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PI MU EPSILON JOURNAL
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FRACTALS: MATHEMATICAL MONSTERS

by Jennifer Zobitz
College of St. Benedict

Which geometry is true? This general question has stumped mathematicians for centuries; the quest to find the one geometric theory that actually describes all of the physical world (if such a theory exists) has as its newest contender, fractal geometry. Originally an attempt to explain "pathological" (not well-behaved) functions, fractal geometry seems to describe common properties of most physical phenomena. This paper is intended as an introduction to the basic concepts of fractal geometry and several of its applications. Although fractal geometry may not be the geometric theory, it certainly appears to be the most effective means of taming the ultimate monster -- the universe.

"Big whorls have little whorls
which feed on their velocity
And little whorls have lesser whorls,
and so on to viscosity."

The above quote by Richardson in Steen's article [4, 123] sounds like a science fiction creation. It does describe a monster -- but of the mathematical variety. Self-similarity, which describes the concept of big whorls having little whorls having lesser whorls, is the basis for a relatively new geometry -- fractal geometry. In 1975, Benoit Mandelbrot coined the word "fractal" to describe the infinite irregularities and fragmentation in nature; hence, fractals were dubbed "a new geometry of nature" [2, 111]. For instance, using strictly Euclidean geometry, one experiences difficulty when measuring the surface area of a charcoal briquette. Upon initial inspection the charcoal appears smooth; magnification reveals that the surface is actually covered with a series of small depressions. At further magnification each depression yields more depressions. The greater the magnification, the greater the resulting surface area. In other words, different scales for Euclidean measure yield varying results. Hence inadequate Euclidean measurements cannot account for new detail revealed under increasing magnification. Fractal measurements and fractal geometry can. As Mandelbrot describes the phenomenon, "The importance of fractals lies in their ability to capture the essential features of very complicated and irregular objects and processes in a way that is susceptible to mathematical analysis" [3, 42]. Thus, fractals are the language of discourse for describing Richardson's whorls, mathematical "monsters," and more concrete problems such as the surface of a charcoal briquette or the curve of a coastline. In essence, fractals describe the structural complexities of nature. What, specifically, is a "fractal" and is fractal geometry consistent with the realm of mathematics? More importantly, how does fractal geometry apply to practical problems in diverse fields such as geology, business, art, and meteorology, as well as to computer science and mathematics?

Underlying fractal geometry is a notion that most people are familiar with but at the same time cannot define -- dimension. Given any function, one can determine the dimensions needed to graph the function by analyzing the variables. We exist in a three-dimensional world; the words on this paper are two-dimensional. But what is "dimension"? In responding to the question, many college mathematics students think of Euclidean n-space; in this sense, "dimension" is the number of coordinate axes in the system or the number of components necessary to distinguish a point in space. Henceforth, we will use the notion of Euclidean space as a reference. "Dimension," however, is more complex. According to Mandelbrot, dimension -- the degree of complexity of an object has two components:

1. topological dimension
2. Hausdorff-Besicovitch (Fractal) dimension.

Furthermore, when the two coincide (as they do in Euclidean geometry) we say that the set involved is dimensionally concordant. When the measures differ the set is dimensionally discordant [1, 15]; fractal geometry basically deals with dimensionally discordant sets.

Topological dimension (often denoted by D_T) is always an integer and can be at most the Euclidean dimension [1, 15]. In 1912 Poincaré intuitively described topological dimension by the properties of points and lines which he generalized into higher dimensions. A condensed

explanation is as follows: when given a continuum of points if a finite number of continuum elements (points) can separate the continuum, then one is its dimension. If points will not separate a different continuum while one or more one-dimensional continua can, then the new continuum is two-dimensional [1, 290-291]. For example, the real number continuum can be separated by points (that is, real numbers); hence, the real line is one-dimensional. A coordinate plane, however, cannot be separated by finitely many points, but can be by a one-dimensional continuum (a line); therefore, the plane has dimension two. In other words, topological dimension moves away from dependence upon coordinate axes and instead utilizes the notion of separating sets of elements (continua).

The second component of dimension -- the Hausdorff-Besicovitch dimension -- essentially is fractal geometry. Often referred to as fractal dimension, this component, according to Weisburd, is "the degree to which the trace fills a space and adds complexity to a straight line," or the degree to which a surface is convoluted [5, 279]. Rather than pursuing the complex mathematical formulas involved in the original calculations of the Hausdorff-Besicovitch dimension we will examine the problem from a geometric/algebraic perspective. Two examples -- a straight line and a curve -- lend results which can be generalized into higher dimensions. Consider a line segment whose length we wish to find. In a Euclidean sense, we can just measure the length. Suppose, however, that our ruler is not long enough; to circumvent this, we divide the line segment into N equal parts and let the total length equal 1 as a relative measure. Each of the N parts, therefore, is reduced in length from the original by some scale factor -- and has a new relative length -- call it r. It follows that the total length of our segment can be calculated as follows: Length = 1 = (number of sub-segments) × (relative length of each sub-segment) = $N \times r$. Hence, $N = 1/r$.

Now consider the area of a square in E^2. Instead of working with a formula for the entire square let us determine its area by dividing the square into parts. Partition the square into N parts -- each side of which is reduced by a scale factor of r. Note that the area factor of each of the N squares is r^2. Again, N is the reciprocal of the area factor -- $N = 1/r^2$. Therefore, Total Area = 1 = (number of sub-squares) × (area of each) = $N \times r^2$. Hence, $N = 1/r^2$.

We can generalize the results from above. Recall the two equations: $N = 1/r$ and $N = 1/r^2$. The first equation deals with a line segment in E^1;

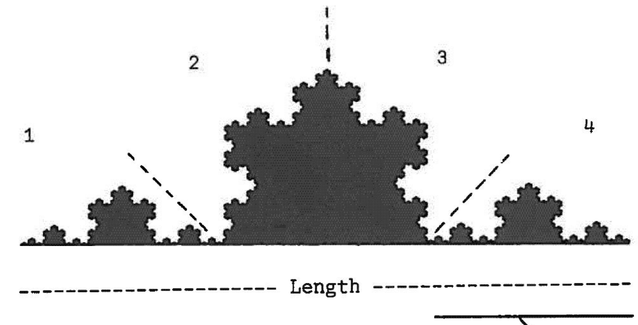
the second deals with a square in E^2. If d equals the dimension, we can rewrite the equations $N = 1/r^d$. Again, N is the number of parts; r is the scaling factor. Solving the equation for the dimension, we obtain the formula:

$$d = [\log N]/[\log(1/r)].$$

Contrary to our intuitive notion of dimension, d may be non-integral, depending on N and r [4, 123]. This number, d, is called the fractal (Hausdorff-Besicovitch) dimension. Structures for which d is non-integral command unusual properties; they "fill the gaps" between dimensions, thus rendering usual Euclidean measuring devices virtually ineffective.

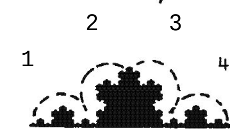
After defining fractal and topological dimension, one can rigorously define "fractal." The following definition, taken from Mandelbrot, explicitly distinguishes a fractal set from any other set: a "fractal" is "a set for which the Hausdorff-Besicovitch dimension strictly exceeds the topological dimension" [1, 15]. This definition seems remote from the intuitive concepts of a fractal mentioned earlier; let us examine the Koch curve to clarify notions of dimensionality and fractionality (Figure 1).

Dimension, contrary to what the preceding paragraphs seemed to say, is not the only important aspect of fractals. Self-similarity determines not only the type of fractal structure but gives us a means for describing the endless fragmentation of a structure. Self-similarity, according to Steen, occurs when exact or random patterns are exhibited at different measuring scales. In other words, changing the gauge has no effect upon the basic pattern. As a result, for a fractal curve of dimension between one and two, length is an insufficient measure of size. In essence, the parts are the same as the whole [4, 122-1231. The frequency of the repeat or the extent of self-similarity helps determine the fractal dimension. The self-similarity characteristics of a structure differentiate fractals into two categories, says McDermott. Geometric fractals exhibit an identical pattern repeated on different scales while random fractals introduce an elements of chance(which is most often the case in nature) [2, 112]. An example of a geometrical fractal is given in Figure 2. A computer-generated random fractal -- a three-dimensional "fractal dragon" -- appeared on the cover of the December, 1983,



Consider the covering of the length.
We have 4 sub-segments, each of which
is 1/3 the length.

Suppose we focus on one sub-segment.
We can also cover this sub-segment
with 4 "balls" each of length equal
to 1/3 the length of the original
sub-segment.



Thus, the number of parts we keep
breaking our segment into is $N = 4$;
the scaling factor is $R = 1/3$.

We expect the dimension to be
$d = (\ln N)/(\ln(1/r)) =$
$(\ln 4)/(\ln 3) \approx 1.2618$.

Figure 1

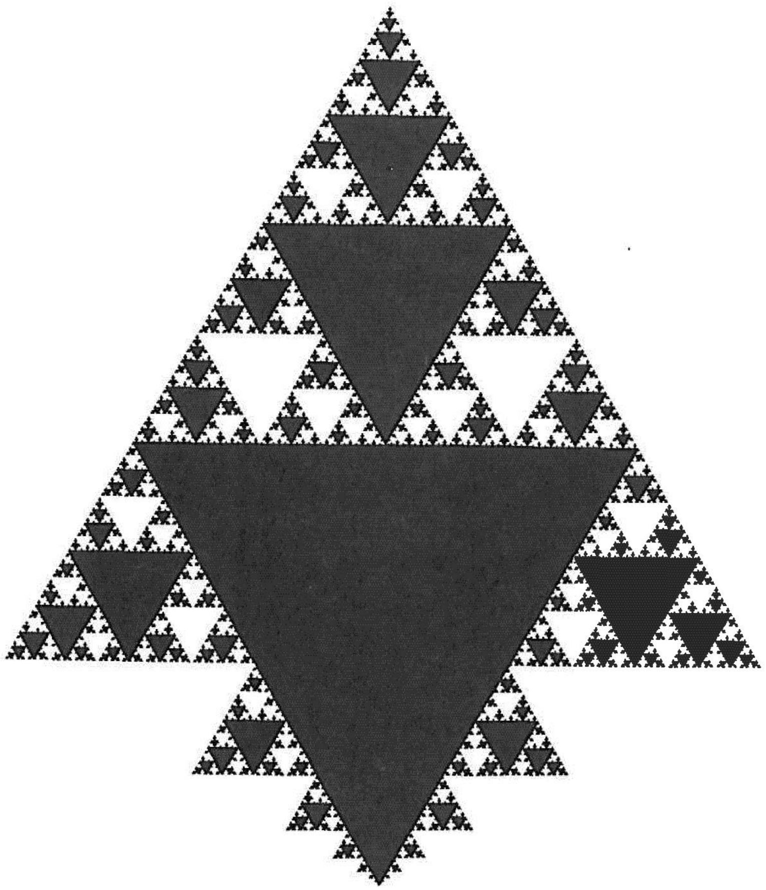


Figure 2

From THE FRACTAL GEOMETRY OF NATURE
by Benoit B. Mandelbrot.

In order to truly appreciate the applicability of fractal geometry one must examine its origins. For two thousand years, Euclidean geometry was the geometry; composed of relatively well-behaved shapes, Euclidean geometry appeared to model nature's designs. Steen suggests that generalizations of Euclidean geometry and other mathematical theories were applied to spaces with dimensions greater than three; however, even in infinitely-dimensional cases, increments between dimensions were always integral [4, 122]. Thus integral dimensions were a "given"; to conjecture otherwise was absurd. How could one talk about structures having a dimension between that of a line and a plane?

Some mathematicians did just that. Peano, Cantor, Weierstrass, Lebesgue, Hausdorff, Koch, Sierpinski, and Besicovitch were among the mathematicians whose work pre-empted the mathematical crisis of 1875. Part of this work was a forerunner of fractal geometry. From 1875 to 1922 a mathematical crisis arose due to the discovery of functions which were nowhere differentiable but everywhere continuous. This foreshadowed the development of fractal sets in the sense that one cannot fix a tangent on a fractal curve due to the constantly evolving detail under magnification. At the same time, though, fractal sets are continuous [1, 2 and 13]. In other words, examples were discovered which tested the extremes of geometry and analysis. Eventually, leading to fractal geometry, these "pathological" functions were monsters -- existing in spite of then hazy mathematical support.

Guiseppe Peano shocked the mathematical world in 1890 with the introduction of his plane-filling curve [4, 123]. This phenomenon mirrors the non-integral fractal dimensions. Peano's curve "fills the gap" between lines and planes -- between one dimension and two. Peano created a curve which meandered sufficiently enough to contact every point in the unit square. Figure 3 illustrates the development of two different Peano curves from their basic compositional patterns. Continued development of the basic pattern yields a curve so contorted that it essentially "covers" the original closed figure. Hence, a curve seemed to cover an area of the unit square. Obviously, the curve was not in the same class as a straight line segment; yet the Peano curve is not two-dimensional. Euclidean geometry offers no solution to Peano's problem.

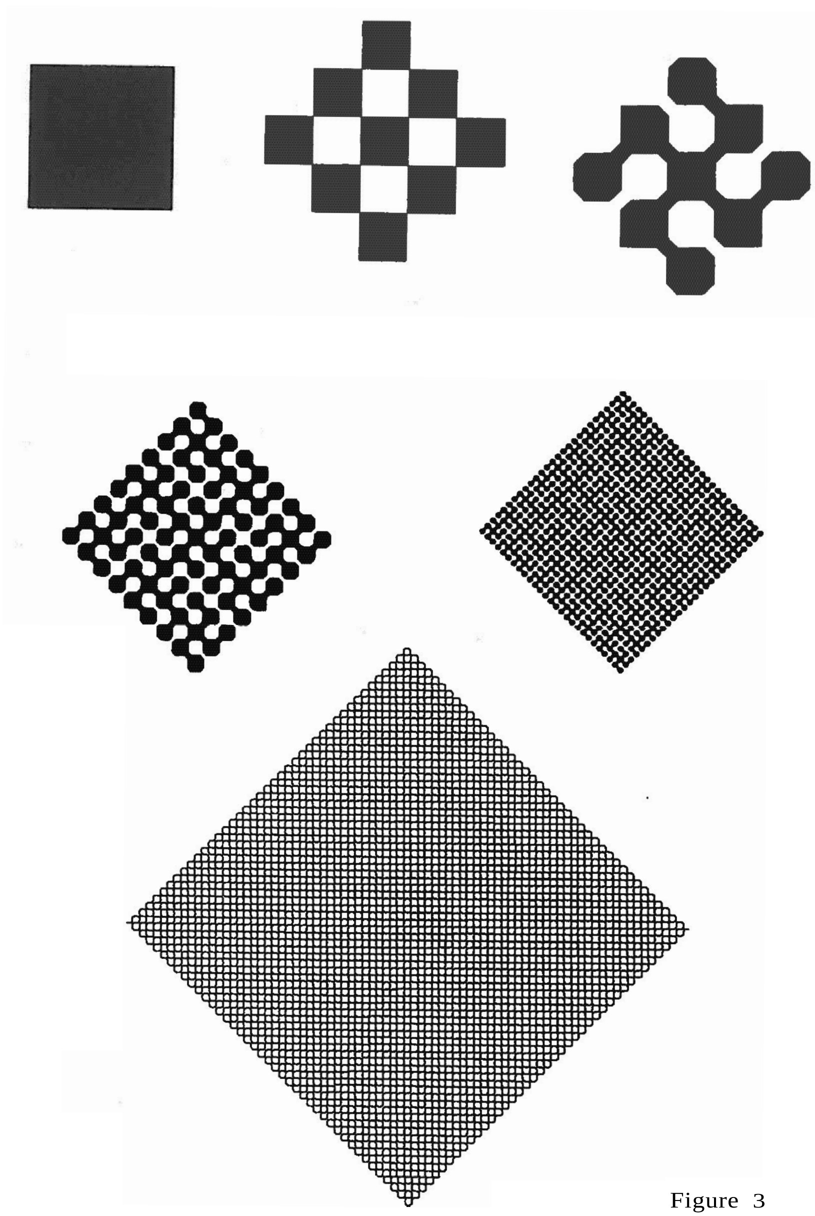


Figure 3

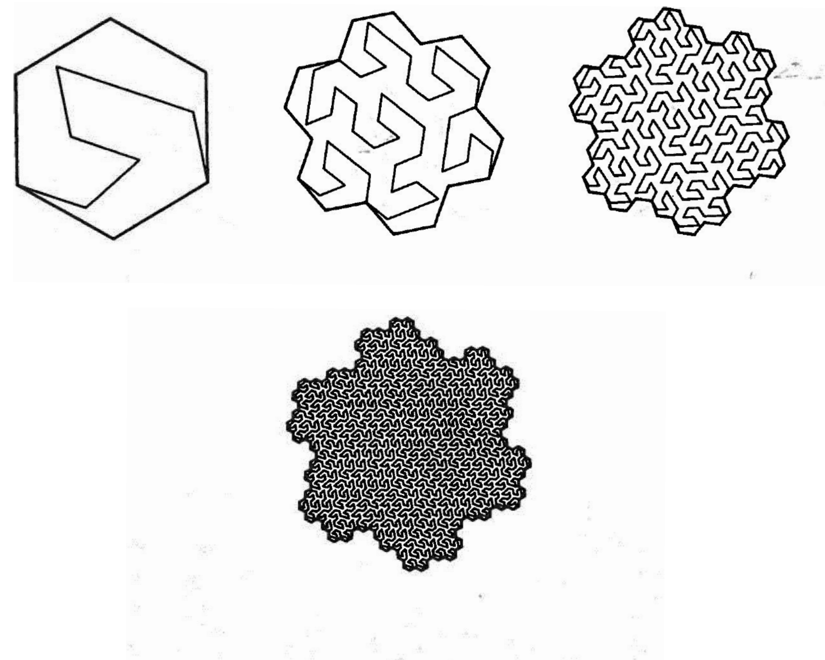


Figure 3

The triadic Koch curve is the most common example of fractal analysis. Figure 4 demonstrates the development of this snowflake curve in closed form. Let us attempt to analyze the notion of self-similarity inherent in its construction from an equilateral triangle. During the second stage each side is trisected and a new equilateral triangle is constructed on the middle third segment of each side. Each consecutive stage trisects sides of equilateral triangles and constructs a new triangle on the middle sector. If this process is continued a limiting structure results; due to continued self-similarity, there exists a sharp corner at virtually every point [1, 36]. One can easily observe the self-similarity of the triadic curve. This curve is truly a fractal for each magnification yields even greater detail.

