=1}^{m} \frac{2n+2-2k}{n+1}\right) = \prod_{k=1}^{m} \frac{2k(2n+2-2k)}{(n+1)^2}$$

$$= \prod_{k=1}^{m} \frac{(n+1)^2 - (n+1-2k)^2}{(n+1)^2} \le 1$$

with equality if and only if $m = n = 1$.

II. A blend of the solutions submitted independently by Bob Prielipp, University of Wisconsin-Oshkosh, and Kenneth M. Wilke, Topeka, Kansas.

By the arithmetic mean-geometric mean inequality we have

$$\frac{n+1}{2} = \frac{n(n+1)/2}{n} = \frac{1+2+\dots+n}{n} \le (n!)^{1/n}.$$

Thus $(n+1)^{3n} > 2^{3n}(n!)^3 = 8^n(n!)^3$, with equality only when $n = 1$, a stronger inequality than that proposed.

Also solved by JIM ARENS, *Columbus*, OH, FRANK P. BATTLES, *Massachusetts Maritime Academy, Buzzards Bay*, CHARLES R. DIMINNIE, *St. Bonaventure University*, NY, RUSSELL EULER, *Northwest Missouri State University, Maryville*, MARK EVANS, *Louisville*, KY, JACK GARFUNKEL, *Flushing*, NY, RICHARD I. HESS, *Rancho Palos Verdes*, CA, JOHN M. HOWELL, *Littlerock*, CA, EDWIN M. KLEIN, *University of Wisconsin, Whitewater*, HENRY S. LIEBERMAN, *John Hancock Mutual Life Insurance Co., Boston*, MA, MICHIEL SMID, *Tilburg, The Netherlands*, WILLIAM STATON, *University of Mississippi, University*, HAO-NHIEN QUI VU, *Purdue University, West Lafayette*, IN, and the PROPOSER.

Other solutions were by mathematical induction, by taking logarithms of each side and applying the differential calculus, and by applying the binomial theorem. Several solvers made use of the inequality $(1 + 1/n)^n \ge 2$.

580. *Proposed by Bob Prielipp, University of Wisconsin-Oshkosh.*

Let a, b, and c be the lengths of the sides of a triangle and let s be its semiperimeter. Prove that

$$(a/2)^a(b/2)^b(c/2)^c \ge (s-a)^a(s-b)^b(s-c)^c.$$

Union of solutions by Edwin M. Klein, University of Wisconsin-Whitewater, and Henry S. Lieberman, John Hancock Mutual Life Insurance Company, Boston, Massachusetts.

The given inequality is equivalent to

$$\left(\frac{b+c-a}{a}\right)^a\left(\frac{c+a-b}{b}\right)^b\left(\frac{a+b-c}{c}\right)^c \le 1.$$

From the extended arithmetic mean-geometric mean inequality, which states that

$$q_1x_1 + q_2x_2 + \dots + q_nx_n \ge x_1^{q_1}x_2^{q_2}\cdots x_n^{q_n}$$

where q_i, $x_i > 0$ and $q_1 + q_2 + \dots + q_n = 1$,

the left side of (*) does not exceed

$$\left[\frac{a\left(\frac{b+c-a}{a}\right) + b\left(\frac{c+a-b}{b}\right) + c\left(\frac{a+b-c}{c}\right)}{a+b+c}\right]^{a+b+c} = 1.$$

Equality holds

if and only if $(b+c-a)/a = (c+a-b)/b = (a+b-c)/c$

if and only if $(b+c)/a = (c+a)/b = (a+b)/c$

if and only if $a = b = c$.

Also solved by FRANK P. BATTLES, *Massachusetts Maritime Academy, Buzzards Bay,* and the PROPOSER. Both BATTLES and KLEIN pointed out that problem 233 in *The College Mathematics Journal* is similar to this problem.

581. *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.*

If a triangle similar to a 3-4-5 right triangle has its vertices at lattice points (points with integral coordinates) in the plane, must its legs be parallel to the coordinate axes?

Amalgam of essentially similar solutions submitted independently by CHARLES R. DIMINNIE, *St. Bonaventure University, New York,* RICHARD I. HESS, *Rancho Palos Verdes, California, and* JOHN PUTZ, *Alma College, Michigan.*

Let a, b, and c be positive integers such that

$$a^2 + b^2 = c^2.$$

Now let p and q be any nonzero integers. Then the triangle with vertices $O(0,0)$, $A(pa,qa)$ and $B(-qb,pb)$ is a right triangle with vertices having integral coordinates, with right angle at the origin and with legs OA and OB proportional to a and b. Thus take $a = 3$ and $b = 4$ to see that the posed question is answered in the negative since the slopes of OA and OB are q/p and $-p/q$. More generally, the figure can be translated to place point O at any lattice point.

Also solved by MARK EVANS, *Louisville, KY,* VICTOR G. FESER, *Mary College, Bismarck, ND,* JOHN M. HOWELL, *Littlerock, CA,* KENNETH M. WILKE, *Topeka, KS, and the* PROPOSER.

