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ON THE LARGEST RAT-FREE SUBSET OF A FINITE SET OF POINTS

by Wah Keung Chan
McGill University

During the summer of 1985, Professor William Moser supervised my activity while I held an NSERC Summer Undergraduate Research Award at McGill University. His now well-known, privately circulated, ***Research Problems in Discrete Geometry*** is a cumulative record of progress made on many unsolved and partially solved problems in discrete geometry. While examining this collection I noticed that I understood many of the problems, and for some I could comprehend the associated published research papers. Several of the problems were featured by Martin Gardner in his Scientific American column "Mathematical Games," and progress in these had been made by readers, that is, by amateur mathematicians using only pre-calculus mathematics. In this paper, I would like to report on one problem (No. 49) in W. Moser's collection [1] which is not completely solved and which I believe to be suitable for readers of this journal to investigate further.

Basically, the problem is as follows. Suppose one is given a set of n points, and the objective is to select as many of them as possible so that no three (of the chosen points) are the vertices of a right-angled triangle. How many can one be assured of choosing no matter how the n points are distributed? Let us phrase the question in a more precise manner. By an ***n-set*** we mean a set of n points in the plane, and we call the vertices of a right-angled triangle an ***RAT***. Let $f(n)$ denote the largest integer for which every ***n-set*** contains an ***RAT-free f(n)-subset.*** The problem is to determine $f(n)$ or, failing this, to establish good upper and lower bounds on $f(n)$. Note that, to establish the lower bound m, that is, to show $f(n) \geq m$, it is necessary to show that every ***n-set*** contains an ***RAT-free m-subset.*** To establish the upper bound l, that is, to show $f(n) < l$, it suffices to exhibit a particular ***n-set*** which does not contain an ***RAT-free l-subset.*** The main result is contained in

Theorem 1.

(1) $$\sqrt{n} \leq f(n) \leq 2\sqrt{n}, \quad n = 4, 5, 6, \ldots .$$

The lower bound was established by Abbott [2] using a famous theorem of Erdős and Szekeres [3]. For the latter theorem we will give a particularly elegant proof due to Seidenberg [4].

A sequence of numbers

(2) $$a_1, a_2, a_3, \ldots, a_r$$

is said to be *monotone increasing* if

(3) $$a_1 \le a_2 \le a_3 \le \cdots \le a_r,$$

monotone decreasing if

(4) $$a_1 \ge a_2 \ge a_3 \ge \cdots \ge a_r,$$

and *monotone* if it is either *monotone increasing* or *monotone decreasing*.

Theorem 2. (Erdős-Szekeres). Let m, n be positive integers. Then every sequence of length $mn + 1$ (but not every sequence of length mn) contains either a monotone increasing subsequence of length $m + 1$ or a monotone decreasing subsequence of length $n + 1$.

Proof. Let

(5) $$y_1, y_2, y_3, \ldots, y_{mn+1},$$

be an arbitrary sequence of length $mn + 1$. To each y_i in (5) we associate the pair (u_i, v_i) of integers, where u_i (resp. v_i) is the length of the longest monotone increasing (resp. decreasing) subsequence of (5) beginning at y_i. Letting $u = \max_i u_i$, $v = \max_i v_i$, we have

(6) $$1 \le u_i \le u, \quad 1 \le v_i \le v, \quad i = 1, 2, \ldots, mn + 1,$$

so the number of distinct points (u_i, v_i) is at most uv. Furthermore, these $mn + 1$ points $(u_1, v_1), (u_2, v_2), \ldots, (u_{mn+1}, v_{mn+1})$ are distinct. For suppose $i \ne j$, say $1 \le i < j \le mn + 1$. If $y_i \le y_j$ then $u_i > u_j$; if $y_i \ge y_j$ then $v_i > v_j$. It follows that

(7) $$mn + 1 \le uv,$$

from which either $u \ge m + 1$ or $v \ge n + 1$.

To construct an example for the "but not every sequence of length mn" part of Theorem 2, we start with the sequence of integers from 1 to mn

$$\underbrace{1, 2, \ldots, n}_{\text{Block } 1}, \underbrace{n+1, n+2, \ldots, 2n}_{\text{Block } 2}, \ldots, \underbrace{(i-1)n+1, (i-1)n+2, \ldots, in}_{\text{Block } i},$$

$$\ldots, \underbrace{(m-1)n+1, \ldots, mn;}_{\text{Block } m}$$

reversing the order in each block, we obtain

(8) $$\underbrace{n, n-1, \ldots, 1}_{\text{Block } 1}, \underbrace{2n, \ldots, n+1}_{\text{Block } 2}, \ldots, \underbrace{in, \ldots, (i-1)n+1}_{\text{Block } i}, \ldots, \underbrace{mn, \ldots, (m-1)n+1}_{\text{Block } m}.$$

Notice that any increasing subsequence of (8) has at most one term in any block, and hence has length at most m. Any decreasing subsequence has length at most n since all terms must be in the same block. These properties are nicely seen in Figure 1 where the i^{th} term of (8) is the ordinate of the point with abscissa i.

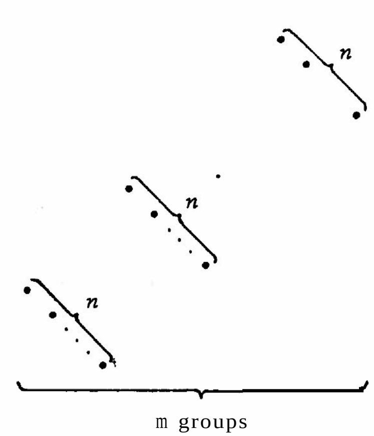


Figure 1

When m and n are equal, say k, we have

Corollary 1. Every sequence of length $k^2 + 1$ (but not every sequence of length k^2) has a monotone subsequence of length $k + 1$.

It is now possible to give a proof of Abbott's Theorem.

Theorem 3.

(9) $$k + 1 \le f(k^2 + 1), \quad k \ge 3.$$

Proof. Suppose we have a set of $l = k^2 + 1$ points. Choose a Cartesian coordinate system whose x-*axis* is not perpendicular to any of the $l(l-1)/2$ segments joining pairs of the points. Then no two points* have the same abscissa. Let the points (x_1,y_1), (x_2,y_2), $\ldots$, (x_l,y_l) be named so that $x_1 < x_2 < \ldots < x_l$. The sequence $y_1, y_2, \ldots , y_l$ contains (by Corollary 1) a monotone subsequence

$$y_{i_1}, y_{i_2}, \ldots , y_{i_{k+1}}$$

of length $k + 1$. It is easy to see that the $k + 1$ points (x_{i_j}, y_{i_j}), $j = 1, 2, \ldots , k + 1$ form an *RAT-free* $(k + 1)$-*subset* of the $k^2 + 1$ given points.

The left inequality of Theorem 1 follows by considering, for a given integer n, the integer k uniquely determined by

$$(k - 1)^2 + 1 \le n \le k^2.$$

Then, using $f(k) \le f(k + 1)$,

$$\sqrt{n} \le k \le f((k - 1)^2 + 1) \le f(n).$$

Erdös [5] remarked that the set $S_k^{(1)}$ of k^2 lattice points

$$S_k^{(1)} = \{(x, y) | x, y \text{ integers}, \ 0 \le x, \ y \le k - 1\}$$

shows

Theorem 4.

(10) $$f(k^2) \le 2k - 2, \quad k \ge 2.$$

Proof. To establish this inequality, let P be an *RAT-free* subset of $S_k^{(1)}$. We call a point $p \in P$ an *a-point* (resp. a *b-point*) if the horizontal row in which it lies contains no (resp, at least one) other point of P. Let a (resp. b) denote the number of *a-points* (resp. *b-points*) in P. No column contains two *b-points*, nor an *a-point* and a *b-point* (together). Clearly, $a \le k$ and $b \le k$. If there are $a = k$ *a-points* then every row contains an *a-point*, allowing no row for a *b-point* $(b = 0)$, and then $a + b = k \le 2k - 2$. Similarly, if there are $b = k$ *b-points* there will be no column for an *a-point*, so $a = 0$, and, again, $a + b = k \le 2k - 2$.

In all other cases, $a \le k - 1$ and $b \le k - 1$, and hence $a + b \le 2k - 2$. Figure 2 shows that $S_k^{(1)}$ does contain an *RAT-free* $(2k - 2)$-*subset*. Indeed, all *RAT-free* $(2k - 2)$-*subsets* of $S_k^{(1)}$ must have the configuration of Figure 2, since the column containing the $(k - 1)$ *a-points* and the row containing the $(k - 1)$ *b-points* meet at a point (of $S_k^{(1)}$) which must be at a corner of the square $S_k^{(1)}$.

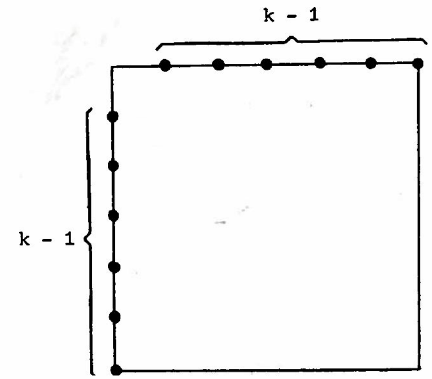


Figure 2

The right inequality of Theorem 1 follows from Theorem 4. If k is the unique integer for which

$$k^2 \le n \le (k + 1)^2 - 1,$$

then

$$f(n) \le f((k + 1)^2 - 1) \le f((k + 1)^2) \le 2k \le 2\sqrt{n}.$$

An improvement of (10) is obtained by removing two adjacent corner points from $S_k^{(1)}$, for example, $S_k^{(2)} = S_k^{(1)} \setminus \{(0,0), (0,k - 1)\}$ yielding

(11) $$f(k^2 - 2) \le 2k - 3.$$

For if P, an *RAT-free* subset of $S_k^{(2)}$, contains $2k - 2$ points, then P must (also) be congruent to Figure 2; but this configuration cannot be found in $S_k^{(2)}$. Therefore, $|P| \le 2k - 3$.

This result was improved by H. L. Abbott [5]. He considered the set

$$S_k^{(3)} = \{(x,y) | x, y \text{ integers}, \ 0 \le x \le k - 1, \ 0 \le y \le k\} \setminus \{(0,0), (0,k), (k - 1,0), (k - 1),k)\}, \quad k \ge 5$$

of a $k \times (k + 1)$ rectangular array of lattice points with corner points removed and found the largest *RAT-free* subset to be $2k - 3$, that is,

(12) $$f(k^2 + k - 4) \le 2k - 3.$$

Based on the same idea, we present the proof of a slightly better result.

Theorem 5.

(13) $$f(k^2 - 4) \le 2k - 4, \quad k \ge 5.$$

Proof. Consider the set

$$S_k^{(4)} = \{(x,y) \mid x, y \text{ integers}, 0 \le x, y \le k-1\} \setminus \{(0,0), (0,k-1), (k-1,0), (k-1,k-1)\}, \quad k \ge 5$$

of a $k \times k$ square array of lattice points with corner points deleted; $k^2 - 4$ points in total. We show that we can find an *RAT-free* subset of $2k - 4$ points while it is not possible to find an *RAT-free* subset of $2k - 3$ points. Once again, following the proof of Theorem 3, we let there be a *α-points* and *b E-points.*

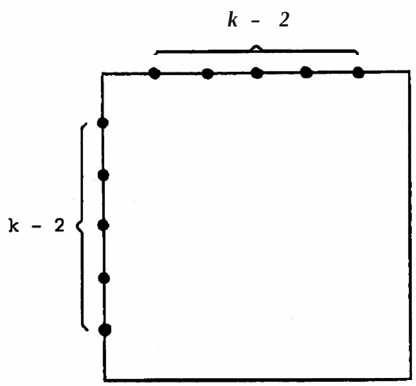


Figure 3

Figure 3 shows an *SAT-free* subset of $2k - 4$ points. Now, assume there is an *SAT-free* subset of $2k - 3$ points, that is, $a + b = 2k - 3$. The number of rows is equal to or greater than a (the number of *a-points)* and at least one row occupied by the *β-points.*

Thus

$$a + 1 \le k.$$

Replacing a,

$$2k - 3 - b + 1 \le k$$

gives

$$b \ge k - 2.$$

It is sufficient to show that there does not exist an *RAT-free* subset of $b = k - 2$ *β-points* and $a = k - 1$ *a-points* since the case $b = k - 1$ and $a = k - 2$ can be reduced to the former.

Since $a = k - 1$ there exists only one row in which the $(k - 2)$ *β-points* can be placed; the *a-points* can occupy two columns. There are four cases to consider, all related to the distribution on the two

"*α-columns*". It can happen that

(i) the columns are not adjacent,

(ii) columns are at c_1 and c_2 (or c_{k-1} and c_k), where c_i denotes the <u>*ith*</u> column,

(iii) columns are at c_2 and c_3 (or c_{k-2} and c_{k-1}), or

(iv) columns are adjacent other than in (ii) and (iii).

Figures 4, 5, 6 and 7 illustrate the impossibility of having an *SAT-free* subset of $a + b = 2k - 3$ points for cases (i), (ii), (iii) and (iv), respectively.

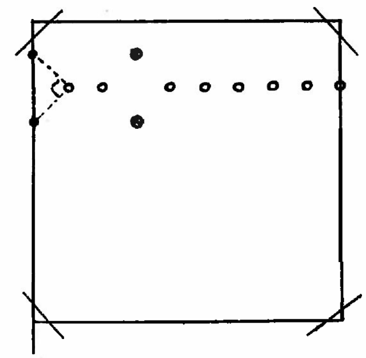


Figure 4

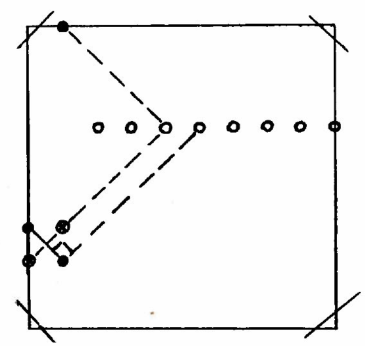


Figure 5

For Figures 4, 5, 6a, 6b, and 7, we have O for beta points. ● for alpha points and ⊗ for illegal points.

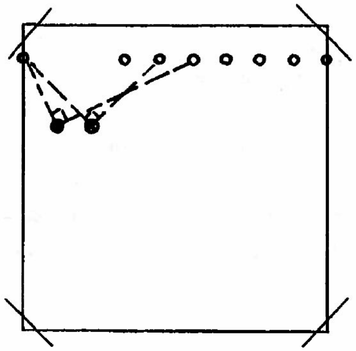
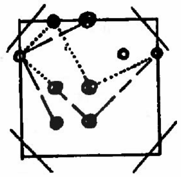


Figure 6a



for k = 5

Figure 6b

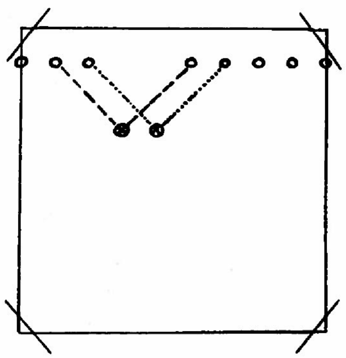


Figure 7

Theorems 2, 3, 4 and 5 restrict $f(n)$ for large n to

(14) $$\sqrt{n} \le f(n) \le 2\sqrt{n}.$$

By considering the set of lattice points contained in a circle, we conjecture that the upper bound can be reduced to

$$f(n) < 1.6\sqrt{n}.$$

It is of interest to find the value of $r = \lim_{n\to\infty} f(n)/\sqrt{n}$ if it exists.

Also of interest is the exact value of $f(n)$ for small n. Using inequalities (1), (9), (10), (12) and (13), we have the following:

$$f(4) = 2$$
$$f(5) = 3$$
$$f(6) = 3$$

$$f(7) = 3$$
$$3 \le f(8) \le 4$$
$$3 \le f(9) \le 4$$
$$f(10) = 4$$
$$4 \le f(i) \le 5, \ i = 11, 12, 13, 14$$
$$4 \le f(i) \le 6, \ i = 15, 16$$
$$5 \le f(i) \le 6, \ i = 17, 18, 19, 20, 21$$
$$5 \le f(i) \le 7, \ i = 22, 23, 24, 25$$
$$6 \le f(26) \le 7$$
$$6 \le f(i) \le 8, \ i = 27, 28, \ldots, 32$$
$$6 \le f(i) \le 9, \ i = 33, 34, 35, 36$$
$$7 \le f(i) \le 9, \ i = 37, 38$$
$$7 \le f(i) \le 10, \ i = 39, 40, \ldots, 45.$$

Theorem 2 gives $f(5) \ge 3$ and $f(10) \ge 4$. $f(5) = f(6) = f(7) = 3$ follows from Figure 8. The configuration in Figure 9 shows $f(10) = 4$. Perhaps a reader can settle some of these cases, for example, is $f(8) = 3$ or 4?

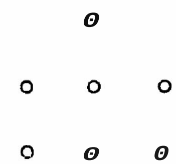


Figure 8

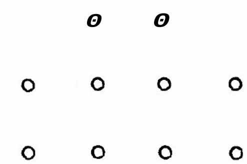


Figure 9

Erdős [2] also asked the same question in three dimensions, that is, for the n^3 lattice points (x,y,z), $0 \le x, y, z \le n - 1$, determine the largest *SAT-free* subset.

For $n = 2$, the largest *RAT-free* subset consists of four (of the eight) points forming the vertices of a regular tetrahedron (see Figure 10).

For $n = 3$, the largest *SAX-free* subset is 6 points, verified by computer analysis. The two groups of configurations are shown in Figures 11 and 12.