Recall our definition of dimension. We said that

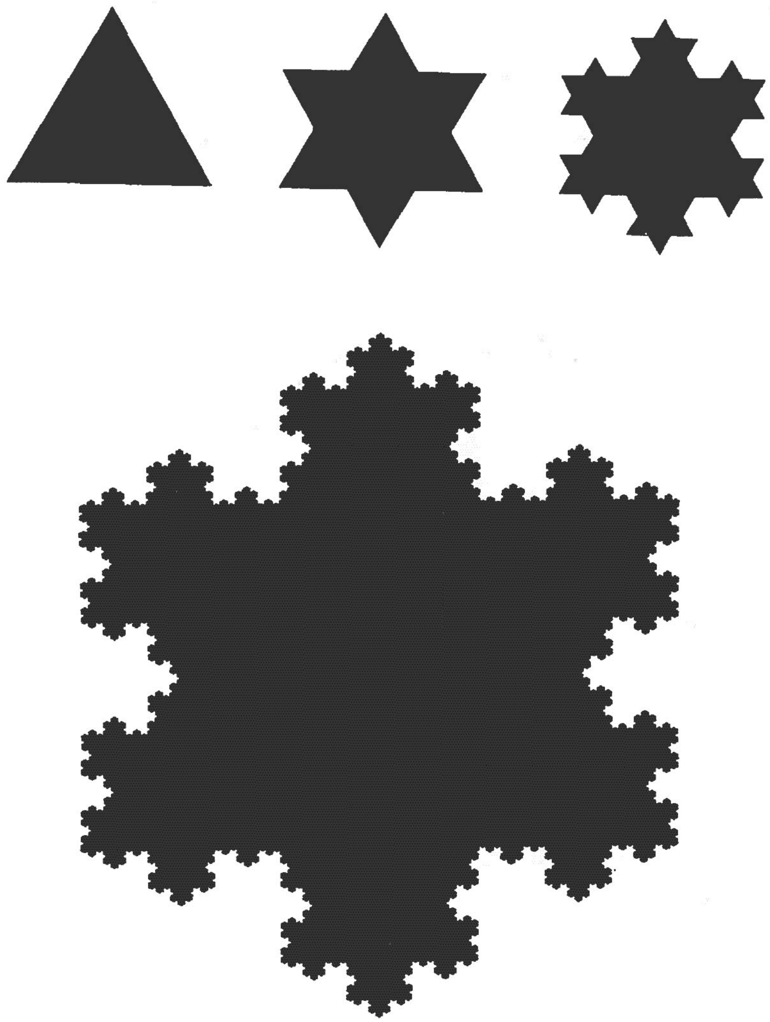


Figure 4

From THE FRACTAL GEOMETRY OF NATURE
by Benoit B. Mandelbrot.

$$d = [\log N]/[\log(1/r)]$$

where N represented the number of parts and r represented the scaling factor. Consider one side of an equilateral triangle on the triadic* curve (Figure 5.1–5.2). Let this length -- call it ℓ -- be the frame of reference. Imagine splitting the length into thirds (Figure 5.3) and constructing another equilateral triangle in the middle one-third (Figure 5.4). Notice that there are now four sides each with length one-third the original length ℓ. Hence, our total length is now equal to:

$$(4 \text{ segments}) \times \ell/(\text{the number of segments}).$$

The dimension of this structure for $N = 4$ and $r = 1/3$ is:

$$d = [\log N]/[\log(1/r)] = [\log 4]/[\log 3].$$

Since the decimal equivalent of the expression is about 1.2618 we have a fractional dimension. Figure 5.5 shrinks the segment to the original scale and places it back where we originally removed the segment. Figure 5.6 demonstrates the completed curve if the preceding process is used on the other segments a, b, and c of the original figure.

Figure 5.1

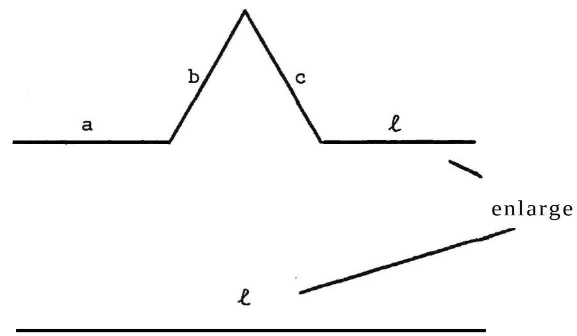


Figure 5.2

Figure 5.3

Figure 5.4

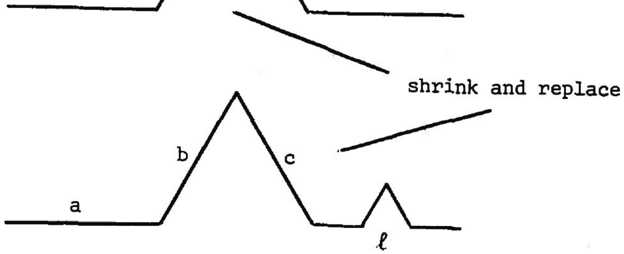


Figure 5.5

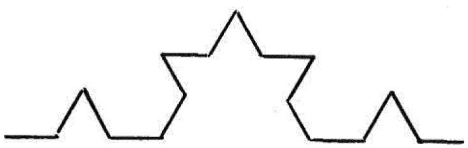


Figure 5.6

If one were to measure the length of part of the snowflake curve there would be some difficulty -- there is no finite length! Self-similarity characteristics dictate that the smaller the unit of measure the more detail released. Therefore, we cannot actually measure the length because we cannot possibly see all the detail.

Another almost "classic" problem in fractal geometry asks the question "How long is the coast of Britain?" Mandelbrot answered, "It depends. As the crow flies the coast is one length. As the person walks, it stretches even longer. As the spider crawls, it stretches still longer. In essence, a coastline with all its microscopic points and inlets is infinitely long" [2, 114]. As a result, coastlines cannot be measured in a Euclidean sense. However, by treating the coastline as a random fractal we can mathematically analyze its properties.

In addition to the so-called "classical" problems, fractals are

now being integrated into virtually all of the sciences. Fractal geometry appropriately describes perceptions of the actual physical world whereas Euclidean geometry, dealing with absolute, ideal shapes, cannot account for the structural intricacies of a fractal form. Weisburd cites an example of fractals used in the geosciences. Researchers studying the San Andreas fault hope that the fractal dimension of the jaggedness of the fault will be useful to seismologists for predicting occurrence and magnitude of earthquakes along the fault. Different fractal dimensions are characteristic of different sections of the fault. After various other studies the researchers concluded that the fractal dimension governs the manner in which fault blocks move over themselves during an earthquake, that is, whether blocks jerk suddenly or move evenly amongst themselves [5, 279]. The significance of this result is that scientists can study each type of earthquake (the ravages of which are extremely different), predict and perhaps eventually alleviate some of the destruction.

Fractals are also invaluable in metallurgy. In Peterson's article, Mandelbrot suggests fractal dimensions be used to characterize the roughness of a surface. It seems that "... roughness is very systematic." Along with this observation is research on the strength of various metals. Fractal dimension remained consistent for different samples of the same metal. Furthermore, fractal dimension changes along with metallic strength when varying heat treatments are applied to samples [3, 42]. Does this mean that fractals could be used to redefine physical properties such as hardness, strength and elasticity of a given product?

The most noticeable application of fractal geometry and the application which has brought fractals into the limelight of mathematical discussion is computer science. Fractals seem to explain how a computer retrieves data from deep within its memory banks. Moreover, McDermott reports on the realistic graphics made possible via fractal dimensions. The new graphics are so natural looking that they are being used in the movie industry to enhance special effects. Lucasfilm, the makers of the *Star Ware* saga, is the first company to specifically employ a computer graphics unit. Loren Carpenter and his crew coax out of the computer not the awkward, synthetic-looking shapes of earlier endeavors, but rather, they create majestic landscapes indistinguishable from actual nature [2, 111]. In this sense, fractals link mathematics and art via the computer. Diversity of graphic images seems unbounded as fractal geometry is paired with computer technology.

The final examples of fractals we will present are easier to comprehend, for they incorporate familiar aspects of everyday life. The first such example is the human circulatory system. In relation to the three-dimensional human body, the millions of blood vessels seem one-dimensional. For all practical purposes, each vessel appears to be a line. Yet, since the entire body needs nourishment via blood vessels in order to survive, the blood vessel "lines" must somehow reach every cell in the body. Hence, we have a system of "lines" intertwined with every point in a three-dimensional space. The "space filling" concept suggests that the circulatory system is based on fractals.

River drainage and oil prospecting are fractal applications similar to the circulatory system. A river draining an area must necessarily have "fingers" of water which seep into the far reaches of the drainage basin. The farther away from the river, the smaller these "'fingers" become; yet they must exist in order for the river to drain the basin.

Oil prospecting via fractals is a relatively new area. Formerly, geologists calculated the amount of oil in a location by the general measurements of the "dome." However, in doing so, they far underestimated the actual amount of oil; fractally, oil seeps into rocks, crevices, and soil surrounding the main dome. As our oil resources deplete themselves, we will begin to rely on these untouched stores of oil. Having an understanding of fractal geometry, one can see that the amount of oil outside the main dome is quite significant.

The final example appears to be a random fractal but is not a fractal at all -- a tree. From a distance one can see the bare, craggy silhouette of the trunk and the primary branches. Come closer and examine one of the principal branches. Upon inspection this branch appears as the whole tree did from the distance, for one can now see new secondary branches shooting from the primary branches. Continual examination of each new part reveals greater branching. Trees seem to exhibit partial self-similarity; however, due to countless biological and environmental influences, trees are not fractal forms. Ongoing research is attempting to discover a relationship between fractal geometry and intricate biological processes.

Biology, geology, computer science, the movie industry, physics, art, ... Fractal geometry, a branch of mathematics, is applicable to

each field in this ever-growing list. By examining the intricacies of nature, fractals provide a language in which to discuss the extent of structural complexities. Euclid's solid shapes are ideal situations; his devices are inadequate for measuring curves that seem to fall between dimensions. Fractals help unravel chaos; before Mandelbrot's exposition the only useful shapes were "... Euclidean shapes, lines, planes, and spheres; all else was chaos. There was order and disorder. Now there is order (simple shapes), manageable chaos (fractals), and unmanageable chaos" [2, 115]. As research escalates on this still relatively unproved theory and more varied applications are discovered, mathematicians, by observing nature from a fractal perspective, may conquer the most intimidating mathematical "monster" of all -- the universe.

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   The first book written on fractals. Mandelbrot includes excellent diagrams and pictures; explanations, however, tend towards complexity. Mandelbrot examines several types of fractals -- both geometric and random.

   (The Fractal Geometry of *Nature*, W. H. Freeman and Company, San Francisco, 1982, is an updated and augmented edition of Fractals by the same author.)

2. McDermott, Jeanne, Geometrical Forms Known as *Fractals* Find Sense *in* Chaos, Smithsonian, December, 1983, 110-117.

   This easy-to-read article provides a good understanding of fractals and a bit of a historical sketch. McDermott indicates the incredible versatility and variety of fractals and includes wonderful color pictures.

3. Peterson, Ivars, Ants in *Labyrinths* and Other Fractal *Excursions*, Science Hews, 21 January, 1984, 42-43.

   Peterson explains how fractals apply to some specific physical processes and properties -- metallic strength, conductivity, and diffusion, for example.

4. Steen, Lynn Arthur, Fractals: A World of *Nonintegral Dimensions*, Science News, 20 August, 1977, 122-123.

   Remarkably easy to comprehend, Steen's article examines the Koch curve in detail, relating it to other "pathological" mathematical situations. Steen nicely combines notions of self-similarity and dimension into an explanation of fractal geometry. In essence, Steen condenses Mandelbrot's theory into a concise, interesting, and easy-to-read article.

5. Weisburd, S., Fractals, Fractures, and Faults, Science News, 4 May 1984, 279.

This article provides an interesting and important application of fractal geometry in the geosciences.

About the author. -

Jennifer Zobitz graduated from the College of, St. Benedict in St. Joseph, Minnesota, in May 1987. At the present time she is teaching mathematics at St. Cloud Cathedral High School in St. Cloud, Minnesota.

About the paper -

Jennifer's paper was written when she was a junior at the. College of St. Benedict. H a paper supervisor was Prof. Thomas Q. Sibley. In 1986, Jennifer presented h a papa at the St. John's University/College of St. Benedict Annual Pi Mu Epsilon Student Conference in Collegeville, Minnesota.

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This annual conference will be held at St. John's University in Collegeville, Minnesota on March 18th and March 19th, 1988.

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THE SEVEN CIRCLES THEOREM

by *Stanley Rabinowitz*
Alliant Computer Systems Corporation
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Start with a circle. Any circle. Draw six more circles inside it, each internally tangent to the original circle and tangent to each, other in pairs. Let A, B, C, D, E, and F be the consecutive points of tangency of the small circles with the outer circle. We wind up with a set of seven circles as shown in Figure 1. The *Seven Circles Theorem* says that no matter what sizes we pick for the seven circles (subject only to certain order and tangency constraints), it will turn out that the lines AD, BE, and CF will meet in a point.

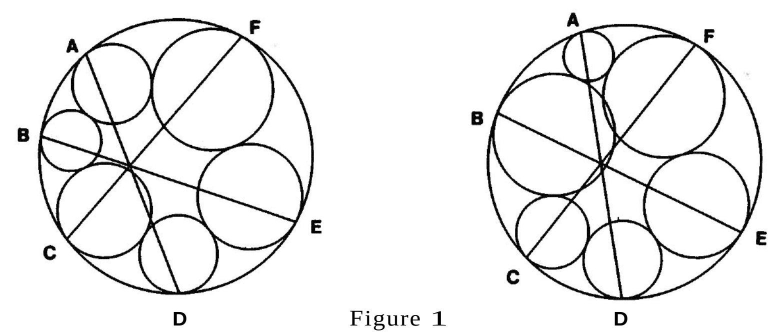

Figure 1

This remarkable theorem is less than fifteen years old. It shows that there are many beautiful relationships involving only lines and circles still waiting to be discovered. Evelyn, Money-Coutts, and Tyrrell [6] first published this theorem in 1974. Since then, other proofs have appeared (see [5]). The purpose of this article is to give a simple proof of this theorem using only elementary geometry.

Since we wish to prove that three lines *concur* (meet in a point), we start by reviewing what is known about three concurrent lines. Various facts about three concurrent lines in a triangle were known to early geometers (like Heron of Alexandria and Archimedes). They knew that the medians concur and that the altitudes concur ([2], pp. 297–298). However, it was not until 1678 that Giovanni–Ceva [1] gave a definitive treatment of such lines. For that reason, a line from a vertex of a triangle to a point on the opposite side is called a *cevian*. Here is a simplified version of Ceva's Theorem.

Ceva's Theorem. Let D, E, and F be points on sides BC, CA, and AB, respectively, of triangle ABC. Then cevians AD, BE, CF concur if and only if $AF \cdot BD \cdot CE = FB \cdot DC \cdot EA$.

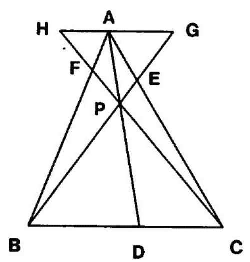

Figure 2

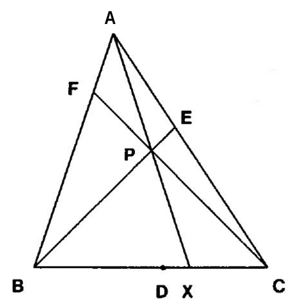

Figure 3

Proof. (i) Suppose AD, BE, CF meet at a point P. Extend BE and CF until they meet the line through A that is parallel to BC at points G and H, respectively (see Figure 2). From similar triangles, we get the proportions:

$$DC/HA = PD/AP$$
$$AG/BD = AP/PD$$
$$AE/EC = AG/BC$$

and $$BF/FA = BC/HA.$$

Multiplying these together gives us the desired result.

(ii) Conversely, suppose

$$AF \cdot BD \cdot CE = FB \cdot DC \cdot EA.$$

Let BE meet CF at P and let AP meet BC at X. Then, by part (i), we have

$$AF \cdot BX \cdot CE = FB \cdot XC \cdot EA.$$

Dividing these two results gives

$$\frac{BD}{DC} = \frac{BX}{XC}.$$

If X does not coincide with D, then without loss of generality, assume X lies on segment DC (see Figure 3). Then $BD < BX$ and $DC > XC$. Conse-

quently, $BD/DC < BX/XC$, a contradiction. Thus X coincides with D.

Remark. If we are a little more careful about signs and use directed line segments, Ceva's Theorem can be generalized to work for any points D, E, F on the sides of the triangle or on the extensions of these sides (see [2]). However, we will not need this extended result here.

Before we can prove the Seven *Circles* Theorem, we must know something about when three chords of a circle concur. Coxeter [3] gives a criterion that we will call *Ceva's* Theorem *for* Chords. Both the statement and proof are remarkably analogous to Ceva's Theorem.