582. *Proposed by Walter Blumberg, Coral Springs, Florida.*

In triangle ABC with sides of lengths a, b, and c, we are given that $bc^2 \cos B = ca^2 \cos C = ab^2 \cos A$. Prove that triangle ABC is equilateral.

Solution by the South Dakota State University Problem Solving Group, Brookings.

Divide through by abc, multiply by 2, and apply the law of cosines to obtain

$$\frac{a^2 - b^2 + c^2}{a^2} = \frac{a^2 + b^2 - c^2}{b^2} = \frac{-a^2 + b^2 + c^2}{c^2}.$$

Now subtract 1 from each side to get

$$\frac{c^2 - b^2}{a^2} = \frac{a^2 - c^2}{b^2} = \frac{b^2 - a^2}{c^2}.$$

If these quantities are all positive, then $c > b > a > c > 0$, which is impossible. Similarly they cannot all be negative. Thus they are all zero and $a = b = c$.

Also solved by FRANK P. BATTLES, *Massachusetts Maritime Academy, Buzzards Bay,* CHARLES R. DIMINNIE, *St. Bonaventure University, NY,* RUSSELL EULER, *Northwest Missouri State University, Maryville,* MARK EVANS, *Louisville, KY,* JACK GARFUNKEL, *Flushing, NY,* JOHN M. HOWELL, *Littlerock, CA,* RALPH KING, *St. Bonaventure University, NY,* HENRY S. LIEBERMAN, *John Hancock Mutual Life Insurance Co., Boston, MA,* PETER A. LINDSTROM, *North Lake College, Irving, TX,* PI MU EPSILON PROBLEM SOLVING TEAM, *Louisiana State University, Baton Rouge,* BOB PRIELIPP, *University of Wisconsin-Oshkosh,* JOHN RUEBUSCH, *St. Xavier High School, Cincinnati, OH,* WADE H. SHERARD, *Furman University, Greenville, SC,* VIS UPATISRINGA, *Humboldt State University, Arcata, CA,* KENNETH M. WILKE, *Topeka, KS, and the* PROPOSER.

583. *Proposed by Joe Van Austin, Emory University, Atlanta, Georgia.*

An urn contains n balls numbered 1 through n, which are drawn one at a time without replacement. Let x be the first number drawn. Let y be the first number drawn that is larger than x if $x < n$ and let $y = 0$ if $x = n$. Let N be the draw which gives the y value if $x < n$ and let $N = n + 1$ if $x = n$. Find $E[y]$ and $E[N]$.

Solution to the first part by MARK EVANS, *Louisville, Kentucky, and to the second part by the* PROPOSER.

Since the probability of drawing ball r on the first draw is $1/n$ and then the expected value of y is $[(r + 1) + n]/2$, we have

$$E[y] = \sum_{r=1}^{n-1} \frac{1}{n} \frac{(r + 1) + n}{2}$$

$$= \frac{1}{2n}\ \frac{2+n+2n}{2}\ (n-1) = \frac{1}{4n}(3n^2 - n - 2).$$

If there are s cards of which t are designated as success cards, the expected number of draws needed to produce the first success is given by $(s+1)/(t+1)$. Note that if $t = 0$, then this expression for the first success still holds if we interpret draw $s + 1$ as giving the first success, which is done in the definition of N. Therefore, since one number has already been drawn, we have

$$E[N] = \sum_{j=1}^{n} \frac{1}{n}\left(1 + \frac{(n-1)+1}{(n-j)+1}\right) = 1 + \sum_{j=1}^{n} \frac{1}{n-j+1} = 1 + \sum_{k=1}^{n} \frac{1}{k}$$

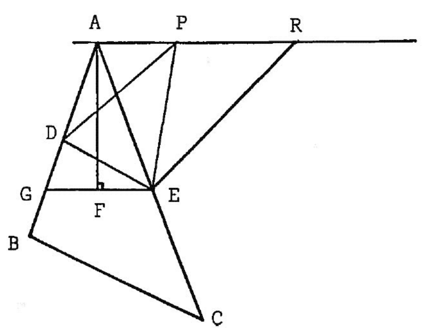
$$= 1 + 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}.$$

Also solved by JOHN M. HOWELL, *Littlerock*, CA. *Partial solution by* RICHARD I. HESS, *Rancho Palos Verdes*, CA.

584. *Proposed by Jack Garfunkel, Flushing, New York.*

Let ABC be any triangle with base BC. Let D be any point on side AB and E any point on side AC. Let PDE be an isosceles triangle with base DE, oriented the same as ABC, and with apex angle P equal to angle A. Find the locus of all such points P.

Solution by M. T. Kopf, Dümmer Lake, West Germany.