For n = 4, *RAT-free* subsets of 10 points were found. The subsets are congruent to the form $\{(0,0,0), (1,1,1), (2,2,2), (3,3,3), (3,1,0), (3,0,1), (0,3,1), (1,3,0), (0,1,3), (1,0,3)\}$. On the other hand, this configuration-

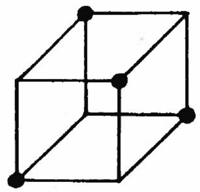


Figure 10

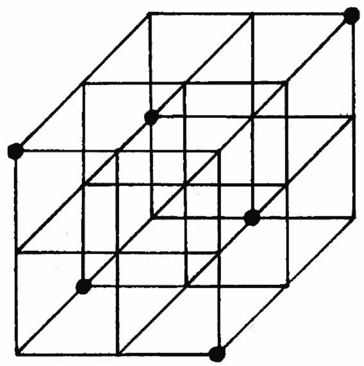


Figure 11

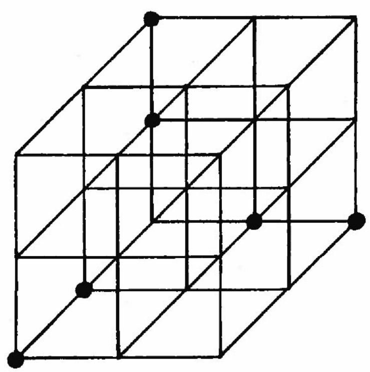


Figure 12

does not work for $n = 5$. It was found that any collection of $3(5) - 2 = 13$ points with points $(0,0,0)$ and $(1,1,1)$ have no *RAT-free* subsets.

Our set of n^3 points determines n^2 columns parallel to each of the coordinate axes. Call the columns c_x, c_y and c_z. In any configuration of an *RAT-free* subset, there are four types of points.

(1) *α-points*, points occupying the same c_x with other points,

(2) *β-points*, points occupying the same c_y with other points,

(3) *γ-points*, points occupying the same c_z with other points,

(4) *δ-points*, points occupying a c_x, a c_y and a c_z alone.

Let there be a, b, c and d of these points, respectively.

We have found two groups of *RAT-free* $(3n - 3)$-*subsets*. One has $d = 0$ and $a = n - 1$, $b = n - 1$ and $c = n - 1$. Another has $a = n - 2$ $\{(1,0,0), \ldots, (n-2,0,0)\}$, $b = n - 2$ $\{(0,1,0), \ldots, (0,n-2,0)\}$, $c = n - 2$ $\{(0,0,1), \ldots, (0,0,n-2)\}$ and $d = 3$ $\{(n-1,0,n-1), (n-1,n-1,0), (0,n-1,n-1)\}$ for $k^2 \nmid n - 1$. Are there any other configurations?

We suspect that for large n the largest *RAT-free* subset is of size $3n - 3$.

We have yet to show (i) that for $a = b = c = 0$ the largest *RAT-free* subset of only *δ-points* has size $d \leq 3n - 3$, and (ii) for $a \neq 0$, $b \neq 0$, $a \neq 0$ and $d \neq 0$, $a + b + a + d \leq 3n - 3$. This would confirm our conjecture.

Acknowledgement. I am *indebted to Professor W. Moser for his valuable advice during the writing of this paper.*

REFERENCES

1. Moser, W. O. J., *100 Research Problems in Discrete Geometry* (1986), Department of Mathematics and Statistics, McGill University.

2. Abbott, H. L., *On a Conjecture of Erdös and Silverman in Combinatorial Geometry*, Journal of Combinatorial Theory, Series A 29 (1980), No. 3, pp. 380-381.

3. Erdös, P. and G. Szekeres, *A Combinatorial Problem in Geometry*, Compositio Math. 2 (1935), pp. 463-470.

4. Seidenberg, A., *A Simple Proof of a Theorem of Erdös and Szekeres*, Journal London Math. Soc. 34 (1959), p. 352.

5. Erdös, P., *Problems and Results in Combinatorial Analysis*, Proceedings, Eighth Southeastern Conference on Combinatorics, Graph Theory and Computing, Congressus Numerantium XIX (F. Hoffman et al, Eds.), (1977), pp. 3-12.

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ON BUFFON'S NEEDLE PROBLEM USING CONCENTRIC CIRCLES

by *H. J. Khamis*
Wright State. *University*

One of the oldest problems in geometrical probability is Buffon's Needle Problem (1777): A board (of large size) is ruled with equidistant parallel lines d units apart. A needle of length $\ell < d$ is dropped at random on the board. What is the probability that the needle will intersect one of the lines?

The answer is that the needle will cross a line with probability $2\ell/\pi d$. It is interesting to note that the answer involves the transcendental number, Pi. In fact, this formula was used in early attempts at approximating Pi; viz, drop the needle onto the board of parallel lines a large number of times, calculate the relative frequency of "crosses", reciprocate, and multiply by $2\ell/d$.

As one might expect for such an important method, a large number of variations of the Buffon Needle Problem have been studied. Duncan (1967) studied the case in which a needle is dropped onto a set of radial lines. Gnedenko (1962) generalized the problem first to n-sided convex polygons with diameter less than d, and then to convex closed curves with diameter less than d by considering such curves as limits of inscribed polygons, giving the probability of a "cross" as $\ell/\pi d$. Other variations of Buffon's Needle Problem have been discussed by Ramaley (1969), Perlman and Wichura (1975), and Robertson and Siegel (1986).

In this paper, we consider the following natural variation: Randomly drop a needle of length ℓ onto a board containing N concentric circles, where the difference in the radii between any two consecutive such circles is a constant d, with $\ell < d$. What is the probability, p_N, that the needle crosses one of the N circumferences? The only mathematical tools that are required for the solution of this problem are integration techniques and elementary probability theory.

To begin, let X represent the distance between the midpoint (M) of the needle and the nearest circumference, as measured along the radius

extending from the center (O) of the concentric circles through the midpoint of the needle. See Figure 1. Let Y represent the acute angle created by OM and the needle. Assume that M falls within the kth annulus, $k = 1, 2, \ldots, N$; that is, $(k-1)d < \overline{OM} < kd$ (k = 1 corresponds to the circle having radius d). Then, there are two cases to consider:

Case 1. $(k - 1/2)d < \overline{OM} < kd$, $k = 1, 2, \ldots, N$, and

Case 2. $(k - 1)d < \overline{OM} < (k - 1/2)d$, $k = 2, 3, \ldots, N$.

The first case corresponds to the event that M falls inside the "outer half" of the kth annulus, and the second case to the event that M falls inside the "inner half" of the kth annulus. Let these two events be represented by I_k' and I_k, respectively. We consider each case separately.

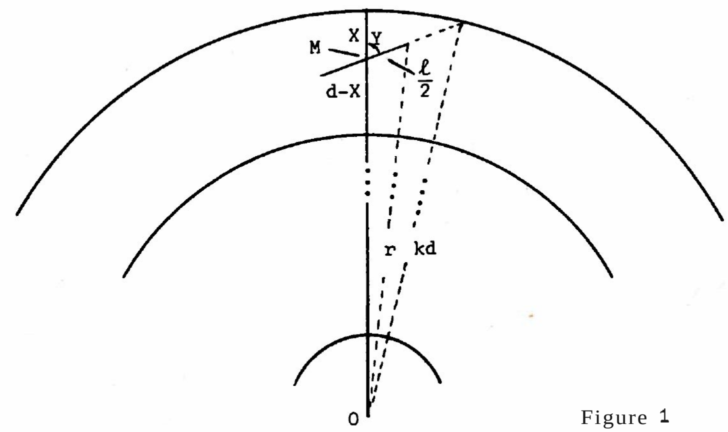


Figure 1

Case 1. First, note that if $X < kd - \sqrt{k^2d^2 - \ell^2/4}$, then the needle crosses the kth circumference regardless of the value of Y. To see this, think about the needle as a chord of the circle (see Figure 2). Using the Pythagorean Theorem, the value of x_0 in Figure 2 can easily be determined to be

$$x_0 = kd - \sqrt{k^2d^2 - \ell^2/4}.$$

Hence, when $X < x_0$, the needle crosses the kth circumference with probability one.

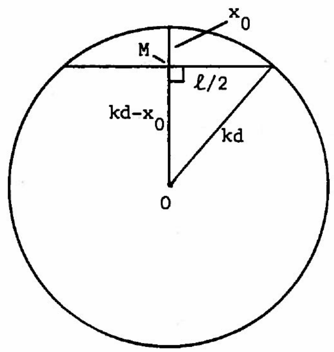


Figure 2

When $x_0 < X < \ell/2$, the probability that the needle crosses the kth circumference depends on Y. By the law of cosines (see Figure 1),

$$r^2 = (kd - X)^2 + \ell^2/4 + (kd - X)\ell\cos Y.$$

Note that the needle crosses the kth circumference if and only if $r > kd$, which is equivalent to

$$Y < \cos^{-1}\left[\frac{(k^2d^2 - \ell^2/4) - (kd - X)^2}{(kd - X)\ell}\right], \quad x_0 < X < \ell/2.$$

Now, Y is treated as a uniform random variable on $[0,\pi/2]$. The density function for X, however, must be derived (it is not uniform!). Note that X takes values in the interval $(0, d/2)$. The probability that M falls at a distance between zero and x from the kth circumference, $F_X(x)$, is the ratio of the area of the annulus having inner radius $kd - x$ and outer radius kd to the area of the annulus having inner radius $(k - 1/2)d$ and outer radius kd; viz,

$$F_X(x) = \frac{\pi(kd)^2 - \pi(kd - x)^2}{\pi(kd)^2 - \pi(kd - d/2)^2}.$$

Hence, the probability density function of X is

$$f_X(x) = \frac{d}{dx}F_X(x) = \frac{2(kd - x)}{(k - 1/4)d^2}, \quad 0 < x < d/2.$$

The random variables X and Y are assumed to be independent.

If we let C_k represent the event that the needle crosses a circumference when M falls inside the kth annulus, then the conditional probability of C_k, given that $(k - 1/2)d < \overline{OM} < kd$, is

$$P(C_k|I'_k) = P[X < x_0] + P\left[x_0 < X < \ell/2,\ Y < \cos^{-1}\left[\frac{(k^2d^2 - \ell^2/4) - (kd - X)^2}{(kd - X)\ell}\right]\right] =$$

$$\int_0^{x_0}\frac{2(kd - x)}{(k - 1/4)d^2}\,dx + \frac{4}{\pi(k - 1/4)d^2}\int_{x_0}^{\ell/2}(kd - x)\cos^{-1}\left[\frac{(k^2d^2 - \ell^2/4) - (kd - x)^2}{(kd - x)\ell}\right]dx.$$

The first integral is straightforward. The second can be evaluated, after a great deal of computation, using integration by parts and a series of substitutions. The final expression is

$$(2.1)\qquad P(C_k|I'_k) = \frac{1}{\pi d^2(k - 1/4)}\left[\ell\sqrt{k^2d^2 - \ell^2/4} + 2k^2d^2\sin^{-1}\frac{\ell}{2kd}\right].$$

This takes care of the case in which M falls inside the outer half of the kth annulus (Case 1).

Case 2. Assume that M falls inside the inner half of the kth annulus. See Figures 3 and 4. By the law of cosines,

$$r^2 = (h + X)^2 + \ell^2/4 - \ell(h + X)\cos Y, \quad \text{where } h := (k - 1)d.$$

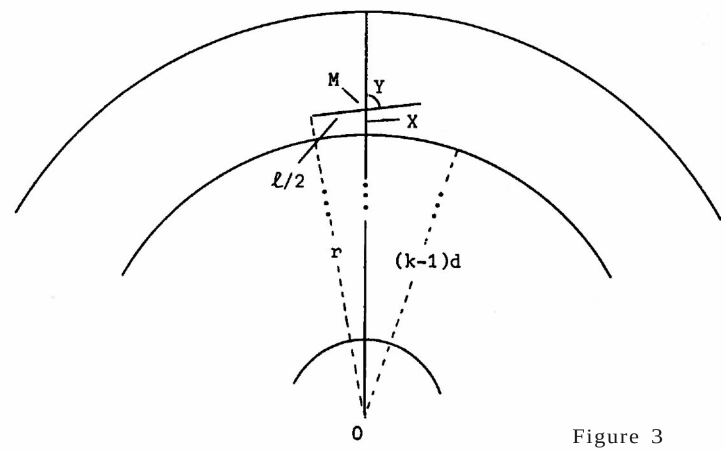


Figure 3

Now, let x_1 represent the value taken on by X when the left end-point of the needle coincides with the point at which the needle is tangent to the circle having radius h (See Figure 5). It can easily be verified, by using the Pythagorean Theorem, that

$$x_1 = \frac{1}{2}\sqrt{4h^2 + \ell^2} - h.$$

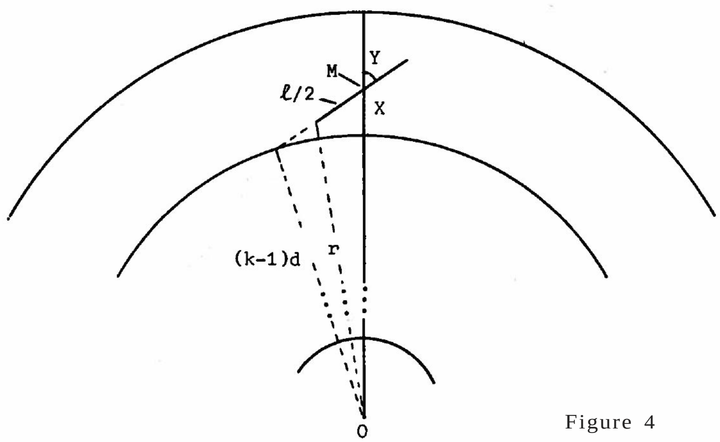
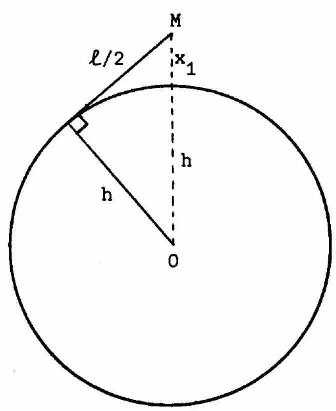


Figure 4



h = (k-1)d

Figure 5

Also, Y will again be assumed to be uniform on $[0,\pi/2]$. Then the probability density function for X can be found to be

$$f_X(x) = \frac{2(x + h)}{(k - 3/4)d^2}, \quad 0 < x < d/2,$$

by expressing the cumulative distribution function of X as a ratio of two annular areas, just as was done in Case 1. Again, X and Y are assumed to be independent.

Case 2 will be handled in two separate parts:

Case 2a. $0 < X < x_1$ (see Figure 3), and

Case 2b. $x_1 < X < \ell/2$ (see Figure 4).

In Case 2a, the needle crosses the $(k - 1)$st circumference if and only if

$$r^2 < (k - 1)^2d^2 + \left[\ell/2 - \sqrt{((k - 1)d + X)^2 - (k - 1)^2d^2}\right]^2.$$

The right-hand side of this inequality is the value of r^2 when the needle in Figure 3 is tangent to the circle of radius h, or, equivalently,

$$Y < \cos^{-1}\left[\sqrt{1 - h^2/(h + X)^2}\right].$$

Then, the joint probability of C_k for the case $0 < X < x_1$, given that $(k - 1)d < \overline{OM} < (k - 1/2)d$, is

$$P(C_k,\ 0 < X < x_1 | I_k) = \frac{4}{\pi(k - 3/4)d^2}\int_0^{x_1}(x + h)\cos^{-1}\left[\sqrt{1 - h^2/(h + x)^2}\right]dx.$$

Once again, this integral can be evaluated using integration by parts. After some lengthy computations, we have

(2.2)
$$P(C_k,\ 0 < X < x_1 | I_k) = \frac{1}{\pi(k - 3/4)d^2}\left[\frac{4h^2 + \ell^2}{2}\cos^{-1}\left(\frac{\ell}{\sqrt{4h^2 + \ell^2}}\right) + h(\ell - \pi h)\right].$$

In Case 2b, $x_1 < X < \ell/2$, and the needle crosses the $(k - 1)$st circumference if and only if $r < h$ (see Figure 4), or, equivalently,

$$Y < \cos^{-1}\left[\frac{(X + h)^2 + (\ell^2/4 - h^2)}{(X + h)\ell}\right].$$

So, the joint probability of C_k for the case $x_1 < X < \ell/2$, given that $(k-1)d < \overline{OM} < (k-1/2)d$, is

$$P(C_k,\ x_1 < X < \ell/2 \mid I_k) =$$

$$(2.3)\qquad \frac{4}{\pi(k-3/4)d^2}\int_{x_1}^{\ell/2}(x+h)\cos^{-1}\left[\frac{(x+h)^2 + (\ell^2/4 - h^2)}{(x+h)\ell}\right]dx =$$

$$\frac{1}{\pi(k-3/4)d^2}\left[h(\ell+\pi h) - \frac{4h^2+\ell^2}{2}\cos^{-1}\frac{\ell}{\sqrt{4h^2+\ell^2}}\right],$$

after, once again, using integration by parts.

Combining the expressions in (2.2) and (2.3), we have that the conditional probability that the needle crosses the $(k-1)$st circumference, given that $(k-1)d < \overline{OM} < (k-1/2)d$, is, upon simplification,

$$(2.4)\qquad P(C_k \mid I_k) = P(C_k,\ 0 < X < x_1 \mid I_k) + P(C_k,\ x_1 < X < \ell/2 \mid I_k) = \frac{2\ell(k-1)}{\pi d(k-3/4)}\,.$$

It is interesting to note that this probability is smaller than in the classical Buffon Needle Problem by a factor of $(k-1)/(k-3/4)$.

The probabilities of the events I_k and I_k' are easily obtained by computing ratios of annular areas:

$$(2.5)\qquad \begin{aligned} P(I_k') &= (k-1/4)/N^2, \quad k = 1, 2, \ldots, N, \text{ and} \\ P(I_k) &= (k-3/4)/N^2, \quad k = 2, 3, \ldots, N. \end{aligned}$$

Then, finally, the probability that the needle crosses a circumference when randomly dropped onto a set of N concentric circles is, by the Law of Total Probabilities,

$$\rho_N = \sum_{k=1}^{N} P(I_k')P(C_k \mid I_k') + \sum_{k=2}^{N} P(I_k)P(C_k \mid I_k),$$

and upon substitution from (2.1), (2.4) and (2.5), we get

$$(2.6)\qquad \rho_N = \frac{\ell(N-1)}{\pi N d} + \frac{1}{\pi N^2 d^2}\sum_{k=1}^{N}\left[\ell\sqrt{k^2d^2 - \ell^2/4} + 2k^2d^2\sin^{-1}\frac{\ell}{2kd}\right].$$

Certain intuitive properties about the formula (2.6) can easily be established. For instance, when $\ell = d$, ρ_N is independent of ℓ and d.

