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ON THE LARGEST RAT-FREE SUBSET OF A FINITE SET OF POINTS

by Wah Kaung Chan
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During the summer of 1985, Professor William Moser supervised my activity while I held an NSERC Summer Undergraduate Research Award at McGill University. His now well-known, privately circulated, Research Problems in Discrete Geometry is a cumulative record of progress made on many unsolved and partially solved problems in discrete geometry. While examining this collection I noticed that I understood many of the problems, and for some I could comprehend the associated published research papers. Several of the problems were featured by Martin Gardner in his Scientific American column "Mathematical Games," and progress in these had been made by readers, that is, by amateur mathematicians using only pre-calculus mathematics. In this paper, I would like to report on one problem (No. 49) in W. Moser’s collection [1] which is not completely solved and which I believe to be suitable for readers of this journal to investigate further.

Basically, the problem is as follows. Suppose one is given a set of \( n \) points, and the objective is to select as many of them as possible so that no three (of the chosen points) are the vertices of a right-angled triangle. How many can one be assured of choosing no matter how the \( n \) points are distributed? Let us phrase the question in a more precise manner. By an \( n \)-set we mean a set of \( n \) points in the plane, and we call the vertices of a right-angled triangle an RAT. Let \( f(n) \) denote the largest integer for which every \( n \)-set contains an RAT-free \( f(n) \)-subset. The problem is to determine \( f(n) \) or, failing this, to establish good upper and lower bounds on \( f(n) \). Note that, to establish the lower bound \( m \), that is, to show \( f(n) \geq m \), it is necessary to show that every \( n \)-set contains an RAT-free \( m \)-subset. To establish the upper bound \( L_n \), that is, to show \( f(n) < L_n \), it suffices to exhibit a particular \( n \)-set which does not contain an RAT-free \( L_n \)-subset. The main result is contained in

**Theorem 1.**

(1) \( \sqrt{n} \leq f(n) \leq 2\sqrt{n}, \ n = 4, 5, 6, \ldots \)
The lower bound was established by Abbott [2] using a famous theorem of Erdős and Szekeres [3]. For the latter theorem we will give a particularly elegant proof due to Seidenberg [4].

A sequence of numbers

\[ a_1, a_2, a_3, \ldots, a_p \]

is said to be monotone increasing if

\[ a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_p \]

monotone decreasing if

\[ a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_p \]

and monotone if it is either monotone increasing or monotone decreasing.

Theorem 2 (Erdős-Szekeres). Let \( m, n \) be positive integers. Then every sequence of length \( mn + 1 \) (but not every sequence of length \( mn \)) contains either a monotone increasing subsequence of length \( m + 1 \) or a monotone decreasing subsequence of length \( n + 1 \).

Proof. Let

\[ y_1, y_2, y_3, \ldots, y_{mn+1} \]

be an arbitrary sequence of length \( mn + 1 \). To each \( y_i \) in (5) we associate the pair \( (u_i, v_i) \) of integers, where \( u_i \) (resp. \( v_i \)) is the length of the longest monotone increasing (resp. decreasing) subsequence of (5) beginning at \( y_i \). Letting \( u = \max_i u_i \) and \( v = \max_i v_i \), we have

\[ 1 \leq u \leq n, \quad 1 \leq v \leq m, \quad i = 1, 2, \ldots, mn + 1, \]

so the number of distinct points \( (u_i, v_i) \) is at most \( uv \). Furthermore, these \( mn + 1 \) points \( (u_1, v_1), (u_2, v_2), \ldots, (u_{mn+1}, v_{mn+1}) \) are distinct.

For suppose \( i \neq j \), say \( 1 \leq i < j \leq mn + 1 \). If \( u_i < u_j \), then \( u_i > u_j \); if \( u_i = u_j \), then \( v_i > v_j \). It follows that

\[ mn + 1 \leq uv, \]

from which either \( u \geq m + 1 \) or \( v \geq n + 1 \).

To construct an example for the "but not every sequence of length \( mn + 1 \)" part of Theorem 2, we start with the sequence of integers from \( 1 \) to \( mn \)

\[ 1, 2, \ldots, n, \quad (n+1), (n+2), \ldots, (mn), \quad (mn+1), (mn+2), \ldots, (mn+n-1), (mn+n), \ldots, (mn+1), \]


Notice that any increasing subsequence of (8) has at most one term in any block, and hence has length at most \( m \). Any decreasing subsequence has length at most \( n \) since all terms must be in the same block. These properties are nicely seen in Figure 1 where the \( k \)th term of (8) is the ordinate of the point with abscissa \( i \).

![Figure 1](image-url)

When \( m \) and \( n \) are equal, say \( k \), we have

**Corollary 1.** Every sequence of length \( k^2 + 1 \) (but not every sequence of length \( k^2 \)) has a monotone subsequence of length \( k + 1 \).

It is now possible to give a proof of Abbott's Theorem.

**Theorem 3.**

\[ k + 1 \leq f(k^2 + 1), \quad k \geq 3. \]

Proof. Suppose we have a set of \( k = k^2 + 1 \) points. Choose a Cartesian coordinate system whose \( z \)-axis is not perpendicular to any of the \( (\ell - 1)/2 \) segments joining pairs of the points. Then no two points*
have the same abscissa. Let the points \((x_1, y_1), (x_2, y_2), \ldots, (x_{k+1}, y_{k+1})\) be named so that \(x_1 < x_2 < \cdots < x_{k+1}\). The sequence \(y_1, y_2, \ldots, y_{k+1}\) contains (by Corollary 1) a monotone subsequence

\[
y_{i_1}, y_{i_2}, \ldots, y_{i_{k+1}}
\]

of length \(k+1\). It is easy to see that the \(k + 1\) points \((x_{i_j}, y_{i_j})\), \(j = 1, 2, \ldots, k + 1\) form an RAT-free \((k + 1)\)-subset of the \(k^2 + 1\) given points.

The left inequality of Theorem 1 follows by considering, for a given integer \(n\), the integer \(k\) uniquely determined by

\[(k - 1)^2 + 1 \leq n \leq k^2.
\]

Then, using \(f(k) \leq f(k+1)\),

\[\sqrt{n} \leq k \leq f((k - 1)^2 + 1) \leq f(n).
\]

Erdős [5] remarked that the set \(S_k^{(1)}\) of \(k^2\) lattice points

\[S_k^{(1)} = \{(x, y) | x, y \text{ integers}, 0 \leq x, y \leq k - 1\}
\]

shows

**Theorem 4.**

\[f(k^2) \leq 2k - 2, \quad k \geq 3.
\]

**Proof.** To establish this inequality, let \(P\) be an RAT-free subset of \(S_k^{(1)}\). We call a point \(p \in P\) an \(a\)-point (resp. a \(6\)-point) if the horizontal row in which it lies contains no (resp. at least one) other point of \(P\). Let \(a\) (resp. \(b\)) denote the number of \(a\)-points (resp. \(6\)-points) in \(P\). No column contains two \(6\)-points, nor an \(a\)-point and a \(6\)-point (together). Clearly, \(a \leq k\) and \(b \leq k\). If there are \(a = k\) \(a\)-points then every row contains an \(a\)-point, allowing no row for a \(6\)-point (\(b = 0\)), and then \(a + b = k \leq 2k - 2\). Similarly, if there are \(b = k\) \(6\)-points there will be no column for an \(a\)-point, so \(a = 0\), and, again, \(a + b = k \leq 2k - 2\).

In all other cases, \(a \leq k - 1\) and \(b \leq k - 1\), and hence \(a + b \leq 2k - 2\). Figure 2 shows that \(S_k^{(1)}\) does contain an RAT-free \((2k - 2)\)-subset. Indeed, all RAT-free \((2k - 2)\)-subsets of \(S_k^{(1)}\) must have the configuration of Figure 2, since the column containing the \((k - 1)\) \(a\)-points and the row containing the \((k - 1)\) \(6\)-points meet at a point (of \(S_k^{(1)}\)) which must be at a corner of the square \(S_k^{(1)}\).

The right inequality of Theorem 1 follows from Theorem 4. If \(k\) is the unique integer for which

\[k^2 \leq n < (k + 1)^2 - 1,
\]

then

\[f(n) \leq f((k + 1)^2 - 1) \leq f((k + 1)^2) \leq 2k \leq 2\sqrt{n}.
\]

An improvement of (10) is obtained by removing two adjacent corner points from \(S_k^{(1)}\), for example, \(S_k^{(2)} = S_k^{(1)} \setminus \{(0,0), (0,k-1)\}\) yielding

\[f(k^2 - 2) \leq 2k - 3.
\]

For if \(P\), an RAT-free subset of \(S_k^{(2)}\), contains \(2k - 2\) points, then \(P\) must (also) be congruent to Figure 2; but this configuration cannot be found in \(S_k^{(2)}\). Therefore, \(|P| \leq 2k - 3\).

This result was improved by H. L. Abbott [5]. He considered the set

\[S_k^{(3)} = \{(x, y) | x, y \text{ integers}, 0 \leq x \leq k - 1, 0 \leq y \leq k\} \setminus \{(0,0), (0,k), (k-1,0), (k-1,k)\}, \quad k \geq 5
\]

of a \(k \times (k + 1)\) rectangular array of lattice points with corner points removed and found the largest RAT-free subset to be \(2k - 3\), that is,

\[f(k^2 + k - 4) \leq 2k - 3.
\]

Based on the same idea, we present the proof of a slightly better result.

**Theorem 5.**

\[f(k^2 - 4) \leq 2k - 4, \quad k \geq 5.
\]
Proof. Consider the set 
\[ S^{(4)}_k = \{(x,y) \mid x, y \text{ integers}, 0 \leq x, y \leq k - 1 \} \setminus \{(0,0), (0,k-1), (k-1,0), (k-1,k-1)\}, \quad k \geq 5 \]
of a \(k \times k\) square array of lattice points with corner points deleted; 
k \(k \times k\) points in total. We show that we can find an RAT-free subset of 
\(2k - 4\) points while it is not possible to find an RAT-free subset of 
\(2k - 3\) points. Once again, following the proof of Theorem 3, we let there be a \(a\)-points and \(b\) \(E\)-points.

Figure 3 shows an SAT-free subset of \(2k - 4\) points. Now, assume 
there is an SAT-free subset of \(2k - 3\) points, that is, \(a + b = 2k - 3\). 
The number of rows is equal to or greater than \(a\) (the number of \(a\)-points) 
and at least one row occupied by the \(E\)-points.

Thus 
\[ a + 1 \leq k. \]
Replacing \(a\), 
\[ 2k - 3 - b + 1 \leq k \]
gives 
\[ b \geq k - 2. \]

It is sufficient to show that there does not exist an RAT-free sub-
set of \(b = k - 2\) \(E\)-points and \(a = k - 1\) \(a\)-points since the case \(b = k - 1\) 
and \(a = k - 2\) can be reduced to the former.

Since \(a = k - 1\) there exists only one row in which the \((k - 2)\) \(E\)-
points can be placed; the \(a\)-points can occupy two columns. There are 
four cases to consider, all related to the distribution on the two

"a-column". It can happen that

(i) the columns are not adjacent, 
(ii) columns at \(a_i\) and \(a_{i+1}\) (or \(a_{k-1}\) and \(a_k\)), 
where \(a_i\) denotes the \(i\)th column, 
(iii) columns are at \(a_i\) and \(a_{i+1}\) (or \(a_{k-1}\) and \(a_{k-2}\)), or 
(iv) columns are adjacent other than in (ii) and (iii).

Figures 4, 5, 6 and 7 illustrate the impossibility of having an SAT-free 
subset of \(a + b = 2k - 3\) points for cases (i), (ii), (iii) and (iv), 
respectively.

For Figures 4, 5, 6a, 6b, and 7, we have \(O\) for beta points. \(\bullet\) for 
alpha points and \(\circ\) for illegal points.
Theorems 2, 3, 4 and 5 restrict $f(n)$ for large $n$ to
\begin{equation}
\sqrt{n} \leq f(n) \leq 2\sqrt{n}.
\end{equation}

By considering the set of lattice points contained in a circle, we conjecture that the upper bound can be reduced to
\[ f(n) < 1.6\sqrt{n}. \]

It is of interest to find the value of $r = \lim f(n) / \sqrt{n}$ if it exists.

Also of interest is the exact value of $f(n)$ for small $n$. Using inequalities (1), (9), (10), (12) and (13), we have the following:
\[ f(4) = 2 \]
\[ f(5) = 3 \]
\[ f(6) = 3 \]

\[ f(7) = 3 \]
\[ 3 \leq f(8) \leq 4 \]
\[ 3 \leq f(9) \leq 5 \]
\[ f(10) = 4 \]
\[ 4 \leq f(i) \leq 5, \ i = 11, 12, 13, 14 \]
\[ 4 \leq f(i) \leq 6, \ i = 15, 16 \]
\[ 5 \leq f(i) \leq 6, \ i = 17, 18, 19, 20, 21 \]
\[ 5 \leq f(i) \leq 7, \ i = 22, 23, 24, 25 \]
\[ 6 \leq f(i) \leq 7 \]
\[ 6 \leq f(i) \leq 8, \ i = 27, 28, \ldots, 32 \]
\[ 6 \leq f(i) \leq 9, \ i = 33, 34, 35, 36 \]
\[ 7 \leq f(i) \leq 9, \ i = 37, 38 \]
\[ 7 \leq f(i) \leq 10, \ i = 39, 40, \ldots, 45. \]

Theorem 2 gives $f(5) \geq 3$ and $f(10) \geq 4$. $f(5) = f(6) = f(7) = 3$ follows from Figure 8. The configuration in Figure 9 shows $f(10) = 4$. Perhaps a reader can settle some of these cases, for example, is $f(8) = 3$ or 4?

\[ 
\begin{array}{ccccccc}
  & & & & o & & \\
  & & o & & & o & \\
  & o & & o & & o & \\
 & o & & o & & o & \\
\end{array}
\]

\[ 
\begin{array}{ccccccc}
  & & & & o & & \\
  & & o & & & o & \\
  & o & & o & & o & \\
 & o & & o & & o & \\
\end{array}
\]

Erdős [2] also asked the same question in three dimensions, that is, for the $n^3$ lattice points $(x, y, z)$, $0 \leq x, y, z \leq n - 1$, determine the largest SAT-free subset.

For $n = 2$, the largest RAT-free subset consists of four (of the eight) points forming the vertices of a regular tetrahedron (see Figure 10).