Take point G on line AB so that AGE is an isosceles triangle with apex angle at A. Let $x = \angle AGE = \angle GEA$. The rotation through angle x, with ratio $r = AE/EG$, and with center E maps D to P. Hence, as D moves along line AB $(=AG)$ with E fixed, then P moves along

another line, the image of AB. This line passes through point A as seen by taking triangle PDE to be triangle AGE. Thus the image line is AP. Now take triangle PDE to be the triangle RAE having AE as base. Of course, R lies on AP. Taking F to be the foot of the perpendicular from A to GE, we have that $\angle FAE$ is complementary to $\angle GEA = x$. Since $\angle RAC = x$, then $\angle RAF$ is a right angle. That is, the locus of P is the external bisector of $\angle BAC$.

585. *Proposed by Victor G. Feser, Mary College, Bismarck, North Dakota.*

The sum of 17 cents can be made up in exactly six ways: (0, 0, 17), (0, 1, 12), (0, 2, 7), (0, 3, 2), (1, 0, 7), and (1, 1, 2), where (d, n, p) denotes the number of dimes, nickels, and pennies, respectively. Find a value of $n > 1$ such that n cents can be made up in exactly n ways and show that that n is unique. You may use pennies, nickels, dimes, quarters, half dollars, and dollars, as needed.

Editorial comment. EDWIN M. KLEIN, *University of Wisconsin-Whitewater, and* KENNETH M. WILKE, *Topeka, Kansas, each found the solution to this problem (50 cents in 50 ways) in* Polya, *How to Solve It*, 2nd edition, Princeton University Press, 1973, pp. 238 and 252-253. HARRY SEDINGER, *St. Bonaventure University, New York, found it in* Notes on Introductory Combinatorics *by Polya et al. The solution is by recursion: there is 1 way to make 0 to 4 cents, 2 ways for 5 to 9 cents, 4 ways for 10 to 14 cents, etc.*

Also solved by MARK EVANS, *Louisville*, KY, RICHARD I. HESS, *Rancho Palos Verdes*, CA, JOHN M. HOWELL, *Littlerock*, CA, *and the* PROPOSER.

586. *Proposed by Robert C. Gebhardt, Hopatcong, New Jersey.*

For what x does $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{(2n)!} = 1 + \frac{x}{2!} + \frac{x^2}{4!} + \frac{x^3}{6!} + \frac{x^4}{8!} + \dots$ converge and what is its sum?

Solution by South Dakota State University Problem Solving Group, Brookings.

Using the well-known series for $\cosh x$ and $\cos x$ we see that the given series converges for all x and is equal to

$\cosh\sqrt{x}$ for $x \geq 0$ **and** $\cos\sqrt{-x}$ for $x \leq 0$.

Also solved by CHARLES R. DIMINNIE, St. *Bonaventure University*, NY, RUSSELL EULER, *Northwest Missouri State University*, *Maryville*, RICHARD I. HESS, Rancho *Palos Verdes*, CA, JOHN M. HOWELL, *Littlerock*, CA, EDWIN M. KLEIN, *University of Wisconsin-Whitewater*, BOB PRIELIPP and N. J. KUENZI, *University* of, *Wisconsin-Oshkosh*, I. PHILIP SCALISI *and* RICHARD QUINDLEY, *Bridgewater State College*, MA, MICHIEL SMID, *Tilburg*, The. *Netherlands*, HAO-NHIEN QUI VU, *Purdue University*, *West Lafayette*, IN, and *the.* PROPOSER. *Partial solutions were submitted* by GEORGE W. BARRATT, *Maryville*, *MO*, FRANK P. BATTLES, *Massachusetts Maritime* Academy, *Buzzards* Bay, VICTOR G. FESER, *Mary College*, *Bismarck*, *ND*, PETER A. LINDSTROM, *North* Lake. *College*, *Irving*, TX, PI MU EPSILON PROBLEM SOLVING TEAM, *Louisiana State University*, Baton Rouge, and VIS UPATISRINGA, *Humboldt State. University*, *Arcata*, CA.

1986 NATIONAL PI MU EPSILON MEETING

There will not be a national meeting of the Mathematical Association of America in the summer of 1986 since the International Congress will be meeting at the University of California, Berkeley from Sunday, August 3 through Monday, August 11, 1986.

Pi Mu Epsilon President, Milton D. Cox, has made tentative arrangements with ICM-86 planners to have members of Pi Mu Epsilon assist at the meetings during the day and participate in student paper sessions in the evenings.

Students that work will have the registration fee waived plus other benefits. Space permitting, students will have opportunities to attend the meetings of the Congress. More detailed information may be obtained from President Cox.

ICM-86 is the first International Congress of Mathematicians to be held in the United States in 36 years. Many internationally known mathematicians will be participating in the meetings and this will be a rare opportunity to see and hear them.

The Congress will focus public attention on the role of mathematics in the modern world and its importance for science and technology.

