PI MU EPSILON JOURNAL

VOLUME 8 SPRING 1988 NUMBER 8

CONTENTS

PI MU EPSILON JOURNAL

VOLUME 8 SPRING 1988 NUMBER 8

CONTENTS

PI MU EPSILON JOURNAL
THE OFFICIAL PUBLICATION OF THE NATIONAL HONORARY MATHEMATICS SOCIETY

EDITOR
Joseph D.E. Konhauser

ASSOCIATE EDITOR
Clayton W. Dodge

OFFICERS OF THE SOCIETY
President: Eilwn L. Poiani, Saint Peter's College
President-Elect: David W. Ballew, Western Illinois University
Secretary-Treasurer: Robert M. Woodside, East Carolina University
Past-President: Milton D. Cox, Miami University

COUNCILORS
Robert C. Eslinger, Hendrix College
J. Douglas Faires, Youngstown State University
Richard A. *Good,* University of Maryland
Richard L. Poss, St. Norbert College

correspondence, including books for review, chapter reports, news items and manuscripts (two copies) should be mailed to EDITOR, PI MU EPSILON JOURNAL, Mathematics and Computer Science Department, Macalester College, St. Paul, MN 55105. Students submitting manuscripts are to identify their college or university and their class or expected graduation date. Others to provide their affiliation, academic a otherwise.

Problems for solution and solutions to problems should be mailed directly to the PROBLEM EDITOR. Puzzle proposals and puzzle solutions should be mailed to the EDITOR.

The **PI MU EPSILON JOURNAL** is published at Macalester College twice a year—Fall and Spring. One volume consists of five years (10 issues) beginning with the Fall 19x4 or Fall 19x9 issue, starting in 1949. For rates, see inside back cover.

THE CHINESE REMAINDER THEOREM: A HISTORICAL ACCOUNT

by *David J. Matkovic*
Iowa State *University*

Indeterminate analysis is a branch of mathematics that has been studied and practiced since the middle of the third century. An indeterminate problem is one which leads to K algebraic equations in more than K unknowns, and very often has an infinite number of solutions. One area in the large subject of indeterminate analysis is the study of linear indeterminate equations, or indeterminate equations of the first degree. These equations are of the following form:

$$ax + by = a.$$

One of the more interesting and applicable aspects of linear indeterminate analysis is the remainder problem. This problem, also known as the "Chinese remainder theorem," has its roots in China, but was also developed in India, the Arabic world, and Europe [7, p. 214].

The oldest known instance of the Chinese remainder theorem is in the work of Chinese mathematician Sun-Tsu. It is not known exactly when Sun-Tsu was alive, but his work on this theorem is believed to be work of the fourth century A.D. [2]. This paper is devoted to the exploration of that idea which Sun-Tsu wrote on sixteen centuries ago. Included is a description of what the Chinese remainder theorem is, the history of its development, a discussion of its proof, and, finally, a look at its useful applications.

The Chinese remainder theorem finds an unknown number given its remainders when divided by various divisors. The problem's general structure is as follows:

$$N \equiv a_1 m_1 + r_1 \equiv a_2 m_2 + r_2 \equiv \dots \equiv a_3 m_3 + r_3 \quad [2].$$

A typical statement of the Chinese remainder theorem found in modern books of algebra would be much like this one taken from W. E. Deskins' *Abstract Algebra* [5, p 140].

"(Chinese Remainder Theorem). If the n positive integers $m_1, m_2, \dots, m_n$ are relatively prime by pairs (i.e., $(m_i, m_j) = 1$ if $i \neq j$), then the set of n congruences

$$x \equiv a_1 \bmod m_1, \ldots, x \equiv a_n \bmod m_n$$

has a solution x'. Moreover the solutions of the set are exactly the elements of congruence class $[x']$ modulo $m_1 m_2 \cdots m_n$."

Over the course of history the Chinese gave several names to the remainder theorem. The different names were taken either from a method of computation or a different application for the theorem. Some of these names are as follows:

1. Ta yen Ch'ie i shu (the method of seeking unity).
2. Chein kuan shu (the method of cutting lengths of tubes).
3. Ch'in Wang an tien ping (the Prince of Ch'in's secret method of counting soldiers).
4. Fu she chih shu (the method of repeated shots).
5. Kuei Ku suan (the computations of Kuei Ku).
6. Ke Ch'iang suan (behind the wall computations) [11, p. 3231.