For $n = 3$, the largest SAX-free subset is 6 points, verified by computer analysis. The two groups of configurations are shown in Figures 11 and 12.

For $n = 4$, RAT-free subsets of 10 points were found. The subsets are congruent to the form \{(0,0,0), (1,1,1), (2,2,2), (3,3,3), (3,1,0), (3,0,1), (0,3,1), (1,3,0), (0,1,3), (1,0,3)\}. On the other hand, this configuration...
does not work for $n = 5$. It was found that any collection of $3(5) - 2 = 13$ points with points $(0,0,0)$ and $(1,1,1)$ have no RAT-free subsets.

Our set of $\mathbf{n}^3$ points determines $\mathbf{n}^n$ columns parallel to each of the coordinate axes. Call the columns $\sigma_x$, $\sigma_y$ and $\sigma_z$. In any configuration of an RAT-free subset, there are four types of points.

1. $a$-points, points occupying the same $\sigma_x$ with other points,
2. $b$-points, points occupying the same $\sigma_y$ with other points,
3. $c$-points, points occupying the same $\sigma_z$ with other points,
4. $d$-points, points occupying a $\sigma_x$, a $\sigma_y$ and a $\sigma_z$ alone.

Let there be $a$, $b$, $c$ and $d$ of these points, respectively.

We have found two groups of RAT-free $(3n - 3)$-subsets. One has $d = 0$ and $a = n - 1$, $b = n - 1$ and $c = n - 1$. Another has $a = n - 2$ $(1,0,0), \ldots, (n - 2,0,0)$, $b = n - 2$ $(0,1,0), \ldots, (0,n - 2,0)$, $c = n - 2$ $(0,0,1), \ldots, (0,0,n - 2)$ and $d = 3$ $(n - 1,0,n - 1)$, $(n - 1,n - 1,0), (0,n - 1,n - 1)$ for $n \geq 3$. Are there any other configurations?

We suspect that for large $n$ the largest RAT-free subset is of size $3n - 3$.

We have yet to show (i) that for $a = b = c = 0$ the largest RAT-free subset of only $d$-points has size $d \leq 3n - 3$, and (ii) for $a \neq 0$, $b \neq 0$, $a \neq 0$ and $d \neq 0$, $a + b + c + d \leq 3n - 3$. This would confirm our conjecture.

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REFERENCES


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ON BUFFON'S NEEDLE PROBLEM USING CONCENTRIC CIRCLES

by H. J. Khambé
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One of the oldest problems in geometrical probability is Buffon's Needle Problem (1777): A board (of large size) is ruled with equidistant parallel lines d units apart. A needle of length \( L < d \) is dropped at random on the board. What is the probability that the needle will intersect one of the lines?

The answer is that the needle will cross a line with probability \( \frac{2L}{\pi d} \). It is interesting to note that the answer involves the transcendental number \( \pi \). In fact, this formula was used in early attempts at approximating \( \pi \); viz, drop the needle onto the board of parallel lines a large number of times, calculate the relative frequency of "crosses", reciprocate, and multiply by \( \frac{2L}{d} \).

As one might expect for such an important method, a large number of variations of the Buffon Needle Problem have been studied. Duncan (1967) studied the case in which a needle is dropped onto a set of radial lines. Gnedenko (1962) generalized the problem first to n-sided convex polygons with diameter less than d, and then to convex closed curves with diameter less than d by considering such curves as limits of inscribed polygons, giving the probability of a "cross" as \( \frac{L}{\pi d} \). Other variations of Buffon's Needle Problem have been discussed by Ramaley (1969), Perlman and Wichura (1975), and Robertson and Siegel (1986).

In this paper, we consider the following natural variation:

Randomly drop a needle of length \( L < d \) onto a board containing \( N \) concentric circles, where the difference in the radii between any two consecutive such circles is a constant \( d \), with \( L < d \). What is the probability, \( P_N \), that the needle crosses one of the \( N \) circumferences? The only mathematical tools that are required for the solution of this problem are integration techniques and elementary probability theory.

To begin, let \( X \) represent the distance between the midpoint \( (M) \) of the needle and the nearest circumference, as measured along the radius extending from the center \( (O) \) of the concentric circles through the midpoint of the needle. See Figure 1. Let \( Y \) represent the acute angle created by \( OM \) and the needle. Assume that \( M \) falls within the \( k \)th annulus, \( k = 1, 2, \ldots, N \); that is, \( (k-1)d < OM < kd \) (\( k = 1 \) corresponds to the circle having radius \( d \)). Then, there are two cases to consider:

**Case 1.** \( (k - 1/2)d < OM < kd, \ k = 1, 2, \ldots, N, \)

**Case 2.** \( (k - 1)d < OM < (k - 1/2)d, \ k = 2, 3, \ldots, N. \)

The first case corresponds to the event that \( M \) falls inside the "outer half" of the \( k \)th annulus, and the second case to the event that \( M \) falls inside the "inner half" of the \( k \)th annulus. Let these two events be represented by \( I_k^+ \) and \( I_k^- \), respectively. We consider each case separately.

![Figure 1](image-url)
When $x_0 < X < \ell/2$, the probability that the needle crosses the kth circumference depends on $Y$. By the law of cosines (see Figure 1),

$$r^2 = (kd - x)^2 + \ell^2/4 + (kd - x)\ell \cos Y.$$ 

Note that the needle crosses the kth circumference if and only if $r > kd$, which is equivalent to

$$\cos^{-1}\left(\frac{(k-1/2)d - x}{kd}ight), \quad x < x < \ell/2.$$ 

Now, $Y$ is treated as a uniform random variable on $[0, \pi/2]$. The density function for $X$, however, must be derived (it is not uniform!). Note that $X$ takes values in the interval $(0, d/2)$. The probability that $M$ falls at a distance between zero and $x$ from the kth circumference is the ratio of the area of the annulus having inner radius $kd - x$ and outer radius $kd$ to the area of the annulus having inner radius $(k - 1/2)d$ and outer radius $kd$; viz,

$$F_X(x) = \frac{\pi(kd)^2 - \pi(kd - x)^2}{\pi(kd)^2 - \pi(kd - d/2)^2}.$$

Hence, the probability density function of $X$ is

$$f_X(x) = \frac{d}{dx} F_X(x) = \frac{2(kd - x)}{(k - 1/4)d^2}, \quad 0 < x < d/2.$$ 

The random variables $X$ and $Y$ are assumed to be independent.

If we let $C_k$ represent the event that the needle crosses a circumference when $M$ falls inside the kth annulus, then the conditional probability of $C_k$, given that $(k - 1/2)d < \theta < kd$, is

$$P(C_k | \theta) = P[X < x_0] +$$

$$P\left[ x_0 < X < \ell/2, \ Y < \cos^{-1}\left(\frac{(k-1/2)d^2 - x^2}{(kd-x)\ell}\right) \right].$$ 

The first integral is straightforward. The second can be evaluated, after a great deal of computation, using integration by parts and a series of substitutions. The final expression is

$$P(C_k | \theta) = \frac{1}{\pi d^2 (k - 1/4)} \left[ \ell\sqrt{k^2d^2 - \ell^2/4} + 2k^2d^2 \sin^{-1}\frac{\ell}{2kd} \right].$$

This takes care of the case in which $M$ falls inside the outer half of the kth annulus (Case 1).

**Case 2** Assume that $M$ falls inside the inner half of the kth annulus. See Figures 3 and 4. By the law of cosines,

$$r^2 = (h + x)^2 - \ell^2/4 - \ell(h + x)\cos Y,$$

where $h := (k - 1)d$.
Now, let $x_1$ represent the value taken on by $X$ when the left end-point of the needle coincides with the point at which the needle is tangent to the circle having radius $h$ (See Figure 5). It can easily be verified, by using the Pythagorean Theorem, that

$$x_1 = \frac{1}{2} \sqrt{4h^2 + \ell^2} - h.$$ 

Also, $Y$ will again be assumed to be uniform on $[0, \pi/2]$. Then the probability density function for $X$ can be found to be

$$f_X(x) = \frac{2(x + h)}{(k - 3/4)d^2}, \quad 0 < x < \frac{d}{2},$$

by expressing the cumulative distribution function of $X$ as a ratio of two annular areas, just as was done in Case 1. Again, $X$ and $Y$ are assumed to be independent.

Case 2 will be handled in two separate parts:

Case 2a. $0 < X < x_1$ (see Figure 3), and

Case 2b. $x_1 < X < \ell/2$ (see Figure 4).

In Case 2a, the needle crosses the $(k - 1)$st circumference if and only if

$$r^2 < (k - 1)^2d^2 + \left[\frac{\ell}{2} - \sqrt{(k - 1)d + x} - (k - 1)^2d^2\right]^2.$$ 

The right-hand side of this inequality is the value of $r^2$ when the needle in Figure 3 is tangent to the circle of radius $h$, or, equivalently,

$$Y < \cos^{-1}\left[1 - h^2/(h + x)^2\right].$$

Then, the joint probability of $C_k$ for the case $0 < X < x_1$, given that $(k - 1)d < \pi d < (k - 1/2)d$, is

$$P(C_k, 0 < X < x_1 | I_{x_1}) = \frac{4}{\pi(k - 3/4)d^2} \int_0^{x_1} (x + h)\cos^{-1}\left[1 - h^2/(h + x)^2\right] dx.$$ 

Once again, this integral can be evaluated using integration by parts. After some lengthy computations, we have

$$P(C_k, 0 < X < x_1 | I_{x_1}) =$$

$$\frac{1}{\pi(k - 3/4)d^2} \left[\frac{4h^2 + \ell^2}{2} \cos^{-1}\left(\frac{\ell}{\sqrt{4h^2 + \ell^2}}\right) + h(\ell - vh)\right].$$

In Case 2b, $x_1 < X < \ell/2$, and the needle crosses the $(k - 1)$st circumference if and only if $r < h$ (see Figure 4), or, equivalently,

$$Y < \cos^{-1}\left[\frac{(x + h)^2 + (\ell^2/4 - h^2)}{(x + h)\ell}\right].$$
So, the joint probability of $C_k$ for the case $x_1 < X < \ell/2$, given that $(k - 1)d < \Omega c (k - 1/2)d$, is

$$P(C_k, x_1 < X < \ell/2 | I_k^1) =$$

$$\frac{4}{\pi(k - 3/4)d^2} \int_{x_1}^{\ell/2} (x + h) \cos^{-1} \left[ \frac{(x + h)^2 + (\ell^2/4 - h^2)}{2x} \right] \, dx =$$

$$\frac{1}{\pi(k - 3/4)d^2} \left[ h(\ell + h) - \frac{h^2 + \ell^2}{2} \cos^{-1} \frac{\ell}{\sqrt{h^2 + \ell^2}} \right].$$

Combining the expressions in (2.2) and (2.3), we have that the conditional probability that the needle crosses the $(k - 1)$st circumference, given that $(k - 1)d < \Omega c (k - 1/2)d$, is, upon simplification,

$$P(C_k | I_k^1) = P(C_k, 0 < X < x_1 | I_k^1) + P(C_k, x_1 < X < \ell/2 | I_k^1) =$$

$$\frac{2\ell(k - 1)}{\pi d(k - 3/4)}. $$

It is interesting to note that this probability is smaller than in the classical Buffon Needle Problem by a factor of $(k - 1)/(k - 3/4)$.

The probabilities of the events $I_k^1$ and $I_k^2$ are easily obtained by computing ratios of annular areas:

$$P(I_k^1) = (k - 1/4)/N^2, \quad k = 1, 2, \ldots, N, \text{ and}$$

$$P(I_k^2) = (k - 3/4)/N^2, \quad k = 2, 3, \ldots, N.$$ (2.5)

Then, finally, the probability that the needle crosses a circumference when randomly dropped onto a set of $N$ concentric circles is, by the Law of Total Probabilities,

$$\rho_N = \sum_{k=1}^{N} P(I_k^1) P(C_k | I_k^1) + \sum_{k=2}^{N} P(I_k^1) P(C_k | I_k^2),$$

and upon substitution from (2.1), (2.4) and (2.5), we get

$$\rho_N = \frac{2(N - 1)}{\pi d} + \frac{1}{\pi d^2} \sum_{k=1}^{N} \left[ \frac{\ell}{k^2d^2 - \ell^2/4 + 2\ell^2 d^2 \sin^{-1} \frac{\ell}{2kd}} \right].$$

Certain intuitive properties about the formula (2.6) can easily be established. For instance, when $\ell = d$, $\rho_N$ is independent of $\ell$ and $d$.

Hence, when the length of the needle is the same as the distance between concentric circles, the probability of a "cross" depends upon neither. One would expect that as the length of the needle gets smaller and smaller (holding $d$ constant), the probability of a "cross" would become small. Indeed, it can easily be seen from (2.6) that $\lim_{\ell \to 0} \rho_N = 0$. One would expect that as $d$ becomes large, $\rho_N$ would become small. In fact, $\lim_{d \to 0} \rho_N = 0$.

Because the curvature of a circle approaches zero as the radius of the circle extends to infinity (so that the circumference of the circle becomes more like a straight line as the radius increases), one would expect $P(C_k | I_k^1)$ and $P(C_k | I_k^2)$ to approach $2\ell/d$, the probability associated with the classical Buffon Needle Problem. As can easily be determined from (2.1) and (2.4),

$$P(C_k | I_k^1) + (2\ell/d)^k$$

With this in mind, one can show from (2.6) (using the fact that (2.6) represents a convex combination of $P(C_k | I_k^1)$ and $P(C_k | I_k^2)$) that

$$\rho_N := \lim_{d \to 0} \rho_N = 2\ell/d,$$

so that for an infinite number of concentric circles on an infinite plane region, the probability of a "cross" is the same as for parallel lines. In Table I, values of $P(C_N | I_N^1)$, $P(C_N | I_N^2)$ and $\rho_N$ are computed for the case $\ell = d$, illustrating convergence of each of these terms to $2\ell/d = 2\ell/77$. 

| $N$ | $P(C_N | I_N^1)$ | $P(C_N | I_N^2)$ | $\rho_N$ |
|-----|----------------|----------------|--------|
| 5   | .66901         | .59917         | .63415 |
| 10  | .65267         | .61941         | .63583 |
| 25  | .64301         | .63006         | .63645 |
| 50  | .63981         | .63339         | .63657 |
| 75  | .63874         | .63448         | .63659 |
| 100 | .63821         | .63502         | .63661 |
| 200 | .63742         | .63582         | .63662 |

Table I
REFERENCES


GRAFFITO

After Cauchy's convergence criterion is covered, advanced calculus texts such as [1] and [4] often include a version of the following exercise: If \( a_1 \) and \( a_2 \) are arbitrary real numbers, and if \( a_n = \frac{a_{n-1} + a_{n-2}}{2} \) for \( n > 2 \), show that the sequence \( \{a_n\} \) converges.