Ceva's Theorem for Chords. Let A, B, C, D, E, and F be six consecutive points around the circumference of a circle. Then chords AD, BE, CF concur if and only if $AB \cdot CD \cdot EF = BC \cdot DE \cdot FA$.

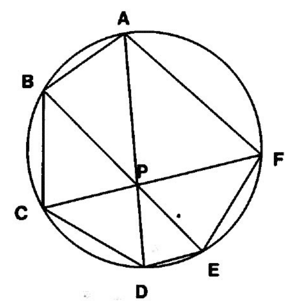

Figure 4

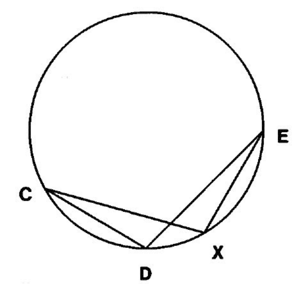

Figure 5

Proof, (i) Suppose AD, BE, CF meet at a point P (see Figure 4). From similar triangles, we get the proportions:

$$AB/DE = PA/PE$$
$$EF/BC = PF/PB$$
$$CD/FA = PC/PA$$

and $$PC/PE = PB/PF.$$

Multiplying these together gives us the desired result.

(ii) Conversely, suppose

(1) $$AB \cdot CD \cdot EF = BC \cdot DE \cdot FA.$$

Of the three arcs, $\overset{\frown}{ABC}$, $\overset{\frown}{CDE}$, $\overset{\frown}{EFA}$, at least one must be smaller than a semicircle. Without loss of generality. assume arc $\overset{\frown}{CDE}$ is smaller than a semicircle. Let BE meet CF at point P and let AP meet the circle again at point X (which must lie on arc $\overset{\frown}{CDE}$). By part (i), we have

$$AB \cdot CX \cdot EF = BC \cdot XE \cdot FA.$$

This combined with (1) gives

$$\frac{CD}{DE} = \frac{CX}{XE}.$$

If X does not coincide with D, then without loss of generality, assume X lies on arc $\widehat{DE}$ (see Figure 5). Then $CD < CX$ and $DE > XE$. Consequently, $CD/DE < CX/XE$, a contradiction. Thus X must coincide with D.

Before proceeding to the Seven *Circles* Theorem, we need one preliminary result.

Lemma. Let two externally tangent circles, P and Q, be internally tangent to circle C at points A and B respectively. If the radii of circles C, P, and Q are R, p, and q, respectively, then

$$AB^2/4R^2 = (p/(R-p)) \cdot (q/(R-q)).$$

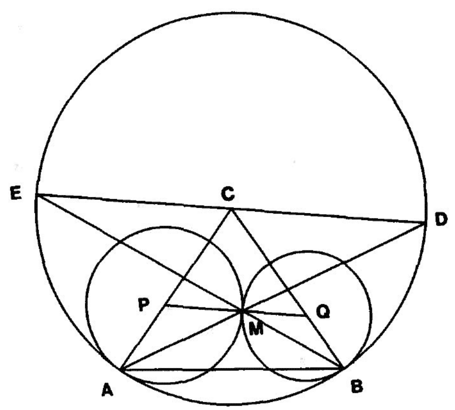

Figure 6

Proof. Let circles P and Q be tangent at point M. Extend AM and BM to meet circle C again at points D and E respectively. Identify the names of circles with their centers. Draw CD and CE. (See Figure 6.) Draw PQ which must pass through M.

$CA = CD$ implies $\angle CAD = \angle CDA$. $PA = PM$ implies $\angle PAM = \angle PMA$. Therefore $\angle PMA = \angle CDA$ and $CD \parallel PM$. Similarly, $CE \parallel QM$. But PMQ is a straight line, so therefore DCE is a straight line also. Note also that $\angle EBA = \angle EDA$ (since both measure half of arc $\widehat{EA}$). We thus have three pairs of similar triangles: $\Delta MDE \sim \Delta MBA$, $\Delta BMQ \sim \Delta BEC$, and $\Delta AMP \sim \Delta ADC$. Then

$$AB/DE = MA/ME = MB/MD.$$

Since $DE = 2R$, we have

$$\frac{AB}{2R} \cdot \frac{AB}{2R} = \frac{MA}{ME} \cdot \frac{MB}{MD} = \frac{MA}{MD} \cdot \frac{MB}{ME} = \frac{PA}{CP} \cdot \frac{QB}{CQ} = \frac{p}{R-p} \cdot \frac{q}{R-q}.$$

We are now ready to prove our main result.

The Seven Circles Theorem. Let $A_0, A_1, A_2, A_3, A_4, A_5$ be six consecutive points around the circumference of a circle O. Suppose circles can be drawn internally tangent to circle O at these six points so that they are also externally tangent to each other in pairs (that is, the circle at A_i is tangent to the circle at A_{i-1} and the circle at A_{i+1}, where subscripts are reduced modulo 6). (See Figure 7.) Then segments A_0A_3, A_1A_4, and A_2A_5 concur.

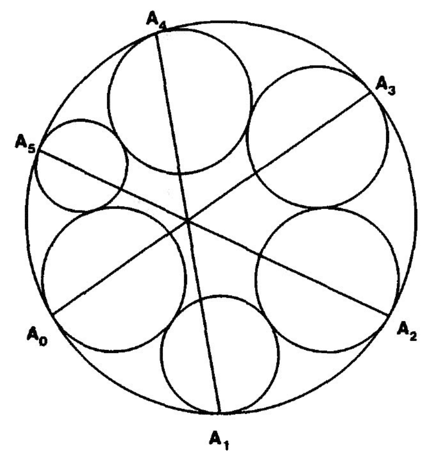

Figure 7

Proof. Let the radius of circle O be R and let the radius of the circle at A_i be r_i. Let us express A_iA_{i+1} in terms of r_i and r_{i+1}. By the lemma, we have

$$A_iA_{i+1} = 2Rf(r_i)f(r_{i+1})$$

where $f(r) = \sqrt{r/(R-r)}$ and the subscripts are reduced modulo 6. Thus

$$A_0A_1 \cdot A_2A_3 \cdot A_4A_5 = 8R^3 f(r_0)f(r_1)f(r_2)f(r_3)f(r_4)f(r_5) = A_1A_2 \cdot A_3A_4 \cdot A_5A_0.$$

So by *Ceva's* Theorem for Chords, A_0A_3, A_1A_4, and A_2A_5 must concur.

The Seven Circles Theorem is true for more general configurations than the one described above. For example, Figure 8 shows the case where the six circles are externally tangent to the original circle rather than internally tangent.

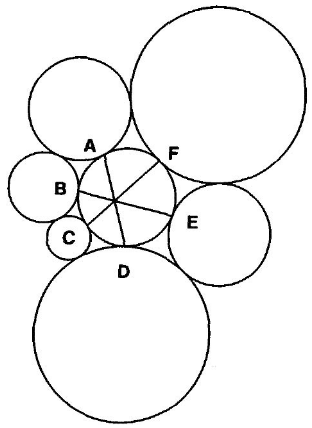

Figure 8

This case can be proved in a manner similar to the previous proof. Using Figure 9, we can derive the formula

$$AB^2/4R^2 = (p/(R + p)) \cdot (q/(R + q))$$

whose proof is analogous to the proof of the preceding lemma. Here, circles P and Q are externally tangent to circle C. R, p, and q denote the radii of circles C, P, and Q, respectively. It then becomes clear that

$$A_0A_1 \cdot A_2A_3 \cdot A_4A_5 = 8R^3 g(r_0)g(r_1)g(r_2)g(r_3)g(r_4)g(r_5) = A_1A_2 \cdot A_3A_4 \cdot A_5A_0$$

where $g(r) = \sqrt{r/(R + r)}$.

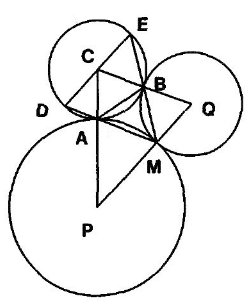

Figure 9

In fact, the Seven Circles Theorem is even more general. The six points of tangency need not occur successively along the circumference of the original circle. Two such cases are shown in Figure 10. A proof for the general configuration can be found in [6].

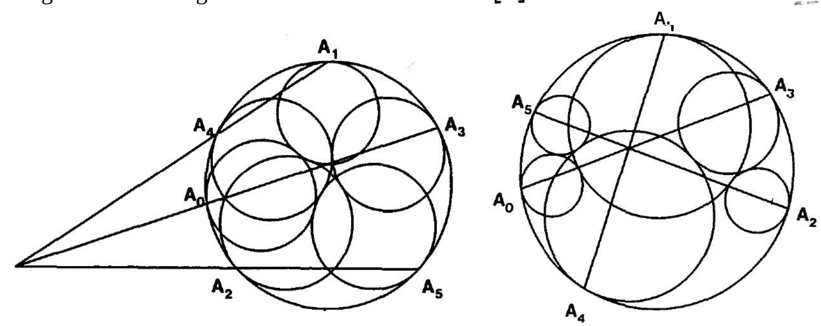

Figure 10

An interesting subtlety occurs when trying to formulate the theorem for the most general configuration. After starting with an initial circle, C, and drawing five circles A_1, A_2, A_3, A_4, and A_5 tangent to C and tangent to themselves in succession, it becomes necessary to draw a sixth circle tangent to C, A_5, and A_1. However, in general, this can be done in two ways (see Figure 11). Of the two choices, one satisfies the conclusion of the Seven Circles Theorem and the other does not. In this sense, the Seven *Circles* Theorem may be thought to hold only 50% of the time.

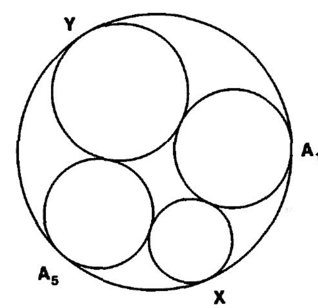

Figure 11

Exercises

We conclude with a few exercises to allow the readers to try their hands on some related problems.

1. A circle is inscribed in triangle ABC. The points of contact with sides BC, CA, and AB are D, E, and F, respectively (see Figure 12). Prove that AD, BE, and CF concur. (The point of concurrence is known as the *Gergonne* point of the triangle; see [2], page 160.) Show further that the conclusion still holds if the circle is replaced by an ellipse.

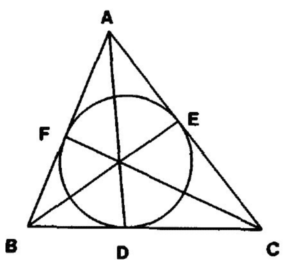

Figure 12

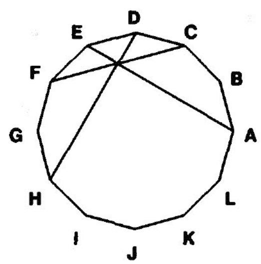

Figure 13

2. Let $ABCDEFGHIJKL$ be a regular dodecagon (see Figure 13). Prove that diagonals AE, CF, and DH concur. (For a proof see [10].)

3. Three circles are situated as shown in Figure 14 so that each meets the others in two points. Prove that AD, BE, CF concur and that $AF \cdot BD \cdot CE = FB \cdot DC \cdot EA$. (This result is due to Haruki, see [9].)

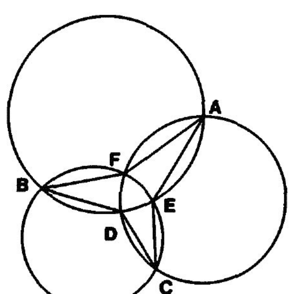

Figure 14

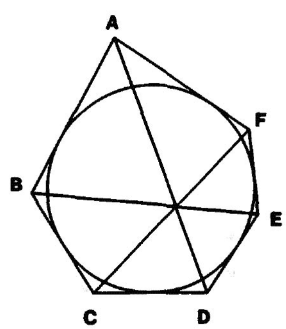

Figure 15

4. Let $ABCDEF$ be a hexagon circumscribed about a circle, as in Figure 15. Prove that AD, BE, CF concur. (This is a special case of *Brianchon's* Theorem, see [4], p. 77.)

5. Let P be a point inside pentagon $ABCDE$ such that the lines AP, BP, CP, DP, EP meet the opposite sides at points F, G, H, I, and J, as shown in Figure 16. Prove that $AI \cdot BJ \cdot CF \cdot DG \cdot EH = BI \cdot CJ \cdot DF \cdot EG \cdot AH$. (See [8], p. 67.)

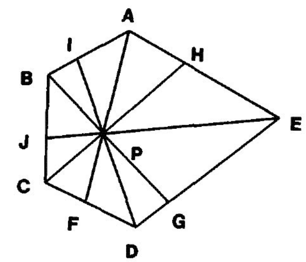

Figure 16

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THE ISOMORPHISM OF THE LATTICE OF CONGRUENCE RELATIONS ON A GROUP AND THE LATTICE OF NORMAL SUBGROUPS OF A GROUP

by *Kelly Ann Chambers*
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A <u>lattice</u> is a nonempty set L, together with a partial order such that the infimum of $\{a, b\}$, denoted by $a \wedge b$, and the supremum of $\{a, b\}$, denoted by $a \vee b$, exist for all $a, b \in L$ [1].

Two lattices $(L, \wedge, \vee)$ and $(L', \wedge', \vee')$ are <u>isomorphic</u> if there is a map $f: L \to L'$ which is one-to-one and onto such that $f(a \wedge b) = f(a) \wedge' f(b)$ and $f(a \vee b) = f(a) \vee' f(b)$ for any $a, b \in L$ [1].

Note that the <u>composition</u> of two binary relations, R and S, on a set A is the binary relation

$R \circ S = \{(x,z)$: there exists $y \in A$ such that xRy and $ySz\}$ [2].

Also, recall that a <u>congruence relation</u>, C, on a group $(G, \cdot)$ is an equivalence relation (binary relation that is reflexive, symmetric, and transitive) which satisfies the following condition for all $a, a', b, b' \in G$: If aCa' and bCb', then $(ab)C(a'b')$, where ab denotes $a \cdot b$.

The fact that there is a one-to-one correspondence between normal subgroups and congruence relations is a result from group theory. If G is a group and C is a congruence relation on G, then the equivalence class containing the identity of G is a normal subgroup of G. Likewise, if N is a normal subgroup of G, then the binary relation C, given by aCb if and only if $Na = Nb$, for all $a, b \in G$, is a congruence relation on G. In this note, we show this bijection forms the basis for the isomorphism between the lattice of congruence relations on a group, $Con(G)$, and the lattice of normal subgroups of a group, $Nor(G)$, shown here.

$Con(G)$ and $Nor(G)$ are both lattices with set-theoretic intersection as the infimum. In order for $Con(G)$ and $Nor(G)$ to be isomorphic, the map must also preserve supremums.

The supremum of two normal subgroups in $Nor(G)$ is the normal subgroup generated by them. It is known from group theory that this is equal to $N_1N_2 = \{n_1n_2: n_1 \in N_1, n_2 \in N_2\}$.

Let $Eq(A)$ be the set of all equivalence relations on a set A. $Eq(A)$ is a lattice with set-theoretic intersection as the infimum and the smallest equivalence relation containing θ_1 and θ_2 as the supremum of $\{\theta_1, \theta_2\}$ for all $\theta_1, \theta_2 \in Eq(A)$. This is not usually the set-theoretic union of θ_1 and θ_2. Rather, $\theta_1 \vee \theta_2 = \theta_1 \cup (\theta_1 \circ \theta_2) \cup (\theta_1 \circ \theta_2 \circ \theta_1) \cup (\theta_1 \circ \theta_2 \circ \theta_1 \circ \theta_2) \cup \ldots$. Since congruence relations on a group are also equivalence relations on a set, $C_1 \vee C_2 = C_1 \cup (C_1 \circ C_2) \cup (C_1 \circ C_2 \circ C_1) \cup (C_1 \circ C_2 \circ C_1 \circ C_2) \cup \ldots$, for all $C_1, C_2 \in Con(G)$.

It is not readily apparent that the congruence relation corresponding to N_1N_2 is $C_1 \cup (C_1 \circ C_2) \cup (C_1 \circ C_2 \circ C_1) \cup (C_1 \circ C_2 \circ C_1 \circ C_2) \cup \ldots$ for $N_1, N_2 \in Nor(G)$ and $C_1, C_2 \in Con(G)$. However, the group structure helps to simplify the supremum for $Con(G)$. For arbitrary sets C_1 and C_2, $C_1 \circ C_2$ is not necessarily a congruence relation. In particular, it is not symmetric or transitive. However, the group structure provides inverses, and the inverses are crucial in the proofs that $C_1 \circ C_2$ is symmetric and transitive. Thus, if C_1 and C_2 are congruences on a <u>group</u>, the supremum is $C_1 \circ C_2$. It is more apparent that the congruence relation corresponding to N_1N_2 is $C_1 \circ C_2$ rather than $C_1 \vee C_2 = C_1 \cup (C_1 \circ C_2) \cup (C_1 \circ C_2 \circ C_1) \cup (C_1 \circ C_2 \circ C_1 \circ C_2) \cup \ldots$. Thus, supremums are indeed preserved, and $Con(G)$ and $Nor(G)$ are isomorphic.

$Con(G)$ denotes the set of all congruence relations on a group G. For all $C_1, C_2 \in Con(G)$, define $C_1 \leq C_2$ if $C_1 \subset C_2$. Also, define $C_1 \wedge C_2 = C_1 \cap C_2$ and $C_1 \vee C_2 = C_1 \circ C_2$.

<u>*Lemma 1*</u>. The set of all congruence relations on a group G, with the partial order of set inclusion, is a lattice.