1985 NATIONAL PI MU EPSILON MEETING

The National Meeting of the Pi Mu Epsilon Fraternity was held at the University of Wyoming in **Laramie** on August 12 through August 14. Highlights were a reception for members and guests, a Dutch Treat Breakfast, a Council Luncheon and the Annual Banquet. The J. Sutherland Frame Lecturer was Professor **Ernst** Snapper, Dartmouth College, whose warmly received talk on "The Philosophy of Mathematics" was interrupted once by a false fire alarm.

Professor Frame has served Pi Mu Epsilon as Associate Editor of the Journal, as Secretary-Treasurer General and as Vice-Director General. He was Director General during the period **1957-1966.** During the Council Luncheon, Professor Frame reviewed the history of the fraternity. He is the author of "Fifty Years in the Pi Mu Epsilon Fraternity," Pi Mu Epsilon Journal, Vol. 3, No. 10, 1964.

1985 STUDENT PAPER PROGRAM

Box-Jenkins Autoregressive Integrated Moving Average Forecasting of **Common** *Stock Prices*	*Milam* W. *Aiken* Oklahoma Alpha *University of Oklahoma*
An Historical Look at Some Interesting Functions	*George* M. *Alexander* *Minnesota Delta* St. *John's University*
A Two-Dimensional Son-Linear Population Model	*Kenneth Mark Alo'* *Texas Nu* *University of Houston-Downtown*
A Shortcut to Solving Polynomial and Rational Inequalities	*Christa Blackwell* *Oklahoma Alpha* *University of* Oklahoma
Create Your Own Geometry	*David Cameron* *Ohio Delta* *Miami University*

Metagame Theory and Political Behavior	Andrew Chin Texas Lambda Univeristy of Texas
Which Came First, the Explicit Formula for the Determinant or its Properties?	Kevin T. Christian California Lambda University of California-Davis
Historic Geometric Problems Involving Compass and Straightedge Construction	Anthony Clacko Ohio Xi Youngstown State University
Non-negative Integer Solutions of $\sum_{i=1}^{n} x_i = k$	Marie Coffin South Dakota Gamma South Dakota State. University
Some Applications of the Gray Code	Henry L. Culver Ohio Xi Youngstown State. University
Using Extrapolation to Obtain Approximations to π	Raymond E. Flannery, Jr. Ohio Xi Youngstown State University
A Report on the Precision of Floating Point Math Functions on Selected Computers	John C. Flaspohler, presenter Georgia Beta Georgia Institute of Technology
A Recursive Descent Scanner in LOGO	Mark E. Frydenberg Connecticut Beta University of Hartford
Monte Carlo Studies in Statistical Research	Kathleen A. Steigelmann Illinois Epsilon Northern Illinois University
Fooling Around with Mother Nature - An Application in Genetics	John Greskovich Ohio Delta Miami University
Mysior's Example of a Regular Topological Space That is Not Completely Regular	Elizabeth Stratton Ohio Delta Miami University
Same Results in the Theory of Amicable Numbers	James 8. Hart Ahkanbah Beta Hendrix College
The Shifted QR Algorithm in Real Matrix Computations	Karalee Howell California Theta Occidental College

Sabermetrics	James D. Johnston New Yonk Omega St. Bonaventure University
Randomness and Complexity	Jorge D. Ochoa-Lions Arizona Alpha University of Arizona
Expanding a System of Propositional Logic	Mitchell Pollack Pennsylvania Beta Bucknell University
Valuations on Monoids	Terry Reilly Montana. Alpha University of, Montana
Mathematical Models in International Relations	Sandra R. Rogers Alabama Beta Auburn University
Hidden Lines, the Unseen Graphic	Henry Rosche, III Louisiana Delta Southeastern Louisiana University
A Microcomputer Application of Karmarkar's Polynomial-Time Algorithm for Linear Programming	Annuar Mohd Saffar Missouri Alpha University of, Missouri-Columbia
From Lead Pipes to Telstar -- History and Applications of Fourier Analysis	James J. Shea Massachusetts Alpha Worcester Polytechnic Institute
Network Modeling in a Transportation Environment	Cynthia M. Stuber Wisconsin Delta St. Norbert College
A General Solution Procedure for the Steady-State Probability of State-Homogeneous Production Line Models	Karim Tofang-Sazi, presenter Missouri Alpha University of Missouri-Columbia
A Phase-Plane Analysis of Coulomb Damping	John Tokar Indiana Gamma Rose-Hulman Institute of, Technology
Dale's Cone of Experience and Improved Math Teaching	Rob Walling Ohio Delta Miami University
Lattices of Periodic Functions	Kirk Weller Michigan Delta Hopi College

Boolean Idempotent Matrices and Applications	Sherry Wheeler Alabama Zeta Alabama State University
Figurate Numbers	Terri Wiencek Ohio Xi Youngstown State University
From Pan-Man to the Dihedral Group, D_4: An Application of Group Theory to Video Games	Erik L. Wynters New Hampshire Alpha University of New Hampshire

Editor's Note.