The standard proof consists of showing that \( |a_n - a_{n-1}| < |a_{n-1} - a_{n-2}|/2 \) for \( n > 2 \), and then using some form of Theorem 1 below. Following the suggestion in Exercise 7b, p. 107 of [3], we consider the extension of this result to a recursive definition in terms of weighted averages of the previous \( k \) terms of the sequence. The proof of this result depends on the following theorem, which can be found in the first chapter of [1].

**Theorem 1.** Let \( A > 0 \) and \( b \) be in \((0,1)\). If \( |k_{n+1} - a_n| < A/n \) for \( n \) sufficiently large, then \( \{a_n\} \) is a Cauchy sequence, and hence converges.

**Theorem 2.** Let \( k \geq 2 \), and let \( w_1, \ldots, w_k \) be in \((0,1)\) satisfying the condition \( \sum w_i = 1 \). If \( a_j, \ldots, a_k \) are arbitrary real numbers, and if \( a_n = \sum w_i a_{n-i} \) for \( n > k \), then \( \{a_n\} \) converges.

**Proof.** For \( n > k \) define \( r_n = \max \{|a_{n-1} - a_{n-j}|; 0 \leq j < k-1\} \). Then for \( n > k \) and \( i = 1, \ldots, k-1 \),

\[
|a_{n+1} - a_{n+1-i}| = \left|\sum w_j (a_{n+1-j} - a_{n+1-j'})\right| \\
\leq \sum w_j |a_{n+1-j} - a_{n+1-j'}| \\
\leq r_n w_j, \text{ with } j \neq i \\
= (1 - w_j) r_n.
\]

If \( w = \max \{1 - w_i; i = 1, \ldots, k-1\} \) then we have that

George Louis Leclerc, Comte de Buffon (1707 - 1788) French naturalist
\[(1) \quad |a_{n+1} - a_{n+1-i}| \leq w_n^i\]

for \(n > k\) and \(i = 1, \ldots, k-1\). Therefore, for \(n > k\), we get
\[(2) \quad r_{n+1} = \max \{ |a_{n+1-i} - a_{n+1-i}| : 0 \leq i < j \leq k-1 \} \leq r_n^i\]

For large values of \(n\), repeated applications of (1) and (2) yield
\[(3) \quad r_n \leq w_n^{(k-1)}\]

For any \(n\), the Division Algorithm implies that there are (unique) non-negative integers \(q\) and \(s\) such that \(n = (k-1)q + s\), with \(0 \leq s < k-1\). Hence, if we use (3) repeatedly for sufficiently large \(n\), we obtain
\[r_n \leq w_n^{(k-1)} \leq w_{n-2}(2k-1) \leq \cdots \leq w^{s-2}_{s-2}(k-1)^s < \cdots \leq w^{s-2}_{s-2}(k-1)^s < w_n^s.\]

If we let \(x = \max \{ r_{2s-(k-1)+s}/(k-1) : 0 \leq s \leq k-2 \}\) and we define \(w = x_{2s-(k-1)+s}/(k-1)\), then \(0 < w < 1\) and, for large \(n\), \(r_n \leq w_n^s\).

Finally, from condition (1), we have that \(|a_n - a_{n+i}| \leq (w^i)/w^n\). The conclusion then follows from Theorem 1.

As an interesting consequence, the result is also true if the weighted arithmetic mean is replaced by the weighted geometric mean. Also, we note that Theorem 2 is valid for sequences in any Banach space, and that the condition on the weights can be relaxed to \(1/w_n < 1\).

**Corollary.** Let \(k \geq 2\) and let \(w_1, \ldots, w_k\) be in \((0,1]\) satisfying the condition that \(\sum w_k = 1\). If \(a_1, \ldots, a_k\) are arbitrary positive numbers, and for all \(n > k\) we have \(a_n = \prod w_k a_k\), then \(a_n\) converges.

**Proof.** For all \(n\), define \(b_n = \prod w_k a_n\). Since \(\{b_n\}\) satisfies the conditions of Theorem 2, it is a convergent sequence. The convergence of \(\{a_n\}\) follows from the continuity of the function \(e^x\).

In the same exercise set of [3], it is further suggested that the arithmetic mean \((a + b)/2\) may be replaced by the harmonic mean \(2/(1/a + 1/b)\). As a project, the reader might wish to consider this problem, as well as those using other generalized means, such as the logarithmic mean \((b - a)/(\ln b - \ln a)\) or the root-mean-square \(\sqrt{(a^2 + b^2)/2}\). As a note of interest, in [2] it is shown that when \(0 < a < b\), we have the following ordering (from smallest to largest):

**REFERENCES**


**1987 NATIONAL PI MU EPSILON MEETING**

The Annual Pi Mu Epsilon National Meeting will be at the University of Utah, Salt Lake City, from Wednesday, August 5, through Saturday, August 8, concurrently with the Joint Summer Meetings of the AMS and MAA.

Student paper presenters and student delegates (non-presenters) are needed. Talks are to be fifteen minutes in length and may include any area of mathematics or its application. Talks may be on either the expository level or on the research level; both are encouraged. Mathematical topics in computing are also welcome.

Each chapter is eligible to apply for air travel support up to a (chapter) total of six hundred dollars ($600) for students presenting papers or up to a (chapter) total of three hundred dollars ($300) for delegates (non-presenters).

Registration for Pi Mu Epsilon and the Joint meetings is $16. In addition to the student contributed paper sessions, activities will include the J. Sutherland Frame Lecture, Banquet, and parties. Lodging will be in a block of rooms in a dorm on the University of Utah campus; rates for doubles will be $8.50 per person per night.

Contact your chapter advisor for detailed information, registration forms and the "Information and Helpful Hints" sheets.
TREE, BRANCH AND ROOT
by Norman Wao
California State University

**Definition 1:** A set of integers \( \{b_i\} \) is called a base for the set of all integers whenever every integer \( n \) can be expressed uniquely in the form
\[
n = \sum_{i=1}^{m} b_i d_i \quad \text{where} \quad e_i = 0 \text{ or } 1 \quad \text{and} \quad \sum_{i=1}^{m} e_i < \infty.
\]

**Theorem 1.** Any base can, by rearrangement, be written in the form \( \{d_1, 2d_2, 2^2 d_3, 2^3 d_4, \ldots \} \) where the \( d_i \)'s are all odd.

*Proof.* This is proved in reference [1]. Note that a sequence \( \{d_1, 2d_2, 2^2 d_3, 2^3 d_4, \ldots \} \) of odd numbers is called a basic sequence whenever \( \{d_1, 2d_2, 2^2 d_3, 2^3 d_4, \ldots \} \) is a base.

**Theorem 2.** Any integer \( x \) can be formally developed into a series
\[
x = \sum_{i=1}^{\infty} e_i d_i \quad \text{where} \quad e_i = 0 \text{ or } 1 \quad \text{and the} \quad d_i \text{'s} \quad \text{are all odd}.
\]

*Proof.* If \( x \) is odd, set \( e_1 = 1 \). If \( x \) is even, set \( e_1 = 0 \). Set
\[
x_1 = \frac{x - e_1 d_1}{2}.
\]
If \( x_1 \) is odd, set \( e_2 = 1 \). If \( x_1 \) is even, set \( e_2 = 0 \). Set
\[
x_2 = \frac{x_1 - e_2 d_2}{2}.
\]
In general, set \( x_i = \frac{x_{i-1} - e_i d_i}{2} \). Set \( e_i = 1 \) if \( x_{i-1} \) is odd and set \( e_i = 0 \) if \( x_{i-1} \) is even.

At the \( k \)th stage, we have the following:
\[
x_1 = \frac{x - e_1 d_1}{2} \quad \text{and} \quad x = 2x_1 + e_1 d_1
\]
\[
x_2 = \frac{x_1 - e_2 d_2}{2} \quad \text{and} \quad x_1 = 2x_2 + e_2 d_2
\]
\[
\vdots
\]

Thus, \( x = \frac{x_{k-1} - e_k d_k}{2} \)
\[
= e_1 d_1 + e_2 2d_2 + e_3 2^2 d_3 + \ldots + e_k 2^{k-1} d_k + 2^k x_k.
\]

Therefore, \( x - \sum_{i=1}^{k} e_i 2^{i-1} d_i = 2^k x_k \equiv 0 \pmod{2^k} \).

Note that all \( e_i \)'s are uniquely determined.

Suppose \( x = \sum_{i=1}^{\infty} e_i 2^{i-1} d_i = \sum_{i=1}^{\infty} e_i 2^{i-1} d_i \).

Then, \( x - e_1 d_1 \equiv x - e_1' d_1 \equiv 0 \pmod{2} \)
\[
e_1 \text{ or } e_1' \pmod{2}.
\]

Since \( \{d_1, d_2\} = 1 \), \( e_1 \equiv e_1' \pmod{2} \) and \( e_1 = e_1' \).

We can use an induction argument to show that \( e_n = e_n' \) for all \( n \).

Of course, if \( \{d_1, 2d_2, 2^2 d_3, 2^3 d_4, \ldots \} \) is a base, then \( \sum_{i=1}^{\infty} e_i 2^{i-1} d_i < \infty \) for any \( x \). It is not difficult to show that \( \{1, -2, 2, -2^2, 2^3, \ldots \} \) is a base or that \( \{1, -1, 1, -1, \ldots \} \) is a basic sequence. In the following example, we will use 14 and express it using \( \{1, -1, 1, -1, \ldots \} \) as our basic sequence.

\[
x_1 = \frac{x - e_1 d_1}{2} = 14 - (0)(1) = 7
\]
\[
x_2 = \frac{x_1 - e_2 d_2}{2} = 7 - (1)(-1) = 8
\]
\[
x_3 = \frac{x_2 - e_3 d_3}{2} = 4 - (0)(1) = 2
\]
\[
x_4 = \frac{x_3 - e_4 d_4}{2} = 2 - (0)(-1) = 1
\]
\[
x_5 = \frac{x_4 - e_5 d_5}{2} = 1 - (1)(1) = 0.
\]

\( 14 = e_1 d_1 + e_2 2d_2 + e_3 2^2 d_3 + e_4 2^3 d_4 + e_5 2^4 d_5 + 2^5 x_k \)
\[ = (0)(1) + (1)(z^1)(-1) + (0)(z^2)(1) + (0)(z^3)(-1) + (1)(z^4)(1) + (z^5)(0) \]
\[ = 0 - 2 + 0 + 0 + 18 + 0 \]
Thus, we have a series of mappings
\[ 14 + 7 + 4 + 2 + 1 + 0 \]
\[ 14 + x_1 + x_2 + x_3 + x_4 + 0 \]
Below is illustrated a series of mappings for the set of integers \{1, 2, 3, \ldots, 50\} using the basic sequence \{1, -1, 1, -1, \ldots\}. It is interesting to observe that this series of mappings forms a \textit{tree}. Each mapping is a \textit{branch}. Each series of \textit{mappings} stops at 0, the \textit{root} of the tree.

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### References


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### YOU CAN’T HURRY LOVE

**OR**

**HOW MUCH MORE MUST ART WAIT?**

by Joseph S. Verducci*

The Ohio State University

1. **INTRODUCTION**

Meet Arthur, a young American male about to fall in love for the first time. Arthur has promised his mother that he will not marry the first girl who bewitches him, but being a romantic fellow he does intend to marry and live happily ever after with the first \textit{amorata} with whom he experiences a quality of love greater than that in his first affair. How many affairs should Arthur expect to have before he finds his intended spouse?

Let us assume that \( X_i \) measures the quality of Arthur's \( i^{th} \) love affair, \( i = 1, 2, \) and so on. Considering the vagaries of love, we might assume that these \( X_i \) are independent, identically distributed random variables from some distribution \( P \). Assuming that Arthur maintains his strategy, he will first propose marriage on his \( T^{th} \) love affair, where \( T \) is the smallest number \( n (= 2, 3, \) \textit{etc.} \) such that \( X_n > X_1 \). What is the expected value of \( T \)?

Four different solutions to this problem are given in the next section. They correspond to different assumptions about the distribution \( P \), the degree of Arthur's knowledge, and, most intriguingly, Arthur's personal attitude toward the future. The last solution raises paradoxical interpretations of expectation that are briefly discussed in Section 3.

*The author thanks David Fairly and Dennis Pearl for enjoyable conversations about this topic.
2. SOLUTIONS

The classical solution found in most introductory textbooks (see, for example, Feller, 1971) takes only a line to present:

\[ E(T) = \sum_{n=1}^{\infty} P(T > n) = \sum_{n=1}^{\infty} (1/n) = \infty, \]

but disturbs some students' intuition, not to mention Arthur's heart. How could it possibly take forever to find a better love? One way this could happen is if the first love affair happened to be the best possible one. For example, if \( F \) were concentrated on the points \( \{k_1, \ldots, k_n \} \) and \( X_1 = k_1 \) then there would be no chance of surpassing the first love affair, and consequently \( T = 1 \). Since this event has positive probability in this example, \( E(T) = \infty \).

2.1 A Merciful Solution

We all know that the quality of love, like mercy, is unbounded, so that \( P(x) < 1 \) for any \( x \). Suppose that \( X_i = x_i \), some arbitrary, but fixed amount. Then

\[ P(T > n) = P(\max(X_{1, \ldots, n}) < x) = [P(x)]^{n-1} \quad (2.1) \]

because the \( \{X_i\} \) are independent. Notice that the classical solution above does not fully use the fact that the \( \{X_i\} \) are independent; for example, the \( \{X_i\} \) might only be exchangeable. In contrast to the classical solution, equation (2.1) implies that

\[ E(T) = \sum_{n=1}^{\infty} P(T > n) = \sum_{n=1}^{\infty} [P(x)]^{n-1} = [1 - P(x)]^{n-1} < \infty, \quad (2.2) \]

because \( P(x) < 1 \) no matter how good the first time was. This new solution is certainly heartening to Art, but to what do we owe this promise of convergence? -- the unlimited nature of love, or some darker assumptions?