Proof. Since $\{(a,a): a \in G\}$ is a congruence relation on G, $Con(G)$ is not empty. Since $\subset$ is reflexive, antisymmetric, and transitive, $\leq$ is a partial order. Clearly, $C_1 \cap C_2$ is an equivalence relation that is contained in both C_1 and C_2.

Assume $a(C_1 \cap C_2)a'$ and $b(C_1 \cap C_2)b'$. Then, aC_1a' and aC_2a' and bC_1b' and bC_2b'. Since C_1 and C_2 are congruences, $(ab)C_1(a'b')$ and $(ab)C_2(a'b')$. Thus, $(ab)(C_1 \cap C_2)(a'b')$, and $C_1 \cap C2$ is a congruence relation.

Let J also be a congruence relation that is contained in both C_1 and C_2. So, $J \subset C_1$ and $J \subset C_2$. $(a,b) \varepsilon J \rightarrow (a,b) \varepsilon C_1$ and $(a,b) \varepsilon C_2$

$\rightarrow (a,b) \varepsilon C_1 \cap C_2$.

Thus, $C_1 \cap C_2$ is the largest such congruence, and $C_1 \cap C_2$ is the infimum of $\{C_1, C_2\}$ in $Con(G)$.

Now, consider C_1 o C_2. Assume $a(C_1 \text{ o } C_2)d$ and $d(C_1 \text{ o } C_2)g$. There exist b, f ε G such that aC_1b and bC_2d and dC_1f and fC_2g. So, aC_1b and $bd^{-1}C_2e$ and $eC_1d^{-1}f$ and fC_2g, where e is the identity of G. Since C_1 and C_2 are congruences, $aC_1(bd^{-1}f)$ and $(bd^{-1}f)C_2g$. Thus, $a(C_1 \text{ o } C_2)g$, and C_1 o C_2 is transitive.

The proofs that C_1 o C_2 is reflexive, symmetric, and a congruence relation follow similarly and easily.

Finally, we verify that C_1 o C_2 is the supremum of $\{C_1, C_2\}$. Let aC_1b. Since C_2 is reflexive, bC_2b. Thus, $a(C_1 \text{ o } C_2)b$, and $C_1 \subset C_1 \text{ o } C_2$. Similarly, $C_2 \subset C_1 \text{ o } C_2$. Let K also be a congruence relation that contains both C_1 and C_2. Thus, $C_1 \subset K$ and $C_2 \subset K$.

$x(C_1 \text{ o } C_2)z \rightarrow$ there exists y ε G such that xC_1y and yC_2z

$\rightarrow xKy$ and $yKz \rightarrow xKz$.

Thus, $C_1 \text{ o } C_2 \subset K$, C_1 o C_2 is the smallest such congruence, and C_1 o C_2 is the supremum of $\{C_1, C_2\}$ in $Con(G)$. Thus, $Con(G)$ is a lattice.

$Nor(G)$ denotes the set of all normal subgroups of a group G. For all $N_1, N_2 \varepsilon Nor(G)$, define $N_1 \leqq N_2$ if $N_1 \subset N_2$. Also, define $N_1 \wedge N_2 = N_1 \cap N_2$ and $N_1 \vee N_2 = N_1N_2 = \{n_1n_2: n_1 \varepsilon N_1, n_2 \varepsilon N_2\}$.

<u>*Lemma 2.*</u> The set of all normal subgroups of a group G, with the partial order of set inclusion, is a lattice.

The proof of Lemma 2 follows that of Lemma 1, and uses standard group theoretic results.

Define f: $Nor(G) \rightarrow Con(G)$ by $f(N) = C_N$ where aC_Nb if and only if $Na = Nb$ for all a, b ε G.

<u>*Theorem.*</u> The mapping f: $Nor(G) \rightarrow Con(G)$ given by $f(N) = C_N$ is a lattice isomorphism.

Proof. Assume $f(N_1) = f(N_2)$. Then, $C_{N_1} = C_{N_2}$. $x \varepsilon N_1 \leftrightarrow N_1x = N_1 = N_1e \leftrightarrow xC_{N_1}e \leftrightarrow xC_{N_2}e \leftrightarrow N_2x = N_2e = N_2 \leftrightarrow x \varepsilon N_2$. So, $N_1 = N_2$, and f is one-to-one.

Let C be a congruence relation on G. Let $[e]_C$ denote the equivalence class of e in C. The fact that $[e]_C$ is a normal subgroup of G is known from group theory. Let N denote $[e]_C$ and let $f(N) = C'$. $aC'b \leftrightarrow Na = Nb \leftrightarrow ab^{-1} \varepsilon N \leftrightarrow ab^{-1}Ce \leftrightarrow (ab^{-1}b)C(eb) \leftrightarrow aCb$. Thus, $C' = C$, and f is onto.

For all a, b ε G:

$$
\begin{aligned}
(a,b) \varepsilon f(N_1 \wedge N_2) &\leftrightarrow (a,b) \varepsilon f(N_1 \cap N_2) \\
&\leftrightarrow aC_{(N_1 \cap N_2)}b \\
&\leftrightarrow (N_1 \cap N_2)a = (N_1 \cap N_2)b \\
&\leftrightarrow ab^{-1} \varepsilon (N_1 \cap N_2) \\
&\leftrightarrow ab^{-1} \varepsilon N_1 \text{ and } ab^{-1} \varepsilon N_2 \\
&\leftrightarrow N_1a = N_1b \text{ and } N_2a = N_2b \\
&\leftrightarrow aC_{N_1}b \text{ and } aC_{N_2}b \\
&\leftrightarrow (a,b) \varepsilon f(N_1) \cap f(N_2) \\
& (a,b) \varepsilon f(N_1) \wedge' f(N_2).
\end{aligned}
$$

Therefore, $f(N_1 \wedge N_2) = f(N_1) \wedge' f(N_2)$, and f preserves infimums.

For all a, b ε G:

$$(a,b) \in f(N_1 \vee N_2) \leftrightarrow (a,b) \in f(N_1N_2) \leftrightarrow aC_{N_1N_2}b$$
$$\leftrightarrow N_1N_2a = N_1N_2b$$
$$\leftrightarrow ab^{-1} \in N_1N_2$$
$$\leftrightarrow ab^{-1} = n_1n_2 \text{ for some } n_1 \in N_1,\ n_2 \in N_2$$
$$\leftrightarrow a = n_1n_2b.$$

Let $d = n_2b$. Then $N_1(a) = N_1(n_1d) = N_1d$ and $N_2d = N_2(n_2b) = N_2b$. So, $aC_{N_1}d$ and $dC_{N_2}b$, and $a(C_{N_1} \circ C_{N_2})b$. Thus, $(a,b) \in f(N_1) \vee' f(N_2)$. On the other hand, let $(a,b) \in f(N_1) \vee' f(N_2)$. So, $a(C_{N_1} \circ C_{N_2})b$, and there exists $g \in G$ such that $aC_{N_1}g$ and $gC_{N_2}b$. Thus, $N_1a = N_1g$ and $N_2g = N_2b$. So, $ag^{-1} \in N_1$ and $gb^{-1} \in N_2$. Thus, $(ag^{-1})(gb^{-1}) \in N_1N_2$, and $ab^{-1} \in N_1N_2$. Therefore, $(a,b) \in f(N_1 \vee N_2)$. So, $f(N_1 \vee N_2) = f(N_1) \vee' f(N_2)$, and f preserves supremums. Thus, $Con(G) \simeq Nor(G)$.

Note. A lattice, L, is distributive if for any $x, y, z \in L$, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$. L is modular if $x \leq z$ implies $x \vee (y \wedge z) = (x \vee y) \wedge z$ [2]. Clearly, distributivity implies modularity, but not conversely. It can be shown that $Nor(G)$ and $Con(G)$ are modular, but not distributive.

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About the paper -

Kelly Ann's paper is baed on her undergraduate thesis.

THE GENERALIZED PRISMATOIDAL VOLUME FORMULA

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In this paper will derive an ancient formula for the volume of certain three-dimensional solids and will give several examples of solids whose volumes we can determine by using the formula.

We begin by noting that the volume of a solid can be determined by the cross-sectional area method (see, for example, Calculus and *Analytic Geometry*, 6th edition, by Thomas and Finney, page 325). Thus, if an x-axis is introduced as in Figure 1, we let $A(x)$ be the cross-sectional area determined by slicing the solid with a plane perpendicular to the x-axis and passing through the axis at x. Then the volume of the solid is $\int A(x)dx$.

A solid whose cross-sectional area function $A(x)$ is a constant is called a prism. If $A(x)$ is a polynomial of degree one, then the solid is usually called a pyramid (or frustum of a pyramid) or cone (or frustum of a cone). If $A(x)$ is a quadratic polynomial, then the solid is called a prismatoid. If $A(x)$ is a polynomial of degree three or less, then the solid is called a generalized prismatoid.

We are concerned here with a volume formula for a generalized prismatoid.

Our main theorem is this:

Theorem. Suppose $A(x)$ is a polynomial of degree three or less, then

$$\int A(x)dx = ((b-a)/6)(A(a) + 4A((a+b)/2) + A(b)).$$

Then, we have the immediate corollary:

Corollary. If the cross-section area function $A(x)$ of a solid is a polynomial of degree three or less, then the volume is

$$V = ((b-a)/6)(A(a) + 4A((a+b)/2) + A(b)).$$

Note: The formula in the corollary is often stated with the hypothesis that $A(x)$ is of degree two or less. Then general form of the corollary may be found in some references (e.g., CRC Mathematical *Tables*, 16th edition, p. 43). Here we will present a somewhat unusual proof, deriving the formula from Simpson's Rule which is usually used for approximating integrals and not for deriving exact formulas. Our proof also suggests why the formula cannot be extended to the situation in which $A(x)$ is a polynomial of degree larger than three.

Before proving the theorem, we will give several examples of how the formula works.

Example 1. If we place our x-axis so that it is perpendicular to the base of a rectangular box (Figure 1) then the cross-sectional area is a rectangle and $A(x) = \ell w$. If we take $h = b - a$, we have

$$V = (h/6)(\ell w + 4\ell w + \ell w) = (h/6)(6\ell w) = \ell wh.$$

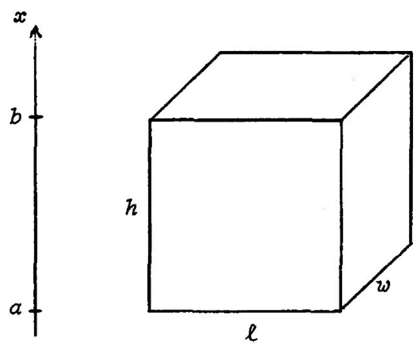


Figure 1

More generally we have:

Example 2. The cross-sectional area of any solid with $A(x) =$ constant $= B$ has volume

$$V = (h/6)(B + 4B + B) = (h/6)(6B) = Bh.$$

Thus, we have the following:

Example 3. The cross-sectional area of a right circular cylinder is $A(x) = \pi r^2$, so the volume is

$$V = (h/6)(\pi r^2 + 4\pi r^2 + \pi r^2) = (h/6)(6\pi r^2) = \pi r^2 h.$$

Example 4. If we place our x-axis along the diameter of a sphere of radius r, it can be easily shown that $A(x)$ is a quadratic function of x. Also, $b - a = 2r$. Thus, the volume of a sphere of radius r is

$$V = (2r/6)(0 + 4\pi r^2 + 0) = (4/3)(\pi r^3)$$

since the cross-sectional area of each end of the sphere is 0 and in the middle is πr^2.

Example 5. Let R be a region in the plane with finite area B. Let P be a point not in the plane of R. Then P and R determine a generalized pyramid or cone consisting of all points on the line segments joining P to the points of R (see Figure 2). If we place our x-axis so that it is perpendicular to the plane of R it is easy to show that $A(x) = (x/h)^2B$, where h is the height of the cone and the origin on the x-axis is placed as in figure 2. Thus, the volume is

$$V = (h/6)(0 + 4(1/4)B + B) = hB/3.$$

In particular, a right circular cone of radius r and height h has volume $\frac{\pi r^2 h}{3}$.

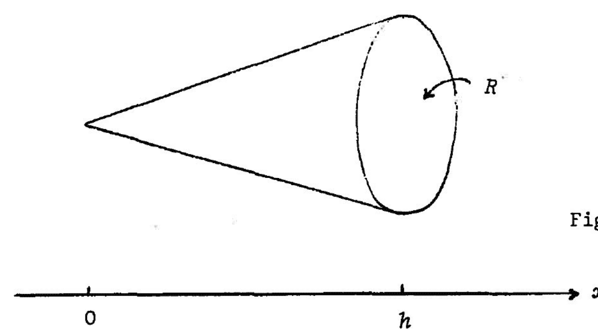


Figure 2

Now we will give the proof of the theorem using Simpson's Rule (see, for example, Thomas and Finney, pp. 308-309).

Proof. Simpson's Rule, a standard method for approximating definite integrals, may be used to approximate

$$\int f(x)dx \text{ where } f(x) = A(x).$$

The approximation is obtained by subdividing $[a,b]$ into n subintervals each of length h (n is even). The error in the approximation is $\mu h^4 f^{(4)}(c)$, where μ is a constant, c is in the interval (a,b) and $f^{(4)}$ is the fourth derivative of f (see Thomas and Finney, p. 309).

If $f(x) = A(x)$ is a polynomial of degree three or less, then $f^{(4)}(x) = 0$. Hence, the error in Simpson's Rule is 0, and we may pick n to be as small as possible, namely we may choose $n = 2$. Since $h = (b-a)/2$, we obtain the following (exact) approximation

$$\int A(x)dx = (h/3)(A(a) + 4A((a+b)/2) + A(b)) =$$
$$((b-a)/6)(A(a) + 4A((a+b)/2) + A(b)).$$

About the author -

Sarah is a senior at the University of Wisconsin-Parkside.

About the paper -

Sarah's paper was submitted by Professor Thomas Fournelle who wrote "... the paper arose out of, a meeting of our local Pi Mu Epsilon chapter after which Sarah became quite interested in formulas for volumes." Sarah was a junior at that time.

IMPORTANT ANNOUNCEMENT

Pi Mu Epsilon's main source of steady income is the National Initiation Fee for new members. The fee covers the cost of a membership certificate and a one-year subscription to the Pi Mu Epsilon Journal.

For the past fourteen years the fee has been set at \$4.00. Effective January 1, 1987, the National Initiation Fee will be \$10.00. After January 1, 1987, any order for membership certificates should be accompanied by the new fee.

Effective July 1, 1987, **all** correspondence regarding membership certificates, posters, stipends for student participants in regional and national meetings, and other financial matters should be directed to Professor Robert Woodside, Department of Mathematics, East Carolina University, Greenville, NC 27858 **[(919) 757-6414].** The names and affilations of other new officers and councilors appear on the inside front cover.

LIES, SPIES, AIDS, AND DRUGS

by Barry W. Brunson
Western Kentucky University

With increasing frequency, there are calls for mandatory testing of large numbers of people for the afflictions mentioned in the title. Some of those making such calls have good intentions, but the effects of such testing would by and large be both unexpected and very unfortunate,

The problem is with the reliability of the tests: no test gives the correct diagnosis 100% of the time, and more to the point, the reliability of the best available test is generally a <u>smaller</u> fraction than that representing the part of the population not afflicted. Whenever this occurs, <u>fewer than half</u> of those **"testing positive"** will in fact have the condition. This is an easy consequence of Bayes' Theorem; most probability texts have at least one example or exercise along these lines, and Problem 627 (Fall '86) of this <u>**Journal**</u> provides another.

We offer a general form of this problem, and point out an aspect of symmetry which makes the folly of such mass testing dramatically obvious.

<u>*Example*</u>. A test for a horrible condition is 95% reliable; that is, if a person has [resp., does not have] the condition, then the test will be positive [resp., negative] with probability **.95.** Suppose that the condition affects 1% of the population. Find the probability that a randomly selected person has the condition, given that the test is positive.

<u>General problem.</u> Replace the 95% reliability of the test by some other $100p\%$ [with p close to **1**], and replace the 1% incidence of the condition by some other $100q\%$ [with $1-q$ even closer to **1**].

Let C denote the event that a randomly selected person has the condition and let T denote the event that a randomly selected person tests positive. We seek $P(C|T)$. By Bayes' Theorem,

$$P(C|T) = \frac{P(C\cap T)}{P(T)} = \frac{P(C)P(T|C)}{P(C)P(T|C) + P(\bar{C})P(T|\bar{C})}$$

$$= \frac{qp}{qp + (1-q)(1-p)}.$$

Note that $P(C|T) < \frac{1}{2}$ if and only if $qp < (1-q)(1-q) \leftrightarrow p+q < 1 \leftrightarrow p < 1-q$, which justifies the assertion made in the second paragraph above.

For the example, the answer is about 16%. But note the symmetry in p and q. Few would even contemplate mass testing under the following circumstances:

> A test for a condition is 1% reliable; the condition affects 95% of the population.

But the proportion of false positives is the same!

Historical remarks. For lie-detector tests in particular, a reliability level of 80% or better is generally acknowledged to be rare.

∫ ∫

Award Certificates

Your Chapter can make use of the PI MU EPSILON Award Certificates available to help you recognize mathematical achievements of your students. Contact Professor Robert Woodside, Secretary-Treasurer.

Matching Prize Fund

If your Chapter presents awards for Outstanding Mathematical Papers or for Student Achievement in Mathematics, you may apply to the National Office for an amount equal to that spent by your Chapter up to a maximum of fifty dollars. Contact Professor Robert Woodside, Secretary-Treasurer.

1988 National Meeting

Negotiations are in progress with the American Mathematical Society regarding summer meetings in Providence, RI about the time of the Society's Centennial Celebration. Look for details in the Spring 1988 issue of the Journal.