Hence, when the length of the needle is the same as the distance between concentric circles, the probability of a "cross" depends upon neither. One would expect that as the length of the needle gets smaller and smaller (holding d constant), the probability of a "cross" would become small. Indeed, it can easily be seen from (2.6) that $\lim_{\ell \to 0} \rho_N = 0$. One would also expect that as d becomes large, ρ_N would become small. In fact, $\lim_{d \to \infty} \rho_N = 0$.

Because the curvature of a circle approaches zero as the radius of the circle extends to infinity (so that the circumference of the circle becomes more like a straight line as the radius increases), one would expect $P(C_k \mid I_k')$ and $P(C_k \mid I_k)$ to approach $2\ell/\pi d$, the probability associated with the classical Buffon Needle Problem. As can easily be determined from (2.1) and (2.4),

$$P(C_k \mid I_k') \to (2\ell/\pi d)^+ \text{ and } P(C_k \mid I_k) \to (2\ell/\pi d)^- \quad \text{as } k \to \infty.$$

With this in mind, one can show from (2.6) (using the fact that (2.6) represents a convex combination of $P(C_k \mid I_k')$ and $P(C_k \mid I_k)$) that $\rho := \lim \rho_N = 2\ell/\pi d$, so that for an infinite number of concentric circles on an infinite plane region, the probability of a "cross" is the same as for parallel lines. In Table I, values of $P(C_N \mid I_N')$, $P(C_N \mid I_N)$ and ρ_N are computed for the case $\ell = d$, illustrating convergence of each of these terms to $2/\pi \approx .63662$.

N	$P(C_N \mid I_N')$	$P(C_N \mid I_N)$	ρ_N
5	.66901	.59917	.63415
10	.65267	.61941	.63583
25	.64301	.63006	.63645
50	.63981	.63339	.63657
75	.63874	.63448	.63659
100	.63821	.63502	.63661
200	.63742	.63582	.63662

Table I

REFERENCES

1. **Buffon, G.,** *L'Histoire Naturelle,* 4 Supplement, 1777.

2. Duncan, R. L., *A Variation of the Buffon Needle Problem,* Mathematics Magazine, 40 (1967), pp. 36–38.

3. Gnedenko, B. V., *Theory of Probability,* Chelsea, New York, 1962.

4. Perlman, M. and Wichura, M., *On Sharpening Buffon's Needle,* American Statistician, 28 (1975), pp. 157–163.

5. Ramaley, J. F., *Buffon's Needle Problem,* American Mathematical Monthly, 76 (1969), pp. 916–918.

6. Robertson, Jack M. and Siegel, Andrew F., *Designing Buffon's Needle for a Given Crossing Distribution,* American Mathematical Monthly, 93 (1986), pp. 116–119.

7. Uspensky, J. V., *Introduction to Mathematical Probability,* McGraw-Hill, New York, 1937.

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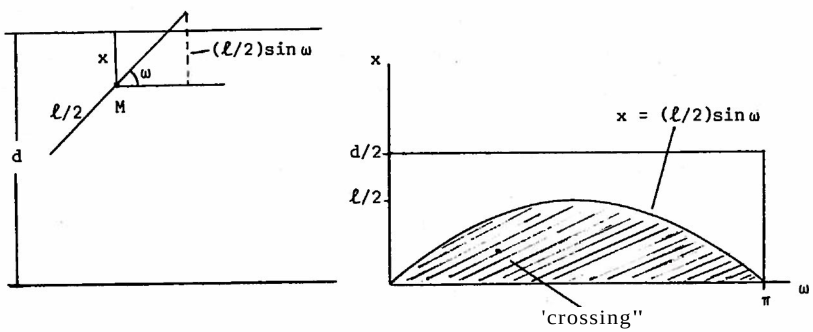
GRAFFITO



$$P(x < \ell \sin\omega/2) = (\ell/2)\int_0^{\pi} \sin\omega \, d\omega \div (\pi d/2)$$

$$= 2\ell/\pi d$$

George Louis Leclerc, Comte de Buffon (1707 – 1788)
French naturalist

CONVERGENCE OF WEIGHTED AVERAGES

by Douglas L. Cashing
and Charles R. Diminnie
St. Bonaventure University

After Cauchy's convergence criterion is covered, advanced calculus texts such as [1] and [4] often include a version of the following excercise: *If a_1 and a_2 are arbitrary real numbers, and if $a_n = (a_{n-1} + a_{n-2})/2$ for $n > 2$, show that the sequence $\{a_n\}$ converges.* The standard proof consists of showing that $|a_n - a_{n-1}| \le |a_{n-1} - a_{n-2}|/2$ for $n > 2$, and then using some form of Theorem 1 below. Following the suggestion in Exercise 7b, p. 107 of [3], we consider the extension of this result to a recursive definition in terms of weighted averages of the previous k terms of the sequence. The proof of this result depends on the following theorem, which can be found in the first chapter of [1].

Theorem 1. Let $A > 0$ and c be in $(0,1)$. If $|a_{n+1} - a_n| \le Ac^n$ for n sufficiently large, then $\{a\}$ is a Cauchy sequence, and hence converges.

Theorem 2. Let $k \ge 2$, and let $w_1, \ldots, w_k$ be in $(0,1)$ satisfying the condition $\sum w_i = 1$. If $a_1, \ldots, a_k$ are arbitrary real numbers, and if $a_n = \sum w_i a_{n-i}$ for $n > k$, then $\{a_n\}$ converges.

Proof. For $n > k$ define $r_n = \max\{|a_{n-i} - a_{n-j}| : 0 \le i < j \le k-1\}$. Then for $n > k$ and $i = 1, \ldots, k-1$,

$$\begin{aligned} |a_{n+1} - a_{n+1-i}| &= |(\textstyle\sum w_j a_{n+1-j}) - a_{n+1-i}| \\ &= |\textstyle\sum w_j (a_{n+1-j} - a_{n+1-i})| \\ &\le \textstyle\sum w_j |a_{n+1-j} - a_{n+1-i}| \\ &\le r_n \textstyle\sum w_j, \text{ with } j \ne i \\ &= (1 - w_i) r_n. \end{aligned}$$

If $w = \max\{1 - w_i : i = 1, \ldots, k-1\}$ then we have that

(1) $$|a_{n+1} - a_{n+1-i}| \le wr_n$$

for $n > k$ and $i = 1, \ldots, k-1$. Therefore, for $n > k$, we get

(2) $$r_{n+1} = max\ \{|a_{n+1-i} - a_{n+1-j}| : 0 \le i < j \le k-1\} \le rn.$$

For large values of n, repeated applications of (1) and (2) yield

(3) $$r_n \le wr_{n-(k-1)}.$$

For any n, the Division Algorithm implies that there are (unique) non-negative integers q and s such that $n = (k-1)q + s$, with $0 \le s < k-1$. Hence, if we use (3) repeatedly for sufficiently large n, we obtain

$$r_n \le wr_{n-(k-1)} \le w^2 r_{n-2(k-1)} \le \cdots \le w^{q-2} r_{2(k-1)+s}.$$

If we let $X = max\ \{ r_{2(k-1)+s} w^{-(2(k-1)+s)/(k-1)} : 0 \le s \le k-2\}$ and we define $W = w^{1/(k-1)}$, then $0 < W < 1$ and, for large n, $r_n \le XW^n$. Finally, from condition (1), we have that $|a_{n+1} - a_n| \le (wX)W^n$. The conclusion then follows from Theorem 1.

As an interesting consequence, the result is also true if the weighted arithmetic mean is replaced by the weighted geometric mean. Also, we note that Theorem 2 is valid for sequences in any Banach space, and that the condition on the weights can be relaxed to $\sum w_i \le 1$.

Corollary. Let $k \ge 2$ and let $w_1, \ldots, w_k$ be in $(0,1)$ satisfying the condition that $\sum w_i = 1$. If $a_1, \ldots, a_k$ are arbitrary positive numbers, and for all $n > k$ we have $a_n = \Pi(a_{n-i})^{w_i}$, then $\{a_n\}$ converges.

Proof. For all n, define $b_n = ln(a_n)$. Since $\{b_n\}$ satisfies the conditions of Theorem 2, it is a convergent sequence. The convergence of $\{a_n\}$ follows from the continuity of the function e^x.

In the same exercise set of [3], it is further suggested that the arithmetic mean $(a + b)/2$ may be replaced by the harmonic mean $2/(1/a + 1/b)$. As a project, the reader might wish to consider this problem, as well as those using other generalized means, such as the logarithmic mean $(b - a)/(lnb - lna)$ or the root-mean-square $\sqrt{(a^2 + b^2)/2}$. As a note of interest, in [2] it is shown that when $0 < a < b$, we have the following ordering (from smallest to largest):

a, 'harmonic, geometric, logarithmic, arithmetic, root-mean-square, b.

REFERENCES

1. Buck, R. C., *Advanced Calculus, 3rd Ed.*, McGraw-Hill, New York, 1978.
2. Burk, F., "Some interesting consequences of a hyperbolic inequality,". *The College Mathematics Journal*, 17 No. 1 (1986), pp. 75-76.
3. Knopp, K., *Theory and Applications of Infinite Series, 2nd Ed.*, Hafner Publishing Co., New York, 1947.
4. Taylor, A. E. and Mann, W. R., *Advanced Calculus, 2nd Ed.*, Xerox, Lexington, MA, 1972.

1987 NATIONAL PI MU EPSILON MEETING

The Annual Pi Mu Epsilon National Meeting will be at the University of Utah, Salt Lake City, from Wednesday, August 5, through Saturday, August 8, concurrently with the Joint Sunnier Meetings of the AMS and MAA.

Student paper presenters and student delegates (non-presenters) are needed. Talks are to be fifteen minutes in length and may include any area of mathematics or its application. Talks may be on either the expository level or on the research level; both are encouraged. Mathematical topics in computing are also welcome.

Each chapter is eligible to apply for air travel support up to a (chapter) total of six hundred dollars ($600) for students presenting papers or up to a (chapter) total of three hundred dollars ($300) for delegates (non-presenters).

Registration for Pi Mu Epsilon and the Joint meetings is $16. In addition to the student contributed paper sessions, activities will include the J. Sutherland Frame Lecture, Banquet, and parties. Lodging will be in a block of rooms in a dorm on the University of Utah campus; rates for doubles will be $8.50 per person per night.

Contact your chapter advisor for detailed information, registration forms and the "Information and Helpful Hints" sheets.

TREE, BRANCH AND ROOT

by *Norman Woo*
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Definition 1: A set of integers $\{b_i\}$ is called a **base** for the set of all integers whenever every integer n can be expressed uniquely in the form

$$n = \sum_{i=1}^{\infty} \varepsilon_i b_i, \text{ where } \varepsilon_i = 0 \text{ or } 1 \text{ and } \sum_{i=1}^{\infty} \varepsilon_i < \infty.$$

Theorem 1. Any base can, by rearrangement, be written in the form $\{d_1, 2d_2, 2^2 d_3, 2^3 d_4, \ldots\}$ where the d_i's are all odd.

Proof. This is proved in reference [1].

Note that a sequence $[d_1, d_2, d_3, d_4, \ldots]$ of odd numbers is called a basic sequence whenever $\{d_1, 2d_2, 2^2 d_3, 2^3 d_4, \ldots\}$ is a base.

Theorem 2. Any integer x can be formally developed into a series $x = \sum_{i=1}^{\infty} \varepsilon_i 2^{i-1} d_i$, where $\varepsilon_i = 0$ or 1, $\sum_{i=1}^{\infty} \varepsilon_i \leq \infty$ and the d_i's are all odd.

Proof. If x is odd, set $\varepsilon_1 = 1$. If x is even, set $\varepsilon_1 = 0$. Set

$$x_1 = \frac{x - \varepsilon_1 d_1}{2}.$$

If x_1 is odd, set $\varepsilon_2 = 1$. If x_1 is even, set $\varepsilon_2 = 0$. Set

$$x_2 = \frac{x_1 - \varepsilon_2 d_2}{2}.$$

In general, set $x_i = \dfrac{x_{i-1} - \varepsilon_i d_i}{2}$. Set $\varepsilon_i = 1$ if x_{i-1} is odd and set $\varepsilon_i = 0$ if x_{i-1} is even.

At the kth stage, we have the following:

$$x_1 = \frac{x - \varepsilon_1 d_1}{2} \text{ and } x = 2x_1 + \varepsilon_1 d_1$$

$$x_2 = \frac{x_1 - \varepsilon_2 d_2}{2} \text{ and } x_1 = 2x_2 + \varepsilon_2 d_2$$

$$\vdots$$

$$x_k = \frac{x_{k-1} - \varepsilon_k d_k}{2}$$

Thus, $x = \varepsilon_1 d_1 + 2x_1$

$$= \varepsilon_1 d_1 + \varepsilon_2 2^1 d_2 + 4x_2$$

$$\vdots$$

$$= \varepsilon_1 d_1 + \varepsilon_2 2^1 d_2 + \varepsilon_3 2^2 d_3 + \cdots + \varepsilon_k 2^{k-1} d_k + 2^k x_k.$$

Therefore, $x - \sum_{i=1}^{k} \varepsilon_i 2^{i-1} d_i = 2^k x_k \equiv 0 \pmod{2^k}$.

Note that all ε_i's are uniquely determined.

Suppose $x = \sum_{i=1}^{\infty} \varepsilon_i 2^{i-1} d_i = \sum_{i=1}^{\infty} \varepsilon'_i 2^{i-1} d_i$.

Then, $x - \varepsilon_1 d_1 \equiv x - \varepsilon'_1 d_1 \equiv 0 \pmod 2$

$$\varepsilon_1 d_1 \equiv \varepsilon'_1 d_1 \pmod 2.$$

Since $(d_1, 2) = 1$, $\varepsilon_1 \equiv \varepsilon_1' \pmod 2$ and $\varepsilon_1 = \varepsilon_1'$.

We can use an induction argument to show that $\varepsilon_n = \varepsilon_n'$ for all n.

Of course, if $\{d_1, 2d_2, 2^2 d_3, 2^3 d_4, \ldots\}$ is a base, then $\sum_{i=1}^{\infty} \varepsilon_1 < \infty$ for any x. It is not difficult to show that $\{1, -2^1, 2^2, -2^3, 2^4, \ldots\}$ is a base or that $[1, -1, 1, -1, \ldots]$ is a basic sequence. In the following example, we will use 14 and express it using $[1, -1, 1, -1, \ldots]$ as our basic sequence.

$$x_1 = \frac{x - \varepsilon_1 d_1}{2} = \frac{14 - (0)(1)}{2} = 7$$

$$x_2 = \frac{x_1 - \varepsilon_2 d_2}{2} = \frac{7 - (1)(-1)}{2} = 4$$

$$x_3 = \frac{x_2 - \varepsilon_3 d_3}{2} = \frac{4 - (0)(1)}{2} = 2$$

$$x_4 = \frac{x_3 - \varepsilon_4 d_4}{2} = \frac{2 - (0)(-1)}{2} = 1$$

$$x_5 = \frac{x_4 - \varepsilon_5 d_5}{2} = \frac{1 - (1)(1)}{2} = 0.$$

$$14 = \varepsilon_1 d_1 + \varepsilon_2 2^1 d_2 + \varepsilon_3 2^2 d_3 + \varepsilon_4 2^3 d_4 + \varepsilon_5 2^4 d_5 + 2^5 x_k$$

$$= (0)(1) + (1)(2^1)(-1) + (0)(2^2)(1) + (0)(2^3)(-1) + (1)(2^4)(1) + (2^5)(0)$$

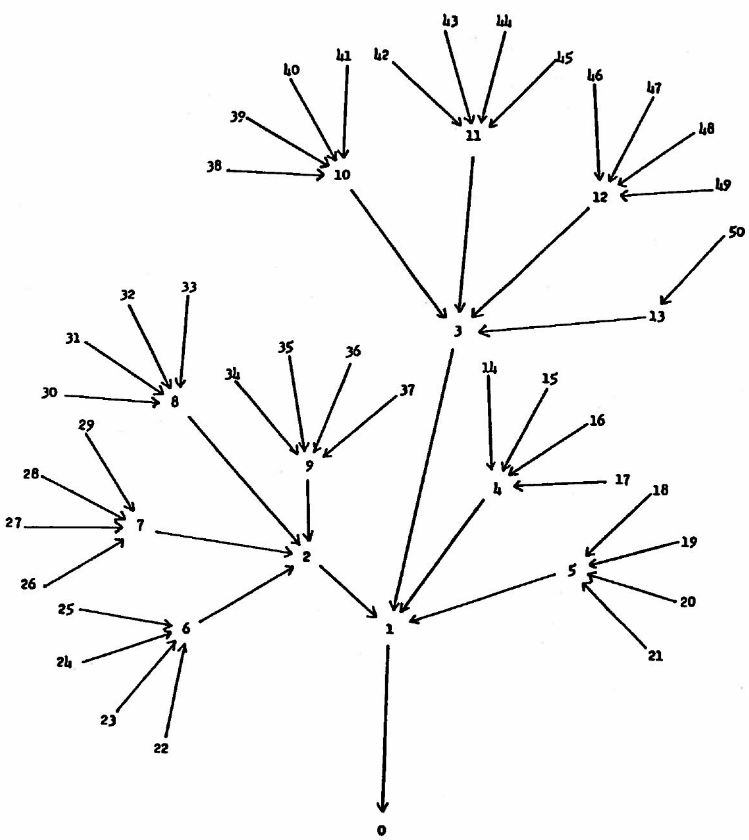
$$= 0 - 2 + 0 + 0 + 16 + 0$$

Thus, we have a series of mappings

$$14 \to 7 \to 4 \to 2 \to 1 \to 0$$

$$14 \to x_1 \to x_2 \to x_3 \to x_4 \to 0$$

Below is illustrated a series of mappings for the set of integers $\{1, 2, 3, \ldots, 50\}$ using the basic sequence $[1, -1, 1, -1, \ldots]$. It is interesting to observe that this series of mappings forms a <u>tree</u>. Each mapping is a <u>branch</u>. Each series of mappings stops at 0, the <u>root</u> of the tree.