2.2 The Devil Laughs

The above analysis works well if Arthur happens to know the quality \( x \) of his first affair. Remember, even if Arthur were finished with this affair, which has yet to happen, he is a romantic and would loathe the crass task of actually measuring the quality of the affair. According to Arthur, "I am confident that I will recognize when a new affair is better than my first, but I could never put a number or a price on any love affair."

A more careful look at equation (2.2) shows that it gives not \( E(T) \) but the conditional expectation \( E[T|X_1 = x] \). To get \( E(T) \), we must integrate as follows:

\[ E(T) = \int E[T|X_1 = x] dP(x) = \int [1 - P(x)]^{-1} dP(x) \]

\[ = \lim_{M \to \infty} \{\log[1 - P(x)] - \log[1 - P(x)] \} = \infty. \]

Therefore, the assumption that \( F \) is unbounded is not, by itself, sufficient to guarantee a finite waiting time.

Having integrated all his knowledge of love, Arthur finds he does not know very much after all. Forever is a long time to wait. Perhaps Arthur should make the devil's bargain, and spend his time measuring the quality of his first love affair, once it is over. In this way, at least he would be spared the despair of expecting to wait forever for a better love. But Arthur is a lover, not a thinker, and so he writes for advice to someone wiser about the prospects of eternity.

2.3 Dear Abbe

The following is reprinted without permission from a short-lived local column entitled Advice to Lovelorn Students of Statistics:

"Dear Arthur,

No sense vegetating. Plan to have 1 love affair this year, 3 love affairs next year, 5 love affairs the year after that, with progressively odd affairs in all the ensuing years. Then see how your expectations change.

Best wishes,

The Abbe"
It is also an instance of Riemann's zeta function $\zeta(s)$. In fact, if Arthur becomes very aggressive and plans to have $\ell$ affairs after $n$ years, for some large $p$, then $E(Z) = \zeta(p)$, which approaches 1 as $p$ increases.

3. THE PARADOX

If he adopts the Abbe's plan, Arthur expects to wait less than two years to find the right woman, during which time he will have had no more than 4 affairs. Does it then follow that he expects to have no more than 4 affairs?

The last analysis seems to contradict the first and third analyses. In these earlier analyses, nothing was assumed about time schedules, so why should the imposition of a time schedule change the results?

The situation is analogous to the fable of Achilles and the tortoise, sometimes called Zeno's paradox. After giving the slow tortoise a head start in a race, the swift Achilles was never able to pass the tortoise because he had to first accomplish the infinite sequence of events of halving the gap between him and the tortoise. Ordinary athletes, on the other hand, have no trouble performing an infinite sequence of such fleeting events in a finite amount of time. However, most ordinary men would find themselves overtaxed by the Abbe's suggested agenda, unless these love affairs themselves became almost as fleeting.

If an ordinary mortal can have no more than a finite number $N$ of love affairs in his life, then, under the given assumptions about the qualities of these affairs, there is a positive probability $N^{-1}$ that $X_j$ will be largest among the $\{X_1, \ldots, X_N\}$. In this case Arthur will never find a suitable bride, and, in effect, will wait forever. In this sense he must expect to wait forever. You can't hurry love, Arthur.

REFERENCES


Note that as \( n \) increases without bound,
\[
\lim_{n \to \infty} \frac{1}{n} = \lim_{n \to \infty} \left( 1 - \left( 1 - \frac{1}{n} \right)^n \right) = 1 - \frac{1}{e}
\]
and thus we can write a question for which the answer is \( 1 - \frac{1}{e} \).

**Problem 2.** An urn contains \( n \) balls numbered from 1 through \( n \). From the urn \( n \) balls are selected one at a time without replacement. A match occurs if ball numbered \( k \) is the \( k \text{th} \) ball selected. (Note that this generates a random permutation of the first \( n \) positive integers.) Find the probability of at least one match, say \( p_n \).

Let \( A \) and \( B \) denote two identical decks of cards, each deck containing \( n \) cards numbered from 1 through \( n \). Shuffle each deck. A match occurs if card numbered \( k \), \( 1 \leq k \leq n \), occupies the same position in each deck. Find the probability of at least one match, say \( p \).

**Solution (when \( n = 4 \)).** Let the event \( A \) denote a match on the \( i \)th draw. Then
\[
P(A_i) = \frac{3!}{4!}
\]
\[
P(A_i \cap A_j) = \frac{2!}{4!}
\]
\[
P(A_i \cap A_j \cap A_k) = \frac{1!}{4!}
\]
\[
P(A_1 \cap A_2 \cap A_3 \cap A_4) = \frac{1}{4!}
\]
The probability of at least one match is
\[
P(A_1 \cup A_2 \cup A_3 \cup A_4) = \sum P(A_i) - \sum P(A_i \cap A_j) + \sum P(A_i \cap A_j \cap A_k) - P(A_1 \cap A_2 \cap A_3 \cap A_4)
\]
\[
= 4 \left( \frac{3!}{4!} \right) - 6 \left( \frac{2!}{4!} \right) + 4 \left( \frac{1!}{4!} \right) - \frac{1}{4!}
\]
\[
= \frac{4!}{4!} - \frac{4!}{4!} + \frac{4!}{4!} + \frac{4!}{4!} - \frac{1}{4!}
\]
\[
= 1 - \frac{1}{4!} + \frac{1}{4!} - \frac{1}{4!}
\]
The solution for any integer \( n \) is
\[
P_n = P(\text{at least one match})
\]
\[
= 1 - \frac{1}{n!} + \frac{1}{n!} - \frac{1}{n!} + \ldots + \left( -\frac{1}{n!} \right)^{n+1}
\]
As \( n \) increases without bound,
\[
\lim_{n \to \infty} p_n = 1 - \left( 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} + \ldots \right) = 1 - \frac{1}{e}
\]
so we can write a second question for which the answer is \( 1 - \frac{1}{e} \).

It is interesting to compare the values of \( p_n \) and \( q_n \) for different \( n \) values of \( n \). We can construct the following table.

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</tr>
<tr>
<td>13</td>
<td>0.63212</td>
<td>0.64474</td>
</tr>
<tr>
<td>14</td>
<td>0.63212</td>
<td>0.64486</td>
</tr>
<tr>
<td>15</td>
<td>0.63212</td>
<td>0.64474</td>
</tr>
</tbody>
</table>

Recall that
\[
1 - \frac{1}{e} \approx 0.63212
\]
We see that the value of \( n \) has little affect on \( p_n \) when \( n > 4 \) and on \( q_n \) when \( n > 15 \). Also for large \( n \), \( p_n \) and \( q_n \) are approximately equal.

An interesting exercise would be to illustrate these probabilities empirically. Either use dice, cards, or balls in an urn or, write a program to simulate these experiments on a computer.
The accent on the a in Turán is not a stress mark; all Hungarian words are stressed on the first syllable. The word syllable should remind one of this. Thus, Doráti, Dochanvay, Kodály, Bartok, Erdős, Radó, Gabor, Fejes-Tóth, Selesremedí.

The distinguished Hungarian mathematician Paul Turán (1910-1976) was active in many fields. One of his most famous works was "A New Method in Analysis," which originally appeared in 1953. An English edition was published by Wiley-Interscience in the 1970's, I believe. (A bibliography of Turán's work appeared in Mat. Lapok (Hungarian) in the June 1977 issue, but some details are missing.) This famous work is an example, but by no means an isolated example, of a significant contribution of a middle-aged or older mathematician.

Turán had interests outside mathematics. Several times, he came to my house to hear string quartet music. He was charmed by some work of Arriaga, a Spanish musician who died at the age of 20 and was not well-known in Hungary. Later, Turán purchased some recordings of the quartets to take home to Hungary.

During the 1966 Moscow Congress I lunches with Turán, his wife, and his wife's sister. I wish I had been able to commit to film the antics of one of the translator-assistants during that luncheon.

Turán showed his knowledge of the Russian Language by ordering the food for all of us: one of the blue-plate special number one and two of the blue-plate special number two. The women were overprotective of their weight, and agreed to share a single order. We then gave the waitress, a business-like middle-aged woman, our meal chits. Shortly afterwards, she returned to the table, waving the chits in her hand. "You don't have enough chits," she remonstrated. This set Paul Turán off into a mixture of English and Russian. The waitress interrupted him.

'I'll go fetch a translator," she said (all in Russian). There was a Latin-looking translator, tall and with a Don Quixote mustache, not far away. He was the one she fetched. They returned together - the Latin undoubtedly in the position of a foreign student, studying in a Russian university, and partly repaying his hosts by offering the courtesy of entertaining what Spanish mathematicians there were by interpreting Russian for them.

When the pair arrived at our table, Turán overlooked the large badge labeled "Español" on the man's lapel, and began explaining, in English, what it was we wanted. This dismayed the Spaniard so greatly that all he could do was put out his tongue, attempting to articulate, but unable to say a word. He pointed repeatedly to his badge, and, began to look like a comedian. The impasse could only be resolved when one of us explained our needs in good Russian to the waitress, who understood perfectly. There is a Russian word, BMECTE, that was very handy to explain that one blue plate special number two was to be shared by the women. It is also a convenient fact that the Russian word for "women" is the same as the Russian word for "wives", so it was irrelevant whether I was or was not married to Mrs. Turán's sister. (She was, and is, actually the wife of the Hungarian finance minister.) So the food was brought serenely.

As the meal was finishing, there was another joke, a quite good one, in view of the fact that English "let alone American slang" is not Turán's native language. His wife, Vera Són, got up abruptly and said, "I have to run." "Yes, you are right," Paul Turán said, "you have Turán!"

He even knew how his name was often mispronounced!

A SUMMATION FORMULA

by John Schue, Macalester College

Let S be the sum \( 1 + 2 + \cdots + (n-1) \). Then \( S \) is the number of elements below the main diagonal of an \( n \times n \) matrix and \( 2S \) is the total number of elements off the main diagonal. Since there are \( n \) diagonal entries we then have \( 2S = n^2 - n \) so \( S = n(n - 1)/2 \).

EXTENDING A FAMILIAR LIMIT

by Norman Schaumberger
Bronx Community College

In a recent issue of this journal [1] we used the mean value theorem to prove the well-known formula

\[(1) \lim_{n \to \infty} \frac{(n!)^{1/n}}{n} = \frac{1}{e}.\]

In this note we prove more generally that if \(a\) and \(b\) are integers with \(a > b\), and \(a + b > 0\), then

\[(2) \lim_{n \to \infty} \frac{1}{n} \left[ (a+b)(2a+b)(3a+b) \ldots (na+b) \right]^{1/n} = \frac{a+b}{e}.\]

Thus, for example, if \(a = 4\) and \(b = -1\), we get

\[\lim_{n \to \infty} \frac{1}{n} \left[ (4-1)(2+1)(3+1) \right]^{1/n} = \frac{3}{e}.\]

Our proof is completely elementary and can be offered immediately after (1). For another approach, see [2].

Since \(a > b\) and \(a + b > 0\) it follows that for all positive integers \(k\), \((k-1)a < (k+1)a\). Furthermore, since \(a\) and \(b\) are integers \(a + b > 1\), hence

\[1 \cdot 2 \cdot a \ldots \left( n - 1 \right)a < \left( a+b \right)(2a+b) \ldots \left( na+b \right) < 2a \cdot 3a \ldots \left( n+1 \right)a.\]

This double inequality can be written as

\[\frac{a^{n+1}}{a^{n-1}} < (a+b)(2a+b) \ldots (na+b) < a^{n+1}(n+1).\]

or

\[(3) \frac{a^{n+1/n}}{a^{1/n}} < \left[ (a+b)(2a+b) \ldots (na+b) \right]^{1/n} < a^{(n+1)/n}.\]

As \(n \to \infty\), \(a^{1/n} \to 1\), \((n+1)/n \to 2\) and \(a^{1/n} \to a\). (2) now follows from (3) by letting \(n \to \infty\) and using (1).

REFERENCES


THE FOURTEENTH ANNUAL Pi Mu Epsilon STUDENT CONFERENCE-,

AT

MIAMI UNIVERSITY

IN

OXFORD, OHIO

OCTOBER 9-10, 1987

WE INVITE YOU TO JOIN US! THERE WILL BE SESSIONS OF THE student conference on FRIDAY EVENING AND SATURDAY AFTERNOON. FREE OVERNIGHT LODGING FOR all STUDENTS WILL BE ARRANGED WITH MIAMI STUDENTS. EACH STUDENT SHOULD BRING A SLEEPING BAG. ALL STUDENT GUESTS are invited to a free FRIDAY EVENING PIZZA PARTY SUPPER, and SPEAKERS WILL BE TREATED TO A SATURDAY NOON PICNIC LUNCH. TALKS MAY BE ON ANY TOPIC RELATED TO MATHEMATICS, STATISTICS OR COMPUTING. WE WELCOME ITEMS RANGING FROM EXPOSITORY TO RESEARCH, INTERESTING APPLICATIONS, PROBLEMS, SUMMER EMPLOYMENT, etc.

PRESENTATION TIME SHOULD BE FIFTEEN OR THIRTY MINUTES.

WE NEED YOUR TITLE, PRESENTATION TIME (15 OR 30 MINUTES), PREFERRED DATE (FRIDAY OR SATURDAY) AND A 50 (APPROXIMATELY) WORD ABSTRACT BY OCTOBER 1, 1987.

PLEASE SEND TO

PROFESSOR MILTON D. COX

DEPARTMENT OF MATHEMATICS AND STATISTICS

MIAMI UNIVERSITY

OXFORD, OHIO 45056

THE STUDENT CONFERENCE IS HELD IN CONJUNCTION WITH THE CONFERENCE on "COMPUTERS AND MATHEMATICS" WHICH BEGINS FRIDAY AFTERNOON, OCTOBER 9. FEATURED SPEAKERS INCLUDE A. K. Dewdney, Anthony Ralston and Robert Tarjan. CONTACT US FOR MORE DETAILS.
PUZZLE SECTION

Edited by

Joseph D. E. Konhauser

The PUZZLE SECTION is for the enjoyment of those readers who are addicted to working on recreational puzzles or who find an occasional mathematical puzzle attractive. We consider mathematical puzzles to be problems whose solutions consist of answers immediately recognizable as correct by simple observation and requiring little formal proof. Material submitted and not used here will be sent to the Problem Editor if deemed appropriate for the PROBLEM DEPARTMENT.

Address all proposed puzzles and puzzle solutions to Professor Joseph D. E. Konhauser, Mathematics and Computer Science Department, Macalester College, St. Paul, Minnesota 55105. Deadlines for puzzles appearing in the Fall Issue will be the next February 15, and for puzzles appearing in the Spring Issue will be the next September 15.