USING AREAS TO OBTAIN THE AM-GM INEQUALITY

by Norman Schaumberger
Bronx Community College

Consider any n positive real numbers $x_1 \le x_2 \le \dots \le x_k \le G \le x_{k+1} \le \dots \le x_n$, where $G = \sqrt[n]{x_1 x_2 \cdots x_n}$ and $A = (x_1 + x_2 + \dots + x_n)/n$.

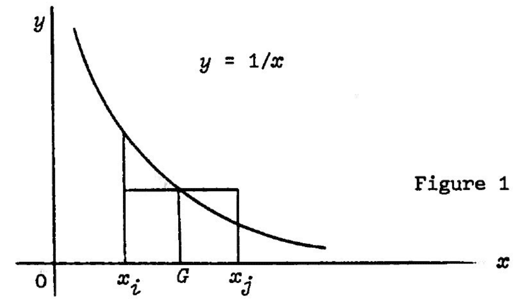


Figure 1

From Figure 1, it follows that

$$(1) \qquad \frac{1}{G}(G - x_i) \le \ln G - \ln x_i$$

with equality if and only if $x_i = G$. Putting $i = 1, 2, \dots, k$ successively in (1) and adding the inequalities gives

$$(2) \qquad k - \frac{x_1 + x_2 + \dots + x_k}{G} \le k \ln G - \ln x_1 x_2 \cdots x_k.$$

Again, using Figure 1, we have

$$(3) \qquad \frac{1}{G}(x_j - G) \ge \ln x_j - \ln G$$

with equality if and only if $x_j = G$. Letting $j = k+1, k+2, \dots, n$ in (3) and adding, we get

$$(4) \qquad \frac{x_{k+1} + x_{k+2} + \dots + x_n}{G} - (n-k) \ge \ln x_{k+1}x_{k+2}\cdots x_n - (n-k)\ln G.$$

Changing signs in (4) and combining with (2) gives

$$n - \frac{x_1 + x_2 + \dots + x_n}{G} \le n \ln G - \ln x_1 x_2 \cdots x_n = 0.$$

Hence, $1 - A/G \le 0$, so that $G \le A$. Furthermore, there is equality in $G \le A$ if and only if each of the substituted values for x_i and x_j is G; that is, if and only if $x_1 = x_2 = \dots = x_n$.

PUZZLE SECTION

Edited by

Jobeph V. E. Konhauser

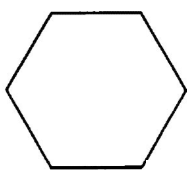
The PUZZLE SECTION is for the enjoyment of those readers who are addicted to working doublecrostics or who find an occasional mathematical puzzle attractive. We consider mathematical pussies to be problems whose solutions consist of answers immediately recognizable as correct by simple observation and requiring little formal proof. Material submitted and not used here will be sent to the Problem Editor if Seemed appropriate for the PROBLEM DEPARTMENT.

Address all proposed puzzles and puzzle solutions to Professor Joseph D. E. Konhauser, Mathematics and Computer Science Department, Macalester College, St. Paul, Minnesota 55105. Deadlines for puzzles appearing in the Fall Issue will be the next February 15, and for the puzzles appearing in the Spring Issue will be the next September 15.

PUZZLES FOR SOLUTION

1. ***Proposed by John M. Howell, Box 669, Littlerock, CA.***

Partition a regular hexagon into four congruent six-sided figures.



2. ***Proposed by the Editor.***

A certain card shuffling device always rearranges the cards in the same way (that is, the card in the ith position always goes into the jth position, and so on). The Ace through King of Clubs are placed into the shuffler in order with the Ace on the top and the King on the bottom. After two shuffles the order of the cards - from top to bottom - is

10, 9, Q, 8, K, 3, 4, A, 5, J, 6, 2, 7.

What was the order of the cards after the first shuffle?

3. ***Phopobed by the Editor.***

Bored in a calculus class, a student started to play with his hand-held calculator. He entered a four-digit number and then pressed the "square" key. To his surprise (and delight) the four terminal digits of the result were the same digits in the same order as those in the number which had been squared. What was that number?

4. ***Proposed by the Editor.***

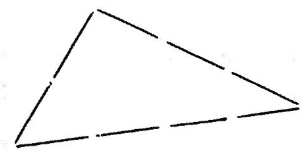
The side lengths of a convex quadrangle are positive integers such that each divides the sum of the other three. Can the four side lengths be different numbers?

5. ***Phopobed by the Editor.***

If the four triangular faces of a tetrahedron have equal areas must the faces be congruent?

6. ***Phopohed by a matchless friend.***

Nine matchsticks are laid end-to-end to enclose a triangular region.



Place two more matchsticks of the same length end-to-end inside the triangle to bisect the triangular region.

7. ***Contributed anonymously.***

In the triangular array

2 6 5
4 1
3

each number not in the top row is equal to the difference of the two numbers above it. Are you able to arrange the integers 1 through 10 in a four-rowed triangular array with the same property? One through 15 in a five-rowed array? One through 21?

* * *

GRAFFITO

No one is born knowing the techniques for solving problems and other dilemmas. It is a <u>learned</u> skill and grows from successful experience. Solving puzzles is one way to get this experience.

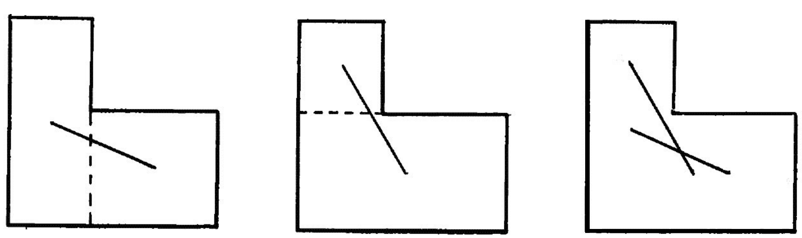
Josephine and Richard V. Andhee
Logic Unlocks

COMMENTS ON PUZZLES 1 - 5, SPRING 1987

Most respondents to *Puzzle* # 1 pointed out that there are infinitely many integer pairs (p,q) different from $(1,2)$ such that

$$a_{n+2} = pa_{n+1} + qa_n, \; n = 1, 2, 3, \ldots$$

generates the sequence 1, 2, 4, 8, 16, ... Any integer pair (p,q) satisfying $2p + q = 4$ is a solution and there are no other integer solutions to the puzzle. Robert Prielipp and Victor Feser pointed out that $(1,2)$ is the only solution in which both p and q are positive integers. Several readers remarked that non-integer solutions also exist. Seven readers submitted the following solution to *Puzzle* # 2: subdivide the L-shaped region into two rectangles in two different ways and in each case draw the line segment joining the centers of the rectangles. The point of intersection of the line segments is the centroid of the L-shaped region. All line segments needed can be drawn using a straight-edge alone.



Eighteen responses were received for Puzzle # 3. All contained the (unique) solution

$$(1)^n + (4)^n + (6)^n + (7)^n + (10)^n + (11)^n + (13)^n + (16)^n =$$
$$(2)^n + (3)^n + (5)^n + (8)^n + (9)^n + (12)^n + (14)^n + (15)^n,$$

$n = 0, 1, 2, 3$, with common sums 8, 68, 748, 9248, respectively. Robert Prielipp found the following generalization in Joe Roberts' *Elementary Number Theory: A Problem Oriented Approach*, The MIT Press, 1977:

$$\sum_{n=1}^{2^{k+1}} (1 - a_{n-1})n^t = \sum_{n=1}^{2^{k+1}} a_{n-1}n^t, \quad 1 \le t \le k,$$

where a_n is 0 if the base 2 representation of n has an even digit sum and is 1 otherwise. For the imprecisely worded *Puzzle* # 4 eleven responses were received. Not all were in agreement. If the vertices of the rectangle are confined to the square and its interior then no movement of the rectangle is possible. If the word move is interpreted to permit lifting and replacing then a non-square inscribed rectangle can be fitted inside a unit square only if the length of the longer side of

the rectangle is between $\sqrt{2}/2$ and unity. If the rectangle is a square with edge length $\sqrt{2}/2$, then the square can be rotated about its center and its vertices will not fall outside the unit square. Ten correct answers, all without detailed explanations, were received for *Puzzle* # 5. Michael J. Taylor did write "by counting ... the missing arrangement would contain an odd number of each color." Most respondents provided either the array below or its transpose

W	B	R	R	W
B	W	R	B	R
W	W	B	R	B
R	R	W	R	B
R	B	W	R	R

and pointed out that the missing 5-tuple was R R W R B.

List of respondents: Steve Ascher *(1,3)*, Charles Ashbacher *(3,4)*, Julia Bednar *(3)*, William Boulger *(1,3,4)*, Russell Euler *(1,3)*, Mark Evans *(1,3,4)*, Victor G. Feser *(1,3,4,5)*, John M. Howell *(1,2,3,4,5)*, Edmund F. Marks, Jr. *(1,3,4,5)*, Glen E. Mills *(1,2,3,4,5)*, John O. Moores *(1,2,3,4,5)*, Stephen Morais *(1,2,3,5)*, Robert Prielipp *(1,3)*, John H. Scott *(1,2,3,4,5)*, Sahib Singh *(1,3,4)*, Emil Slowinski *(1,2,3,4,5)*, Michael J. Taylor *(5)* and Marc Whinston *(1,3,4)*. One unsigned response contained solutions to all five puzzles. Receipt was acknowledged by postal card before the envelope was discarded. Sorry. Ed.

Solution to Mathacrostic No. 24. (See Spring 1987 Issue).

Words:

A. deferent	J. eyeholes	S. ramose
B. alpestrine	K. rounded	T. Trojan horse
C. vara	L. showboated	U. episode
D. impress	M. hefty	V. swallowtail
E. stereotomy	N. Dirichlet cell	W. doubly elliptic
F. aleatoric	O. edgeways	X. replete
G. Norwegian taco	P. sieve	Y. equably
H. diehard	Q. constructivist	Z. aestival
I. hartshorn	R. asafetida	a. monohedral

Quotation: *Is probability real or is it just a cover-up dot ignorance? The question of what is real is seldom easy. Is the Devil a real aspect of the (real) would? In centuries gone by, the answer was clearly yes. Today, in the developed world cut least, the Devil has receded to a more modest and metaphorical role.*

Solved by: Jeanette Bickley, Webster Groves High School, MO; Betsy Curtis, Saegertown, PA; Robert Forsberg, Lexington, MA; Joan Jordan, Indianapolis, IN; Dr. Theodor Kaufman, Winthrop-University Hospital, Mineola, NY; Henry S. Liebennan, John Hancock Mutual Life Insurance Co., Boston, MA; Charlotte Maines, Caldwell, NJ; Robert Prielipp, University of Wisconsin - Oshkosh, WI; Stephanie Sloyan, Georgian Court College, Lakewood, NJ; and Michael J. Taylor, Indianapolis Power and Light Co., IN.

Mathacrostic No. 25

Proposed by Joseph D. E. Konhauser

The 302 letters to be entered in the numbered spaces in the grid will be identical to those in the 31 keyed *Words* at the matching numbers. The key numbers have been entered in the diagram to assist in constructing the solution. When completed, the initial letters of the *Words* will give the name(s) of the author(s) and the title of a book; the completed grid will be a quotation from that book.

The solution to *Mathacrostic No. 24* is given elsewhere in the PUZZLE SECTION.

1 T	2 S	3 G		4 F	5 P	6 Z	7 X	8 V	9 O	10 N	11 J		12 Z	13 c
14 W	15 D	16 e	17 V	18 S	19 L	20 C	21 G		22 F	23 I		24 B	25 T	26 Y
	27 M	28 S	29 H	30 Q	31 C		32 J	33 X	34 Y	35 B	36 N	37 G		38 F
39 P		40 R	41 D	42 b	43 U	44 Y	45 Z	46 B	47 Q	48 N	49 G	50 d		51 d
52 R	53 G	54 M	55 a	56 b	57 V	58 C	59 c	60 Q		61 E	62 M		63 Z	64 R
65 d	66 B	67 a	68 J		69 P	70 X		71 G	72 E	73 I	74 R	75 H	76 Y	77 N
78 P	79 U	80 O	81 b		82 L	83 D	84 V	85 Z	86 Y	87 U	88 C	89 E	90 d	91 Q
92 T	93 S		94 A		95 a	96 c	97 K	98 F	99 N	100 V	101 H	102 E		103 P
104 b	105 U	106 X	107 Q	108 B		109 I	110 M		111 F		112 L	113 C	114 K	115 S
	116 R	117 A	118 G		119 J	120 K	121 P	122 W	123 d		124 e	125 X	126 R	
127 Z	128 a	129 M	130 d	131 X	132 A	133 H	134 S	135 G	136 K	137 F		138 b	139 P	140 S
141 N	142 V	143 U	144 Z	145 G	146 I	147 a	148 D		149 I	150 C	151 B	152 T	153 J	
154 c	155 E	156 F	157 Y	158 Z	159 D	160 L	161 P		162 O	163 T	164 Q		165 S	166 b
167 G	168 T		169 E	170 M	171 S	172 X	173 Z	174 D		175 T	176 Y		177 d	178 C
179 O		180 b	181 c	182 V	183 J	184 d		185 X	186 W	187 L	188 O	189 E	190 C	191 H
	192 I	193 S	194 c		195 H	196 P	197 N		198 d	199 B	200 e	201 Z	202 L	
203 Y	204 G	205 B		206 E	207 W	208 L	209 Z		210 I	211 B		212 D	213 T	214 S
215 V	216 C	217 Z		218 J	219 V	220 U		221 M	222 Y	223 I	224 c	225 d	226 a	227 G
228 P	229 F	230 Z	231 B		232 F	233 e	234 W	235 C	236 a	237 V		238 V	239 M	
240 G	241 R	242 L	243 d	244 X	245 F	246 Z	247 I	248 O	249 A		250 S	251 I	252 O	253 Q
254 A	255 H		256 b	257 W	258 T		259 U	260 V	261 L	262 D	263 E	264 c	265 I	266 M
	267 C	268 Y	269 G	270 T	271 V	272 a	273 A	274 Q		275 O	276 B	277 P		27B V
279 I	280 C	281 Y	282 d	283 T	284 F	285 A		286 F	287 b		28B P	289 W	290 I	
291 Q	292 V	293 C	294 P		295 E	296 L	297 S	298 D	299 K	300 P	301 U	302 I		

Definitions	*Words*
A. a principal way of specifying an infinite formal language by finite means	249 285 94 273 132 117 254
B. the only knot with four alternating over and under crossings (2 wds.)	211 151 46 66 276 108 205 35 231 199 24
C. rigorously just	20 178 113 267 88 280 235 216 293 150 58 190 31
D. the study of inflection and word order as grammatical devices	212 15 83 159 148 41 262 298 174
E. one who is admitted to court to sue as the representative of a minor or other person under legal disability (2 wds.)	**89 206 72 295 169 155 189 102 61 263**
F. "Master of Space! Hero of Science!", June 27, 1949 (2 wds.)	245 111 98 286 284 22 137 4 229 232 156 38
G. condition for maximum transfer of energy from source to load (2 wds.)	135 37 53 71 21 49 118 227 3 167 269 145 240 204
H. the central line in the Greek letter epsilon	255 133 191 29 195 101 75
I. $X \cup (X \cap Y) = X$ and its dual (2 wds.)	279 223 302 210 251 73 146 247 109 265 290 192 149 23
J. an instrument for describing ellipses	68 183 218 119 32 153 11
K. a system of writing peculiar to an early Irish alphabet	136 114 299 120 97
L. a mock moon	112 19 242 82 208 261 187 202 160 296
M. a product	62 129 239 221 27 54 170 110 266
N. blank spaces	99 77 48 36 141 10 197
O. narrow (comp.)	80 248 179 162 252 275 9 188
P. considered by many to have been Sam Loyd's greatest puzzle, 1896 (4 wds.)	**161 300 121 69 39 103 78 294 5 277 288 139 228 196**
Q. a continuous mapping from one space to a higher dimensional space	91 291 30 164 107 274 47 253 60
R. sometime synonym for "set"	116 241 64 40 74 126 52
S. engaged in dispute (2 wds.)	297 134 193 18 171 93 115 28 2 250 140 214 165
T. deft (sometimes hyphenated)	152 283 270 25 1 163 175 92 258 168 213
U. model or pattern	87 259 301 43 220 79 105 143
V. sponge weaves, satinette, duck cloth, figured twill (2 wds.)	8 142 238 219 100 260 182 237 278 292 17 271 215 84 **57**
W. the sequence $\{(2n)!/n!(n+1)!\}$, $n = 1, 2, \ldots$ bears the name of this Belgian mathematician	122 207 234 186 289 14 257
X. " ________ isn't what it used to be." Arthur C. Clarke (2 wds.)	185 125 33 70 172 7 106 131 244
Y. absolute	222 76 86 268 176 281 44 157 203 26 **34**
Z. what the gumball machine was an early example of (2 wds.)	173 45 12 85 158 6 209 246 63 144 201 127 230 217
a. the annual Commemoration of founders and benefactors at Oxford University	55 67 95 226 147 128 236 272
b. a stick or cudgel	180 166 56 138 104 81 256 42 287
c. a cactus-like tree of Mexico and the southwestern U.S. having clusters of scarlet tubular flowers	59 154 **96** 224 264 13 194 181
d. transcendental	65 177 123 51 225 198 130 243 50 184 **282** 90
e. it originated in China (about 1200 B.C.) as a military signaling device	16 200 124 233

PROBLEM DEPARTMENT

Edited by Clayton IS. Dodge
University of, Maine

This department welcomes problems believed to be new and at a level appropriate for the readers of, this journal. Old problems displaying novel and elegant methods of, solution are also invited. Proposals should be accompanied by solutions if available mid by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

All communications should be addressed to C. IS. Dodge, Math. Dept., University of Maine, Orono, ME 04469-0122. Please submit each proposal and solution preferably typed oh clearly written on a separate sheet (one side only) properly identified with name and address. Solutions to problems in this issue should be mailed by July 1, 1988.