REFERENCES

1. de **Bruijn,** N. G., "On Bases for the Set of Integers," *Publ.* Math. *Debrecen*, Vol. I, 1950, pp. 232-242.
2. Woo, Norman, "A Note on Basic M-tuples," The *Fibonacci Quarterly*, Vol. 17, No. 2, 1979, pp. 165-168.

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YOU CAN'T HURRY LOVE
OR
HOW MUCH MORE MUST ART WAIT?

by *Joseph S. Verducci**
The *Ohio State University*

1. INTRODUCTION

Meet Arthur, a young American male about to fall in love for the first time. Arthur has promised his mother that he will not marry the first girl who bewitches him, but being a romantic fellow he does intend to marry and live happily ever after with the first *amorata* with whom he experiences a quality of love greater than that in his first affair. How many affairs should Arthur expect to have before he finds his intended spouse?

Let us assume that X_i measures the quality of Arthur's i^{th} love affair, $i = 1, 2$, and so on, Considering the vagaries of love, we might assume that these X_i are independent, identically distributed random variables from some distribution P. Assuming that Arthur maintains his strategy, he will first propose marriage on his T^{th} love affair, where T is the smallest number n (= 2, 3, *etc.*) such that $X_n > X_1$. What is the expected value of T?

Four different solutions to this problem are given in the next section. They correspond to different assumptions about the distribution F, the degree of Arthur's knowledge, and, most intriguingly, Arthur's personal attitude toward the future. The last solution raises paradoxical interpretations of expectation that are briefly discussed in Section 3.

* The author thanks David Fairly and Dennis Pearl for enjoyable conversations about this topic.

2. SOLUTIONS

The classical solution found in most introductory textbooks (see, for example, Feller, 1971) takes only a line to present:

$$E(T) = \sum_{n=1}^{\infty} P(T > n) = \sum_{n=1}^{\infty} (1/n) = \infty,$$

but disturbs some students' intuition, not to mention Arthur's heart. How could it possibly take forever to find a better love? One way this could happen is if the first love affair happened to be the best possible one. For example, if F were concentrated on the points $\{1, \ldots, k\}$ and $X_1 = k$; then there would be no chance of surpassing the first love affair, and consequently $T = \infty$. Since this event has positive probability in this example, $E(T) = \infty$.

2.1 A Merciful Solution

We all know that the quality of love, like mercy, is unbounded, so that $F(x) < 1$ for any x. Suppose that $X_1 = x$, some arbitrary, but fixed amount. Then

$$P(T > n) = P(\max\{X_2, \ldots, X_n\} < x) = [F(x)]^{n-1} \quad (2.1)$$

because the $\{X_i\}$ are independent. Notice that the classical solution above does not fully use the fact that the $\{X_i\}$ are independent; for example, the $\{X_i\}$ might only be exchangeable. In contrast to the classical solution, equation (2.1) implies that

$$E(T) = \sum_{n=1}^{\infty} P(T > n) = \sum_{n=1}^{\infty} [F(x)]^{n-1} = [1 - F(x)]^{n-1} < \infty, \quad (2.2)$$

because $F(x) < 1$ no matter how good the first time was. This new solution is certainly heartening to Art, but to what do we owe this promise of convergence? -- the unlimited nature of love, or some darker assumptions?

2.2 The Devil Laughs

The above analysis works well if Arthur happens to know the quality x of his first affair. Remember, even if Arthur were finished with this affair, which has yet to happen, he is a romantic and would loathe the crass task of actually measuring the quality of the affair. According to Arthur, "I am confident that I will recognize when a new affair is better than my first, but I could never put a number or a price on any love affair."

A more careful look at equation (2.2) shows that it gives not $E(T)$ but the conditional expectation $E[T|X_1 = x]$. To get $E(T)$, we must integrate as follows:

$$E(T) = \int E[T|X_1 = x]dF(x) = \int [1-F(x)]^{-1}dF(x)$$
$$= \lim_{a\to\infty}\{\log[1-F(-a)] - \log[1-F(a)]\} = \infty.$$

Therefore, the assumption that F is unbounded is not, by itself, sufficient to guarantee a finite waiting time.

Having integrated all his knowledge of love, Arthur finds he does not know very much after all. Forever is a long time to wait. Perhaps Arthur should make the devil's bargain, and spend his time measuring the quality of his first love affair, once it is over. In this way, at least he would be saved the despair of expecting to wait forever for a better love. But Arthur is a lover, not a thinker, and so he writes for advice to someone wiser about the prospects of eternity.

2.3 Dear Abbe

The following is reprinted without permission from a short-lived local column entitled *Advice to Lovelorn Students of Statistics:*

"Dear Arthur,

No sense vegetating. Plan to have 1 love affair this year, 3 love affairs next year, 5 love affairs the year after that, with progressively odd affairs in all the ensuing years. Then see how your expectations change.

Best wishes,
The Abbe"

How will this advice help Arthur? Let Z be the number of years that Arthur must wait before finding a better love, and let X_i be defined as before. Notice that after n years Arthur plans to have had $1 + 3 + \ldots + (2n-1) = n^2$ love affairs. Thus

$$P\{Z > n\} = P(X_1 \text{ is largest among } \{X_1, \ldots, X_{n^2}\} = n^{-2},$$

and so

$$E(Z) = \sum n^{-2} = (\pi^2)/6,$$

less that 20 months. This last summation is a special case of a convergent Fourier series [see for example Abramowitz and Stegun (1965)].

It is also an instance of Riemann's zeta function $\zeta(z)$. In fact, if Arthur becomes very aggressive and plans to have *if* affairs after n years, for some large p, then $E(Z) = \zeta(p)$, which approaches 1 as p increases.

3. THE PARADOX

If he adopts the Abbe's plan, Arthur expects to wait less than two years to find the right woman, during which time he will have had no more than 4 affairs. Does it then follow that he expects to have no more than 4 affairs?

The last analysis seems to contradict the first and third analyses. In these earlier analyses, nothing was assumed about time schedules, so why should the imposition of a time schedule change the results?

The situation is analogous to the fable of Achilles and the tortoise, sometimes called Zeno's paradox. After giving the slow tortoise a head start in a race, the swift Achilles was never able to pass the tortoise because he had to first accomplish the infinite sequence of events of halving the gap between him and the tortoise. Ordinary athletes, on the other hand, have no trouble performing an infinite sequence of such fleeting events in a finite amount of time. However, most ordinary men would find themselves overtaxed by the Abbe's suggested agenda, unless these love affairs themselves became almost as fleeting.

If an ordinary mortal can have no more than a finite number N of love affairs in his life, then, under the given assumptions about the qualities of these affairs, there is a positive probability N^{-1} that X_1 will be largest among the $\{X_1, \ldots, X_N\}$. In this case Arthur will never find a suitable bride, and, in effect, will wait forever. In this sense he must expect to wait forever. You can't hurry love, Arthur.

REFERENCES

1. Abramowitz, A., and Stegun, I. A., *Handbook of Mathematical Functions*, Dover Press, New York, NY, 1965.

2. Feller, W., *An Introduction to Probability Theory and Its Applications, 2nd edition*, vol 2, John Wiley and Sons, New York, NY, 1971.

THE ANSWER IS 1 - 1/e. WHAT IS THE QUESTION?

(Two matching problems that have the same limiting answer.)

by *Elliot A. Tanis*
Hope College

An urn contains n balls numbered from 1 through n. The balls are selected randomly from the urn, one at a time, until n balls have been selected. A match occurs if ball numbered k is the k^{th} ball that is selected. We are interested in finding the probability that there is at least one match.

Does selecting the balls with replacement or selecting without replacement affect the probability of at least one match? Does the number of balls in the urn affect the probability of at least one match? You should be able to answer these questions after you have read this paper.

Problem 1. An urn contains n balls numbered 1 through n. From the urn n balls are selected one at a time *with replacement*. A match occurs if ball numbered k is the k^{th} ball that is selected. Find the probability of at least one match, say q_n.

Problem 1'. Roll an n-sided die n times. A match occurs if side k is the outcome on the k^{th} roll, $k = 1, 2, \ldots, n$. Find the probability of at least one match during the n rolls of the die, say q_n.

Solution. We find the probability of no matches and subtract this answer from 1. The probability of no matches is easy to calculate because the trials are independent. We have

$$
\begin{aligned}
q &= P(\text{at least one match}) \\
&= 1 - P(\text{no matches}) \\
&= 1 - \left(\frac{n-1}{n}\right)\left(\frac{n-1}{n}\right)\cdots\left(\frac{n-1}{n}\right) \\
&= 1 - \left(\frac{n-1}{n}\right)^n = 1 - \left(1 - \frac{1}{n}\right)^n
\end{aligned}
$$

Note that as n increases without bound,

$$\lim_{n\to\infty} q_n = \lim_{n\to\infty} \left[1 - \left(1 - \frac{1}{n}\right)^n \right] = 1 - \frac{1}{e}$$

and thus we can write a question for which the answer is $1 - 1/e$.

Problem 2. An urn contains n balls numbered from 1 through n. From the urn n balls are selected one at a time without *replacement*. A match occurs if ball numbered k is the k^{th} ball selected. (Note that this generates a random permutation of the first n positive integers.) Find the probability of at least one match, say p_n.

Problem 2'. Let A and B denote two identical decks of cards, each deck containing n cards numbered from 1 through n. Shuffle each deck. A match occurs if card numbered k, $1 \le k \le n$, occupies the same position in each deck. Find the probability of at least one match, say p_n.

Solution (when $n=4$). Let the event A_i denote a match on the i^{th} draw. Then

$$P(A_i) = \frac{3!}{4!}$$

$$P(A_i \cap A_j) = \frac{2!}{4!}$$

$$P(A_i \cap A_j \cap A_k) = \frac{1!}{4!}$$

$$P(A_1 \cap A \cap A \cap A_4) = \frac{1}{4!}$$

The probability of at least one match is

$$P(A_1 \cup A_2 \cup A_3 \cup A_4) = \sum P(A_i) - \sum P(A_i \cap A_j) + \sum P(A_i \cap A_a \cap A_k) - P(A_1 \cap A \cap A \cap A_4)$$

$$= 4\left(\frac{3!}{4!}\right) - \binom{4}{2}\left(\frac{2!}{4!}\right) + \binom{4}{3}\left(\frac{1}{4!}\right) - \frac{1}{4!}$$

$$= \frac{4!}{4!} - \frac{4!}{2!2!}\frac{2!}{4!} + \frac{4!}{3!1!}\frac{1}{4!} - \frac{1}{4!}$$

$$= 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!}$$

The solution for any integer n is

$$p_n = P(\text{at least one match})$$

$$= 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \dots + \frac{(-1)^{n+1}}{n!}$$

As n increases without bound,

$$\lim_{n\to\infty} p_n = 1 - \left[1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \dots \right] = 1 - \frac{1}{e}$$

so we can write a second question for which the answer is $1 - 1/e$.

It is interesting to compare the values of p_n and q_n for different values of n. We can construct the following table.

n	p_n	q_n
1	1.00000	1.00000
2	0.50000	0.75000
3	0.66667	0.70370
4	0.62500	0.68359
5	0.63333	0.67232
6	0.63194	0.66510
7	0.63214	0.66008
8	0.63212	0.65639
9	0.63212	0.65356
10	0.63212	0.65132
11	0.63212	0.64951
12	0.63212	0.64800
13	0.63212	0.64674
14	0.63212	0.64566
15	0.63212	0.64474

Recall that

$$1 - \frac{1}{e} \approx 0.63212$$

We see that the value of n has little affect on p_n when $n > 4$ and on q_n when n > IS. Also for large n, p_n and q_n are approximately equal.

An interesting exercise would be to illustrate these probabilities empirically. Either use dice, cards, or balls in an urn - or, write a program to simulate these experiments on a computer.

PAUL TURÁN - SOME ANECDOTES

by *J. L. Brenner*
10 Phillips Road.
Palo Alto, CA 94303

The accent on the **a** in **Turán** is not a stress mark; all Hungarian words are stressed on the first syllable. The word **"syllable"** should remind one of this. Thus, **Doráti, Dochnanyi, Kodály, Bartok, Erdös, Radó, Gabor, Fejes-Tóth, Szemeredi.**

The distinguished Hungarian mathematician Paul **Turán** (1910-1976) was active in many fields. One of his most famous works was "A New Method in Analysis," which originally appeared in 1953. An English edition was published by Wiley-Interscience in the **1970's, I** believe. **(A** bibliography of **Turán's** work appeared in Mat. Lapok (Hungarian) in the June 1977 issue, but some details are missing.) This famous work is an example, but by no means an isolated example, of a significant contribution of a middle-aged or older mathematician.

Turán had interests outside mathematics. Several times, he came to my house to hear string quartet music. He was charmed by some work of **Arriaga,** a Spanish musician who died at the age of 20 and was not well-known in Hungary. Later, **Turán** purchased some recordings of the quartets to take home to Hungary.

During the 1966 Moscow Congress I lunched with **Turán,** his wife, and his wife's sister. I wish I had been able to commit to **film** the antics of one of the translator-assistants during that luncheon.

Turán showed his knowledge of the Russian Language by ordering the food for all of us: one of the blue-plate special number one and two of the blue-plate special number two. The women were overprotective of their weight, and agreed to share a single order. We then gave the waitress, a business-like middle-aged woman, our meal chits. Shortly afterwards, she returned to the table, waving the chits in her hand. "You don't have enough chits," she remonstrated. This set Paul **Turán** off into a mixture of English and Russian. The waitress interrupted him.

'I'll go fetch a translator," she said (all in Russian). There was a Latin-looking translator, tall and with a Don Quixote mustache, not far away. He was the one she fetched. They returned together - the Latin undoubtedly in the position of a foreign student, studying in a Russian university, and partly repaying his hosts by offering the courtesy of entertaining what Spanish mathematicians there were by interpreting Russian for them.

When the pair arrived at our table, **Turán** overlooked the large badge labeled **"Español"** on the **man's** lapel, and began explaining, in English, what it was we wanted. This dismayed the Spaniard so greatly that all he could do was put out his tongue, attempting to articulate, but unable to say a word. He pointed repeatedly to his badge, and, began to look like a comedian. The impasse could only be resolved when one of us explained our needs in good Russian to the waitress, who understood perfectly. There is a Russian word, ВМЕСТЕ, that was very handy to explain that one blue plate special number two was to be shared by the women. It is also a convenient fact that the Russian word **for** "women" is the same as the Russian word for "wives", so it was irrelevant whether I was or was not married to Mrs. **Turán's** sister. (She was, and is, actually the wife of the Hungarian finance minister.) So the food was brought serenely.

As the meal was finishing, there was another joke, a quite good one, in view of the fact that English - let alone American slang - is not **Turán's** native language. His wife, Vera **Sós,** got up abruptly and said, "I have to run." "Yes, you are right," Paul **Turán** said, "you have **Turán."**

He even knew how his name was often mispronounced!

ʃ ʃ

A SUMMATION FORMULA

by John Schue
Macalester College

Let S be the sum $1 + 2 + \dots + (n-1)$. Then S is the number of elements below the main diagonal of an $n \times n$ matrix and $2S$ is the total number of elements off the main diagonal. Since there are n diagonal entries we then have $2S = n^2 - n$ so $S = n(n-1)/2$.

~~Editor's note~~. - *for several other proofs, see "A Discrete Look at $1 + 2 + \dots + n$," by LoW C. Larson, The College Mathematics Journal, Vol. 16, No. 5 (1985), pp. 369-382.*

EXTENDING A FAMILIAR LIMIT

by *Norman Schaumberger*
Bronx Community College

In a recent issue of this journal [1] we used the mean value theorem to prove the well-known formula

$$\lim_{n\to\infty} \frac{(n!)^{1/n}}{n} = \frac{1}{e}. \tag{1}$$

In this note we prove more generally that if a and b are integers with $a > b$, and $a + b > 0$, then

$$\lim_{n\to\infty} \frac{1}{n} [(a+b)(2a+b)(3a+b) \ldots (na+b)]^{1/n} = \frac{a}{e}. \tag{2}$$

Thus, for example, if $a = 4$ and $b = -1$, we get

$$\lim_{n\to\infty} \frac{1}{n} [3\cdot 7\cdot 11 \ldots (4n-1)]^{1/n} = \frac{4}{e}.$$

Our proof is completely elementary and can be offered immediately after (1). For another approach, see [2].

Since $a > b$ and $a + b > 0$ it follows that for all positive integers k, $(k-1)a < ka+b < (k+1)a$. Furthermore, since a and b are integers $a + b \geq 1$, hence

$$1\cdot a\cdot 2a \ldots (n-1)a < (a+b)(2a+b) \ldots (na+b) < 2a\cdot 3a \ldots (n+1)a.$$

This double inequality can be written as

$$\frac{a^n\cdot n!}{a\cdot n} < (a+b)(2a+b) \ldots (na+b) < a^n\cdot n!\cdot (n+1)$$

or

$$\frac{a\cdot (n!)^{1/n}}{a^{1/n}\cdot n^{1/n}\cdot n} < \frac{[(a+b)(2a+b) \ldots (na+b)]^{1/n}}{n} < \frac{a\cdot (n+1)^{1/n}\cdot (n!)^{1/n}}{n}. \tag{3}$$

As $n\to\infty$, $n^{1/n} \to 1$, $(n+1)^{1/n} \to 1$ and $a^{1/n} \to 1$. (2) now follows from (3) by letting $n\to\infty$ and using (1).