PUZZLES FOR SOLUTION

1. Proposed by the Editor.

The numbers in the sequence 1, 2, 4, 8, 16, 32, 64, ... satisfy the recurrence relation

\[ a_{n+2} = a_{n+1} + 2a_n, \quad n = 1, 2, 3, \ldots \]

Are you able to find an integer pair \((p, q)\) different from \((1, 2)\) such that

\[ a_{n+2} = pa_{n+1} + qa_n, \quad n = 1, 2, 3, \ldots \]

2. Proposed by the Editor.

Using only a straightedge (unmarked ruler) locate the centroid of the L-shaped region in the sketch.

3. Proposed by the Editor.

Note that

\[ 1 + 4 = 2 + 3, \]

and that

\[ 1 + 4 + 6 + 7 = 2 + 3 + 5 + 8, \]

Fill the blanks with the numbers 1 through 16 so that equality holds for \(n = 1, 2\) and 3.

\[
\begin{align*}
(1^n + (1^n + (1^n + (1^n + (1^n + (1^n + (1^n + (1^n + (1^n + (1^n + (1^n + (1^n + (1^n + (1^n + (1^n + (1^n + (1^n + (1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1^n + 1
\end{align*}
\]

4. Proposed by the Editor.

If the four vertices of a rectangle lie on a square (one vertex on each side of the square) is it possible to move the rectangle so that all four vertices of the rectangle become interior points of the square?

5. Proposed by the Editor.

The 25 squares of a 5×5 array have been colored red, white and blue. Reading the rows from left to right and the columns from top to bottom, the color arrangements in nine of the rows and columns are

\[
\begin{array}{ccccccc}
R & R & B & W & W & W & W \\
R & B & R & R & R & R & R \\
R & B & W & R & R & R & R \\
W & B & W & R & R & R & R \\
W & B & B & R & R & R & R \\
W & B & R & W & R & R & R \\
B & W & W & R & R & R & R \\
B & W & R & B & R & R & R \\
\end{array}
\]

What is the remaining color arrangement?
Seven readers responded to Puzzle \#4, the labelling of a square array of sixteen points so that all fourteen squares with horizontal and vertical sides have equal "vertex sums." All seven gave at least one example and stated that there were many more. With some machine help, John M. Howell found 384 solutions. One is (below left)

\[
\begin{pmatrix}
1 & 8 & 10 & 15 \\
12 & 13 & 3 & 6 \\
7 & 2 & 16 & 9 \\
14 & 11 & 5 & 4
\end{pmatrix}
\]

The magic constant is, of course, 34. Doug Desbski, Jason Pinkney and Glen E. Mills submitted equivalent solutions to the pentagon dissection puzzle (above right).

List of responders: Doug Desbski (4,5), Mark Evans (1), Victor G. Feser (1,2,4), James Hansen (1), John M. Howell (4), Joseph Jackson (2), Edmund F. Marks, Jr. (1,2,4), Glen E. Mills (1,3,4,5), John D. Moores (1,2,3,4), Jason Pinkney (1,5), Robert Prielipp (2), John H. Scott (1,3,3,4).

Solution to Mathacrostic No. 23, (See Fall 1986 Issue).

Words:

A. J J Sylvester  
B. earless  
C. fractal  
D. floccule  
E. wrangler  
F. en cabochon  
G. entropy  
H. klystron  
I. scuttlebutt  
J. top banana  
K. Heine-Borel  
L. eversion  
M. synclinal  
N. harmonic tetrad  
O. alligation  
P. Peaucellier cell  
Q. Eulerian path  
R. overboard  
S. Fields medal  
T. serendipity  
U. petty-pat  
V. antorse  
W. caboodle  
X. environ

Quotation: Orientability is an intrinsic property ... of a surface ... Topologists on a projective plane couldn't tell locally that they weren't on a sphere, but they could tell globally because a projective plane is nonorientable, and long-distance travellers can come back mirror-reversed.

Solved by: Jeanette Bickley, Webster Groves High School, MO; Victor G. Feser, University of Mary, Bismarck, ND; Robert Forsberg, Lexington, MA; Dr. Theodor Kaufman, Winthrop-University Hospital, Mineola, NY; Henry S. Lieberman, John Hancock Mutual Life Insurance Co., Boston, MA; Charlotte Maines, Caldwell, NJ; Don Pfaff, University of Nevada, Reno, NY; Robert Prielipp (with help from Pat Collier and John Oman), University of Wisconsin-Oshkosh; Stephanie Sloyan, Georgian Court College, Lakewood, NJ; Jeffrey Weeks and Nadia Marano, Ithaca College, Ithaca, NY; and Barbara Zeeberg, Denver, CO.

Late solution: A late solution to Mathacrostic No. 22 was received from Barbara Zeeberg, Denver, CO.
Definitions

A. in the Ptolemaic system the circle around the earth in which a celestial body or the center of the epicycle of its orbit was thought to move

8. growing at high elevations but not above the timber line

C. a Texas unit equal to 33.33 inches

D. stamp

E. the art or technique of cutting solids

G. leafse (2 wds.)

H. completely and determinedly fixed (comp.)

I. formerly a chief source of ammonia

J. orbits

K. balanced; complete

L. sought attention by ostentatious behavior

M. vehement or vigorous; violent or rough

N. a polygon whose interior consists of all the points in the plane which are nearer to a particular lattice point than to any other lattice point (2 wds.)

O. obliquely or downward to one side

P. Viggo Brun’s prime counter

Q. one who asserts that only those mathematical objects have real existence and are meaningful which can be explicitly exhibited

R. onetime antispasmodic

S. branching

T. an unexpected or malicious side-effect of a program that otherwise operates correctly (2 wds.)

U. in music, a passage between statements of a main subject or theme

V. the catastrophe which can be interpreted as a split or furrow

W. in this kind of geometry any two straight lines meet in two points (2 wds.)

X. abounding

Y. uniformly in action or intensity

Z. of or relating to the summer

a. a kind of tiling in which all the tiles are the same size and shape

Word

203 137 60 174 14 95 163 106
150 171 3 142 186 55 125 91 41 200
169 197 43 138
39 222 173 33 121 99 54
128 206 132 234 166 5 103 114 209 154
111 97 184 225 123 7 55 125 91 41 200
139 18 178 113 110 40 22 7 58 187 232 201 37
88 118 190 107 9 4 76
196 44 229 27 2 86 214 109 119
223 101 89 62 130 179 237 81
211 104 124 45 180 204 167
73 189 207 143 177 219 12 68 117
136 155 36 188 82
175 1 94 230 120 165 152 47 182 102 32 70 92
74 191 129 87 176 160 13 66
146 56 127 193 50
251 172 220 156 64 67 28 147 85 71 31 20 140 164
69 198 28 105 15 133 191 221 145
38 208 213 158 26 170
157 199 77 24 98 122 49 228 115 217 57
149 226 126 84 39 215 192
21 147 80 148 195 210 61 218 96 11 183
205 235 52 8 233 134 168 75 116 162 100 23 83 46
79 79 15 10 108 225 207
212 51 34 16 6 112 153
63 216 72 35 65 90 187 177
78 42 131 30 227 101 159 143 185 123

Words

203 137 60 174 14 95 163 106
128 206 132 234 166 5 103 114 209 154
111 97 184 225 123 7 55 125 91 41 200
139 18 178 113 110 40 22 7 58 187 232 201 37
88 118 190 107 9 4 76
196 44 229 27 2 86 214 109 119
223 101 89 62 130 179 237 81
211 104 124 45 180 204 167
73 189 207 143 177 219 12 68 117
136 155 36 188 82
175 1 94 230 120 165 152 47 182 102 32 70 92
74 191 129 87 176 160 13 66
146 56 127 193 50
251 172 220 156 64 67 28 147 85 71 31 20 140 164
69 198 28 105 15 133 191 221 145
38 208 213 158 26 170
157 199 77 24 98 122 49 228 115 217 57
149 226 126 84 39 215 192
21 147 80 148 195 210 61 218 96 11 183
205 235 52 8 233 134 168 75 116 162 100 23 83 46
79 79 15 10 108 225 207
212 51 34 16 6 112 153
63 216 72 35 65 90 187 177
78 42 131 30 227 101 159 143 185 123

Methoc~ottic.

Proposed by Joseph V. E. Konhauser

The 237 letters to be entered in the numbered spaces in the grid will be identical to those in the 27 keyed Words at the matching numbers. The key numbers have been entered in the diagram to assist in constructing the solution. When completed, the initial letters of the Words will give the name(s) of the author(s) and the title of a book; the completed grid will be a quotation from that book.

The solution to Methoc~ottic No. 24 is given elsewhere in the PUZZLE SECTION.
This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

All communications should be addressed to C. W. Dodge, Math. Dept., University of Maine, Orono, ME 04469. Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly identified with name and address. Solutions to problems in this issue should be mailed by December 15, 1987.

Problems for Solution

639. Proposed by Charles W. Trigg, San Diego, California.
Find the smallest SLICE the KNIFE can cut from the CAKE if CAKE + KNIFE = SLICE.

*640. Proposed by John M. Howell, Littlerock, California.
Find the largest value of \( S(n) \) and the limit of \( S(n) \) as \( n \to \infty \) if
\[
S(n) = \sum_{x=1}^{n-1} \frac{n}{x}.
\]

Let \( f_1 = f_2 = 1 \) and \( f_{k+2} = f_k + f_{k+1} \) for \( k > 0 \) define the Fibonacci sequence. It is known that \( f_k f_{k+1} = \sum_{i=1}^{k} f_i^2 \) for any positive integer \( k \). Find a similar formula for the generalized Fibonacci sequence \( g_k \), where \( g_1 \) through \( g_n \) are given and for \( k > 0 \)
\[
g_k = \sum_{i=1}^{k} g_{k-i+1}.
\]
642. Proposed by Dmitry P. Mavlo, Moscow, USSR.
Let \( \pi, \mu, \epsilon > 0 \). Prove that
\[
\frac{1}{\pi} \left( \frac{1}{\mu + \epsilon} + \frac{1}{\mu + \epsilon^2} + \frac{1}{\mu + \epsilon^3} \right) \geq 3,
\]
with equality if and only if \( \pi = \mu = \epsilon = 1 \).

643. Proposed by M. S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.
If \( a, b, c, d > 0 \), prove that
\[
\left( \frac{a}{b} \right)^2 - \left( \frac{b}{c} \right)^2 + \left( \frac{c}{d} \right)^2 - \left( \frac{d}{a} \right)^2 \geq 1.
\]

In the figure below prove that regions A and B have equal areas.

645. Proposed by Dmitry P. Mavlo, Moscow, USSR.
Let \( M \) be an arbitrary point on segment CD of trapezoid ABCD having sides AD and BC parallel. Let \( S, S_1, \) and \( S_2 \) be the areas of triangles ABM, BCM, and ADM respectively. Prove that
\[
S \geq 2 \min \{ S_1, S_2 \}.
\]

646. Proposed by Dick Field, Santa Monica, California.
Find the smallest \( k \) for which there is only one \( k \)-digit palindrome that is the square of an integer.

647. Proposed by Robert C. Gebhardt, Hopatcong, New Jersey.
For each positive integer \( n \) find the earliest row of Pascal's triangle in which the first \( n \) terms have the property that each term after the first is an integral multiple of its predecessor.

If \( A, B, C \) are the angles of a triangle \( ABC \), prove that
\[
\sum \cos \frac{A}{3} = \frac{5}{2} \cos \frac{B}{3} \cos \frac{C}{3}.
\]

649. Proposed by Edward J. Arismendi, Jr., California State University, Long Beach, California.
How far beyond the edge of a table can a deck of cards be stacked without the pile falling off the table?

The 1980 Wimbleton final between Borg and McEnroe involved a tiebreak game that went to 18-16. Given that the server has a 70% chance of winning the point, what is the probability that the two players reach a 16-16 tie in a tiebreak game? (In a tiebreak game the first player serves one point. Thereafter players alternate serving 2 points each. The first player reaching 7 or more points with an advantage of 2 or more points wins the game.)

651. Proposed by At Terenzo, Malden, Massachusetts.
Professor E. P. Umbagio has recently been strutting around because he hit upon the solution of the fourth degree equation which results when the radicals are eliminated from the equation
\[
x = \left( \frac{2}{3} + \frac{1}{3} \right)^{1/2} + \left( \frac{1}{3} - \frac{1}{3} \right)^{1/2}.
\]
Deflate the professor by solving it using nothing higher than quadratic equations. [From Robinson's Mathematical Recreations. 1851.]

Solutions

Use a bit of number theory to solve this alphametic that pays homage to geometry, algebra and analysis. Find that solution in base 7 yielding a prime ANAL.

In base 7 there are 23 primes that fit the form ANAL with \( A \) at least 2 \( 2021, 2326, 2623, \ldots, 6562 \) (corresponding to the numbers 701, 853, 997, \( \ldots \), 2347 in base ten). Ten of these primes can be

An additional 12 primes are eliminated quickly since $G = A - 1$ determines $E = 0$, and $M$. For example, if $ANAL = 2326$, then $G = 1$, so $M = 1$. Now $O = 3$ which contradicts $H = 3$. We have the unique solution

$$4216 \div 534 = 5053.$$  

II. Solution by Frank P. Battles, Massachusetts Maritime Academy, Buzzards Bay, Massachusetts.

It is intuitively obvious to the most casual observer that

$$4216 + 534 = 5053$$

is the unique solution to this base 7 alphametric.

III. Comment by Elizabeth Andy, New Limerick, Maine.

A brilliant professor named Battles

With problem solutions just rattles.

His craftsmanship "obvious"

Yields work for the mobs of us

We only can think in small prattles.

Also solved by CHARLES ASHBACHER, Mount Mercy College, Cedar Rapids, IA, JAMES E. CAMPBELL, University of Missouri, Columbia, VICTOR G. FESER, University of Mary, Bismarck, ND, RICHARD I. HESS, Rancho Palos Verdes, CA, JOHN M. HOWELL, Little Rock, CA, GLEN E. MILLS, Valencia Junior College, Orlando, FL, JOHN H. SCOTT, Macalaster College, Saint Paul, MN, and the PROPOSER.

614. [Spring 1986] Proposed by Leon Bankoff, Los Angeles, California, and the editor.

A 10000-meter section of straight railroad track expands 1 meter and buckles into a circular arc. How high above ground is the middle of the arc? [This is an old problem and easy to solve using ordinary trigonometry. It is repeated here because the answer is so surprisingly large.]