Problems for Solution

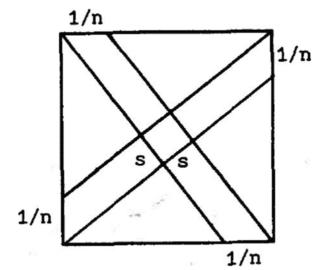
652. *Proposed by John M. Howell, Littlerock, California.*

Most people get their news from radio and television. Hence, solve this base 8 addition alphametric for the greatest NEWS.

$$\begin{array}{r} ABC \\ NBC \\ \underline{CBS} \\ NEWS \end{array}$$

***653.** *Proposed independently by Robert C. Gebhardt, County College of Norris, Randolph, New Jersey, and Clifford H. Singer, Great Neck, New York.*

A small square is constructed inside a square of area 1 by marking off segments of length $1/n$ along each side as shown in the figure at the top of the next page. For $n = 4$ the side s of the small square is $1/5$. For what other positive integral n is s the reciprocal of an integer? (This proposal is based on a 1985 AIME problem.)



654. *Proposed by Richard 1. Hess, Rancho Palos Verdes, California.*

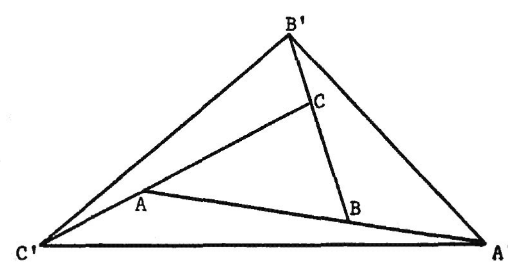
In the game of Rouge et Noir, cards are dealt one at a time from a large number of well-shuffled decks until the total pip count is in the range 31 to 40. (Face cards each count 10.) *Hoyle Complete* (by Foster, 1916) gives the relative probabilities of arriving at the sums 31, 32, 33, ..., 40 as 13, 12, 11, ..., 4, respectively. Find a more accurate set of probabilities.

655. *Proposed by R. S. Luthar, University of, Wisconsin Center, Janesville, Wisconsin.*

In triangle ABD, $\angle B = 120^\circ$. Furthermore, there is a point C on side AD such that $\angle ABC = 90^\circ$, $AC = \sqrt[3]{2}$, and $BD = 2/AC$. Find the lengths of AB and CD.

656. *Proposed by Jack Garfunkel, Flushing, New Yohk.*

Let ABC be any triangle and extend side AB to A', side BC to B', and side CA to C' so that B lies between A and A', etc., and $BA' = \lambda\cdot AB$, $AC' = \lambda\cdot CA$, and $CB' = \lambda\cdot BC$. Find the value of λ so that the area of triangle $A'B'C'$ is four times the area of triangle ABC. See the figure below.



657. *Proposed by R. S. Luthar, University of Wisconsin Center, Janesville, Wisconsin.*

Evaluate the trigonometric sum

$$\sin^6\frac{\pi}{8} + \sin^6\frac{3\pi}{8} + \sin^6\frac{5\pi}{8} + \sin^6\frac{7\pi}{8}.$$

658. *Proposed by M. S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.*

Factor $(x + y + z)^7 - x^7 - y^7 - z^7$ into a product of real polynomials, each having degree not to exceed four.

659. *Proposed by Harry Sedinger and Albert White, St. Bonaventure University, St. Bonaventure, New York.*

If $0 < x < 1$, $p > 1$, and $q = p/(p - 1)$, then prove that

$$2^{p-2}(x^p + 1) \le (x^q + 1)^{p-1}.$$

660. *Proposed by Stanley Rabinowitz, Alliant Computer Systems Corp., Littleton, Massachusetts.*

Recently the elderly numerologist E. P. B. Umbugio read the life of Leonardo Fibonacci and became interested in the Fibonacci numbers 1, 1, 2, 3, 5, 8, 13, ..., where each number after the second one is the sum of the two preceding numbers. He is trying to find a 3 x 3 magic square of distinct Fibonacci numbers (but $F_1 = 1$ and $F_2 = 1$ can both be used), but has not yet been successful. Help the professor by finding such a magic square or by proving that none exists.

661. *Proposed by John M. Howell, Littlerock, California.*

a) How close to a cubical box can you get if the sides and the diagonal of a rectangular parallelopiped are all integral?

*b) How close can you get to a cube if all the face diagonals must be integral, too?

662. *Proposed by R. S. Luthar, University of Wisconsin Center, Janesville, Wisconsin.*

Solve the equation

$$10^{2y-4} - 2\cdot 10^{y-2} - 10^{y-1} + 20 = 0.$$

663. *Proposed by M. S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.*

Find a series expansion for the integral

$$\int_0^{\pi/2} \frac{x\,dx}{\sin x}.$$

664. *Proposed by William M. Snyder, Jr., University of Maine, Orono, Maine.*

In this sentence the number of occurrences of the digit 0 is _____, of 1 is _____, 2 is _____, 3 is _____, 4 is _____, 5 is _____, 6 is _____, 7 is _____, 8 is _____, and of the digit 9 is _____.

a) Fill in the blanks to make the sentence true.

*b) How many solutions are there?

(This problem appeared on the bulletin board of a community college in Maryland.)

Solutions

595. [Spring 1985, Spring 1986, Fall 1986] *Proposed by Harry Nelson, Livermore, California.*

If the integers from 1 to 5000 are listed in equivalence classes according to the number of written characters (including blanks and hyphens) needed to write them out in full in correct English, there are exactly forty such non-empty classes. For example, class "4" contains 4, 5, and 9, since FOUR, FIVE, and SINE are the only such numbers that can be written out with exactly four characters. Similarly, class "42" contains 3373, 3377, 3378, 3773, 3777, 3778, 3873, 3877, and 3878. Find the unique class "n" that contains just one number.

IV. *Comment by Leroy F. Meyers, The Ohio State University, Columbus, Ohio.*

On February 13, 1977, *The New York Times* in a book review printed this interesting misinterpretation: "All of Apple's best characters are fanatics, each with one eye open a 30-second of an inch too far." I also object to The Times' use of "three-thousandths of an inch" for 0.003 inch, since I consider "twenty three-thousandths" to be 20/3000, rather than 23/1000. Of course, the editors of *The Times* might object that nobody would use 20/3000, since the fraction is easily reducible to 2/300. But would I write this as "two three-hundredths" or "two three hundredths?" I think the former. What about 200/23000 versus 223/1000 versus 220/3000? Incidentally, Fowler advises The *Times'* way.

615. [Spring 1986, Spring 1987] *Proposed by William S. Cariens, Lorain County Community College, Elyria, Ohio.*

Although several years into retirement, Professor Euclide Pasquale **Bombasto Umbugio** still practices mathematics with his usual prowess and efficiency. His native country, Guayazuala, still cannot afford a computer, but they do have a pocket **four-**function calculator to which he has occasional access. His latest project is to find the sum of the abscissas of the points of intersection of the seventh-degree polynomial

$$f(x) = x^7 - 3x^6 - 13x^5 + 55x^4 - 36x^3 - 52x^2 + 48x$$

with its derivative polynomial. So far he has laboriously found one of the intersections at $x = 1.3177227$. Help the kindly, old professor to find his sum without resorting to a computer.

II. ***Comment by Michael W. Ecker, Pennsylvania State University, Wilkes-Barre Campus, Lehman, Pennsylvania.***

The solution is unsatisfactory. True, the answer of 10 is correct; the solution, which incorporates the all-important reasoning, however, is incomplete or defective. The sum of the zeros of a **monic** polynomial is indeed the coefficient of the second term, counting by descending powers. However, this counts all the complex zeros and this problem seeks only the **sum** of the real zeros. Thus one must show that there are no imaginary zeros or one must prove that the sum of all the imaginary zeros of the polynomial is zero. In this case a Newton's method program verified the former situation by showing that the seven roots are approximately -3.582, -0.726, 0.326, 1.318, 2, 2.694, and 7.971.

626. *[Fall 19861* ***Proposed by Charles W. Trigg, San Diego, California.***

Reconstruct this doubly true German alphametric where, of course, DREI and *SECHS* are divisible by 3. They also have the same digit SUB.

$$\text{EINS} + \mathit{ZWEI} + \mathit{DREI} = \mathit{SECHS}.$$

Composite of solutions submitted independently by Glen E. Mills, Orange County Public Schools, Orlando, Florida, and John V. Moores, Cambridge, Massachusetts.

From the units column we have $I = 0$ or 5. From the thousands column we have that $S = 1$. Also we have

$$D + R + E + I \equiv 2 + E + C + H \equiv 0 \pmod 3.$$

By trial and success we substitute for each pair N, E, which determines H. Then C is chosen subject to the displayed congruence. Next, from the remaining digits, take Z and D so that $Z + D$ is 8, 9, or 10. Juggle the remaining digits to obtain compatible W, R, and C. Finally, apply the displayed congruence. We arrive at the unique restoration

$$3591 + 2835 + 7035 = 13461.$$

Also solved by CHARLES ASHBACHER, Mount Mercy College, Cedar Rapids, IA, MARK EVANS, Louisville, KY, VICTOR G. FESER, University of Mary, Bismarck, ND, RICHARD I. HESS, Rancho Palos Verdes, CA, JOHN M. HOWELL, Littlerock, CA, and the PROPOSER.

627. *[Fall 19861* ***Proposed by Robert C. Gebhardt, Hopatcong, New Jersey.***

This problem has interesting applications for anyone who is asked to take a lie-detector test, a drug-use test, an AIDS test, oh. any similar test where the percentages are of the order shown in the question. It is known, let us say, that 0.1% of the general population are liars. When people known to be liars take lie-detector tests, the test results are correct 95% of the time. When people known to be truthful take lie-detector tests, the test results are correct 99% of the time. To get a certain job, you are asked to take a lie-detector test. Its results indicate you are a liar. What is the probability that you actually are a liar?

I. ***Solution by Russell Euler, Northwest Missouri State University, Maryville, Missouri.***

Let A be the event that you actually are a liar and let B be the event that the lie-detector test indicates that you are a liar. Then

$$p(A|B) = \frac{p(A \cap B)}{p(B)} = \frac{(0.001)(0.95)}{(0.001)(0.95) + (0.999)(0.01)} = 0.0868372.$$

II. ***Solution by the Proposer.***

Consider a population of 100,000 people. Then 100 of them will be liars and the other 99,900 will be truthful. When the 100 liars are given lie-detector tests, they will indicate that 95 of them are liars and 5 are not. If the 99,900 truthful people are tested, the test will wrongly show 999 of them to be liars. Thus a test result indicating that a person is a liar is correct only $95/(999 + 95) = 0.086837\ldots$ of the time, less than 9%!

III. *Comment by Barry Brunson, Western Kentucky University, Bowling Green, Kentucky.*

The answer to the question as posed is either 0 or 1, depending on whether or not you are, in fact, a liar; presumably "you" know which.

The *other* answer is 95/1094, or less than 9%. The question to which this is the answer is: Given the assumptions of the first three sentences of the problem, suppose a person is chosen at random from the general population and tested. If the test should be positive, what is the probability that the person is a liar?

IV. *Comment by Peter Geiser, St. Cloud State University, St. Cloud, Minnesota.*

The more interesting result is that the probability a person is a truth-teller given that the machine has labeled him a liar is $1 - 0.0868 = 0.9132$. That is, the probability that a person is actually a truth-teller given that the machine has labeled him a liar is greater than 0.90:

Also solved by CHARLES ASHBACHER, Mount Mercy College, Cedar Rapids, IA, JAMES E. CAMPBELL, University of, Missouri, Columbia, MARK EVANS, Louisville, KY, RICHARD I. HESS, Rancho Palos Verdes, CA, HENRY S. LIEBERMAN, Waban, MA, JOHN D. MOORES, Cambridge, MA, HARRY SEDINGER, St. Bonaventure University, NY, TIMOTHY SIPKA, Alma College, Alma, MI, and WADE H. SHERARD, Furman University, Greenville, SC. All these solvers used Bayes' theorem, the method of, Solution 1. GEORGE P. EVANOVICH, Edward Williams College, Hackensack, NJ, VICTOR G. FESER, University of Mary, Bismarck, ND, and THOMAS F. SWEENEY, Russell Sage College, Thoy, NY, all submitted solutions that had minor errors of one sort or another.

628. [Fall 19861 *Proposed by Al Terego, Malden, Massachusetts.*

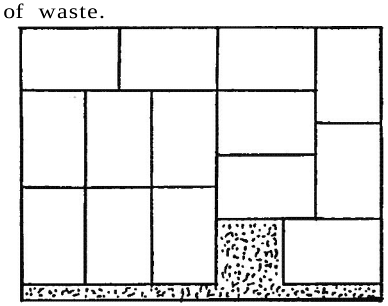
a) How many 4 x 6 cards can a paper wholesaler cut from a standard 17 x 22-inch sheet of card stock?

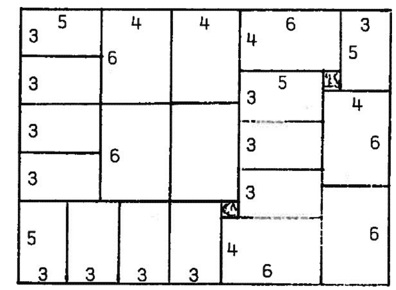
b) Can the waste be eliminated if one is allowed to cut both 3 x 5 and 4 x 6 cards from the same sheet?

I. *Solution by William D. McIntosh, Central Methodist College, Fayette, Missouri.*

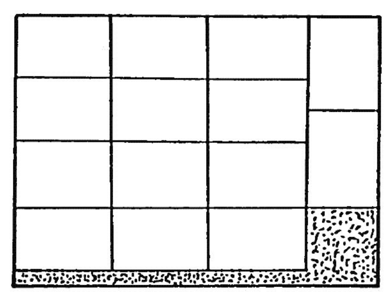
a) Since both dimensions of the cards are even, the number of cards that can be cut from a 17 x 22-inch sheet is the same as the number that can be cut from a 16 x 22 sheet. Since the latter sheet contains 352 square inches and 15 cards would require 360 square inches, it is clear that at most 14 cards can be cut from one sheet. The left diagram below shows one way to cut fourteen 4 x 6 cards from a 17 x 22 sheet.

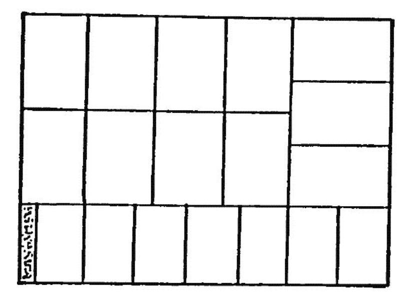
(b) The waste cannot be eliminated, but can be greatly reduced. Since the areas of the cards are 15 and 24 square inches respectively, both of which are multiples of 3, then the total area of the cards must be a multiple of 3. Since the area of the sheet is 374 square inches, there must be at least 2 square inches of waste. The right-hand figure shows a way to cut eight 4 x 6 cards and twelve 3 x 5 cards from a 17 x 22-inch sheet, with exactly 2 square inches of waste.





II. *The figure (below left) for part (a) was submitted independently by Victor G. Feser, University of, Mary, Bismarck, North Dakota., Robert C. Gebhardt, Hopatcong, New Jersey, John V. Moores, Cambridge, Massachusetts, and Wade H. Sherard, Furman University, Greenville, South. Carolina.*





III. ***The figure (preceding page, right) for part (b) was submitted independently by Victor G. Feser, University of, Mary, Bismarck, North Dakota, Richard I. Hut, Rancho Palos Verdes, California, John M. Howell, Littlerock, California, and John H. Scott, Macalester College, Saint Paul, Minnesota.***

IV. ***Comment by the Proposer.***

A commercial paper knife makes only straight cuts across the entire sheet of paper. Hence solution I(b) cannot be done with such a knife. Each of the other three figures can be so cut, those of solutions III and IV being perhaps easier for the operator of the knife. The question remains whether there is a commercial knife solution to (b) that wastes only 2 square inches. The answer is 'no"; solution IV is indeed the best one can do.

To find the minimum waste for part (b), we form a waste table. Recall that the 17 x 22 sheet must be cut into two smaller pieces, then each piece must be cut again, repeating the process until only 3 x 5 cards, 4 x 6 cards, and scrap pieces remain. We form the waste table by "rebuilding" the original sheet, using all possible combinations. Clearly any $1 \times n$ and $2 \times n$ sheets are all waste. A $3 \times n$ sheet has waste 3, 6, 9, 12, 0, 3, 6, ... square inches for n = 1, 2, 3, ... A $4 \times n$ sheet has 4, 8, 12, 16, 5, 0, 4, 8, 12, 10, 5, 0, ... square inches of waste by cutting 3 x 5 or 4 x 6 cards as appropriate. Continue in this fashion through $n = 6$. Note that an $m \times n$ sheet has the same waste as an $n \times m$ sheet. Then, whenm and n are both larger than 6, we find the waste by taking the minimum of the sums of the wastes of two smaller sheets that combine to form the $m \times n$ sheet. Thus consider $k \times n$ and $(m - k) \times n$ sheets for $k = 1$ to $m - 1$ and also $m \times j$ and $m \times (n - j)$ sheets for $j = 1$ to $n - 1$. For example, a 9 x 11 sheet has waste 0 because the wastes for 9 x 5 and 9 x 6 sheets are both 0, even though no combination of k x 11 and $(9 - k) \times 11$ will give the zero sum. This tedious process, which can be programmed into a computer, eventually shows that a 17 x 22 sheet must have waste 5.