REFERENCES

1. Schaumberger, N., *Three Familiar Results via the Mean Value Theorem*, Vol. 8, No. 4, Spring 1986.

2. Problem #275, *The College Mathematics Journal*, Vol. 17, No. 1, January 1986, p. 97.

THE FOURTEENTH ANNUAL PI MU EPSILON STUDENT CONFERENCE,-

AT

MIAMI UNIVERSITY

IN

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OCTOBER 9–10, 1987

WE INVITE YOU TO JOIN US! THERE WILL BE SESSIONS OF THE STUDENT CONFERENCE OH FRIDAY EVENING AND SATURDAY AFTERNOON. FREE OVERNIGHT LODGING FOR **ALL** STUDENTS WILL BE ARRANGED WITH MIAMI STUDENTS. EACH STUDENT SHOULD BRING A SLEEPING BAG. ALL STUDENT GUESTS **ARE** INVITED TO A FREE FRIDAY EVENING PIZZA PARTY SUPPER AND SPEAKERS WILL BE TREATED TO A SATURDAY NOON PICNIC LUNCH. TALKS MAY BE ON ANY TOPIC RELATED TO MATHEMATICS, STATISTICS OR COMPUTING. WE WELCOME ITEMS RANGING FROM EXPOSITORY TO RESEARCH, INTERESTING APPLICATIONS, PROBLEMS, SUMMER EMPLOYMENT, **ETC.** PRESENTATION TIME SHOULD BE FIFTEEN OR THIRTY MINUTES.

WE NEED YOUR TITLE, PRESENTATION TIME (15 OR **30** MINUTES), PREFERRED DATE (FRIDAY OR SATURDAY) AND A **50** (APPROXIMATELY) WORD ABSTRACT BY OCTOBER 1, 1987.

PLEASE SEND TO

PROFESSOR MILTON D. COX
DEPARTMENT OF MATHEMATICS AND STATISTICS
MIAMI UNIVERSITY
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THE STUDENT CONFERENCE IS HELD IN CONJUNCTION WITH THE CONFERENCE ON "COMPUTERS AND MATHEMATICS" WHICH BEGINS FRIDAY AFTERNOON, OCTOBER 9. FEATURED SPEAKERS INCLUDE A. **K. DEWDNEY,** ANTHONY RALSTON AND ROBERT TARJAN. CONTACT US FOR MORE DETAILS.

PUZZLE SECTION

Edited by

Joseph D. E. Konhauser

The PUZZLE SECTION is for the enjoyment of those readers who are addicted to working doublecrostics or who find an occasional mathematical puzzle attractive. We consider mathematical puzzles to be problems whose solutions consist of answers immediately recognizable as correct by simple observation and requiring little formal proof. Material submitted and not used here will be sent to the Problem Editor if deemed appropriate for the PROBLEM DEPARTMENT.

Address all proposed puzzles and puzzle solutions to Professor Joseph D. E. Konhauser, Mathematics and Computer Science Department, Macalester College, St. Paul, Minnesota 55105. Deadlines for puzzles appearing in the Fall Issue will be the next February 15, and for puzzles appearing in the Spring Issue will be the next September 15.

PUZZLES FOR SOLUTION

1. *Proposed by the Editor.*

The numbers in the sequence 1, 2, 4, 8, 16, 32, 64, ... satisfy the recurrence relation

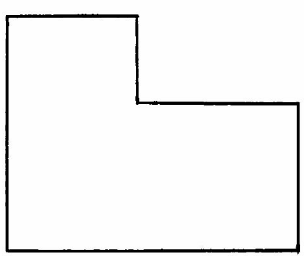
$$a_{n+2} = a_{n+1} + 2a_n, \quad n = 1, 2, 3, \ldots .$$

Are you able to find an integer pair (p,q) different from (1,2) such that

$$a_{n+2} = pa_{n+1} + qa_n, \quad n = 1, 2, 3, \ldots ?$$

2. *Proposed by the Editor.*

Using only a straightedge (unmarked ruler) locate the centroid of the L-shaped region in the sketch.



3. *Proposed by the Editor.*

Note that

$$1 + 4 = 2 + 3,$$

that $$1 + 4 + 6 + 7 = 2 + 3 + 5 + 8$$

and that $$1^2 + 4^2 + 6^2 + 7^2 = 2^2 + 3^2 + 5^2 + 8^2.$$

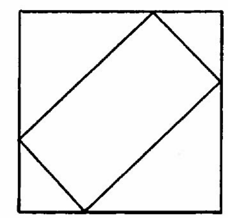
Fill the blanks with the numbers 1 through 16 so that equality holds for n = 1, 2 and 3.

$$(\ )^n + (\ )^n + (\ )^n + (\ )^n + (\ )^n + (\ )^n + (\ )^n + (\ )^n =$$

$$(\ )^n + (\ )^n + (\ )^n + (\ )^n + (\ )^n + (\ )^n + (\ )^n + (\ )^n.$$

4. *Proposed by the Editor.*

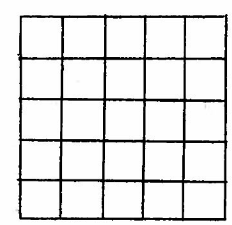
If the four vertices of a rectangle lie on a square (one vertex on each side of the square) is it possible to move the rectangle so that all four vertices of the rectangle become interior points of the square?



5. *Proposed by the Editor.*

The 25 squares of a 5×5 array have been colored red, white and blue. Reading the rows from left to right and the columns from top to bottom, the color arrangements in nine of the rows and columns are

R R B W W
R B R R R
R B W R R
W W B R B
W B W R R
W R B B R
W B R R W
B W W R B
B W R B R



What is the remaining color arrangement?

COMMENTS ON PUZZLES 1 - 5, FALL 1986

What is the most money one can have in pennies, nickels, dimes, quarters, half-dollars, $1 bills, $2 bills, $5 bills and $10 bills without being able to make change for a $20 bill? **Glen E. Mills**, jointly with **James Hansen**, and **Jason Pinkney** observed that if we assume that one must have at' least one of each type of coin and bill, then the maximum is $19.99 as follows: one $10 bill, one $5 bill, one $2 bill, two $1 bills, one half-dollar, one quarter, one dime, two nickels and four pennies. Otherwise, most respondents came up with the answer $24.19. One solution is having one $10 bill, one $5 bill, four $2 bills, one half-dollar, one quarter, four dimes and four pennies. Several solvers found more than one. **Edward F. Marks, Jr.** found eight other solutions and claims that there are no more. *Puzzle* # 2 asked for an expression for 71 using four 1's and standard mathematical symbols. **Victor G. Feser** offered

$$\left[\sqrt{\sqrt{11!}}\,(1 - .1)\right].$$

Edmund F. Marks, Jr. and **Robert Prielipp** submitted

$$\phi(111) - 1,$$

where $\phi(111) = \phi(3)\times\phi(37) = 2\times 36 = 72$. **John D. Moores** submitted

$$\left[(\sqrt{\sqrt{11!}}) \div 1.1\right].$$

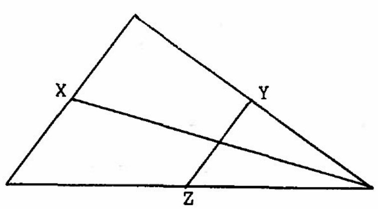
Robert Prielipp also submitted

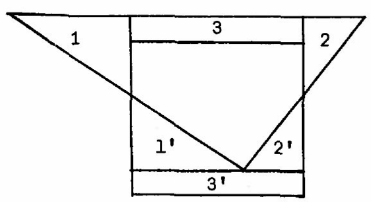
$$(.\bar{1})^{-2} - 11 + 1 = 81 - 11 + 1 = 71$$

and

$$(.\bar{1})^{-2} - 1 - (1/.\bar{1}) = 81 - 1 - 9 = 71.$$

John H. Scott wrote "I believe that zero is the 'standard mathematical symbol' used to ensure that the digits of a number are in the right columns. So the binary number 1000111 = 71 decimal fits the condition." For Puzzle # 3, **Glen E. Mills** submitted the following figure (left).

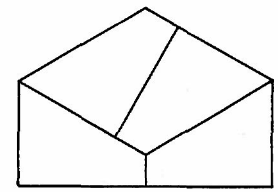




X, Y and Z are side midpoints. Make the joins, as shown, and cut along the line segments interior to the triangle. The four pieces can be reassembled to form a parallelogram. Try it. It is an easy but nice puzzle. The figure on the right (above) was submitted by **John H. Scott.** Pieces 1 and 2 are half-turned into 1' and 2', respectively. Piece 3 is translated to 3'. Again, no part of the boundary of the resulting quadrilateral is part of the original boundary of the triangle. **John D. Moores** sent still a different solution and suggested that readers see the haberdasher's puzzle in Henry E. Dudeney's *The Canterbury Puzzles.*

Seven readers responded to *Puzzle* # 4, the labelling of a square array of sixteen points so that all fourteen squares with horizontal and vertical sides have equal "vertex sums." All seven gave at least one example and stated that there were many more. With some machine help, **John M. Howell** found 384 solutions. One is (below left)

1	8	10	15
12	13	3	6
7	2	16	9
14	11	5	4



The magic constant is, of course, 34. **Doug Debski, Jason Pinkney** And **Glen E. Mills** submitted equivalent solutions to the pentagon dissection puzzle (above right).

List of responders: **Doug Debski** *(4,5)*, **Mark Evans** *(1)*, **Victor G. Feser** *(1,2,4)*, **James Hansen** *(1)*, **John M. Howell** *(4)*, **Joseph Jackson** *(2)*, **Edmund F. Marks, Jr.** *(1,2,4)*, **Glen E. Mills** *(1,3,4,5)*, **John D. Moores** *(1,2,3,4)*, **Jason Pinkney** *(1,5)*, **Robert Prielipp** *(2)*, **John H. Scott** *(1,2,3,4)*.

Solution to Mathacrostic No. 23, (See Fall 1986 Issue).

Words:

A. J J Sylvester	I. scuttlebutt	Q. Eulerian path
B. earless	J. top banana	R. overboard
C. fractal	K. Heine-Borel	S. Fields medal
D. floccule	L. evection	T. serendipity
E. wrangler	M. synclinal	U. pitty-pat
F. en cabochon	N. harmonic tetrad	V. antrorse
G. entropy	O. alligation	W. caboodle
H. klystron	P. Peaucellier cell	X. environ

Quotation: *Orientability is an intrinsic property ... of a surface ... flatlanders on a projective plane couldn't tell locally that they weren't on a sphere, but they could tell globally because a projective plane is nonorientable, and long-distance travellers can come back mirror-reversid.*

Solved by: **Jeanette Bickley**, Webster Groves High School, MO; **Victor G. Feser,** University of Mary, Bismarck, ND; **Robert Forsberg,** Lexington, MA; **Dr. Theodor Kaufman,** Winthrop-University Hospital, Mineola, NY; **Henry S. Lieberman,** John Hancock Mutual Life Insurance Co., Boston, MA; **Charlotte Maines,** Caldwell, NJ; **Don Pfaff,** University of Nevada, Reno, NV; **Robert Prielipp** (with help from **Pat Collier** and **John Oman**), University of Wisconsin-Oshkosh; **Stephanie Sloyan,** Georgian Court College, Lakewood, NJ; **Jeffrey Weeks** and **Nadia Marano,** Ithaca College, Ithaca, NY; and **Barbara Zeeberg,** Denver, CO.

Late solution: A late solution to Mathacrostic No. 22 was received from **Barbara Zeeberg,** Denver, CO.

Definitions		*Words*
A.	In the Ptolemaic system the circle around the earth in which a celestial body or the center of the epicycle of its orbit was thought to move	203 137 60 174 14 95 163 106
8.	growing at high elevations but not above the timber line	150 171 3 142 186 55 125 91 41 200
C.	a Texas unit equal to 33.33 inches	169 197 43 138
D.	stamp	39 222 173 33 121 99 54
E.	the art or technique of cutting solids	128 206 132 234 166 5 103 114 209 154
F.	unpredictable	111 97 184 225 123 57 151 9 29
G.	lefse (2 wds.)	139 18 178 113 110 40 22 7 58 187 232 201 37
H.	completely and determinedly fixed (comp.)	88 118 190 107 93 4 76
I.	formerly a chief source of ammonia	196 44 229 27 2 86 214 109 119
J.	orbits	223 161 89 62 130 179 237 81
K.	balanced; complete	211 104 124 45 180 204 167
L.	sought attention by ostentatious behavior	73 189 207 144 133 177 219 12 68 117
M.	vehement or vigorous; violent or rough	136 155 36 188 82
N.	a polygon whose interior consists of all the points in the plane which are nearer to a particular lattice point than to any other lattice point (2 wds.)	175 1 94 230 120 165 152 47 182 102 32 70 92
0.	obliquely or downward to one side	74 191 129 87 176 160 13 66
P.	Viggo Brun's prime counter	146 56 127 193 50
Q.	one who asserts that only those mathematical objects have real existence and are meaningful which can be explicitly exhibited	231 172 220 156 64 67 25 147 85 71 31 20 140 164
R.	onetime antispasmodic	69 198 28 105 15 135 194 221 145
S.	branching	38 208 213 158 26 170
T.	an unexpected or malicious side-effect of a program that otherwise operates correctly (2 wds.)	157 199 77 24 98 122 49 228 115 217 53
U.	in music, a passage between statements of a main subject or theme	149 226 126 84 59 215 192
V.	the catastrophe which can be interpreted as a split or furrow	21 141 80 148 195 210 61 218 96 11 183
W.	in this kind of geometry any two straight lines meet in two points (2 wds.)	205 235 52 8 233 134 168 75 116 162 100 23 83 46
X.	abounding	19 79 35 10 108 224 202
Y.	uniformly in action or intensity	212 51 34 16 6 112 153
Z.	of or relating to the summer	63 216 72 48 65 90 181 17
a.	a kind of tiling in which all the tiles are the same size and shape	78 42 131 30 227 101 159 143 185 236

Mathacrostic No. 24

Proposed by Joseph V. E. Konhauser

The 237 letters to be entered in the numbered spaces in the grid will be identical to those in the 27 keyed *Words* at the matching numbers. The key numbers have been entered in the diagram to assist in constructing the solution. When completed, the initial letters of the *Words* will give the name(s) of the author(s) and the title of a book; the completed grid will be a quotation from that book.

The solution to *Mathacrostic No.* 23 is given elsewhere in the PUZZLE SECTION.

1 N	2 I		3 B	4 H	5 E	6 Y	7 G	8 W	9 F	10 X	11 V	12 L	13 O	
14 A	15 R	16 Y	17 Z		18 G	19 X		20 Q	21 V		22 G	23 W		24 T
25 Q	26 S	27 I		28 R		29 F	30 a	31 Q	32 N	33 D		34 Y	35 X	
36 M	37 G	38 S		39 D	40 G	41 B	42 a	43 C	44 I	45 K	46 W	47 N		48 Z
49 T	50 P		51 Y	52 W	53 T	54 D	55 B	56 P	57 F	58 G		59 U	60 A	
61 V	62 J	63 Z	64 Q		65 Z	66 O		67 Q	68 L	69 R	70 N		71 Q	72 Z
	73 L	74 O	75 W	76 H	77 T	78 a		79 X	80 V	81 J	82 M		83 W	84 U
	85 Q	86 I	87 O		88 H	89 J	90 Z	91 B	92 N		93 H		94 N	95 A
96 V	97 F		98 T	99 D	100 W	101 a	102 N	103 E		104 K	105 R		106 A	107 H
108 X		109 I	110 G	111 F	112 Y		113 G	114 E	115 T	116 W	117 L		118 H	119 I
	120 N	121 D	122 T	123 F	124 K	125 B	126 U	127 P	128 E		129 O	130 J	131 a	132 E
	133 L	134 W		135 R	136 M	137 A		138 C	139 G	140 Q	141 V	142 B	143 a	
144 L	145 R	146 P		147 Q	148 V	149 U	150 B	151 F	152 N	153 Y		154 E	155 M	156 Q
	157 T	158 S	159 a	160 O	161 J		162 W	163 A		164 Q	165 N	166 E		167 K
168 W	169 C	170 S	171 B	172 Q	173 D	174 A	175 N		176 O	177 L	178 G	179 J	180 K	
181 Z	182 N		183 V	184 F	185 a	186 B	187 G		188 M	189 L	190 H		191 O	192 U
193 P	194 R	195 V		196 I	197 C	198 R		199 T	200 B	201 G	202 X	203 A	204 K	205 W
	206 E	207 L		208 S		209 E	210 V	211 K	212 Y		213 S	214 I	215 U	216 Z
217 T	218 V		219 L	220 Q	221 R		222 D	223 J	224 X	225 F	226 U	227 a	228 T	229 I
230 N	231 Q	232 G	233 W		234 E	235 W	236 a	237 J						

ΠΜΕ

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PROBLEM DEPARTMENT

Edited by Clayton W. Dodge
University of Maine

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

All communications should be addressed to C. W. Dodge, Math. Dept., University of Maine, Orono, ME 04469. Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly identified with name and address. Solutions to problems in this issue should be mailed by December 15, 1987.

Problems for Solution

639. *Proposed by Charles W. Trigg, San Diego, California.*

Find the smallest SLICE the KNIFE can cut from the CAKE if

$$\text{CAKE} + \text{KNIFE} = \text{SLICE}.$$

***640.** *Ptopobed by John M. Howell, Littlerock, California.*

Find the largest value of $S(n)$ and the limit of $S(n)$ as $n \to \infty$ if

$$S(n) = \sum_{x=1}^{n-1} \frac{n}{\binom{n}{x}}.$$

***641.** *Proposed by Paul A. McKlueen, Raleigh, North Carolina.*

Let $f_1 = f_2 = 1$ and $f_{k+2} = f_k + f_{k+1}$ for $k > 0$ define the Fibonacci sequence. It is known that $f_k f_{k+1} = \sum_{i=1}^{k} f_i^2$ for any positive integer k. Find a similar formula for the generalized Fibonacci sequence g_k, where g_1 through g_n are given and for $k > 0$

$$g_{k+n} = \sum_{i=1}^{n} g_{k+i-1}.$$

642. *Proposed by Dmitry P. Mavlo, Moscow, U.S.S.R.*

Let $\pi, \mu, \varepsilon > 0$. Prove that

$$(1+\pi\mu\varepsilon)\left(\frac{1}{\pi(1+\mu)}+\frac{1}{\mu(1+\varepsilon)}+\frac{1}{\varepsilon(1+\pi)}\right) \ge 3,$$

with equality if and only if $\pi = \mu = \varepsilon = 1$.