I. Solution by Wade H. Sherard, Furman University, Greenville, South Carolina.

Assuming that the track was originally a straight line KB with midpoint $M$, let $O$ be the center of the circular arc it now forms, let $\mathbf{OM}$ cut the arc at $S$, and let $x = OM$, $h = OS$, and $y = OA = OB = OS$.

Let $\theta = \angle BOM = \angle SOB$. Then we have

$$r = x + h, \quad x = 5000 \cot \theta, \quad y = 5000, \quad \text{and} \quad \theta = 5000 \cot \theta.$$  

Therefore $(5000)(\theta) = (5000.5) \sin \theta$. An application of Newton's method to this equation yields $0.0245$ radians. Hence

$$h = r - x = 5000 \sin 0.0245 \approx 61.23 \text{ meters}.$$  

This result appears to be much too large and may be due to the approximations made. We therefore proceed with a more exact solution.

Let $R_e (= 6.371 \times 10^6 \text{ m})$ denote the earth's radius. Prior to buckling we place an xy-coordinate system at the center of the earth so that the coordinates of the ends of the track are at $(t^2x_0', y_0')$ and the center of the track is at $(0, y_0 + h_0)$. Let $\theta_0$ be the half angle subtended by the track at the center. Then we have

$$R_e \theta_0 = 5000, \quad x_0 = R_e \sin \theta_0, \quad y_0 = R_e \cos \theta_0, \quad \text{and} \quad h_0 = R_e^2 (1 - \cos \theta_0).$$  

Then $\theta_0 \approx 0.0007848$ radians and $h_0 \approx 1.96 \text{ m}$.

After the track has buckled we place an xy-coordinate system at the center of the circle defined by the track so that the coordinates of the ends of the track are at $(t^2x_1', y_1')$ and the center of the track is at $(0, y_1 + h_1)$. Let $\theta_1$ be the half angle subtended by the track at $(0, 0)$. Then we have

$$x_1 = R_1 \sin \theta_1 = R \sin \theta_1 \quad \text{and} \quad R_1 \theta_1 = 5000 \cdot 5.$$  

Eliminating $\theta$ we get

$$R_1 \sin \left( \frac{5000}{R_1} \right) = R \sin \theta_0 = 4999.999487.$$  

By approximate means we obtain that $R_1 \approx 204060 \text{ m}$ and $\theta_1 \approx 0.0245$.
radians. Then \( h_2 = R_x (1 - \cos \theta_x) \approx 61.3 \text{ m} \). Finally
\[
h = h_2 - h_0 \approx 59 \text{ m}
\]
to the nearest meter.

III. Comment by John H. Scott, Macalester College, Saint Paul, Minnesota.

This result is shocking. I was all set for a tiny number. I still find it hard to believe. I would like to see the spikes that could hold the two ends in place.

IV. Comment by Jack Garfunkel, Flushing, New York.

Another dramatic way of illustrating that intuition cannot be trusted is the following. Imagine a giant standing somewhere in space and encircling the earth at the equator with a huge rope. He then lengths the rope by just 40 feet and this slack is distributed equally around the globe, creating a distance between the earth and the globe. What is this distance? Although intuition indicates that 40 feet distributed over 25,000 miles would produce an infinitesimal distance, elementary circle geometry shows it to be over 6 feet, more than enough for an average man to walk underneath.


Although several years in retirement, Professor Euclide Pasqualetto Umbagio still practices mathematics with his usual prowess and efficiency. His native country, Guayaquil, still cannot afford a computer, but they do have a pocket four-function calculator to which he has occasional access. His latest project is to find the sum of the abscissas of the points of intersection of the seventh-degree polynomial
\[
f(x) = x^7 - 3x^6 - 13x^5 + 55x^4 - 36x^3 - 52x^2 + 48x
\]
with its derivative polynomial. So far he has laboriously found one of the intersections at \( x = 1.3177227 \). Help the kindly, old professor to find his sum without resorting to a computer.

Solution by Henry S. Lieberman, Waban, Massachusetts.

The eminent numerologist should subtract the derivative of \( f(x) \) from \( f(x) \) and take the negative of the coefficient of \( x^6 \) as the required sum. Since the derivative of \( f(x) \) is
\[
f'(x) = 7x^6 - 18x^5 - 65x^4 + 230x^3 - 108x^2 - 104x + 48,
\]
the coefficient of \( x^6 \) in \( f(x) - f'(x) \) is -10. Thus the sum of the abscissas of the seven points of intersection is 10.

Also solved by James E. Campbell, University of Missouri, Columbia, Russell Euler, Northwest Missouri State University, Maryville, George P. Evanovich, Edward Williams College—Fairleigh Dickinson University, Hackensack, NJ, Robert C. Gebhardt, Hopatcong, NJ, Richard H. Hess, Rancho Palos Verdes, CA, Glen E. Mills, Colonial Senior High, Orlando, FL, Laura L. Kelleher and Frank P. Battles, Massachusetts Maritime Academy, Buzzards Bay, Ralph King, St. Bonaventure University, NY, John H. Scott, Macalester College, Saint Pout, MN, Harry Sedinger, St. Bonaventure University, NY, Kenneth M. Wilke, Topeka, KS, and the PROPOSER. Gebhardt noted that \( x - 2 \) is a factor of the expression \( f(x) - f'(x) \).

616. [Spring 1986] Proposed by Dimitry P. Vavlo, Moscow, U.S.S.R.

Prove that in any triangle
\[
\frac{\tan A + \tan B + \tan C}{\cot A + \cot B + \cot C} = \frac{s}{a} + \frac{s}{b} + \frac{s}{c}
\]
with equality if and only if the triangle is equilateral.

Solution by Jack Garfunkel, Flushing, New York.

Let \( R, r, \) and \( s \) denote the triangle’s circumradius, inradius, and semiperimeter. We use the known identities
\[
\sum \tan \frac{A}{2} = \frac{4R + r}{2} \quad \text{and} \quad \sum \cot \frac{A}{2} = a.
\]
Then we have to show that
\[
\frac{4R + r}{2} \cdot \frac{r}{s} \leq \frac{s}{a} + \frac{r^2}{s} + \frac{r}{s},
\]
which simplifies to \( 2r^2 \geq 27Rr \), a known inequality: see O. Bottema.
**Geometric Inequalities**, page 52, item 5.12. The proof given there utilizes the inequality \((a + b + c)^3 > 27abc\), which in turn is readily proved by the arithmetic mean–geometric mean inequality, showing that equality holds if and only if \(a = b = c\), that is, if and only if the triangle is equilateral.

Also solved by BARRY BRUNSON, Western Kentucky University, Bowling Green, HENRY S. LIEBERMAN, Waban, MA, BOB Prielipp, University of Wisconsin-Oshkosh, JIN H. SCOTT, Macalester College, Saint Paul, MN, and the PROPOSER.


It is known (The Two-Year College Mathematics Journal, problem 226, September 1982, page 277) that a \(7 \times 7 \times 7\) box can be packed with a maximum of forty \(1 \times 2 \times 4\) bricks, requiring 23 cubic units of unoccupied space. How many such bricks can be packed into a \(5 \times 5 \times 5\) cubic box?

I. Partial solution by VICTOR G. FESER, University of Mary, Bismarck, North Dakota.

The volume of the box is 125 cubic units; of the brick, 8. Therefore the absolute maximum is 15 bricks. It is easy to get 14: the bricks readily stand on edge or on end to fill all but one corner cube in each of the first four layers. Then two bricks can be laid flat in the top layer leaving another 9 unfilled unit cubes. Can one get 15? It seems so, since otherwise the problem would be a bit anticlimactic.

II. Solution by the proposer.

It is easy to pack 14 bricks into the box and, if we are allowed to cut just two bricks into two \(1 \times 1 \times 4\) subbricks, the 15th brick is readily inserted, along with another subbrick, for 15 1/2 bricks. To best follow the proof that it is impossible to pack 15 bricks into the box, it is suggested that you obtain or make a set of \(1 \times 2 \times 4\) blocks and a \(5 \times 5 \times 5\) cardboard box. (I used an ordinary board and cut 16 blocks \(11/16 \times 11/8 \times 11/4\) inches, an easy size to work with.)

Number each unit cube in each layer as shown above. Then any brick with faces parallel to the sides of the box fills exactly one cube of each number. Now there are 18 cubes numbered 1, 17 numbered 3, and 15 each numbered 2, 4, 5, 6, 7, and 8. To be able to pack 15 bricks into the box, every space numbered 2, 4, 5, 6, 7, and 8 must be occupied. In any level, only a 1 or a 3 can be uncovered. Since each brick covers an even number of cubes in each level, one cube must be uncovered. Since this last statement must be true no matter which face of the box is taken as the base, it follows that there must be exactly one empty cube in each level, one in each row, and one in each column. That is, the same row or column cannot contain an empty cube in each of two different levels. We write \(1AR\) to denote the unit cube in the level 2 that occupies row \(A\) and column \(R\).

In level 1, since a \(1_{-}\) or a \(3\)-cube must be uncovered, then the empty cube must be in row \(A\), \(C\), or \(E\) and column \(P\), \(R\), or \(T\). Without loss of generality we need consider only the three cases that \(1AP\), \(1AR\), or \(1CT\) is empty. At least one brick must lie flat in level 1 in order that the empty cube in level 2 be different from that in level 1. Clearly no more than two bricks can lie flat in any level.

Case 1. Suppose \(1AP\) is empty. If bricks on edge cover \(1AQ-1AT\) and \(1BP-1ET\), then \(2AP\) must be empty, a contradiction since \(2AP\) is numbered 7 and not 1 or 3. Clearly one brick on edge or on end must cover a cube abutting \(1AP\) and we may take bricks on edge covering \(1BP-1ET\) and \(1AQ-1ET\). Flat bricks cover the rest of level 1. Now at least one brick on end or on edge must cover \(2AP\) and there is room for only one brick flat in level 2. In any case, no matter which of \(2BQ, 2BS, 2DQ, \text{ or } 2DS\) is empty at least one additional-brick on end must stand next to the empty cube, forcing a flat brick or one on
edge above that empty cube and lying in level \( J \). Then there is a block of cubes in level \( J \) that must be covered and are less than 4 units long. Hence they can be covered only by bricks on end, which then protrude out of the top of the box. 'Having just one flat brick in level \( I \) only compounds the difficulty.' Hence case 1 cannot occur with 15 bricks in the box.

Case 2. If 1CR is empty, then at least two bricks on end must abut that empty cube. We may suppose that a flat brick lies in level \( I \) covering 1BP-1BR. Then one or two bricks on end cover 1DR-1BR. Also 1AR-1BR are similarly covered. If 2BQ or 2DQ is empty, then a flat brick cannot cover the rest of the region from 2BP-2BR, so we are forced to use bricks on end or on edge. Thus cube 3BQ or 3DQ must be empty, also. But then we cannot have 15 bricks in the box. A similar situation occurs if 2BS or 2DS is left empty.

Case 3. Finally, take 1AR empty. If bricks on end cover 1AP-1AQ and 1AS-1AT and bricks on end or on edge cover 1BR-1BR, then both levels 1 and 2 have cube AS empty. If bricks on end cover 1AP-1AQ and 1AS-1AT and a brick on edge covers 1BP-1BT, then a brick on end or on edge covers 2AR-2BR. Then 2BQ or 2DQ or 2DS empty necessitates the corresponding cube empty in level \( J \). If 2BS is empty, a flat brick can be placed to cover 2CT-2DS, but 2BP-2BR must be covered by a brick on edge or bricks on end. Then it is impossible to cover cube 1BS.

If bricks on end cover 1AP-1AQ and 1ES-1ET and bricks on end or a brick on edge covers 1BR-1BR, then a brick on end must cover 2AS-3AT and a brick on edge or 2 bricks on end cover 2AR-3BT. If 2BQ or 2DQ is empty, then the same cube in level \( J \) must be empty. If 2BS or 2DS is empty, then the other cube and 2CS must be covered by a brick on end. A flat brick can cover 2BP-2EQ and a brick on edge can cover 3DP-3DS or 3BP-3BS, but the remaining cubes in level \( J \) cannot be covered.

Let bricks on end cover 1AP-1AQ and 1ES-1ET and a brick on edge or two bricks on end cover 1BP-1EP. If 2BQ is to be empty, then a brick that is flat or on edge must cover 3BQ-3BT or 3BQ-3EQ. In the former situation there is no way to cover the three cubes 2BR-2BT and in the latter we have the same difficulty with 2BQ-2EQ. If 2DQ is to be empty, a similar situation occurs. Now a flat brick can cover 2BQ-2ER.

Then, whichever of 2BS or 2DS is left empty, the same cube in level \( J \) must be uncovered.

Our proof is complete. We have shown that no matter which allowable cube is empty in level \( I \), then it is impossible to leave an allowable cube empty in level 2 and also in level 3 and to fill all other cubes in those levels. It appears that the same situation occurs if we use \( I \times 2 \times 2 \) bricks, but I have not pursued that case.

III. Comment by Elizabeth Andy, New Limerick, Maine.

When a problem is anticlimactic,

With a proof that is antidramactic,

Then the better disposal

Of such a proposal

Is to relegate it to an anticactic.

Packings of 14 bricks were also given by Richard I. Hess, Rancho Palos Verdes, CA, and John H. Scott, Macalester College, St. Paul, MN. Scott argued that one cube must be empty 'in each level, no matter, which side the box rests on, but this argument guarantees only five empty cubes, if they are (or can be) properly placed.


(i) Find when the sum of the squares of four consecutive integers is divisible by 3.

(ii) Repeat part (i) for the sum of the squares of four consecutive odd or four consecutive even integers.

Solution by David E. Penny, The University of Georgia, Athens, Georgia.

We generalize by finding when the sum of the squares of \( p + 1 \) integers in arithmetic progression is divisible by the prime \( p \).

For \( p = 2 \) or \( p = 3 \) it is when the first term and common difference are both divisible by \( p \) or neither is divisible by \( p \). For \( p > 3 \), it is when \( p \) divides the first term in the progression.