The Pi Mu Epsilon Journal was founded in 1949 and is dedicated to undergraduate and beginning graduate students interested in mathematics. Submitted articles, announcements and contributions to the Puzzle Section and Problem Department of the Journal should be directed toward this group.

Undergraduate and beginning graduate students are strongly urged to submit papers to the Journal for consideration and possible publication. Student papers will be given top priority.

Expository articles by professionals in all areas of mathematics are especially welcome.

Matching Prize Fund

If your Chapter presents awards for Outstanding Mathematical Papers or for Student Achievement in Mathematics, you may apply to the National Office for an amount equal to that spent by your Chapter up to a maximum of fifty dollars.

Posters

A supply of 10" by 14" Fraternity Crests are available. One in each color combination will be sent free to each Chapter upon request. Additional posters are available at the following rates

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GLEANINGS FROM CHAPTER REPORTS

ARKANSAS BETA (Hendrix College); The chapter's ninth year was a very active and fulfilling one. Several chapter members attended the Arkansas-Oklahoma MAA meeting in Tulsa in March. The Annual Hendrix-Sewanee-Rhodes Math Symposium was hosted by Hendrix. Kevin Shirley, Tammy Tucker, Scott Roberts and Jim Hart presented papers. Jim and Tammy also presented papers at The Conference of Undergraduate Mathematics at Guilford College in April. Guest speakers during the school year included Dr. Peter Pella, Hendrix College, on "The Simple Pendulum and Perturbation Theory," Dr. John Jobe, Oklahoma State University - Stillwater, on "Teaching Experiential Applied Mathematics," Dr. Tommy Leavelle, John Brown University, on "How Do You Add Together Infinitely Many Numbers," Craig Arnold, Southern Methodist University, on "Operations Research," David Sutherland, Hendrix graduate, on "Simple Codes that Detect/Correct Errors," Karen Shirley on "B.A. in Mathematics -- What Next?," Tammy Tucker, Hendrix senior, on "Applying Prime Factorization: Amicable and Perfect Numbers," and Dr. Gordon Johnson, University of Houston, on "Prediction of Solar Activity." At the Honors Convocation, Samuel J. Clark and Leigh Ann Stewart received McHenry-Lane Freshmen Math Awards, Kevin Shirley and Dick Wofford shared the Hogan Senior Math Award, and Jim Hart received the Phillip Parker Undergraduate Research Award.

CALIFORNIA LAMBDA (University of California - Davis). Chapter social activities included a pot luck dinner and a Pi Mu Epsilon versus Graduate Students Volleyball Game. Lectures were given by Calvin Johnson on "Demystifiying the Vector Integral Theorems," by Kevin Zumbrun on "The Penrose Tiles," and by Dr. Tanya Deretsky on "Magic Squares, Projective Planes and Other Strange Things." The Annual Spring Barbeque closed the school year.

FLORIDA EPSILON (University of Southern Florida). Fall 1983 talks included the Presidential Address by Rocky Rathgeber on "Curvature of Paths and Surfaces," a lecture on "Applications of Physics to Mathematics" by Dr. A. David Snider, a lecture on "Density Estimation" by Dr. Ibrahim A. Ahmad, a lecture on "The Relationship Between the Number of Holes in a Surface and the Curvature of that Surface" by Dr. W. E. Clark, and a talk by Christian Schwindt on "The Magic Cube and Orthogonal Matrices." In the Spring, Jonathan Frisco lectured on "Aspects of the Ternary Cantor Set," Dr. Mary Parrott discussed "What's the Delay?", Dr. Ernest Thieleker presented "It's All Done With Mirrors or Reflections on Groups Generated by Reflections." Chapter members aided the Florida Section of the MAA in organizing the Student Papers Session. Students from five area colleges gave talks. In April, the chapter hosted a banquet in honor of new members. Dr. and Mrs. Witold Kosmala presented a violin recital. At the last meeting of the year, Student Correspondent, Christian Schwindt spoke on "The Tchebysheff Transform, a Radon Transform of Radially Symmetric Functions."

GEORGIA BETA (Georgia Institute of Technology). In May, at the 1985 Honors Program, *Lori Greenberg O'Neal* was recognized and was awarded a book of her choice. Requirements for recognition include a grade point average of at least 3.7 (A = 4.0) in all mathematics courses taken.

KANSAS GAMMA (Wichita State University). Chapter activities included eight talks, a picnic, a banquet and pizza parties. *Dr. Richard V. Andree*, The University of Oklahoma, spoke on "Some Problems Neither My Computer Nor I Can Solve - YET," *Tim Hiebert* on "Questions in the Philosophy of Mathematics," *W. M. Perel* on "Two Plus Two Equals Four," *Hossein Oloomi* on "Stability of Linear Lumped-Parameter Systems," *D. V. Chopra* on "Mathematics and Pi Mu Epsilon," *Jan M. Zytkow* on "On Artificial Intelligence," *Leonid Krop* on " Quaternions, Rotations and the Hurwitz Problem," and *Kirk Lancaster* on "On The Boundary Behavior of Minimal Surfaces and Soap Bubbles."