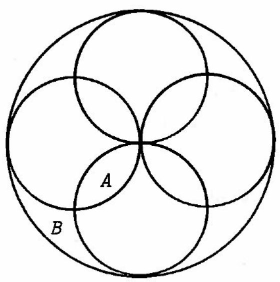
643. *Proposed M. S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.*

If $a, b, c, d > 0$, prove that

$$\left(\frac{a}{b}\right)^{2a-b+d}\left(\frac{b}{c}\right)^{2b-c+a}\left(\frac{c}{d}\right)^{2c-d+b}\left(\frac{d}{a}\right)^{2d-a+c} \ge 1.$$

644. *Proposed by Richard I. Hess, Rancho Palos Verdes, California.*

In the figure below prove that regions A and B have equal areas.



645. *Proposed by Dmitry P. Mavlo, Moscow, U.S.S.R.*

Let M be an arbitrary point on segment CD of trapezoid $ABCD$ having sides AD and BC parallel. Let S, S_1, and S_2 be the areas of triangles ABM, BCM, and ADM respectively. Prove that

$$S \ge 2\min\{S_1, S_2\}.$$

646. *Proposed by Dick Field, Santa Monica, California.*

Find the smallest k for which there is only one k-digit palindrome that is the square of an integer.

647. *Proposed by Robert C. Gebhardt, Hopatcong, New Jersey.*

For each positive integer n find the earliest row of Pascal's triangle in which the first n terms have the property that each term after the first is an integral multiple of its predecessor.

648. *Proposed by Jack Garfunkel, Flushing, New York.*

If A, B, C are the angles of a triangle ABC, prove that

$$\prod \cos\frac{A}{2} \le \frac{\sqrt{3}}{6}\sum \cos^2\frac{A}{2}.$$

649. *Proposed by Edward J. Arismendi, Jr., California State University, Long Beach, California.*

How far beyond the edge of a table can a deck of cards be stacked without the pile falling off the table?

650. *Proposed by Richard I. Hess, Rancho Palos Verdes, California.*

The 1980 Wimbleton final between Borg and McEnroe involved a tiebreak game that went to 18-16. Given that the server has a 70% chance of winning the point, what is the probability that the two players reach a 16-16 tie in a tiebreak game? (In a tiebreak game the first player serves one point. Thereafter players alternate serving 2 points each. The first player reaching 7 or more points with an advantage of 2 or more points wins the game.)

651. *Proposed by Al Tarego, Malden, Massachusetts.*

Professor E. P. Umbugio has recently been strutting around because he hit upon the solution of the fourth degree equation which results when the radicals are eliminated from the equation

$$x = (x - 1/x)^{1/2} + (1 - 1/x)^{1/2}.$$

Deflate the professor by solving it using nothing higher than quadratic equations. [From Robinson's Mathematical Recreations, 1851.]

Solutions

613. [Spring 1986] *Proposed by Martha Matticks, Veazie, Maine.*

Use a bit of number theory to solve this alphametric that pays homage to geometry, algebra and analysis. Find that solution in base 7 yielding a prime $ANAL$.

$$\begin{array}{r} GEOM \\ ALG \\ ANAL \end{array}$$

I. Composite of solutions by Mark Evans, Louisville, Kentucky, and Robert C. Gebhardt, Hopatcong, New Jersey.

In base 7 there are 23 primes that fit the form $ANAL$ with A at least 2 2021, 2326, 2623, ..., 6562 (corresponding to the numbers 701, 853, 997, ..., 2347 in base ten). Ten of these primes can be eliminated because $G = A - 1$ and therefore $N \neq A - 1$ and $L \neq A - 1$. An additional 12 primes are eliminated quickly since $G = A - 1$ determines E, O, and M. For example, if $ANAL = 2326$, then $G = 1$, so $M = 1$. Now $O = 3$ which contradicts $N = 3$. We have the unique solution

$$\begin{array}{r} 4216 \\ \underline{534} \\ 5053. \end{array}$$

II. *Solution by Frank P. Battles, Massachusetts Maritime Academy, Buzzards Bay, Massachusetts.*

It is intuitively obvious to the most casual observer that

$$4216 + 534 = 5053$$

is the unique solution to this base 7 alphametric.

III. *Comment by Elizabeth Andy, New Limerick, Maine.*

A brilliant professor named Battles
With problem solutions just rattles.
His craftsmanship "obvious"
Yields work for the mobs of us
Who only can think in small prattles.

Also solved by CHARLES ASHBACHER, *Mount Mercy College, Cedar Rapids, IA,* JAMES E. CAMPBELL, *University of Missouri, Columbia,* VICTOR G. FESER, *University of Mary, Bismarck, ND,* RICHARD I. HESS, *Rancho Palos Verdes, CA,* JOHN M. HOWELL, *Littlerock, CA,* GLEN E. MILLS, *Valencia Junior College, Orlando, FL,* JOHN H. SCOTT, *Macalester College, Saint Paul, MN, and the* PROPOSER.

614. [Spring 1986] *Proposed by Lion. Bankoff, Los Angeles, California, and the editor.*

A 10000-meter section of straight railroad track expands 1 meter and buckles into a circular arc. How high above ground is the middle of the arc? [This is an old problem and easy to solve using ordinary trigonometry. It is repeated here because the answer is so surprisingly large.]

I. *Solution by Wade. H. Sherard, Furman University, Greenville, South Carolina.*

Assuming that the track was originally a straight line KB with midpoint M, let O be the center of the circular arc it now forms, let

OM cut the arc at S, and let $x = OM$, $h = MS$, and $r = OA = OB = OS$. Let $\theta = \angle AOS = \angle SOB$. Then we have

$$r = x + h, \qquad x = 5000 \cot \theta,$$
$$r \sin \theta = 5000, \qquad \text{and} \qquad r\theta = 5000.5.$$

Therefore $(5000)(\theta) = (5000.5) \sin \theta$. An application of Newton's method to this equation yields $\theta \approx 0.0245$ radians. Hence

$$h = r - x \approx \frac{5000.5}{0.0245} - 5000 \cot 0.0245 \approx 61.23 \text{ meters.}$$

II. *Solution by Frank P. Battles and Laura Kelleher, Massachusetts Maritime Academy, Buzzards Bay, Massachusetts.*

Let h be the desired height. If we assume that the curvature of the track after bending is negligible, we readily obtain

$$h = \sqrt{5000.5^2 - 5000^2} \approx 71 \text{ meters.}$$

This result appears to be much too large and may be due to the approximations made. We therefore proceed with a more exact solution.

Let R_e $(= 6.371 \times 10^6\ m)$ denote the earth's radius. Prior to buckling we place an xy-coordinate system at the center of the earth so that the coordinates of the ends of the track are at $(\pm x_0, y_0)$ and the center of the track is at $(0, y_0 + h_0)$. Let θ_0 be the half angle subtended by the track at the center. Then we have

$$R_e\theta_0 = 5000, \qquad x_0 = R_e \sin \theta_0,$$
$$y_0 = R_e \cos \theta_0, \qquad \text{and} \qquad h_0 = R_e(1 - \cos \theta_0).$$

Then $\theta_0 \approx 0.0007848$ radians and $h_e \approx 1.96$ m.

After the track has buckled we place an xy-coordinate system at the center of the circle defined by the track so that the coordinates of the ends of the track are at $(\pm x_1, y_1) = (\pm x_0, y_1)$ and the center of the track is at $(0, y_1 + h)$. Let θ_1 be the half angle subtended by the track at $(0,0)$. Then we have

$$x_1 = R_1 \sin \theta_1 = R_e \sin \theta_0 \qquad \text{and} \qquad R_1\theta_1 = 5000.5.$$

Eliminating θ we get

$$R_1 \sin\left(\frac{5000.5}{R_1}\right) = R_e \sin \theta_0 = 4999.999487.$$

By approximate means we obtain that $R_1 \approx 204060$ m and $\theta_1 \approx 0.0245$

radians. Then $h_1 = R_1(1 - \cos \theta_1) \approx 61.3$ m. Finally

$$h = h_1 - h_0 \approx 59 \text{ m}$$

to the nearest meter.

III. *Comment by John H. Scott, Macalester College, Saint Paul, Minnesota.*

This result is shocking. I was all set for a tiny number. I still find it hard to believe. I would like to see the spikes that could hold the two ends in place.

IV. *Comment by Jack Garfunkel, Flushing, New York.*

Another dramatic way of illustrating that intuition cannot be trusted is the following. Imagine a giant standing somewhere in space and encircling the earth at the equator with a huge rope. He then lengthens the rope by just 40 feet and this slack is distributed equally around the globe, creating a distance between the earth and the globe. What is this distance? Although intuition indicates that 40 feet distributed over 25000 miles would produce an infinitesimal distance, elementary circle geometry shows it to be over 6 feet, more than enough for an average man to walk underneath.

Also wived by MARK EVANS, *Louisville, KY,* ROBERT C. GEBHARDT, *Hopatcong, NJ,* RICHARD I. HESS, *Rancho Palos Verdes, CA,* JOHN M. HOWELL, *Littlerock, CA,* RALPH KING, *St. Bonaventure University, NY,* JOHN H. SCOTT, *Macalester College, Saint Pout, MN,* KENNETH M. WILKE, *Topeka, KS, and the.* PROPOSERS.

615. [Spring 1986] *Proposed by William S. Cariens, Lorain County Community College, Elyria, Ohio.*

Although several years into retirement, Professor Euclide Pasquale Bombasto Umbugio still practices mathematics with his usual prowess and efficiency. His native country, Guayazuala, still cannot afford a computer, but they do have a pocket four-function calculator to which he has occasional access. His latest project is to find the sum of the abscissas of the points of intersection of the seventh-degree polynomial

$$f(x) = x^7 - 3x^6 - 13x^5 + 55x^4 - 36x^3 - 52x^2 + 48x$$

with its derivative polynomial. So far he has laboriously found one of the intersections at $x = 1.3177227$. Help the kindly, old professor to find his sum without resorting to a computer.

Solution by Henry S. Lieberman, Waban, Massachusetts.

The eminent numerologist should subtract the derivative of $f(x)$ from $f(x)$ and take the negative of the coefficient of x^6 as the required sum. Since the derivative of $f(x)$ is

$$f'(x) = 7x^6 - 18x^5 - 65x^4 + 220x^3 - 108x^2 - 104x + 48,$$

the coefficient of x^6 in $f(x) - f'(x)$ is -10. Thus the sum of the abscissas of the seven points of intersection is 10.

Also solved by JAMES E. CAMPBELL, *University of Missouri, Columbia,* RUSSELL EULER, *Northwest Missouri State University, Maryville,* GEORGE P. EVANOVICH, *Edward Williams College-Fairleigh Dickerson University, Hackensack, NJ,* ROBERT C. GEBHARDT, *Hopatcong, NJ,* RICHARD I. HESS, *Rancho Palos Verdes, CA,* GLEN E. MILLS, *Colonial Senior High, Orlando, FL,* LAURA L. KELLEHER *and* FRANK P. BATTLES, *Massachusetts Maritime Academy, Buzzards Bay,* RALPH KING,. *St. Bonaventure University, NY,* JOHN H. SCOTT, *Macalester College, Saint Pout, MN,* HARRY SEDINGER, *St. Bonaventure University, NY,* KENNETH M. WILKE, *Topeka, KS, and the.* PROPOSER. *Gebhardt noted that $x - 2$ is a factor of the expression $f(x) - f'(x)$.*

616. [Spring 1986] *Proposed by Dmitry P. Mavlo, Moscow, U.S.S.R.*

Prove that in any triangle

$$\frac{\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2}}{\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}} \le \frac{8}{27} + \left(\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}\right)^2$$

with equality if and only if the triangle is equilateral.

Solution by Jack Garfunkel, Flushing, New Yolk..

Let R, r, and s denote the triangle's circumradius, inradius, and semiperimeter. We use the known identities

$$\sum \tan\frac{A}{2} = \frac{4R + r}{s} \quad \text{and} \quad \sum \cot\frac{A}{2} = \frac{s}{r}.$$

Then we have to show that

$$\frac{4R + r}{s} \cdot \frac{r}{s} \le \frac{8}{27} + \frac{r^2}{s^2},$$

which simplifies to $2s^2 \ge 27Rr$, a known inequality: see O. Bottema, *Geometric Inequalities*, page 52, item 5.12. The proof given there utilizes the inequality $(a + b + c)^3 \geq 27abc$, which in turn is readily proved by the arithmetic mean-geometric mean inequality, showing that equality holds if and only if $a = b = c$, that is, if and only if the triangle is equilateral.

Also solved by BARRY BRUNSON, *Western Kentucky University, Bowling Green,* HENRY S. LIEBERMAN, *Waban, MA,* BOB PRIELIPP, *University of Wisconsin-Oshkosh,* JOHN H. SCOTT, *Macalester College, Saint Paul, MN, and the* PROPOSER.

617. [Spring 1986] *Proposed by Titus Canby, Adjustable Wrench Company, Buffalo, New Yotk.*

It is known (*The Two-Year College Mathematics Journal*, problem 226, September 1982, page 277) that a 7 x 7 x 7 box can be packed with a maximum of forty 1 x 2 x 4 bricks, requiring 23 cubic units of unoccupied space. How many such bricks can be packed into a 5 x 5 x 5 cubic box?

I. *Partial solution by Victor G. Feser, University of Mary, Bismarck, North Dakota.*

The volume of the box is 125 cubic units; of the brick, 8. Therefore the absolute maximum is 15 bricks. It is easy to get 14; the bricks readily stand on edge or on end to fill all but one corner cube in each of the first four layers. Then two bricks can be laid flat in the top layer leaving another 9 unfilled unit cubes. Can one get 15? It seems so, since otherwise the problem would be a bit anticlimactic.

II. *Solution by the proposer.*

It is easy to pack 14 bricks into the box and, if we are allowed to cut just two bricks into two 1 x 1 x 4 subbricks, the 15th brick is readily inserted, along with another subbrick, for 15 1/2 bricks. To best follow the proof that it is impossible to pack 15 bricks into the box, it is suggested that you obtain or make a set of 1 x 2 x 4 blocks and a 5 x 5 x 5 cardboard box. (I used an ordinary board and cut 16 blocks 11/16 x 11/8 x 11/4 inches, an easy size to work with.)

	P	Q	R	S	T
A	1	2	3	4	1
B	6	7	8	5	6
C	3	4	1	2	3
D	8	5	6	7	8
E	1	2	3	4	1

Levels 1, 5

	P	Q	R	S	T
A	7	8	5	6	7
B	2	3	4	1	2
C	5	6	7	8	5
D	4	1	2	3	4
E	7	8	5	6	7

Level 2

	P	Q	R	S	T
A	3	4	1	2	3
B	8	5	6	7	8
C	1	2	3	4	1
D	6	7	8	5	6
E	3	4	1	2	3

Level 3

	P	Q	R	S	T
A	5	6	7	8	5
B	4	1	2	3	4
C	7	8	5	6	7
D	2	3	4	1	2
E	5	6	7	8	5

Level 4

Number each unit cube in each layer as shown above. Then any brick with faces parallel to the sides of the box fills exactly one cube of each number. Now there are 18 cubes numbered 1, 17 numbered 3, and 15 each numbered 2, 4, 5, 6, 7, and 8. To be able to pack 15 bricks into the box, every space numbered 2, 4, 5, 6, 7, and 8 must be occupied. In any level, only a 1 or a 3 can be uncovered. Since each brick covers an even number of cubes in each level, one cube must be uncovered. Since this last statement must be true no matter which face of the box is taken as the base, it follows that there must be exactly one empty cube in each level, one in each row, and one in each column. That is, the same row or column cannot contain an empty cube in each of two different levels. We write *2AR* to denote the unit cube in the level 2 that occupies row *A* and column *R*.

In level 1, since a 1- or a 3-cube must be uncovered, then the empty cube must be in row *A*, *C*, or *E* and column *P*, *R*, or *T*. Without loss of generality we need consider only the three cases that *1AP*, *1AR*, or *1CR* is empty. At least one brick must lie flat in level 1 in order that the empty cube in level 2 be different from that in level 1. Clearly no more than two bricks can lie flat in any level.

Case 1. Suppose *1AP* is empty. If bricks on edge cover *1AQ-1AT* and *1BP-1EP*, then *2AP* must be empty, a contradiction since *2AP* is numbered 7 and not 1 or 3. Clearly one brick on edge or on end must cover a cube abutting *1AP* and we may take bricks on edge covering *1BP-1EP* and *1EQ-1ET*. Flat bricks cover the rest of level 1. Now at least one brick on end or on edge must cover *2AP* and there is room for only one brick flat in level 2. In any case, no matter which of *2BQ*, *2BS*, *2DQ*, or *2DS* is empty at least one additional brick on end must stand next to the empty cube, forcing a flat brick or one on edge above that empty cube and lying in level *3*. Then there is a block of cubes in level *3* that must be covered and are less than 4 units long. Hence they can be covered only by bricks on end, which then protrude out of the top of the box. "Having just one flat brick in level *1* only compounds the difficulty.' Hence case 1 cannot occur with *15* bricks in the box.

Case 2. If *ICR* is empty, then at least two bricks on end must abut that empty cube, We may suppose that a flat brick lies in level *1* covering **1BP-1EQ.** Then one or two bricks on end cover **1DR-1ER.** Also **1AR-1BR** are similarly covered. If **2BQ** or **2DQ** is empty, then a flat brick cannot cover the rest of the region from **2BP-2ER,** so we are forced to use bricks on end or on edge. Thus cube *3BQ* or *3DQ* must be empty, also. But then we cannot have IS bricks in the box. A similar situation occurs if **2BS** or **2DS** is left empty.