Proof. Suppose \( p \) is a prime, \( p \) and \( d \) are integers, and \( d > 0 \). The arithmetic progression with first term \( n \), difference \( d \), and containing \( p + 1 \) terms has the sum of squares
\[ S = n^2 + (n + d)^2 + (n + 2d)^2 + \ldots + (n + pd)^2 \]
\[ = (p + 1)n^2 + 2d(n + 2 + p) + d^2(1^2 + 2^2 + \ldots + p^2) \]
\[ = (p + 1)n^2 + dnp(p + 1) + \frac{p(p + 1)(2p + 1)d^2}{6} \]
\[ = n^2 + \frac{p(p + 1)(2p + 1)d^2}{6} \quad (\text{mod} \, p). \]

If \( p \geq 3 \), then 6 is a divisor of \((p + 1)(2p + 1)\), so \( S \equiv n^2 \quad (\text{mod} \, p) \). Consequently, \( p \) will be a divisor of \( S \) exactly when \( p \) divides \( n \). If \( p = 2 \) or \( p = 3 \), then \( S \equiv n^2 - d^2 \quad (\text{mod} \, p) \). Hence \( p \) divides \( S \) if and only if \( p \) divides both \( n \) and \( d \) or neither \( n \) nor \( d \).

In particular, the sum of the squares of four consecutive integers, or of four consecutive odd or even integers, is divisible by 3 if and only if the first of the four integers is not divisible by 3.

Also solved by Phillip Abbott, Macalester College, Saint Paul, MN; Charles Ashbacher, Mount Mercy College, Cedar Rapids, IA; Frank P. Battles, Massachusetts Maritime Academy, Buzzards Bay; James E. Campbell, University of Missouri, Columbia; Russell Euler, Northwest Missouri State University, Maryville; Mark Evans, Louisville, KY; Victor G. Feser, University of Mary, Bismarck, ND; Robert C. Gebhardt, Hope College, NJ; Richard A. Gibbs, Fort Lewis College, Durango, CO; Richard H. Hess, Rancho Palos Verdes, CA; Francis C. Leary, Saint Bonaventure University, NY; Henry S. Lieberman, Wayland, MA; Glen E. Mills, Valencia Junior College, Orlando, FL; Oxford Running Club, University of Mississippi, University; Bob Prielipp, University of Wisconsin-Oshkosh; John H. Scott, Macalester College, Saint Paul, MN; Harry Sedinger, St. Bonaventure University, NY; Kenneth M. Wilke, Topeka, KS, and the proposer.


Find the largest value of \( x \) such that \( x = \sin x = \tan x \), correct to 3, 4, 5, 6, 7, and 8 decimal places.

Composite of solutions submitted by Mark Evans, Louisville, Kentucky, and Russell Euler, Northwest Missouri State University, Maryville, Missouri.

For small positive \( x \) we have that

\[ \sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \ldots \]

and

\[ \tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots \]

so that

\[ \tan x - x = x \sin x > 0. \]

Thus we need only check \( x > x \sin x \) to ensure the correct accuracy. So for \( n \)-digit accuracy we set \( 0.5 \times 10^{-n} = x^3/3 \) and solve for \( x \). We obtain the following table.

<table>
<thead>
<tr>
<th>Decimal accuracy</th>
<th>Largest permissible ( x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.24</td>
</tr>
<tr>
<td>3</td>
<td>0.114</td>
</tr>
<tr>
<td>4</td>
<td>0.0531</td>
</tr>
<tr>
<td>5</td>
<td>0.02466</td>
</tr>
<tr>
<td>6</td>
<td>0.011446</td>
</tr>
<tr>
<td>7</td>
<td>0.0053132</td>
</tr>
<tr>
<td>8</td>
<td>0.00246621</td>
</tr>
</tbody>
</table>

Solutions were also submitted by Richard H. Hess, Rancho Palos Verdes, CA; Ralph King, St. Bonaventure University, NY; John H. Scott, Macalester College, Saint Paul, MN, and the proposer. Most of the solutions differed considerably from that printed above.

621. [Spring 1986] Proposed by R. S. Lutah, University of Wisconsin Center at Janesville, Wisconsin.

(i) Characterize all triangles whose angles and whose sides are both in arithmetic progression.

(ii) Characterize all triangles whose angles are in arithmetic progression and whose sides are in geometric progression.

I. Solution by Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin.

Let \( \triangle ABC \), with \( A \leq B \leq C \), be a triangle having its angles in arithmetic progression. Then \( a < b < c \). Because the angles are in arithmetic progression, then \( A + B + C = 180^\circ \). Hence \( B = 60^\circ \) since \( A + B + C = 180^\circ \).

(i) If the sides are in arithmetic progression, then \( b = (a + c)/2 \). Thus, by the law of cosines,
\[ a^2 + 2ac + c^2 = a^2 + c^2 - 8a \cos 60^\circ, \]

making \( a = c \). It follows that triangle \( ABC \) is equilateral.

(ii) If the sides are in geometric progression, then \( b^2 = ac \).

Again by the law of cosines we have

\[ a^2 = a^2 + c^2 - 2ac \cos 60^\circ \]

again making \( a = c \). Again triangle \( ABC \) must be equilateral.

II. Comment by John H. Scott, Macalester College, Saint Paul, Minnesota.

I suspect that the proposer is a humorist wondering how surprised people would be. In both cases the answer is all equilateral triangles with an arithmetic difference of zero for the angles and an arithmetic difference of zero and a ratio of one respectively for the sides.

Also solved by FRANK P. BATTLES and LAURA KELLEHER, Massachusetts Maritime Academy, Buzzards Bay, MA; JAMES E. CAMPBELL, University of Missouri, Columbia, RICHARD I. HESS, Rancho Palos Verdes, CA; JOHN M. HOWELL, Littlerock, CA; OXFORD RUNNING CLUB, University of Mississippi, University, JOHN H. SCOTT, Macalester College, Saint Paul, MN; KENNETH H. WILKE, [2 solutions], Topeka, KS, and the PROPOSER.


Let point \( P \) be the center of an equilateral triangle \( ABC \) and let \( Q \) be any circle centered at \( P \) and lying entirely within the triangle. Let \( BR \) and \( CS \) be tangents to the circle such that point \( R \) is closer to \( C \) than to \( A \) and \( S \) is closer to \( A \) than to \( B \). Prove that line \( RS \) bisects side \( BC \).

Solution by William E. Hoff, Princeton, West Virginia.

The theorem is true more generally for point \( P \) the circumcenter of any given triangle \( ABC \), so that is what we assume here. Let \( \psi \) be the circumcircle of triangle \( ABC \). Let \( \theta \) be the angle such that the rotation about \( P \) through angle \( \theta \) maps \( B \) to \( B' \) and \( C \) to \( C' \), so that \( B' \) lies on \( BR \). Then \( C' \) lies on \( PS \). Let \( X \) and \( X' \) be the midpoints of \( BC \) and \( B'C' \). Then angle \( \angle FX' \) equals \( \theta \). Now the central dilation mapping \( B'C' \) to \( RS \) also maps \( Q \) to \( M \), the point where \( RS \) meets \( PQ \). We have

\[ PX = PX' = PQ \cos \theta = PQ \frac{PR}{PF} = PQ \frac{PR}{PF} = PQ \frac{FM}{FQ} = FM. \]

Thus \( X = M \), so \( RS \) bisects \( BC \).

Also solved by RICHARD A. GIBBS, Fort Lewis College, Durango, CO; RICHARD I. HESS, Rancho Palos Verdes, CA; LAURA L. KELLEHER and FRANK P. BATTLES, Massachusetts Maritime Academy, Buzzard's Bay, RALPH KING, St. Bonaventure University, NY, JOHN H. SCOTT, Macalester College, Saint Paul, MN; LASZLO ZUKECS, Fort Lewis College, Durango, CO, and the PROPOSER.


A 30-foot ladder and a longer ladder are crossed in an alley. The longer one breaks just 20 feet from its foot and the top falls back to the other side of the alley and just touches the top of the 30-foot ladder. If the ladders cross just 10 feet above the ground, find the original length of the longer ladder. (This variation of the old "crossed ladders" problem cost an aircraft company thousands of dollars in lost time during World War II by engineers and other technical people trying to solve it. I finally circulated a solution that probably saved the company thousands more, but alas, I received no credit for it.)

![Diagram](image_url)
20p, FA = 20q, FE = 30p, and FB = 30q. From the Pythagorean theorem and the similarity of triangles ABD and FCD we find that
\[
\frac{BC}{CD} \cdot \frac{20q}{20p} = \frac{600q^2 - 100}{400p^2 - 100},
\]
which reduces to
\[
(p + q)^2 = (p - q)^2 + 4pq
\]
since \(p + q = 1\). We let \(r = p - q\) and \(s = pq\), so that
\[
(p + q)^2 = (p - q)^2 + 4pq
\]
or
\[
1 = r^2 + 4s.
\]
Now equation (1) becomes \(5s^2 = r\) which when inserted into equation (2) yields
\[
25s^4 + 4s = 1.
\]
The only positive root, found by Newton's method, is \(s = 0.2310189\), whence \(r = 0.2689319\), \(p = 0.634466\), and \(q = 0.365534\). Now from the Pythagorean theorem we get \(BC = 4.5003994\) and \(CD = 7.8114558\), so \(BD = 12.311855\). Then \(ED = 27.357233\) and \(AB = 15.761289\). Finally we have
\[
\frac{AE^2}{(ED - AB)^2} + BD^2, \quad \text{so} \quad AE = 16.912945
\]
and the original ladder length was 36.912945 ft \(\approx 36 \text{ ft 11 in.}

II. Solution by Ralph King, St. Bonaventure University, St. Bonaventure, New York.

In the figure above let \(a = BC\), \(b = BD\), \(a = DE\), and \(d = AB\). Then
\[
\frac{d}{b} = \frac{10}{b - a} \quad \text{and} \quad \frac{a}{b} = \frac{10}{a}, \quad \text{so} \quad a = \frac{10d}{d - 10}.
\]
Also \(b^2 + a^2 = 10^2\) and \(b^2 + d^2 = 20^2\), so then \(a^2 - d^2 = 500\). Eliminate \(a\) to get
\[
d^4 - 500d^2 + 5000d^2 - 10000d + 50000 = 0,
\]
which has real roots \(d_1 = 7.008882\) and \(d_2 = 15.761287\). Since \(d > 10\), then \(d = 15.761287\), so \(a = 27.357233\), \(b = 12.311857\), and \(x = 16.912947\). Thus the original ladder length was 36.912947 feet.

Also solved by GREG DUKEMAN, Tuscola, IL, MARK EVANS, Louisville, KY, RICHARD R. HESS, Rancho Palos Verdes, CA, JOHN H. SCOTT, Macalester College, Saint Paul, MN, WADE H. SHERARD, Furman University, Greenville, NC, and the PROPOSER.


It is known and easy to prove that
\[
\sum_{i=1}^{n} i! = (n + 1)! - 1.
\]
Find a closed expression for \(S(n)\) and prove that for \(n > 1\), \(S(n)\) is divisible by 3 where
\[
S(n) = \sum_{i=1}^{n} i! = 1! + 2! + 3! + \ldots + n!.
\]

Solution by Frank E. Battles and Laura L. Kelleher, Massachusetts Maritime Academy, Buzzards Bay, Massachusetts.

That \(S(n)\) is divisible by 3 for \(n > 1\) follows readily since \(1! + 2! = 3\) and each succeeding term is divisible by 3.

A "closed form" expression can be obtained from the integral definition of \(i!\), i.e., the gamma function \(i! = \int_0^\infty x^i e^{-x} \, dx\). Thus
\[
S(n) = \int_0^\infty \sum_{i=1}^{n} i x^i e^{-x} \, dx = \int_0^\infty (\frac{e^x - 1}{x}) e^{-x} \, dx.
\]

Proofs of the divisibility by 3 were also submitted by VICTOR G. FESER, University of Mary, Bismarck, ND, RICHARD R. HESS, Rancho Palos Verdes, CA, JOHN M. HOWELL, Littleton, CA, BOB PRIELIPP, University of Wisconsin-Oshkosh, and the PROPOSER.

625. [Spring 1986] Proposed by Sam Pearsall, Loyola Marymount University, Los Angeles, California.

Let \(G\) be a group in which there is a unique element \(x\) such that generates a cyclic subgroup of order 2. Show that \(x\) commutes with every element of \(G\).

II. Solution by Stephanie Dumiski, Occidental College, Los Angeles, California.

Consider any \(g \in G\). We know that the order of \(g^{-1}xg\) equals that of \(x\) since conjugates have the same order. Thus \(g^{-1}xg = x\) since \(x \neq e\) (where \(e\) is the group identity element) and \(x\) is unique. Hence \(xg = gx\), so \(x\) commutes with every element of \(G\).

II. Solution by Harry Sedinger, St. Bonaventure University, St.
Bonaventure, N Y o k .

Let \( a \) be in \( G \) and consider \( b = a x a^{-1} \). Then

\[
b^2 = a x a^{-1} a x a^{-1} = a x^2 a^{-1} = a x - a,
\]

so either \( b = a \) or \( b = x \). If \( b = a \), then \( a x a^{-1} = e \), which implies that \( x = a^{-1} a = e \), a contradiction. Thus \( b = x \), so \( a = a x a^{-1} \) or \( a x = x a \). Since \( a \) was arbitrary, \( x \) commutes with every element of \( G \).

Also solved by CHARLES ASHBACHER, Mount Mercy College, Cedar Rapids, IA, RICHARD A. GIBBS, Fout Lewis College, Durango, CO, FRANCIS C. LEARY, Saint Bonaventure University, NY, HENRY S. LIEBERMAN, Babson, MA, MASSACHUSETTS GAMMA. Bridgewater State College, OXFORD RUNNING CLUB, University of Mississippi, University, BOB PRIELIPP, University of Wisconsin-Oshkosh, ARTHUR H. SIMONSON, Eastern Texas State University at East Texas, PHILLIP J. SLOAN, Pembroke State University, Statesville, NC, and the PROPOSER. One incorrect solution was also received.

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**LETTERS TO THE EDITOR**

Dear Editor:

(Here) is a better solution to Puzzle #1 in the Spring 1986 issue.

*What is the closest value to 22/7 that you can obtain by using the usual arithmetic symbols and the digits 1, 2, 3, 4 and 5 in that order both from left to right and from right to left?*

**Answer:** \[
\left\lfloor \frac{5! + \sqrt{4}}{3!} \right\rfloor + 2! = 66/21 = 22/7, \text{ and}
\]
\[
\left\lfloor \frac{-1 + 23}{\sqrt{4} + 5} \right\rfloor = 22/7, \text{ or}
\]
\[
\left\lfloor \left( \left\lfloor 0(1) \right\rfloor \right) + 0(4) \right\rfloor + \left\lfloor \left( \left\lfloor 3 \times 2 + 1 \right\rfloor \right) + \left\lfloor 0\left(23\right) \right\rfloor + \left\lfloor 0(4) + 5 \right\rfloor \right\rfloor = 22/7.
\]

**Notes:** \( \left\lfloor 3 \times 2 + 1 \right\rfloor \) can be replaced by \( 3! + 2 - 1 \); \( 0(4) \) can be replaced by \( A \) (and vice-versa).