MASSACHUSETTS GAMMA (Bridgewater State College). During the Spring, chapter members met regularly with Faculty Advisor, *Professor Thomas E. Moore*, for problem-solving sessions. Chapter-sponsored invited talks included "Hilbert's Third Problem" by *Thomas E. Moore* and "A Use of Symbolic Logic in Medical Diagnosis" by *Professor Thomas Koshy*. The Initiation Address was given by *Joseph B. Chiccarelli*, Vice-president of Administration and Finance of Bridgewater State College.

MINNESOTA GAMMA (Macalester College). Fall activities began with a get-acquainted picnic with faculty. In October, *Dr. Ron Jacobson*, Quality Control Division, Metro Waste Commission, spoke on "Linear Models and Statistical Procedures." In November, film programs featured a Canadian-produced movie on sorting and several of the College Geometry productions. A Game Night and a Christmas Bash closed out the Fall Semester. In the Spring, *Jerry Degerness*, St. Paul Companies, spoke to club members and guests on the actuarial profession and opportunities therein. The Invited Speaker for the Initiation Day Program was *Professor Dale Varberg*, Hamline University, who presented a new proof of Pick's Theorem. Chapter President, *David Warland*, was awarded the Ezra J. Camp Prize as Outstanding Mathematics Student. In April, *David* spoke on "Morley's Theorem and Complex Numbers" at the Annual Pi Mu Epsilon Student Conference at St. John's University. A last-day-of-classes picnic closed the school year activities.

MINNESOTA ZETA (St. Mary's College). In addition to mathematics colloquium talks, the chapter members (jointly with the Commerce Club, the Computer Science Club and the Career Services Center) sponsored a panel discussion on "Undergraduate Education and the Outside World: Things I Did Right - Things I Did Wrong." The Annual Induction Ceremony for new members was on April 19. *Timothy Doffing*, chapter Vice-President spoke on the "Sheffer Stroke." The Mathematics and Statistics Club and Pi Mu Epsilon jointly support awards for outstanding freshmen mathematics students. A Prisoner's Dilemma Game is being organized on campus.

MISSISSIPPI ALPHA (The University of Mississippi). Four invited talks highlighted the year's activities. These were "A Simplistic Approach to the Simplex Method" by *Dr. William R. Trott*, faculty advisor, "Classroom Computer Applications for Microcomputers" by *Dr. David E. Cook*, "Fibonacci Numbers" by *Dr. William A. Staton* and "A Historical Viewpoint of Fermat's Theorem" by *Dr. Przemo Kranz*. *Miss Elizabeth Hayes* was the winner of the Annual Pi Mu Epsilon Award.

MISSOURI GAMMA (Maryville College and St. Louis University). *Sister Harriet Padberg*, chairperson and Professor of Mathematics, reported the detah of *Sister Marie P. Kernaghan* who headed the Science Department at Maryville College for many years. *Sister Marie* was the first woman to receive a doctorate from St. Louis University and the first person to receive a doctorate in physics at the university. *Sister Marie* was a charter member of Pi Mu Epsilon at that institution. *John Painter* was winner of the Pi Mu Epsilon Senior Contest and *Hil Kin Wong* was Junior Contest winner. Cash and book prizeswere awarded. Projected club activities include a tutoring service. At the Annual Initiation Ceremony, *Richard H. Austing*, University of Maryland, gave the 40th annual James E. Case, S.J. Memorial Lecture. His talk was on the history of computer science as a discipline and on computer science in society.

NEW JERSEY DELTA (Seton Hall University). During the Fall of 1984, members met weekly with *Dr. John Masterson* for problem-solving sessions. Lectures during the school year included "Magic Squares" by *Dr. John Saccoman* of the Seton Hall University Mathematics Department, "Mathematics and Operations Research" by *Dr. John Klincewicz*, AT & T Bell Laboratories, and "Can You Trisect a Given Angle with Ruler and Compass?" by *Dr. Philip Schwartau*, Seton Hall. At the 18th Annual Induction Ceremony and Luncheon, *Dr. John Saccoman*, faculty advisor, spoke on "The History of Pi Mu Epsilon."

NEW JERSEY EPSILON (Saint Peter's College). Recipient of the Collins Award, given to a student who has completed the sophomore year and who has the highest average in courses in the areas of mathematics, natural sciences, and computer science, was *Theresa Jean Grant*. *Theresa* was presented with a copy of ***The Mathematical Experience*** by Davis and Hersh.

NEW YORK PHI (Potsdam College). Invited speaker at the Fall Induction was *Dr. Kathryn Weld*, a Potsdam College graduate, class of 1977. Her lecture was "Why Do Mathematics?" At the Spring Induction, *Dr. Joyce Scott* spoke on "The Liberal Arts Education." In April, the chapter co-sponsored Career Day with Career Services and the Mathematics Department. Five speakers from the Potsdam area gave talks on careers in mathematics. Fund raisers included Down Town Night and a T-shirt Sale. Social activities were an Open House Mixer and a picnic with other student organizations. The chapter lost its softball game with Computer Science.