Case 3. Finally, take **1AR** empty. If bricks on end cover **1AP-1AQ** and **1AS-1AT** and bricks on end or on edge cover **1BR-1ER,** then both levels *1* and *2* have cube **AS** empty.

If bricks on end cover **1AP-1AQ** and **1AS-1AT** and a brick on edge covers **1BT-1ET,** then a brick on end or on edge covers **2AR-2BR.** Then **2BQ** or **2DQ** or **2DS** empty necessitates the corresponding cube empty in level *3*. If **2BS** is empty, a flat brick can be placed to cover *2CP-***2DS,** but **2EP-2ES** must be covered by a brick on edge or bricks on end. Then it is impossible to cover cube *3BS.*

If bricks on end cover **1AP-1AQ** and **1ES-1ET** and bricks on end or a brick on edge covers **1BR-1ER,** then a brick on end must cover *2AS-*SAT and a brick on edge or *2* bricks on end cover **2AT-2DT.** If **2BQ** or **2DQ** is empty, then the same cube in level *3* must be empty. If **2BS** or *2DS* is empty, then the other cube and *2CS* must be covered by a brick on end. A flat brick can cover **2BP-2EQ** and a brick on edge can cover *3DP-3DS* or **3BP-3BS,** but the remaining cubes in level *3* cannot be covered.

Let bricks on end cover **1AP-1AQ** and **1ES-1ET** and a brick on edge or two bricks on end cover **1BP-1EP.** If **2BQ** is to be empty, then a brick that is flat or on edge must cover *3BQ-3BT* or **3BQ-3EQ.** In the former situation there is no way to cover the three cubes **2BR-2BT** and in the latter we have the same difficulty with **2BQ-2EQ.** If **2DQ** is to be empty, a similar situation occurs. Now a flat brick can cover

2BQ-2ER. Then, whichever of **2BS** or **2DS** is left empty, the same cube in level *3* must be uncovered.

Our proof is complete. We have shown that no matter which allowable cube is empty in level *1*, then it is impossible to leave an allowable cube empty in level *2* and also in level *3* and to fill all other cubes in those levels. It appears that the same situation occurs if we use *1* x *2* x *2* bricks, but *I* have not pursued that case.

III. ***Comment by Elizabeth Andy, New Limerick, Maine.***

When a problem is anticlimactic,
With a proof that is antidramactic,
Then the better disposal
Of such a proposal
Is to relegate it to an **acttic.**

Packings of 14 bricks were also given by RICHARD I. HESS, ***Rancho Palos Verdes, CA, and*** JOHN H. SCOTT, ***Macalester College, St. Paul, MN. Scott argued that one cube mutt be empty 'in each level, no matter, which side the. box rests on, but this argument guarantees only five empty cubes, if they are {or. can be} properly placed.***

618. [Spring **1986]** ***Proposed by John M. Howell, Littlerock, California.***

(i) Find when the sum of the squares of four consecutive integers is divisible by *3*.

(ii) Repeat part **(i)** for the sum of the squares of four consecutive odd or four consecutive even integers.

Solution by David E. Penney, The. University of Georgia, Athens, Georgia.

We generalize by finding when the sum of the squares of $p+1$ integers in arithmetic progression is divisible by the prime p.

For $p = 2$ or $p = 3$ it is when the first term and common difference are both divisible by p or neither is divisible by p. For $p > 3$, it is when p divides the first term in the progression.

Proof. Suppose p is a prime, n and d are integers, and $d > 0$. The arithmetic progression with first term n, difference d, and containing $p+1$ terms has the sum of squares

$$S = n^2 + (n + d)^2 + (n + 2d)^2 + \dots + (n + pd)^2$$
$$= (p + 1)n^2 + 2dn(1 + 2 \dots + p) + d^2(1^2 + 2^2 + \dots + p^2)$$
$$= (p + 1)n^2 + dnp(p + 1) + \frac{p}{6}(p + 1)(2p + 1)d^2$$
$$\equiv n^2 + \frac{p}{6}(p + 1)(2p + 1)d^2 \pmod p.$$

If $p > 3$, then 6 is a divisor of $(p + 1)(2p + 1)$, so $S \equiv n^2 \pmod p$. Consequently p will be a divisor of S exactly when p divides n. If $p = 2$ or $p = 3$, then $S \equiv n^2 - d^2 \pmod p$. Hence p divides S if and only if p divides both n and d or neither n nor d.

In particular, the sum of the squares of four consecutive integers, or of four consecutive odd or even integers, is divisible by 3 if and only if the first of the four integers is not divisible by 3.

Also solved by PHILLIP ABBOTT, *Macalester College, Saint Paul, MN*, CHARLES ASHBACHER, *Mount Mercy College, Cedar Rapids, IA*, FRANK P. BATTLES, *Massachusetts Maritime Academy, Buzzards Bay*, JAMES E. CAMPBELL, *University of Missouri, Columbia*, RUSSELL EULER, *Northwest Missouri State University, Maryville*, MARK EVANS, *Louisville, KY*, VICTOR G. FESER, *University of Mary, Bismarck, ND*, ROBERT C. GEBHARDT, *Hopatcong, NJ*, RICHARD A. GIBBS, *Fort Lewis College, Durango, CO*, RICHARD I. HESS, *Rancho Palos Verdes, CA*, FRANCIS C. LEARY, *Saint Bonaventure University, NY*, HENRY S. LIEBERMAN, *Waban, MA*, GLEN E. MILLS, *Valencia Junior College, Orlando, FL*, OXFORD RUNNING CLUB, *University of Mississippi, University*, BOB PRIELIPP, *University of Wisconsin-Oshkosh*, JOHN H. SCOTT, *Macalester College, Saint Paul, MN*, HARRY SEDINGER, *St. Bonaventure University, NY*, KENNETH M. WILKE, *Topeka, KS*, *and the* PROPOSER.

619. [Spring 1986] *Proposed by Victor G. Feser, Mary College, Bismarck, North Dakota.*

Find the largest value of x such that $x = \sin x = \tan x$, correct to 3, 4, 5, 6, 7, and 8 decimal places.

Composite of solutions submitted by Mark Evans, Louisville, Kentucky, and Russell Euler, Northwest Missouri State University, Maryville, Missouri.

For small positive x we have that

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \dots$$

and

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots$$

so that

$$\tan x - x > x - \sin x > 0.$$

Thus we need only check $\tan x - x \approx x^3/3$ to ensure the correct accuracy. So for *n-digit* accuracy we set $.5 \times 10^{-n} = x^3/3$ and solve for x. We obtain the following table.

Decimal accuracy	Largest permissible x
2	0.24
3	0.114
4	0.0531
5	0.02466
6	0.011446
7	0.0053132
8	0.00246621

Solutions were also submitted by RICHARD I. HESS, *Rancho Palos Verdes, CA*, RALPH KING, *St. Bonaventure University, NY*, JOHN H. SCOTT, *Macalester College, Saint Pout, MN*, *and the* PROPOSER. *Most of the solutions differed considerably from that printed above.*

621. [Spring 1986] *Proposed by R. S. Luthar, University of Wisconsin Center at Janesville, Wisconsin.*

(i) Characterize all triangles whose angles and whose sides are both in arithmetic progression.

(ii) Characterize all triangles whose angles are in arithmetic progression and whose sides are in geometric progression.

I. *Solution by Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin.*

Let ABC, with $A \le B \le C$, be a triangle having its angles in arithmetic progression. Then $a \le b \le c$. Because the angles are in arithmetic progression, then $A + C = 2B$. Hence $B = 60^o$ since $A + B + C = 180^o$.

(i) If the sides are in arithmetic progression, then $b = (a + c)/2$. Thus, by the law of cosines,

$$\frac{a^2 + 2ac + c^2}{4} = a^2 + c^2 - \text{Sac} \cos 60^o,$$

making $a = c$. It follows that triangle *ABC* is equilateral.

(ii) If the sides are in geometric progression, then $b^2 = ac$. Again by the law of cosines we have

$$ac = a^2 + c^2 - 2ac \cos 60^o$$

again making $a = c$. Again triangle ABC must be equilateral.

II. *Comment by John H. Scott, Macalester College, Saint Paul, Minnesota.*

I suspect that the proposer is a humorist wondering how surprised people would be. In both cases the answer is all equilateral triangles with an arithmetic difference of zero for the angles and an arithmetic difference of zero and a ratio of one respectively for the sides.

Also solved by FRANK P. BATTLES *and* LAURA KELLEHER, *Massachusetts Maritime Academy, Buzzards Bay,* JAMES E. CAMPBELL, *University of, Missouri, Columbia,* RICHARD I. HESS, *Rancho Palos Verdes, CA,* JOHN M. HOWELL, *Littlerock, CA,* OXFORD RUNNING CLUB, *University of, Mississippi, University,* JOHN H. SCOTT, *Macalester College, Saint Paul, MN,* KENNETH M. WILKE, (2 *solutions*), *Topeka, KS, and the* PROPOSER.

622. [Spring 1986] *Proposed by Walter Blumberg, Coral Springs, Florida.*

Let point *P* be the center of an equilateral triangle *ABC* and let *c* be any circle centered at *P* and lying entirely within the triangle. Let *BR* and *CS* be tangents to the circle such that point *R* is closer to *C* than to A and *S* is closer to A than to *B*. Prove that line *RS* bisects side *BC*.

Solution by William E. Hoff, Princeton, West Virginia.

The theorem is true more generally for point *P* the circumcenter of any given triangle *ABC*, so that is what we assume here. Let ψ be the circumcircle of triangle *ABC*. Let 9 be the angle such that the rotation about *P* through angle 9 maps *B* to *B'* and *C* to C' so that *B'* lies on *PR*. Then *C'* lies on *PS*. Let X and X' be the midpoints of *BC* and *B'C'*. Then angle *XPX'* equals 8. Now the central dilation mapping *B'C'* to *RS* also maps *Q* to *M*, the point where *RS* meets *PQ*. We have

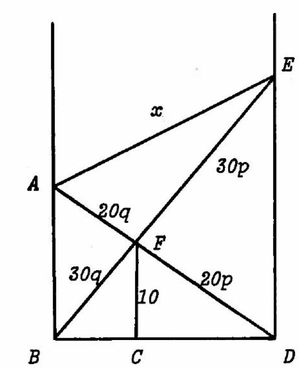
$$PX = PX' = PQ \cos\theta = PQ\frac{PR}{PB} = PQ\frac{PR}{PB'} = PQ\frac{PM}{PQ} = PM.$$

Thus X = *M*, so *RS* bisects *BC*.

Also solved by RICHARD A. GIBBS, *Fort Lewis College, Durango, CO,* RICHARD I. HESS, *Rancho Palos Verdes, CA,* LAURA L. KELLEHER *and* FRANK P. BATTLES. *Massachusetts Maritime Academy, Buzzard* Boy,* RALPH KING, *St. Bonaventure University,* NY, JOHN H. SCOTT, *Macalester College, Saint Paul, MN,* LASZLO SZUECS, *Fort Lewis Cottage., Durango, CO, and the.* PROPOSER.

623. [Spring 1986] *Proposed by John M. Howell, Littlerock, California.*

A 30-foot ladder and a longer ladder are crossed in an alley. The longer one breaks just 20 feet from its foot and the top falls back to the other side of the alley and just touches the top of the 30-foot ladder. If the ladders cross just 10 feet above the ground, find the original length of the longer ladder. (This variation of the old "crossed ladders" problem cost an aircraft company thousands of dollars in lost time during World War II by engineers and other technical people trying to solve it. I finally circulated a solution that probably saved the company thousands more, but alas, I received no credit for it.)



I. *Solution by Kenneth M. Wilke, Topeka, Kansas.*

Label the vertices of the figure as shown and let AD and BE divide each other in the ratio p/q, where $p + q = 1$, so that FD =

$20p$, $FA = 20q$, $FE = 30p$, and $FB = 30q$. From the Pythagorean theorem and the similarity of triangles ABD and FCD we find that

$$\frac{BC}{CD} = \frac{20q}{20p} = \frac{\sqrt{900q^2 - 100}}{\sqrt{400p^2 - 100}},$$

which reduces to

(1) $$5p^2q^2 = p^2 - q^2 = (p + q)(p - q) = p - q$$

since $p + q = 1$. We let $r = p - q$ and $s = pq$, so that

$$(p + q)^2 = (p - q)^2 + 4\,pq$$

or

(2) $$1 = r^2 + 4s.$$

Now equation (1) becomes $5s^2 = r$ which when inserted into equation (2) yields

$$25s^4 + 4s = 1.$$

The only positive root, found by Newton's method, is $s = 0.2310189$, whence $r = 0.2689319$, $p = 0.634466$, and $q = 0.365534$. Now from the Pythagorean theorem we get $BC = 4.5003994$ and $CD = 7.8114558$, so $BD = 12.311855$. Then $ED = 27.357233$ and $AB = 15.761289$. Finally we have

$$AE^2 = (ED - AB)^2 + BD^2, \quad \text{so} \quad AE = 16.912945$$

and the original ladder length was 36.912945 ft ≈ 36 *ft* 11 *in*.

II. *Solution by Ralph King, St. Bonaventure University, St. Bonaventure, New York.*

In the figure above let $a = BC$, $b = BD$, $c = DE$, and $d = AB$. Then

$$\frac{d}{b} = \frac{10}{b - a} \quad \text{and} \quad \frac{a}{b} = \frac{10}{c}, \quad \text{so} \quad c = \frac{10d}{d - 10}.$$

Also $b^2 + a^2 = 30^2$ and $b^2 + d^2 = 20^2$, so then $c^2 - d^2 = 500$. Eliminate a to get

$$d^4 - 20d^3 + 500d^2 - 10000d + 50000 = 0,$$

which has real roots $d_1 = 7.008982$ and $d_2 = 15.761287$. Since $d > 10$, then $d = 15.761287$, so $a = 27.357233$, $b = 12.311857$, and $x = 16.912947$. Thus the original ladder length was 36.912947 feet.

Also solved by GREG DUKEMAN, *Tuscola, IL*, MARK EVANS, *Louisville, KY*, RICHARD I. HESS, *Rancho Palos Verdes, CA*, JOHN H. SCOTT, *Macalester College, Saint Paul, MN*, WADE H. SHERARD, *Furman University, Greenville, NC*, and the PROPOSER.

624. [Spring 1986] *Proposed by Robert C. Gebhardt, Hopatcong, New Jersey.*

It is known and easy to prove that

$$\sum_{i=1}^{n} (i)(i!) = (n + 1)! - 1.$$

Find a closed expression for $S(n)$ and prove that for $n > 1$, $S(n)$ is divisible by 3 where

$$S(n) = \sum_{i=1}^{n} i! = 1! + 2! + 3! + \ldots + n!.$$

Solution by Frank P. Battles and Laura L. Kelleher, Massachusetts Maritime Academy, Buzzards Bay, Massachusetts.

That $S(n)$ is divisible by 3 for $n > 1$ follows readily since $1! + 2! = 3$ and each succeeding term is divisible by 3.

A "closed form" expression can be obtained from the integral definition of $i!$, i.e. the gamma function $i! = \int_0^\infty x^i e^{-x}\,dx$. Thus

$$S(n) = \sum_{i=1}^{n} \int_0^\infty x^i e^{-x}\,dx = \int_0^\infty \left(\frac{x^n - 1}{x - 1}\right) x e^{-x}\,dx.$$

Proofs of the divisibility by 3 was also submitted by VICTOR G. FESER, *University of Mary, Bismarck, ND*, RICHARD I. HESS, *Rancho Palos Verdes, CA*, JOHN M. HOWELL, *Littlerock, CA*, BOB PRIELIPP, *University of Wisconsin-Oshkosh, and the* PROPOSER.

625. [Spring 1986] *Proposed by Sam Pearsall, Loyola Marymount University, Los Angeles, California.*

Let G be a group in which there is a unique element x such that generates a cyclic subgroup of order 2. Show that x commutes with every element of G.

I. *Solution by Stephanie Dumoski, Occidental College, Los Angeles, California.*

Consider any $g \in G$. We know that the order of $g^{-1}xg$ equals that of x since conjugates have the same order. Thus $g^{-1}xg = x$ since $x \neq e$ (where e is the group identity element) and x is unique. Hence $xg = gx$, so x commutes with every element of G.

II. *Solution by Harry Sedinger, St. Bonaventure University, St.*

Bonaventure, N w Yolk.

Let a be in G and consider $b = axa^{1}$. Then

$$b^2 = axa^{-1}axa^{-1} = ax^2a^{-1} = aa^{-1} = e,$$

so either b = e or b = x. If b = e, then $axa^{-1} = e$, which implies that $x = a^{-1}a = e$, a contradiction. Thus b = x, so $a; = axa^{-1}$ or $ax = xa$. Since a was arbitrary, x commutes with every element of G.

Also solved by CHARLES ASHBACHER, *Mount Mercy College, Cedar Rapids, IA,* RICHARD A. GIBBS, *Fout Lewis College, Durango, CO,* FRANCIS C. LEARY, *Saint Bonaventure University, NY,* HENRY S. LIEBERMAN, *Waban, MA,* MASSACHUSETTS GAMMA, *Bridgewater State College,* OXFORD RUNNING CLUB, *University of Mississippi, University,* BOB PRIELIPP, *University of Wisconsin-Oshkosh,* ARTHUR H. SIMONSON, *East Texas State University at Texarkana,* PHILLIP J. SLOAN, *Pembroke State University, Statesville, NC,* and the PROPOSER. One *incorrect solution* wail *received.*

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LETTERS TO THE EDITOR

Dear Editor:

(Here) is a *better solution* to *Puzzle # 1* An the *Spring* 1986 *issue.*

"*What is* the *closest value to 22/7* that you can *obtain* by *using the usual arithmetic symbols* and *the digits* 1, 2, 3, 4 and 5 in that *order* both *from left to right* and. *from right* to *left?*"

Answer: $[(5! \div \sqrt{4}) + 3!] \div 21 = 66/21 = 22/7$, and

$(-1 + 23) \div (\sqrt{4} + 5) = 22/7$, *or*

$[(\phi(5))! - \phi(4)] \div [(3\times2) + 1] = 22/7$, *and*

$[1\times\phi(23)] \div [\phi(4) + 5] = 22/7$.