Edmund F. Marks, Jr.
Massachusetts Delta
University of Lowell

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Dear Editor:


The proof hinges upon consideration of the ratio \( x = BS/SC \). Now, this simply is \( AB/AC \) since "Each angle bisector divides the opposite side into segments proportional to the adjacent sides." The problem of showing \( AC > AB \) implies \( SC > BS \) is thereby settled.

Moreover, the reader will notice that if the function \( x(x) = BS/SC \) where \( x = AB \) is just the identity function \( x(x) = x \) in case \( AC = 1 \) as in the text.

The useful proposition above (which can be augmented) is proved, for example, in Coxeter and Greitzer, Geometry Revisited, WAD, MML, Vol. 19, page 9, using the law of sines: \( SC/\sin(\overrightarrow{A}/2) = AC/\sin(\overrightarrow{A}) \). BS/\sin(\overrightarrow{A}/2) = AB/\sin(\overrightarrow{A}) \). So, since the angles at \( S \) one supplementary, having equal bineb, \( SC/AC = BS/AB \) or \( BS/SC = AB/AC \).

For a proof that avoids, sines draw \( BD/SA \). Then \( SC/AC = BS/DA \) and \( 2BD = 2BA + 3AB = 2BA + 3AB = 4AB \) so that \( PA = AB \). Hence \( BS/SC = AB/AC \).

Or, thirdly, we may take at the area ratio \( A(AB)A(A) \). It equals the ratio of the bases \( BS/SC \) since the height from \( A \) is the same. Now turn over triangle \( ABS \), that is, make \( E \) such that \( EA = BS \). Then \( ES/AC = 3ASE = 3AC \). So this time the area ratio is \( ES/AC = AB/AC \). Whence the result.

Yours sincerely,

J. Suck
Rohmstrasse 140
04300 ESSEN 12
GLEANINGS FROM CHAPTER REPORTS

ARKANSAS BETA (Hendrix College). The chapter's tenth year (1985-86) was an active one. In April several students attended the Arkansas-Oklahoma MAA Meeting at Arkansas College, where Karen Billings, Jim Hart, Bruce Hulsey, and Travis Williams presented papers on their work in the Undergraduate Research Program. In May, Karen and Jim again presented theirs at the annual Hendrix-Rhodes-Sewanee Undergraduate Mathematics Symposium. In April, Bruce and Travis discussed their papers at the MAA conference on Undergraduate Mathematics at Purchase. At the AR-OK Section Meeting, the team of Gary Thacher, Pauline Bello and Jim Hart received an honorable mention in the Mathematical Competition in Modelling. At the Honors Convocation in May, the McHenry-Lane Freshman Mathematics Award was shared by George Fisher and Randy Pate. The Phillips Parker Undergraduate Research Award was given to Jim Hart. Speakers during the school year included Silke Hufnagel Allen (Hendrix) on "The Cantor Set," John Merrill (CCX Corporation) on "A Potted History of Real-Life Job Topics," Paul Fjeldstad (St. Olaf College) on "Investigating the Calculus," Dr. Joe Diestel (Kent State) on "Sets of Discontinuities," Dr. Joel Haack (Oklahoma State) on "Aspects of Operads," Dr. Ralph Scott on "Applications of Calculus in Economics," Dr. Robert Swan (University of Central Arkansas) on "Self Mappings of Polynomials," and Dr. William R. Ingram (University of Houston) on "Approximations of Functions."

CALIFORNIA LAMDA (University of California - Davis). Guest lecturer at the fall initiation was Professor Henry Alder who spoke on "How to Discover and Prove Theorems: A Demonstration with Partitions." At the winter initiation, Professor Kenneth Joy (Division of Computer Science) spoke on "Geometric Continuity." In May, Kim Mix (Department of Civil Engineering) spoke on "Earthquakes and Linear Algebra."

GEORGIA BETA (Georgia Institute of Technology). In June, the chapter presented a book award to Kamel Haddad. The award is given to students receiving the degree B.S. in Applied Mathematics with a grade point average of at least 3.7 in all mathematics courses taken.

KANSAS GAMMA (The Wichita State University). Activities during the 1985-86 school year included talks by S. Eley on "Notions of Stability and Dichotomy in Ordinary Differential Equations," P. G. Nahlkeck on "Connecting Mathematical Models with Experiments," and D. V. Chopra on "Some Aspects of Graph Theory." In December, a panel consisting of Dr. L. Arteaga, Dr. B. Friedman, and Dr. Paramasivam, with Dr. W. Perel as moderator, discussed mathematics in India, Spain and the Soviet Union.

MINNESOTA GAMMA (Macalester College). Chapter activities during 1985-86 included the showing of several films produced by the College Geometry Project and the following guest lecturers: Seymour Schuster on "Mathematics and Painting," Jeff Parker on "Constructing Binary Trees," Tom Myers on "What's Left of Computer Science if Automatic Program Generation Ever Works," Hsing Bin Zou on "Greatest Common Subgraphs," and Cherl Shabban on "Fractal Geometry." Loren Larson (St. Olaf College) was invited speaker for the annual initiation. He spoke on "A Discrete Look at 1 + 2 + ... + n." Social activities included fall and spring picnics and a game night.

MINNESOTA ZETA (Saint Mary's College). The chapter formulated the following resolution which has been forwarded to the national officers of Pi Mu Epsilon: "WHEREAS women as well as men are members of Pi Mu Epsilon BE IT RESOLVED THAT the Minnesota Zeta Chapter of Pi Mu Epsilon proposes that the name of the corporation be changed from the Pi Mu Epsilon Fraternity to the Pi Mu Epsilon Mathematics Honor Society or to the Pi Mu Epsilon Honor Society.

In March, Faculty Advisor Louis Guilleon spoke on "Integer Programming." At the initiation ceremony in April, Jay Flaherty presented his honors project "Riemann-Stieltjes Integrals." Also, in April, Tim Matecha reported on his honors project "Notherian Rings."

MISSOURI GAMMA (Maryville College and St. Louis University). The 1985-86 James E. Case Memorial Lecture was presented by Dr. William E. Perrault at the 49th annual initiation banquet. Dr. Perrault spoke on "The Missouri Lottery: Applications of Mathematics." The James W. Garneau Mathematics Award, for the outstanding senior in mathematics at SLU, went to John Swirsky. The Francis O. Began Graduate Service Award, for a first-year graduate student at SLU, went to Joel Haddad. The Missouri Gamma Undergraduate Award, for graduating seniors at Fontbonne, Lindenwood or Maryville Colleges, went to John Brown. At St. Louis University, the Missouri Gamma Graduate Award, for outstanding senior in mathematics at SLU, went to Teresa Hurley. The John J. Andrews Graduate Service Award, for a graduate student at SLU who took an active part in departmental affairs, went to Mohamad Azarian. The Beradino Family Fraternity Award, for outstanding participation in affairs of the fraternity, friendliness, and concern for its members, went to William Coplin.

NEBRASKA ALPHA (University of Nebraska - Lincoln). A freshman scholarship program has been created. Awards consist of cash prizes and/or gift certificates from the Nebraska Bookstore. Recipients in 1985-86 were Terry Clements, Allen George, Larry McConville and Maeric Morrell. Awards were given according to the results of a 25-question multiple-choice examination covering concepts up to those in second-year calculus. The Missouri Gamma Graduate Award, for a first-year graduate student at SLU, went to Teresa Hurley. The John J. Andrews Graduate Service Award, for a graduate student at SLU who took an active part in departmental affairs, went to Mohammad Azarian. The Beradino Family Fraternity Award, for outstanding participation in affairs of the fraternity, friendliness, and concern for its members, went to William Coplin.

NEW JERSEY DELTA (Seton Hall University). Weekly problem-solving sessions were led by Vh. John Masterson. Other activities during the 1985-86 year included a talk on "Actuarial Mathematics" by Gary Strunk...
A.S.A., a talk on "Artificial Intelligence" by Dr. David H. Opp, and the showing of the film "Pits, Peaks and Passages." At the 19th Annual Induction ceremony, Dr. John J. Saccoman spoke on "The History of Pi Mu Epsilon."

NEW YORK ALPHA BETA (LeMoyne College). Professor Norman J. Pullman (Queen's University) spoke on "Scheduling a Golf Tournament: An Application of Finite Geometry." The spring address "Cryptography: From Caesar Ciphers to Public-Key Cryptosystems" was given by Professor Dennis M. Luciano of Western New England College.

NEW YORK PHI (State University of New York at Potsdam). At the Fall Induction, Dr. Charles Mosier (Clarkson University) spoke on "The Group Technology Clustering Problem." The speaker at the Spring Induction was Dr. Guy Johnson, Jr. (Syracuse University) who spoke on "How to be a Halley Watcher." "Chapter students members Virginia Filici and William Martin gave talks at the Seaway Section Spring Meeting of the MA at Ithaca College. In April, Cheri Brunner and Career Services cosponsored the Second Annual Career Night Event, which featured six speakers representing careers in Operations Research (Mary Charles, BMO, Software Engineering (Teresa Kochbrooner, GE), Insurance (Gary Bissomette), Public School Teaching (Anthony Vaccaro), Banking (Susan Abbott) and Graduate Studies in Mathematics (Daryl Weatherly and Tim Jones). The Pi Mu Epsilon Senior Award was given to Susan L. Piotrak, a B.A./M.A. candidate and the ranking graduating mathematics major.

NORTH CAROLINA LAMDA (Wake Forest University). An excellent mix of student, faculty and visitor talks during the 1985-86 academic year included Dr. Rick Heatley, Office of Educational Planning and Placement, on "Employment Opportunities in Mathematics and Computer Science." Dr. Blaok on "The Institute for Retraining in Computer Science," student Mark Roberson on "The Fractional Calculus," Dr. Fred Howard on "Stirling Numbers," student Salvador R. Sonar on "Generalizing Fermat's Little Theorem," Dr. Walter Rudin on "Sets of Distances," student Muriel McLean on "Decimal Fractions that can be Represented in Terms of Fibonacci and Lucas Numbers," student Cynthia Bacou on "Formal Language Theory," Dr. John Frame on "The Funhouse Mirror," and student Helen Rogers on "The Ongoing Study of Continued Fractions."

OHIO DELTA (Miami University). In August, 1984, six students and Dr. Melton Cott, national president of Pi Mu Epsilon, attended the national meetings at the University of Oregon at Eugene. Two students presented papers: Leslie Youngdale spoke on "Complements - Mathematically Speaking," and Saguana Poppo on "Samuelson's Interaction Between the Accelerator and the Multiplier." The 116th Annual Student Conference in September included a record 29 student speakers, eight from Miami. Saguana and Leslie repeated their talks. Nick Short spoke on "Artificial Intelligence," David Cameron on "A Twenty-Five Point Geometry Reviewed." Carol Richard on "Spiromania!" Michael Heflin on "Fun with Sound."

The highlight of the year was the 12th Annual Pi Mu Epsilon Student Conference at Miami University. Greg Bishop spoke on "A Practical Estimate of Standard Deviation," Ken Bloch discussed "Orthogonality in Quantum Mechanics," Kelly Ann Chambers presented "Two Fundamental Results from Lattice Theory," Gary Johnson discussed "Basic Results from Group Theory," John Steenwalt explained how to "Escape the Markov Chains of Monte Carlo Jails."

At the chapter's annual banquet, Jeff Diller was presented the Sophomore Class Award of Excellence. In April, Mark Liatti presented "The Weighing of the Coins" at the MA meeting at John Carroll University. He was awarded a one-year membership in the MA.


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OHIO NU (University of Akron). Sam Selby Scholarship Awards in the amount $125 were given to John Keblesh and to Michael Quinn. Evan Wett, winner of the Mathematics Category in the Akron Public Schools Science Fair (First Place), was awarded a check for $50 by the chapter. Michael Quinn, Ron Zanoglia, and John Keblesh received one-year memberships in the American Mathematical Society. Roma Felli was awarded one-year memberships in the American Statistical Association.

OHIO XII (Youngstown State University). Six chapter members attended the 1985 summer meeting and presented "Larson Company, Insurance, Tony Clack, Terry Wiener and Henry Gable gave papers. At the Fall Quarter Initiation, Visiting Professor Andrzej Szymanski was guest speaker. At the Spring Quarter Initiation, graduate student Gene Santos and Dr. Eric..."
Wingler were guest speakers. At the Annual Spring Banquet, guest speaker was Dr. John J. Buoni. During the 1985-86 school year several students attended MAA Meetings at Miami University and at John Carroll University.

OHIO OMEGA (Mount Union College). In September, 1985, the chapter sponsored a trip to the Twelfth Annual Pi Mu Epsilon Conference at Miami University. In April, Joe Smith, a graduate of Mount Union, discussed the actuarial profession and the mathematics necessary for it. Also, in April, Chapter President, Mark Larson, summarized his mathematics independent study project.

PENNSYLVANIA NU (Edinboro University). At the Fall Initiation, Dr. Peter Weidner spoke on mathematical paradoxes. At the Spring Initiation a Professor James Watson spoke on difference equations. In April, a group of eight students attended the Annual Meeting of the Allegheny Mountain Section of the MAA at Clarion University.

PENNSYLVANIA OMEGA (Moravian College). The chapter sponsored a presentation on the actuarial profession by Etienne Livote and Judy Weaver of Mutual Benefit Life. Professor Johanna O'F, who retired at the end of the school year, gave a talk reflecting on her years at Moravian College. Permanent Faculty Correspondent and Chapter Advisor, Ross Gingrich, presented a talk on "Some Geometrical Algebra," the historical use of classical geometry to produce or prove algebraic results.

**IMPORTANT ANNOUNCEMENT**

Pi Mu Epsilon’s main source of steady income is the National Initiation Fee for new members. The fee covers the cost of a membership certificate and a one-year subscription to the Pi Mu Epsilon Journal.

For the past fourteen years the fee has been set at $4.00. Effective January 1, 1987, the National Initiation Fee will be $10.00. After January 1, 1987, any order for membership certificates should be accompanied by the new fee.