NEW YORK OMEGA (Saint Bonaventure University). Problem-solving participation was encouraging. Lectures included *Professor Erik Hemmingsen*, Syracuse University, on "The Strange World of the Iteration of Functions," *Professor* Lent *Friedland*, St. Bonaventure University, on "Fixed Point Theorems - A Survey of Some Elementary Results," *James Johnston*, chapter president, on "Sabermetrics - the Statistical Analysis of Baseball Records," and *Professor Douglas Cashing*, St. Bonaventure University, on 'Arrow's Paradox: Why Democracy Can't Work." In November, chapter members helped with arrangements for the Fall meeting of the Seaway Section of the MAA. At the Spring Meeting of the Seaway Section, *James Johnston* presented a shortened version of his talk "Sabermetrics." At the annual induction ceremony, the Pi Mu Epsilon Award (a check for $50.00) was presented to *James Johnston*. *Taut Catalano* and *James Dougherty* received honorable mention.

NEW YORK ALPHA ALPHA (Queen's College of the City University of New York). *Dr.* Eugene *Don*, Queen's College, spoke on "Mathamagic" and explained how mathematics is at the basis of many magic tricks. The film "Isometries," a College Geometry production, was shown at the Annual Initiation Ceremony. Other highlights were end-of-term parties each semester and several organizational and social meetings. *Lisa Katz* and *Karen Stern* were the recipients of the 1985 Pi Mu Epsilon Prize (cash awards) for excellence in mathematics and service to the chapter.

NEW YORK ALPHA GAMMA (Mercy College). At the annual induction meeting in May, *Dr. Mariano Garcia*, Hostos Community College, spoke on "Amicable Numbers." Other chapter activities included a lecture on "Mathematical Puzzles" by *Dr.* John *Tucciarone*, Mercy College, and a talk on "Interesting Problems in Mathematics" by *Dr.* Gordon *Feathers*, Mercy College.

NORTH CAROLINA LAMBDA (Wake Forest University). At the Installation and Initiation Banquet, National President, *Professor Milton D.* COX, lectured on "A Geometry Problem Requiring a Calculus Solution." Senior *Doug Lee*, winner of the Raynor Scholarship in mathematics, talked about "An Intriguing Series." In April, *Dr.* John W. *Kenelly*, Clemson University, lectured on "Do You Know M. Pascal?" Chapter members assisted the Southeastern Section of the MAA in its April Meeting at Wake Forest by recruiting helpers, setting up rooms, registering conferees and serving as guides. A social activity in February was a pizza party.

OHIO DELTA (Miami University). The Fall 1983 initiation was held during the Tenth Annual Pi Mu Epsilon Student Conference. New conference records were set: 21 student talks and 53 off-campus student guests. In October, *Dr. Catherine Rudin* of Miami's English Department spoke on "A Formal Model of Natural Language." In November, *Dr. David Groggel* answered the question "What Does A Statistician Do?" The Annual Holiday Party was in December. *Dr. Richard Laatsch* gave a short talk on recreational mathematics. At a regional conference at the University of Louisville in January, talks were given by chapter members Lee Ann *Schollenberger* and Renee *Tehi*. At the winter initiation, Vh. Tom *Bengston* discussed "Redshift and Distance." At March meetings, Miami's *Dr.* Bob *Deckhart* discussed "Wedderburn's Theorem" and *Beth Barnes* of Miami's Marketing Department talked about "Advertising and Statistics." *Dr. David Kullman's* Spirolateral of 175 Sides (in celebration of Miami's 175th anniversary) won the Student Conference Program Cover Design Contest. At April's Awards Banquet, Pi Mu Epsilon Examination Prizes were awarded to John R. *Knauer*, *Leslie* L. *Youngdahl* and John F. *Greskovich*. At the state MAA meeting at Bowling Green State University, talks were given by *Lee* Ann *Shollenberger*, *Mark Russell* and *David Cameron*. Several" chapter members attended the National Pi Mu Epsilon Meeting at the University of Oregon in August.

OHIO NU (The University of Akron). Awards to students for outstanding achievement in mathematical academics at the April Initiation and Awards Banquet included MAA memberships to *Carol Hoover*, *Cheryl Picciotti*, *Mike Kirchner*, *Lisa Oster* and *George Woods*, AMS memberships to *Jim Anderson*, *Jeff Watson* and *Mark Mahoney*. *Lisa Oster* and *Chris Bolinger* received the Samuel Selby Mathematics Scholarship Award for 1985-86 and *Christie Mazu* was the recipient of the Annual Akron Regional Science Fair Award. A picnic, a faculty versus students volleyball game, Career Night (with speakers from industry in the fields of mathematics and computer science) rounded out the year. *Dr. Douglas Cameron*, The University of Akron, presented "an interesting and very challenging talk entitled 'South of the Border.'"