Also solved by MARK EVANS (who furnished the same figure as the left one of, Solution I), Louisville, KY, and all those listed above. All solvers answered correctly the question in part (b), bat only those listed in solutions I and 111 provided figures for that part.

629. **[Fall 19861** ***Proposed by Jack Garfunkel, Flushing, New York.***

If A, B, C are the angles of a triangle, prove that

$$\cos A \cos B \cos C \le (1 - \cos A)(1 - \cos B)(1 - \cos C).$$

I. Solution by George P. Evanovich, Edward Williams College, Hackensack, New Jersey.

In any triangle, the distance between the **incenter** I and the orthocenter H, according to **Hobson,** A Treatise on *Plane* **and** Advanced Trigonometry, page 200, is given by

$$IS = 4R^2[(1 - \cos A)(1 - \cos B)(1 - \cos C) - \cos A \cos B \cos C],$$

where R is the circumradius of the triangle. Hence the displayed quantity is nonnegative and the theorem follows. Furthermore, equality occurs if and only if $H = I$, that is, if and only if triangle ABC is equilateral.

II. Comment by Richard I. Hess, Rancho Palos Verdes, California.

This problem has appeared in *Crux Mathematicorum* as problem 836 **[1983, 113]** and **[1984, 228].** A very similar problem, proposed by Jack Garfunkel, appeared there as problem 974 **[1984, 262]** and **[1985, 328].**

Also solved by MARK EVANS, Louisville, KY, RICHARD I. HESS, Rancho Palos Verdes, CA, BOB PRIELIPP, University of, Wisconsin-Oshkosh, and the. PROPOSER. Prielipp also pointed out the Crux Mathematicorum problem number S36.

630. **[Fall 1986]** ***Proposed by Russell Euler, Northwest Missouri State University, Maryville, Missouri.***

Evaluate

$$\prod_{m=1}^{a} \sin \frac{m\pi}{2j+1}.$$

Solution by Kenneth M. Wilke, Topeka, Kansas.

Recall that $exp(xi) = cis\ a = \cos x + i \sin x$, $exp(2\pi i) = 1$, $exp(a) \cdot exp(b) = exp(a + b)$ and that the roots of $x^r - 1 = 0$ are $exp(2\pi mi/r)$ for $m = 1, 2, \ldots, r$ for positive integral r. Then

$$\frac{x^r - 1}{x - 1} = x^{r-1} + x^{r-2} + \ldots + x + 1 = \prod_{m=1}^{r-1} (x - exp(\frac{2\pi mi}{r})).$$

Now set $a = 1$ in this equation to get

(1) $$r = \prod_{m=1}^{r-1} (1 - exp(\frac{2\pi mi}{r})).$$

By standard trigonometric double-angle formulas we get that

$$\begin{aligned} 1 - exp(2ti) &= 1 - \cos 2t - i \sin 2t \\ &= 1 - (1 - 2\sin^2 t) - 2i \sin t \cos t \\ &= (-2i \sin t)(\cos t + i \sin t) \\ &= (-2i \sin t)\, exp(ti). \end{aligned}$$

Now substitute this result into Equation (1) to get

(2) $$r = \prod_{m=1}^{r-1} (2 \sin \frac{m\pi}{r}) \cdot \prod_{m=1}^{r-1} (-i)\, exp(\frac{m\pi i}{r}).$$

Since

$$\prod_{m=1}^{r-1} exp(\frac{m\pi i}{r}) = exp(i \sum_{m=1}^{r-1} \frac{m\pi}{r}) = exp(\frac{\pi(r-1)i}{2}) = i^{r-1}$$

and since $\sin(\pi m/r) = \sin(\pi(r - m)/r)$, then we take $r = 2j + 1$ in Equation (2) to get that

(3) $$r = 2j + 1 = \prod_{m=1}^{2j} 2 \sin \frac{\pi m}{r} = \left(\prod_{m=1}^{j} 2 \sin \frac{\pi m}{2j + 1} \right)^2.$$

Now take square roots, noting that each factor in the product is positive, to get

$$\prod_{m=1}^{j} \sin \frac{m\pi}{2j + 1} = \frac{\sqrt{2j + 1}}{2^j}.$$

Also solved by SEUNG JIN BANG, *Seoul, Korea,* BARRY BRUNSON, *Western Kentucky University, Bowling Green,* BOB PRIELIPP, *University of, Wisconsin-Oshkosh,* JOHN H. SCOTT, *Macalester College, Saint Paul, MN, and the* PROPOSER. *Prielipp located this problem in Shklarsky, Chentzov, and Yaglom,* **The USSR Olympiad Problem Book** *(revised and edited by Irving Sussman and translated by John Maykovitch), Freeman 1962, problem 232(a). Brunson noted that the second equality in Equation (3) is found in* **The American Mathematical Monthly**, *vol. 69 (1962), pp. 217-218.*

631. *[Fall 1986] Proposed by Sam Pearsall, Pomona, California.*

Let

$$y_{n+1} = k(1 - y_n)$$

for $n = 0, 1, 2, \ldots$ and k a given constant. If the initial value y_0 has an absolute error $\varepsilon = y_0 - y$, where y is the true value, show that the formula is unstable for $|k| > 1$ and stable for $|k| < 1$.

Solution by Richard I. Hess, Rancho Palos Verdes, California.

Let the error in y_n be e_n. Then

$$e_{n+1} = k(1 - (y_n + e_n)) - k(1 - y_n) = -ke_n,$$

so that $e_{n+1} = (-k)^n e_0$. Hence the formula is unstable (the error grows without bound) for $|k| > 1$ and is stable (the error shrinks to zero) for $|k| < 1$.

Also solved by FRANK P. BATTLES, *Massachusetts Maritime Academy, Buzzards Bay,* MARK EVANS, *Louisville, KY,* JOHN H. SCOTT, *Macalester College, Saint Paul, MN, and the* PROPOSER.

632. *[Fall 1986] Proposed by R. S. Luthar, University of, Wisconsin Center, Janesville, Wisconsin.*

Show that

$$\int_0^1 x^x (x + (x - 1) \ln x)\, dx = 1.$$

Solution by Richard I. Hess, Rancho Palos Verdes, California.

Let

$$u = (x - 1)x^x = (x - 1)e^{x \ln x},$$

so that

$$du = x^x\, [x + (x - 1) \ln x]\, dx$$

and

$$\lim_{x \downarrow 0} x \ln x = \lim_{x \downarrow 0} \frac{\ln x}{1/x} = \lim_{x \downarrow 0} \frac{1/x}{-1/x^2} = \lim_{x \downarrow 0}(-x) = 0$$

by L'Hospital's rule. Then $u(1) = 0$ and $\lim_{t \downarrow 0} u(t) = u(0) = -1$, so

$$\int_0^1 x^x\, [x + (x - 1)\ln x]\, dx = \int_{-1}^{0} du = 1.$$

Also solved by FRANK P. BATTLES, *Massachusetts Maritime Academy, Buzzards Bay,* BARRY BRUNSON, *Western Kentucky University, Bowling Green,* GEORGE P. EVANOVICH, *Saint Peters College, Jersey City, NJ,* MARK EVANS, *Louisville, KY,* JACK GARFUNKEL, *Flushing, NY,* ROBERT C. GEBHARDT, *Hopatcong, NJ,* PETER GIESER, *Sauk Rapids, MN,* RALPH KING, *St. Bonaventure University, NY,* HENRY S. LIEBERMAN, *Waban, MA,* PETER A. LINDSTROM, *North Lake College, Irving, TX,* JOHN D. MOORES, *Cambridge*, MA, BOB PRIELIPP, *University of Wisconsin-Oshkosh*, JOHN PUTZ, *Alma College*, MI, WADE H. SHERARD, *Furman University*, *Greenville*, SC, TIMOTHY SIPKA, *Alma College*, MI, and the PROPOSER.

633. [Fall 1986] *Proposed by Dmitry P. Mavlo, Moscow, U.S.S.R.*

Let $a, b, c > 0$, $a + b + c = 1$, and $n \in N$. Prove that

$$\left(\frac{1}{a^n} - 1\right)\left(\frac{1}{b^n} - 1\right)\left(\frac{1}{c^n} - 1\right) \ge (3^n - 1)^3,$$

with equality if and only if $a = b = c = 1/3$.

Solution by Seung Jin Bang, Seoul, Korea.

We use the method of **Lagrange** multipliers. Thus let

$$F(a,b,c) = (a^{-n} - 1)(b^{-n} - 1)(c^{-n} - 1) - \lambda(a + b + c - 1).$$

Then we have

$$\frac{\partial F}{\partial a} = -na^{-n-1}(b^{-n} - 1)(c^{-n} - 1) - \lambda$$

and similar expressions for the other two partial derivatives. Setting the three partials equal to zero, we get that

$$(1) \qquad a - a^{n+1} = b - b^{n+1} = c - c^{n+1} \quad \text{and} \quad a + b + c = 1.$$

Since the graph of $f(x) = x - x^{n+1}$ is 0 at $x = 0$ and at $x = 1$, concave downward in this interval, and has a maximum at $x = 1/(n + 1)^{1/n} \ge 1/2$, then at the equality above, some two of a, b, and c must be equal. Hence we assume $a = c$. Then we have

$$a - a^{n+1} = b - b^{n+1} \quad \text{and} \quad 2a + b = 1 \quad (0 < a < \tfrac{1}{2}).$$

Now the graph of $g(x) = (1 - 2x) - (1 - 2x)^{n+1}$ is 0 at $x = 0$ and at $x = 1/2$, concave downward in this interval, and has a maximum at $x = 1/2 - 1/2(n + 1)^{1/n} \le 1/4$. Furthermore the maximum values of $f(x)$ and $g(x)$ are both equal to $1/(n + 1)^{1+1/n}$. Hence the only positive intersection of $f(x)$ and $g(x)$ is at $x = 1/3$.

Therefore the only solution to (1) is at $a = b = c = 1/3$. Since $F(a,b,c)$ tends to infinity as a, b, or c tends to zero, then

$$F(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}) = (3^n - 1)^3$$

is the minimum value of $(a^{-n} - 1)(b^{-n} - 1)(c^{-n} - 1)$, with equality if and only if $a = b = c = 1/3$.

Also solved by RICHARD I. HESS, Rancho *Palos Verdes*, CA, and the PROPOSER (*two solutions*).

634. [Fall 1986] *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.*

Find the condition for one root of the cubic equation

$$x^3 - px^2 + qx - r = 0$$

to be equal to the sum of the other two roots.

I. *Solution by Charles R. Diminnie, St. Bonaventure University, St. Bonaventure, New York.*

A necessary and sufficient condition is that $p/2$ be a root of the equation, since p is the sum of the three roots.

II. *Solution by Oxford Running Club, University of Mississippi, University, Mississippi.*

Since the sum of the roots of this cubic is p, then one of the roots must be $p/2$. Substituting $x = p/2$ into the equation yields

$$\frac{p^3}{8} - \frac{p^3}{4} + \frac{pq}{2} - r = 0,$$

or

$$8r = 4pq - p^3.$$

III. *Solution by the Proposer.*

The condition is $(a + b - c)(b + c - a)(c + a - b) = 0$, where a, b, and c are the roots of the equation. Expanding out this equation into a symmetric polynomial and then expressing it in terms of elementary symmetric polynomials gives us the result $p(4q - p^2) = 8r$, although the computation is a bit messy.

Also solved by BARRY BRUNSON, *Western Kentucky University*, *Bowling Green*, JAMES E. CAMPBELL, *University of Missouri*, *Columbia*, RUSSELL EULER, *Northwest Missouri State University*, *Maryville*, MARK EVANS, *Louisville*, KY, VICTOR G. FESER, *University of Mary*, *Bismarck*, ND, JACK GARFUNKEL, *Flushing*, NY, ROBERT C. GEBHARDT, *Hopatcong*, NJ, RICHARD I. HESS, Rancho *Palos Verdes*, CA, JOHN M. HOWELL, *Littlerock*, CA, XIAN SHAN HUI, *James Madison High School*, *Brooklyn*, NY, GLEN E. MILLS, *Orange County Public Schools*, *Orlando*, FL, JOHN D. MOORES, *Cambridge*, MA, NORTHWEST MISSOURI STATE UNIVERSITY MATHEMATICS CLUB, *Maryville*, BOB PRIELIPP, *University of Wisconsin-Oshkosh*, JOHN PUTZ, *Alma College*, MI, JOHN H. SCOTT, *Macalester College*, *Saint Paul*, MN, WADE H. SHERARD, *Furman University*, *Greenville*, SC, ARTHUR H.

SIMONSON, East Texas State University at Texarkana, KENNETH M. WILKE, Topeka, KS, and the PROPOSER (second solution). Prielipp found this problem as Exercise 39 on page 446 of Chrystal's *Textbook of Algebra*, vol. 1, 7th ed; Chelsea, 1964.

635. [Fall 1986] *Proposed by John M. Howell, Littlerock, California.*

Our old friend Professor Euclide Pasquali Bombasto Umbugio has been amusing himself in his retirement with problems about infinite series, continued fractions, and other nonterminating expressions. He says that now he has the time to follow through with such computations. So far he has found that $y = \sqrt{x}$ and $y = 1 + x$ do not intersect, and he is working on finding the intersections of the curves $y = (x + \sqrt{x})^{1/2}$ and $y = 1 + x/(1 + x)$. Proceed to the limit and help the good Professor by finding all intersections of the curves defined by the continued expressions

$$y = (x + (x + (x + \ldots)^{1/2})^{1/2})^{1/2}$$

and

$$y = 1 + \cfrac{x}{1 + \cfrac{x}{1 + \cfrac{x}{1 + \ldots}}}$$

for $x > 0$.

Solution by John V. Moores, Cambridge, Massachusetts.

Clearly $x > 0$ implies $y > 0$ for each continued expression. The first expression is equivalent to

$$y = (x + y)^{1/2}, \qquad \text{or} \qquad y^2 = x + y.$$

The continued fraction is

$$y = 1 + \frac{x}{y}, \qquad \text{or} \qquad y^2 = y + x.$$

Hence both expressions represent the same parabola:

$$x = y^2 - y, \qquad x, y > 0.$$

Similar solutions were submitted by JAMES E. CAMPBELL, University of Missouri, Columbia, RUSSELL EULER, Northwest Missouri State University, Maryville, GEORGE P. EVANOVICH, Saint Peters College, Jersey City, NJ, ROBERT C. GEBHARDT, Hopatcong, NJ, RICHARD I. HESS, Rancho Palos Verdes, CA, PETER A. LINDSTROM, North Lake College, Irving, TX, BOB PRIELIPP, University of Wisconsin-Oshkosh, JOHN H. SCOTT, Macalester College, Saint Paul, MN, KENNETH M. WILKE, Topeka, KS, and the PROPOSER.

636. [Fall 1986] *Proposed by Walter Blumberg, Coral Springs, Florida.*

a) Prove that if p is an odd prime, then $1 + p + p^2$ cannot be a perfect square or a perfect cube.

*b) Is part (a) true when p is not prime?

I. *Solution for the square case by Robert C. Gebhardt, Hopatcong, New Jersey.*

Assume that $1 + p + p^2$ is a square, say k^2. Then

$$p^2 + p + (1 - k^2) = 0, \qquad \text{so} \qquad p = -\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{4k^2 - 3}.$$

The only square that is three less than another square is 1, which occurs under the radical here when $k = 1$ or -1, in which case $p = 0$ or -1. Thus $p^2 + p + 1$ is never a square for prime p and is a square for integral p only when $p = -1$ or 0.

II. *Solution for the cube case by Kenneth M. Wilke, Topeka, Kansas.*

Clearly $p^2 + p + 1 = s^3$ has integral solutions for $p = -1$ and for $p = 0$. Let us assume the equation holds for any other integers p and s. Then $s > 1$ and odd, and $|p| > s$. Thus $t = s - 1$ is a positive integer and, substituting $t + 1$ for s, we get that

$$(1) \qquad p(p + 1) = t(t^2 + 3t + 3).$$

Since $|p| > s > t$, then t divides p or $p + 1$. If $t \mid p$, then we write $tj = p$ for some integer j. Then Equation (1) reduces to

$$tj^2 + j = t^2 + 3t + 3,$$

and applying the quadratic formula, we find that

$$t = \frac{-(3 - j^2) \pm \sqrt{(3 - j^2)^2 - 4(3 - j)}}{2}.$$

We let $F_1 = (3 - j^2)^2 - 4(3 - j)$.

If, on the other hand, $t \mid p + 1$, we take $tk = p + 1$, so that $p = tk - 1$. Again substitute into Equation (1) and simplify to get

$$tk^2 - k = t^2 + 3t + 3,$$

whose solution is

$$t = \frac{-(3 - k^2) \pm \sqrt{(3 - k^2)^2 - 4(3 + k)}}{2}.$$

Here we let $F_2 = (3 - k^2)^2 - 4(3 + k)$.

The two cases yield equivalent expressions for F_1 and F_2, as is seen by replacing j by $-k$. Thus we look at $F = F_2$, without loss of generality. We readily check that $F = (3 - k^2)^2 - 4(3 + k)$ is not a square for any integer k from −2 through 4. For k > 4,

$$(k^2 - 3)^2 > F > (k^2 - 4)^2.$$

Since F lies strictly between two adjacent squares, F cannot be a square. For k < −3,

$$(k^2 - 3)^2 < F < (k^2 - 3)^2,$$

and again F cannot be a square. If k = −3, then t = 0 or 6, and we have the solutions $(p, s) = (-1, 1)$ or $(-19, 7)$. Similarly taking $j = 3$, again t = 0 or 6 and $(p, s) = (0, 1)$ or $(18, 7)$. We see that

$$p^2 + p + 1 = s^3$$

has just four integral solutions, no solution where p is a positive prime, although some texts do allow −19 to be called prime.