Notes: $(3\times2) + 1$ can be *replaced* by $3! + 2 - 1$; $\phi(4)$ can be *replaced* by $\sqrt{4}$ [and *vice-versa*].

Edmund F. Marks, Jr.
Massachusetts Delta
University of Lowell

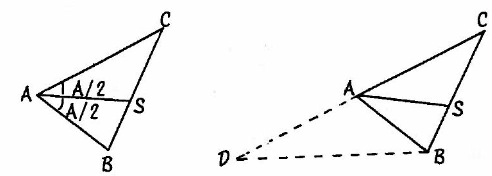
Dear Editor:

Re: A *New Proof of* a *Familiar Result,* PA Mu *Epsilon Journal, Vol.* 8, No. 3 (1985), 164-168.

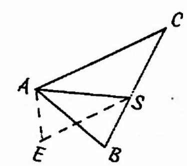
The *proof hinges* upon *consideration of* the. *ratio* $r = BS/SC$. Now, *this simply is AB/AC since* "Each *angle bisector divides* the *opposite side into segments proportional* to the adjacent *sides."* The *problem of showing* AC > AB *implies* SC > BS *is thereby settled.*

Moreover, the. *reader will notice* that the *function* $r(x) = BS/SC$, *where* $x := AB$, *is just* the *identity function* $r(x) = x$ *in case* $AC = 1$ *as in* the *text.*

The *useful proposition* above (*which* can be *augmented*) *is proved, for* example, An *Coxeter* and *Greitzer, Geometry Revisited,* MAA, NML, Vol. 19, page 9, *using* the *law of sines:* $SC/\sin(A/2) = AC/\sin CSA$, $BS/\sin(A/2) = AB/\sin ASB$, *so, since* the *angles at* S *one. supplementary, having equal* bineb, $SC/AC = BS/AB$ *or* $BS/SC = AB/AC$.



For a *proof* that avoids, *sines draw* $BD \parallel SA$. Then $SC/AC = BS/DA$ and $\angle BDA = \angle BAC - \angle ABD = 2\angle BAS - \angle ABD = \angle ABD$ *so* that $PA = AB$. Hence $BS/SC = AB/AC$.



Or, thirdly, we may took *at* the *area ratio* $A(ABS)/A(ASC)$. It *equals* the *ratio of* the *bases* BS/SC *since* the *height from* A *is* the *same.* Now *turn over triangle* ABS, that *is,* make E buch that $EA = BS$, $ES = AB$. Then $ES \parallel AC$ *since* $\angle ASE = \angle SAC$. So *this* time the. *area ratio is* $ES/AC = AB/AC$. *Whence* the *result.*

Yours sincerely,

J. Suck
Rahmstrasse 140
04300 ESSEN 12

GLEANINGS FROM CHAPTER REPORTS

ARKANSAS BETA (Hendrix College). The chapter's tenth year (1985-86) was an active one. In April several students attended the Arkansas-Oklahoma MAA Meeting at Arkansas College, where *Karen Billings, Jim Hart, Bruce Hulsey* and *Travis Williams* presented papers on their work in the Undergraduate Research Program. In May, *Karen* and *Jim* again presented theirs at the annual Hendrix-Rhodes-Sewanee Undergraduate Mathematics Symposium. In April, *Bruce* and *Travis* discussed their papers at the 9th Conference on Undergraduate Mathematics at SUNY at Purchase. At the AR-OK MAA Section Meeting, the team of *Gary Thacker, Pauline Bello* and *Jim Hart* received an honorable mention in the Mathematical Competition in Modelling. At the Honors Convocation in May, the McHenry-Lane Freshman Mathematics Award was shared by *Kim Eherdt* and *Randy Pate*. The Hogan Senior Mathematics Award was given to *Jim Hart*. The Phillip Parker Undergraduate Research Award went to *Travis Williams*. Speakers during the school year included *Silke Hufnagel Allen* (Hendrix) on "The Cantor Set," *John Merrill* (CCX Corporation) on "A Potpourri of Real-Life Job Topics," *Paul Fjelstad* (St. Olaf College) on "Inventing the Calculus," *Dr. Joe Diestel* (Kent State) on "Sets of Discontinuities," *Dr. Joel Haack* (Oklahoma State) on "Aspects of Escher Prints," *Dr. Ralph Scott* on "Applications of Calculus in Economics," *Dr. Robert Serven* (University of Central Arkansas) on "Self Mappings of Polynomials," and *Dr. William T. Ingram* (University of Houston) on "Approximations of Functions."

CALIFORNIA LAMBDA (University of California - Davis). Guest lecturer at the fall initiation was *Professor Henry Alder* who spoke on "How to Discover and Prove Theorems: A Demonstration with Partitions." At the winter initiation, *Professor Kenneth Joy* (Division of Computer Science) spoke on "Geometric Continuity." In May, *Kim Mish* (Department of Civil Engineering) spoke on "Earthquakes and Linear Algebra."

GEORGIA BETA (Georgia Institute of Technology). In June, the chapter presented a book award to *Kamel Haddad*. The award is given to students receiving the degree B.S. in Applied Mathematics with a grade point average of at least 3.7 in allmathematics courses taken.

KANSAS GAMMA (The Wichita State University). Activities during the 1985-86 school year included talks by *S. Elaydi* on "Notions of Stability and Dichotomy in Ordinary Differential Equations," *P. G. Wahlbeck* on "Connecting Mathematical Models with Experiments," and *D. V. Chopra* on "Some Aspects of Mathematics." In December, a panel consisting of *Dr. L. Arteaga, Dr. B. Fridman* and *Dr. Paramasivam*, with *Dr. W. Perel* as moderator, discussed mathematics in India, Spain and the Soviet Union.

MINNESOTA GAMMA (Macalester College). Chapter activities during 1985-86 included the showing of several films produced by the College Geometry Project and the following guest lecturers: *Seymour Schuster* on "Mathematics and Painting," *Jeff Parker* on "Enumerating Binary Trees," *Tom Myers* on "What's Left of Computer Science if Automatic Program Generation Ever Works," *Hung Bin Zou* on "Greatest Common Subgraphs," and *Cheri Shakiban* on "Fractal Geometry." *Loren Larson* (St. Olaf College) was invited speaker for the annual initiation. He spoke on "A Discrete Look at 1 + 2 + ... + n." Social activities included fall and spring picnics and a game night.

MINNESOTA ZETA (Saint Mary's College). The chapter formulated the following resolution which has been forwarded to the national officers of Pi Mu Epsilon:

> *WHEREAS women as well as men are members of Pi Mu Epsilon*
>
> *BE IT RESOLVED THAT the Minnesota Zeta Chapter of Pi Mu Epsilon proposes that the name of the corporation be changed from the PI MU EPSILON FRATERNITY to the PI MU EPSILON MATHEMATICS HONOR SOCIETY or to the PI MU EPSILON HONOR SOCIETY.*

In March, Faculty Advisor *Louis Guillou* spoke on "Integer Programming." At the initiation ceremony in April, *Jay Flaherty* presented his honors project "Riemann-Stieltjes Integrals." Also, in April, *Tim Malecha* reported on his honors project "Noetherian Rings."

MISSOURI GAMMA (Maryville College and St. Louis University). The 1986 James E. Case Memorial Lecture was presented by *Dr. William E. Perrault* at the 49th annual initiation banquet. *Dr. Perrault* spoke on "The Missouri Lottery - Applications of Mathematics." The James W. Garneau Mathematics Award, for the outstanding senior in mathematics at SLU, went to *William Coplin*. The Francis Regan Scholarship, for a graduating senior active in Missouri Gamma activities, went to *Gary Menard*. The Missouri Gamma Undergraduate Award, for graduating seniors at Fontbonne, Lindenwood or Maryville Colleges, went to *Bitten Dill* and *Penny Havlic*. The Missouri Gamma Graduate Award, for a first-year graduate student at SLU, went to *Teresa Huether*. The John J. Andrews Graduate Service Award, for a graduate student at SLU who took an active part in departmental affairs, went to *Mohammad Azarian*. The Beradino Family Fraternityship Award, for active participation in affairs of the fraternity, friendliness, and concern for its members, went to *William Coplin*.

NEBRASKA ALPHA (University of Nebraska - Lincoln). A freshman scholarship program has been created. Awards consist of cash prizes and/or gift certificates from the Nebraska Bookstore. Recipients in 1985-86 were *Terry Clements, Allen George, Larry McConville* and *Moez Miraoui*. Awards were given according to the results of a 25-question multiple-choice examination covering concepts up to those in second-year calculus. To help finance the scholarship the chapter sells copies of old mathematics finals. At the annual new members initiation, the film FLY LORENZ was shown. *Dr. Steve Dunbar* provided insight and explanations.

NEW JERSEY DELTA (Seton Hall University). Weekly problem-solving sessions were led by *Dr. John Masterson*. Other activities during the 1985-86 year included a talk on "Actuarial Mathematics" by *Gary Strunk*, *A.S.A.*, a talk on "Artificial Intelligence" by *Dr. David H. Copp*, and the showing of the film "Pits, Peaks and Passes." At the 19th Annual Induction ceremony, *Dr. John J. Saccoman* spoke on "The History of Pi Mu Epsilon."

NEW YORK ALPHA BETA (LeMoyne College). *Professor Norman J. Pullman* (Queen's University) spoke on "Scheduling a Golf Tournament: An Application of Finite Geometry." The spring address "Cryptology: From Caesar Ciphers to Public-Key Cryptosystems" was given by *Professor Dennis M. Luciano* of Western New England College.

NEW YORK PHI (State University College of New York at Potsdam). At the Fall Induction, *Dr. Charles Mosier* (Clarkson University) spoke on "The Group Technology Clustering Problem." The speaker at the Spring Induction was *Dr. Guy Johnson, Jr.* (Syracuse University) who spoke on "How to be a Halley Watcher." Chapter student members *Virginia Filiaci* and *William Martin* gave talks at the Seaway Section Spring Meeting of the MAA at Ithaca College. In April, *Cheri Brunner* and Career Services cosponsored the Second Annual Career Nights Event, which featured six speakers representing careers in Operations Research (*Mary Charles*, IBM), Software Engineering (*Teresa Kohlbrenner*, GE), Insurance (*Gary Bissonette*), Public School Teaching (*Anthony Vaccaro*), Banking (*Susan Abbott*) and Graduate Study in Mathematics (*Darryl Weatherly* and *Tom Jones*). The Pi Mu Epsilon Senior Award was given to *Susan L. Doriski*, a BA/MA candidate and the ranking graduating mathematics major.

NORTH CAROLINA LAMBDA (Wake Forest University). An excellent mix of student, faculty and visitor talks during the 1985-86 academic year included *Dr. Rick Heatley*, Office of Educational Planning and Placement, on "Employment Opportunities in Mathematics and Computer Science," *Dr. Elmer K. Hayashi* on "The Institute for Retraining in Computer Science," student *Mark Roberson* on "The Fractional Calculus," *Dr. Fred Howard* on "Stirling Numbers," student *Salman Azhar* on "Generalizing Fermat's Little Theorem," *Dr. Walter Rudin* on "Sets of Distances," student *Muriel McLean* on "Decimal Fractions that can be Represented in Terms of Fibonacci and Lucas Numbers," student *Christie Baucom* on "Formal Language Theory," *Dr. John Franke* on "The Funhouse Mirror," and student *Helen Rogers* on "The Ongoing Study of Continued Fractions."

OHIO DELTA (Miami University). In August, 1984, six students and *Dr. Milton Cox*, national president of Pi Mu Epsilon, attended the national meetings at the University of Oregon at Eugene. Two students presented papers: *Leslie Youngdahl* spoke on "Complements - Mathematically Speaking," and *Saguna Pappu* on "Samuelson's Interaction Between the Accelerator and the Multiplier." The 11th Annual Student Conference in September included a record 29 student speakers, eight from Miami. *Seguna* and *Leslie* repeated their talks, *Nick Short* spoke on "Artificial Intelligence," *David Cameron* on "A Twenty-Five Point Geometry Revisited," *Carol Rickard* on "Spiromania!," *Michael Heflin* on "Fun With Sound," *Mark Russell* on "Manufacturing Reflector Lamps in the Age of Micros," and *Jeff Ziegler* "On Fitting a Hyperbolic Decline Curve Using SAS." At the first regular chapter meeting in October, *Dr. Daniel Pritikin* lectured on "Pseudo-Mathematical Recreations." In November, alumna *Vicki Stover-Hertzburg* (now a bio-statistician at the University of Cincinnati) spoke on "Yes, Virginia, There is Life Aftermath." In February, *Dr. Douglas Ward* lectured on "Operations Research, Life and Mathematics in Canada." In March, *Jerry McAllister* of Union Central Life Insurance Company gave a talk entitled "That's Incredible." Sophomores *Robert J. Walling* and *Darrin R. Gaines* were prize-winners in the Pi Mu Epsilon Examination which was prepared and graded by the juniors and seniors. Trips to the MAA Ohio Section meeting at the University of Akron and the Undergraduate Conference at Rose-Hulman Institute in Terre Haute rounded out the year. At Akron, *David Cameron* spoke on "Create Your Own Geometry," *Mike Grinkemeyer* on "Modeling: Vented Loud Speaker Design," and *Karen Trageser* on "Polygonal Residue Patterns." At Terre Haute papers presented included *Nick Short's* "Expert Systems and Their Role in AI" and *David Cameron's* "Create Your Own Geometry."

OHIO ZETA (The University of Dayton). Activities in 1985-86 featured student and faculty member talks. *Dr. Joseph Mashburn* lectured on "Proving the Intermediate Value Theorem: An Introduction to Topology." Senior *Gary Johnson* gave a talk on "Some Surprising Differences Between the Calculus of Real Numbers and the Calculus of Complex Numbers." *Ken Bloch* spoke on "The Probability of Election Reversal." *Laura Augustine* explained "Runge-Kutta Runs." *Rafe Donahue* pointed out "Geometrical Fallacies and other things that go wrong with Mathematics." *Mark Liatti* investigated "The Weighing of the Coins," and solved the problem with one less weighing than the standard solution. *John Feller* explained "A Unique Method to Solve Large-Order Determinants on a Microcomputer." *Dave Aubel* presented and solved Question A-6 of the 1985 Putnam Competition, *Dave Geis* spoke on "Numerical Interpolation," and *Mary Beth Anderson* gave a presentation. *Lora Eohn* displayed a proof of the Duality Law from Logic and *Laura Konerman* demonstrated how any 2^n by 2^n checkerboard with one square removed can be tiled with trominos. The highlight of the year was the 12th Annual Pi Mu Epsilon Student Conference at Miami University. *Greg Bishop* spoke on "A Practical Estimate of Standard Deviation," *Ken Bloch* discussed "Orthogonality in Quantum Mechanics," *Kelly Ann Chambers* presented "Two Fundamental Results from Lattice Theory," *Gary Johnson* discussed "Basic Results from Group Theory," *John Sengewalt* explained how to "Escape the Markov Chains of Monte Carlo Jails." At the chapter's annual banquet, *Jeff Diller* was presented the Sophomore Class Award of Excellence. In April, *Mark Liatti* presented "The Weighing of the Coins" at the MAA meeting at John Carroll University. He was awarded a one year's membership in the MAA.

OHIO NU (University of Akron). Sam Selby Scholarship Awards in the amount $125 were given to *John Keblesh* and to *Michael Quinn*. *Evan Witt*, winner of the Mathematics Category in the Akron Public Schools Science Fair (Junior High) was awarded a check for $50 by the chapter. *Michael Quinn*, *Ron Bartfai* and *John Keblesh* received one-year memberships in the American Mathematical Society. *Rima Felfli* was awarded a one-year membership in the American Statistical Association.

OHIO XI (Youngstown State University). Six chapter members attended the 1985 summer meetings in Laramie, Wyoming, where *Raymond Flannery*, *Tony Clacko*, *Terri Wiencek* and *Henry Culver* gave papers. At the Fall Quarter Initiation, *Visiting Professor Andrzej Szymanski* was guest speaker. At the Spring Quarter Initiation, graduate student *Gene Santos* and *Dr. Eric*

Wingler were guest speakers. At the Annual Spring Banquet, guest speaker was *Dr. John J. Buoni.* During the 1985-86 school year several students attended MAA Meetings at Miami University and at John Carroll University.

OHIO OMICRON (Mount Union College). In September, 1985, the chapter sponsored a trip to the Twelfth Annual Pi Mu Epsilon Conference at Miami University. In April, *Joe Smith*, a graduate of Mount Union, discussed the actuarial profession and the mathematics necessary for it. Also, in April, Chapter President, *Mark Harker*, summarized his mathematics independent study project.

PENNSYLVANIA NU (Edinboro University). At the Fall Initiation, *Dr. Peter Weidner* spoke on mathematical paradoxes. At the Spring Initiation, *Professor James Watson* spoke on difference equations. In April, a group of eight students attended the Annual Meeting of the Allegheny Mountain Section of the MAA at Clarion University.

PENNSYLVANIA OMICRON (Moravian College). The chapter sponsored a presentation on the actuarial profession by *Elayne Livote* and *Judy Weaver* of Mutual Benefit Life. *Professor Johanna Ott*, who retired at the end of the school year, gave a talk reflecting on her years at Moravian College. Permanent Faculty Correspondent and Chapter Advisor, *Ross Gingrich*, presented a talk on "Some Geometrical Algebra," the historical use of classical geometry to produce or prove algebraic results.

IMPORTANT ANNOUNCEMENT

Pi Mu Epsilon's main source of steady income is the National Initiation Fee for new members.

The fee covers the cost of a membership certificate and a one-year subscription to the Pi Mu Epsilon Journal.

For the past fourteen years the fee has been set at $4.00. Effective January 1, 1987, the National Initiation Fee will be $10.00. After January 1, 1987, any order for membership certificates should be accompanied by the new fee.

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