OHIO OMICRON (Mount Union College). In September, chapter members were invited to attend the Eleventh Annual Pi Mu Epsilon Conference at Miami University. *Curt Blaisiman*, chapter president, spoke on "Math's Role in Converting Analog Signals to Digital Signals." In March, *Paul Andaloro*, a Mount Union College alumnus and Ohio State Ph. D. candidate, spoke on "The 3n+l Problem." Initiation ceremonies were held in April.

OHIO XI (Youngstown State University). Four chapter members attended the 1984 National Pi Mu Epsilon Meeting in Eugene, Oregon. *Jeff Kubina* and *Debbie Whitfield* presented papers. *Jeff* gave his paper again at the Pi Mu Epsilon Student Conference at Miami University in September. At the April 1985 MAA Meeting in Akron, *Paul Mullins*, *Ray Flannery* and *Theresa Weincek* gave talks. Other highlights were a field trip to Reliance Electric in Cleveland, a sled riding party, the Fall Quarter Initiation, at which *Dr. David C. Buchtal*, University of Akron, was guest speaker, the Winter Quarter Initiation, at which *Dr. James Hall*, Westminster College, lectured, and the Spring Quarter Initiation, at which *Dr. Piotrowski* of Youngstown State was guest speaker. *Dr. Milt* COX, National President, was the invited lecturer at the Annual Spring Banquet.

OHIO ZETA (The University of Dayton). Twelve meetings were held during, the school year 1984-85. Principal activities included eleven student talks. *Ken Bloch* spoke on "Applications of Orthogonality to Quantum Mechanics" and on "Matrix Solutions in Quantum Mechanics." *Gary Johnson* spoke on an application of linear algebra to chemistry and on a topic from group theory. *John Sengewalt* spoke on Markov chains and on the European Roulette Wheel. *Brain Donahue* spoke on "Cube Mania at its

Best." *Anne Schmid* spoke on "Queueing Theory." *Sandi Jurcak* spoke on "Euler's Konigsberg Bridge Problem." *Greg Bishop* spoke on "The Shewhart Control Chart." *Kelly Ann Chambers* spoke on "Paradoxes, Past and Present." *Dr. Paul Eloe*, chapter advisor, gave a presentation entitled "An Application of the Riemann-Stieltjes Integral to Infinite Series." Several chapter members attended the Eleventh Annual Pi Mu Epsilon Conference at Miami University in September, at which *Brian Donahue* spoke on "Some Algebraic Structures on Rubik's Cube" and *David Goloff* spoke on "A Line Thought From Rings of Continuous Functions." In March, chapter members assisted the University of Dayton Mathematics Department in preparing for the Twelfth Biennial Alumni Seminar on Employment Opportunities in the Mathematical Sciences. At the chapter's annual banquet, *Mary Beth Anderson* was presented the Sophomore Class Award of Excellence.

PENNSYLVANIA NU (Edinboro University). At initiation ceremonies in November, *Dr. Richard H. Reese* gave a presentation on mathematical puzzles. At initiation ceremonies in February, *Jewell* spoke on the infinity of even primes. Fall and Spring pizza parties and a social get-together at the home of the chapter advisor were other highlights.

TEXAS IOTA (The University of Texas at Arlington). The chapter sponsored a series of eight lectures by members of the mathematics department, area companies and faculty from other universities in the Dallas-Fort Worth area. Average attendance was 40 (Remarkable turnouts - Ed.) The Dunsworth Lectures were given by *Dr. Esenfeld* on "Why Do Elephants Walk?" Donut sales were fund raisers. Other activities were an annual picnic and assisting the UTA Mathematics Department in hosting the High School Seminar. Chapter service awards went to *Rebecca Pierce, Steve Cain* and *Irma Almazan.*

VIRGINIA GAMMA (James Madison University). Club activities began in September with a well-attended Farm Party/Picnic at the University Farm. In October, *Dr. George W. Marrah*, JMU, spoke on "Job Opportunities and the Job Search." A booksale was held to raise funds for the club. Initiation ceremonies were held in November. Guest speaker was *Lois Mansfield*, University of Virginia, who spoke on "The World of Scientific Computing." A "Get-To-Know-the-Faculty" series added much interest to chapter meetings and attendance improved. In February, *Ken Guthrie*, Electronic Data Systems, spoke on "Resume Writing and the Job Search." A ski trip to Wintergreen Resort was sponsored by the JMU Math Club. Series speaker at regular meetings in March was *Mr. Fairfield*. The Pi Mu Epsilon Award to the most outstanding senior was given to *Roberta Cochrane*. Last speaker for the year was *Dr. J. A. N. Lee*, Virigina Polytechnic Institute, who spoke on "Computer Security." Another Farm Party closed out a most successful year. The final project was a fund raiser/social which netted $200, most of which was donated to a mathematics scholarship fund.

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