III. ***Solution by David E. Penney, The University of Georgia, Athens, Georgia.***

The first of these assertions is easy to establish. Use the inequality

$$x^2 < x^2 + x + 1 < (x + 1)2$$

for $x > 0$, together with the observation that no square lies properly between the squares of two consecutive integers. Similarly the inequality

$$(x - 1)^2 < x^2 - x + 1 \leq x2$$

for $x < 0$ shows that there are no solutions other than $x = 0$ and -1.

Now we turn our attention to the equation

$$1 + x + x^2 = y^3.$$

Suppose that it holds for some integers x and y. Then

$$64x^2 + 64x + 64 = 64y^3$$

and

$$(8x + 4)^2 + 48 = (4y)^3.$$

This equation is of the form $u^2 + 48 = v^3$ and is discussed at length on pages 246 - 247 of L. J. Mordell's *Diophantine Equations* (New York: Academic Press, 1969). He states that this equation has "only the solutions $(u, v) = (\pm 4, 4), (\pm 148, 28)$ and is of particular interest." Mordell refers us to W. Ljunggren, "Einige Gleichungen von der Form $ay^2 + by + c = dx^3$," Vid. Akad. Skrifter Oslo, Nr. 7 (1930). The solutions Mordell lists give rise to the complete list of solutions we obtained above.

IV. ***Comment by H. Abbott and M. S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.***

In "*Note sur l'equation indeterminée* $(x^n - 1)/(x - 1) = y^2$," Norsk Matematisk Tidsscrift 2 (1920) 75–76, Trygve Nagel has shown that the equation $(x^n - 1)/(x - 1) = y^2$ has no integral solutions for n = 7, 9, 11, and 25; that there are solutions for n = 4 (e.g. x = 7) and n = 5 (e.g. x = 3). He has also shown that there are no solutions to $x^2 + x + 1 = 3y^q$ for $|y| > 1$ and $q > 3$. In "A Note on the equation $n^2 + n + 1 = p^r$," Math. Mag. 37 (1964) 339–340, J. P. Hurling and V. H. Keiser consider solutions where p is prime. They determine various conditions on n, p, and r for solutions to exist. By means of a computer, they have shown that for $r > 1$, there is only one solution for $n < 180{,}000$, namely $n = 18$, $p = 7$, and $r = 3$ (corresponding to the one given above).

V. ***Comment by Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin.***

It may be of interest to note that $1 + 3 + 3^2 + 3 + 3^4 = 11^2$.

Also solved by H. ABBOTT ***and*** M. S. KLAMKIN, ***University of Alberta, Canada.***, MARK EVANS ***(solution for squares), Louisville,*** KY, VICTOR G. FESER ***(solution for squares), University of Mary, Bismarck, ND,*** ROBERT C. GEBHARDT ***(answer for cubes), Hopatcong, NJ,*** RICHARD I. HESS ***(solution for squares, answer for cubes), Rancho Palos Verdes, CA,*** BOB PRIELIPP, ***University of Wisconsin-Oshkosh,*** KENNETH M. WILKE, ***Topeka, KS, and the*** PROPOSER ***(solution to part (a)).***

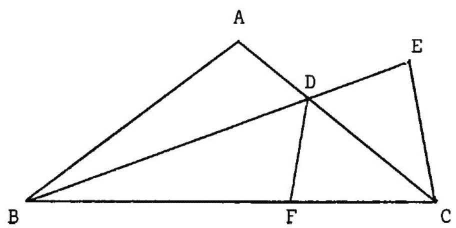
637. [Fall 1986] ***Proposed by R. S. Luthar, University of Wisconsin Center, Janesville, Wisconsin.***

Let ABC be a triangle with $\measuredangle ABC = \measuredangle ACB = 40^\circ$. Let BD be the bisector of $\measuredangle ABC$ and produce it to E so that DE = AD. Find the measure of $\measuredangle BEC$. See the figure on the next page.

Solution by Harry Sedinger *and* Charles R. Diminnie, *St. Bonaventure University, St. Bonaventure, New York.*

Choose point F on BC such that $\sphericalangle BDF = \sphericalangle BDA$. Then, by ASA, triangles ABD and PBD are congruent. Therefore $DF = AD = DE$. We see that $\sphericalangle ADB = \sphericalangle CDE = 60^\circ$. Then $\sphericalangle CDF = 60^\circ$, too. Thus, by SAS, triangles CDP and CDE are congruent and we get that

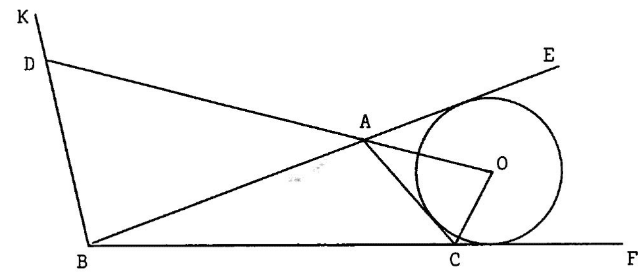
$$\sphericalangle BEC = \sphericalangle DEC = 120^\circ - WE = 120^\circ - \sphericalangle DCF = 120^\circ - \sphericalangle ACB = 80^\circ.$$



Also solved by RUSSELL EULER, *Northwest Missouri State University, Maryville,* JACK GARFUNKEL, *Flushing, NY,* ROBERT C. GEBHARDT, *Hopatcong,* NJ, RICHARD A. GIBBS, *Fort Lewis College, Durango, CO,* PETER GEISER, *St. Ctoud State University, MN,* RICHARD I. HESS, *Rancho Palos Verdes, CA,* RALPH KING, *St. Bonaventure University, NY,* JOHN D. MOORES, *Cambridge, MA,* NORTHWEST MISSOURI STATE UNIVERSITY MATHEMATICS CLUB, *Maryville,* JOHN H. SCOTT, *Macalester College, Saint Paul, MN,* WADE H. SHERARD, *Furman University, Greenville, SC,* ARTHUR H. SIMONSON, *East Texas State University at Texarkana,* KENNETH M. WILKE, *Topeka, KS, and the* PROPOSER. *Some of the solvers used the law of sines.* GLEN E. MILLS, *Orange County Public Schools, Orlando, FL, using decimal values, found the angle to within* 0.05'. *One other submission assumed that D is the midpoint of AC and a final paper erred in applying the law of sines, each obtaining a wrong answer.*

638. *[Fall 19861 Proposed by* R. S. Luthar, *University of Wisconsin Center, Janesville, Wisconsin.*

In the figure on the next page, the circle with center O is an excircle of triangle ABC. Then BK is drawn so that $\sphericalangle KBA = \sphericalangle AOC$, and OA is produced to meet BK in D. Prove that $OCBD$ is a cyclic quadrilateral.



Solution by Jack Garfunkel, *Flushing, New York.*

Since AE and AC are tangent to the circle, then OA bisects $\sphericalangle CAE$. Similarly OC bisects $\sphericalangle ACF$. Hence

$$\sphericalangle OAC = \frac{1}{2}\sphericalangle CAE = \frac{1}{2}(180^\circ - \sphericalangle BAC) = 90^\circ - \frac{1}{2}\sphericalangle BAC$$

and similarly

$$\sphericalangle ACO = 90^\circ - \frac{1}{2}\sphericalangle BCA.$$

Hence

$$\sphericalangle AOC = 180^\circ - (WAC + \sphericalangle ACO) = \frac{1}{2}\sphericalangle BAC + \frac{1}{2}\sphericalangle BCA = 90^\circ - \frac{1}{2}\sphericalangle ABC.$$

Now we have that

$$\begin{aligned} \sphericalangle DBC + \sphericalangle DOC &= \sphericalangle DBA + \sphericalangle ABC + \sphericalangle AOC = \sphericalangle ABC + 2\sphericalangle AOC \\ &= \sphericalangle ABC + 180^\circ - \sphericalangle ABC = 180^\circ, \end{aligned}$$

so quadrilateral $DBCO$ is cyclic, since a pair of opposite angles are supplementary.

Similar solutions were submitted by RICHARD A. GIBBS, *Fort Lewis College, Durango, CO,* RALPH KING, *St. Bonaventure University, NY,* HENRY S. LIEBERMAN, *Waban, MA,* JOHN D. MOORES, *Cambridge, MA,* NORTHWEST MISSOURI STATE UNIVERSITY MATHEMATICS CLUB, *Maryville,* JOHN H. SCOTT, *Macalester College, Saint Paul, MN,* MADE H. SHERARD, *Furman University, Greenville, SC,* ARTHUR H. SIMONSON, *East Texas State University at Texarkana, and the* PROPOSER.

In Memoriam

Léo Sauvé taught a backbreaking load of trigonometry and algebra at Algonquin College in Ottawa, Ontario, Canada. Nevertheless, to keep from becoming mathematically stale, in 1975 he founded a small problem journal called **Eureka**, later renamed **Crux Mathematicorum**, "a puzzle or problem for mathematicians." Singlehandedly he was soon editing 10 issues a year, each 30 pages in length, a monumental task for any person. Rarely could you find an error on its pages, so careful was its editor. He personally verified every mathematical statement that appeared on its 300 pages each year. His wit and fluency in both French and English made the journal lively and exciting, as well as informative. Early in 1986 Léo retired because of ill health, after having built *Crux* into a journal with subscribers and contributors throughout the world. He died in June, 1987. Therefore it is our privilege to dedicate this issue of the Problem Department to the memory of Léo Sauvé, a true scholar and friend, and to express our hope that we will always be guided by his spirit in our editing of these pages.

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THE RICHARD V. ANDREE AWARDS

Richard V. Andree, Professor Emeritus of the University of Oklahoma, died on May 8, 1987, at the age of 67.

Professor Andree was a Past-President of Pi Mu Epsilon. He also served the fraternity as Secretary-Treasurer General and as Editor of the Pi Mu Epsilon Journal.

At the summer meetings in Salt Lake City the fraternity Council voted to designate the prizes in the national student paper competition as Richard V. Andree Awards.

First prize winner for 1986-1987 is Wah Keung Chan, McGill University, for his paper "On the Largest RAT-free Subset of a Finite Set of Points." Wah's paper appeared in the Spring 1987 issue. Wah will receive $200.

Second prize winner is Jennifer Zobitz, College of St. Benedict, for her paper "Fractals: Mathematical Monsters." Jennifer's paper is the lead article in this issue of the Journal. Jennifer will receive $100.

Third prize winner is Kelly Ann Chambers, University of Dayton, for her paper "The Isomorphism of the Lattice of Congruence Relations on a Group and the Lattice of Normal Subgroups of a Group." Kelly Ann's paper also appears in this issue of the Journal. Kelly Ann will receive $50.

Congratulations Wah, Jennifer and Kelly Ann.

1987 NATIONAL PI MU EPSILON MEETING

The National Meeting of the Pi Mu Epsilon Fraternity was held at the University of Utah in Salt Lake City on August 5 through August 8. Highlights included a reception for students, faculty advisors, and alumni, a Council Luncheon and business meeting, the Annual Banquet, and informal student parties. The J. Sutherland Frame Lecturer was Professor Clayton W. Dodge, editor of the Journal's Problem Department. He entertained his listeners with "Reflections of a Problem Editor."

The program of student papers included:

Deterministic and Probabilistic Fire Models	Warren E. Blaisdell Massachusetts Alpha Worcester Polytechnic Institute
The Epidemiology of the AIDS Virus	Aaron Klebanoff California Lambda University of California, Davis
The Strangely Attracted Bouncing Ball	Stephanie Ruth Land Texas Lambda University of Texas
Applications of Signal Processing	Debra Shale Massachusetts Delta University of Lowell
Self-Calibration of Complex Visibility Data from a Very Large Array of Antennas	Ali Safaei-nili Iowa Alpha Iowa State. University
Representations and Characters of Groups	Ken Chick Ohm Delta Miami University
An Algebraic Construction of a Projective Geometry	Stephanie Dumoski California Theta Occidental College
The RSA Public Key Cryptosystem: An Application for Modem Algebra	Stephen Fiete West Virginia Alpha West Virginia University
16/64 = 1/4 and Other ii-digit Canoellations	David A. Messineo Connecticut Beta University of Hartford

An Algebraist's View of Competitive Games	Erlan Wheeler II Virginia Beta Virginia Polytechnic Institute
Perpendicular Least Square Estimators	Brian Anderson Kentucky Beta Western Kentucky University
The Isoperimetric Inequality	Jeff Diller Ohio Zeta University of, Dayton
Inverting a Pin in R^2	Russell Godwin Arkansas Alpha University of, Arkansas
Fourier Series and the "Best" Mean Square Approximation	Margaret M. Lineberger North Carolina Delta East Carolina University
Dynamic Programming Applied to Computer Voice Recognition	Thomas Eugene Gibbons Minnesota Delta St. John's University
A Formal Sum Method Approach to the Traveling Salesman Problem	Jeffrey Horn Wisconsin Alpha Marquette University
A Bit of Checking and Correcting	Summer Quimby Wisconsin Delta St. Norbert College
A Graphical Illustration of the Covergence of Karmarkar's Linear Programming Algorithm	Josef S. Crepeau Montana Atpha University of Montana
The Existence of Eulerian and Hamiltonian Circuits in Graphs and Their Line Graphs	Carol Parker Arkansas Beta Hendrix College
Circuit Spaces and Cut-Spaces of a Connected Graph	Michael Tackett Ohio Delta Miami University
Yet Another Discussion of Graceful Graphs	Andrew P. Ferreira Massachusetts Alpha Worcester Polytechnic Institute
Sub-families of Venn Diagrams	Philip Beymer O-tegon Alpha University of Oregon
Soap Films as Minimal Surfaces	George Mader Minnesota Delta St. John's University

Theology, Mathematics and Meaning	Mary Ehle Wisconsin Delta St. Norbert College
Games of Timing with Two or Three Players	Timothy P. Ronan Pennsylvania Omicron Moravian College
A Glimpse at the Theory of Restricted Choice	Scott Krutsch Indiana Gamma Rose-Hulman Institute of Technology

The following papers were presented by students in the Mathematical Association of America Student Paper Session, held jointly with the Pi Mu Epsilon Paper Session.

Curves Length Minimizing Modulo v in R^n	Jeff Abrahamson Massachusetts Institute of Technology
NP-Completeness and the Traveling Salesman Problem	Melanie K. Breaker Northeast Missouri State University
The Classical 'Problems of Antiquity in the Hyperbolic Plane	Robert Curtis University of California, Santa Cruz
A Physical Derivation of the Well Tempered Musical Scale	Timothy Koponen Aquinas College

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Editor's Note

The Pi Mu Epsilon Journal was founded in 7949 and is dedicated to undergraduate and beginning graduate students interested in mathematics. Submitted articles, announcements and contributions to the Puzzle Section and P-tobtem Department of, the Journal should be directed toward this group.

Undergraduate and beginning graduate students are urged to submit papers to the Journal for consideration and possible publication. Student papers m e given top' priority.

Expository articles by professionals in all areas of mathematics are especially welcome.

This issue contains three student papers. Each year, the National Paper Competition awards prizes of $200, $100 and $50, p-tovided that at least five student papers have. bun submitted to the. Editor. All students who have not yet received a Master's Degree, or higher, are eligible for these awards. Awards for 1986-1987 are announced on page 488 of this issue.

GLEANINGS FROM CHAPTER REPORTS

CONNECTICUT GAMMA (Fairfield University). *Barri Schoch,* a charter member, represented the chapter at the National Pi Mu Epsilon Meeting held during the International Congress of Mathematicians in Berkeley, CA. She presented the paper "The Mathematical World of Cryptology."

During the Fall semester the chapter sponsored two lectures. *Robert Bolger,* Fairfield University, spoke on "The Influence of the Mathematishe Institut of the Georg-August Universität (in Göttingen, West Germany) on the 20th Century Mathematics and Physics: An Incredible Scenario." *David Burry,* Perkin-Elmer, described his "Experiences as a Software Engineer."

In the Spring, as part of the annual initiation ceremony, *Stuart J. Sidney,* University of Connecticut, spoke on "The Pigeon Hole Principle and Geometry."

During the Annual Arts and Sciences Awards Ceremony, three members, *Tatiana Foroud, Sandra Jacopian* and *Patricia Jarzabek* received recognition for their outstanding performance in mathematics. Each was given a Pi Mu Epsilon certificate of achievement, a copy of Hofstadter's *Gödel, Escher, Bach: An Eternal* Golden *Braid,* and a one-year membership in the Mathematical Association of America.

GEORGIA BETA (Georgia Institute of Technology). At the 1987 Honors Program *Tracey Redding* received a mathematics book of her choice. Each year book awards are presented to students receiving the B.S. degree in Applied Mathematics with a grade point average of at least 3.7 (A = 4.0) in all mathematics courses taken.

KANSAS GAMMA (The Wichita State University). From mid-August through the end of June, 1986-1987, sixteen lectures/talks were sponsored. The talks ranged from magic squares to wind tunnel computations, from math humor on a Sunday afternoon to the real projective plane, from Rubik's cube to superstrings, from Fibonacci numbers to unsolved problems in fluid dynamics. At the 119th annual meeting of the Kansas Academy of Sciences, held at Wichita State University in April 1987, *G.B. Ross, P.S. Mangat* and *S. Shah* contributed papers. The chapter dedicated its activities in 1987 to thememory of the Indianmathematician *S. Ramanujan* in honor of his 100th birthday. *Dr. J.S. Rao* spoke on "Renaissance of Science in India in the Early 20th Century and Contributions by Ramanujan." In November 1986 sixty students participated in a bi-level competition in college algebra.

Editor's Note.

Additional gleanings from chapter reports will be published in the Spring 1988 issue of the Pi Mu Epsilon Journal.

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