Over the span of time that the remainder theorem was being developed, it attracted the attention of many great mathematicians, the first being Sun-Tsu. As stated earlier, Sun-Tsu's work on the remainder theorem is thought to have been done in the fourth century A.D. Although most historians agree that this general span of time is correct, some consider Sun-Tsu to be the same person as Sun-Wu, the celebrated tactician in the sixth century B.C. Sun-Wu was usually referred to as Sun-Tsu by way of respect. Archaeologist Chu I-tsun (1629-1709) held this belief. The view of mathematician Tai Cheng (1722-1777) is more widely accepted. Cheng believed that Sun-Wu and Sun-Tsu were not one and the same because the work of Sun-Tsu contains some words which are definitely known not to have been used in Sun-Wu's time. Yuan Yuan (1764-1849), a mathematical historian, agrees with Cheng and strengthens his argument by stating that we cannot take Sun-Tsu's work to be older than the Arithmetic in Nine Sections, which was written by an unknown author around 200 B.C. This is because Sun-Tsu greatly surpasses the Nine Sections in the elaborateness of his descriptions. So the opinion of Chu I-tsun is generally thought to be incorrect. the mathematician Sun-Tsu was a different person than Sun-Wu, and Sun-Tsu was active in approximatelythe fourth century A.D. [8, pp. 25-30].

Now having the answer to when the work of Sun-Tsu took place, let's look into the actual piece of work which contains the Chinese remainder theorem. Sun-Tsu wrote a book called the Sun-Tsu *Suan-Ching* which literally means the Arithmetical Classic of Sun-Tsu. This work is one of the earliest preserved Chinese textbooks on arithmetic. It is from this book we get our insight on how the ancient Chinese had treated the arithmetical operations of multiplication and division [8, p. 351. The Sun-Feu *Suan-Ching* consists of three volumes. In its twenty-sixth problem, third volume, Sun-Tsu wrote the following:

> "We have things of which we do not know the number; if we count them by threes, the remainder is 2; if we count them by fives, the remainder is 3; if we count them by sevens, the remainder is 2. How many things are there:
>
> Answer 23
>
> Method:
>
> If you count by threes and have remainder 2, put 140.
> If you count by fives and have remainder 3, put 63.
> If you count by sevens and have remainder 2, put 30.
> Add these numbers and you get 233.
> From this subtract 210 and you have the result.
> For each unity as remainder when counting by threes, put 70.
> For each unity as a remainder when counting by fives, put 21.
> For each unity as a remainder when counting by seven, put 15.
> If the sum is 106 or more, subtract 105 from this and you get the result." [2]

In more modern algebraic symbols, the description above would be:

$N \equiv 2 \pmod 3$, $3 \equiv 3 \pmod 5$, $N \equiv 2 \pmod 7$.
$70 \equiv 1 \pmod 3 \Rightarrow 140 \equiv 2 \pmod 3$
$21 \equiv 1 \pmod 5 \Rightarrow 63 \equiv 3 \pmod 5$
$15 \equiv 1 \pmod 7 \Rightarrow 30 \equiv 2 \pmod 7$
$N = 140 + 63 + 30 - n \cdot 105 = 23$ [7, pp. 269-270].

From this problem of Sun Tsu's, the meaning behind the remainder theorem was established. Notice that there are an infinite number of solutions to the last equation based on the value given to n. In this case, Sun-Tsu was looking for only the smallest positive solution. As is easily seen, it is a highly structured procedure and if the procedure is followed, the calculations are quite simple. The above is essentially the ta-yen rule of Sun-Tsu, which is equivalent to the use of congruences. This book did not contain a generalization of this theorem and did not give a proof. However, the Chinese of this era were not generally known to practice the art of proving theorems. Now that the reader understands what the Chinese remainder theorem is and is also informed about its roots in the time of Sun-Tsu, let's look into the history of the further development of the Chinese remainder theroem.

Between the time of Sun-Tsu and the time of Ch'in Chiu-shao (a great thirteenth century Chinese mathematician whose work will be discussed later), the progress in the area of indeterminate analysis, including the remainder theorem, was relatively small in China. There were many Chinese mathematicians who attempted to make gains, but none were terribly successful. The ta-yen rule, which Sun-Tsu used in a non-general form, to solve some of his problems, was not expanded until the time of Ch'in Chiu-shao eight centuries later. This leaves a gap of approximately eight centuries when progress on the remainder problem in China was stagnated. The lack of progress in China did not, however, stop the neighboring Hindus of India from picking up the slack. Between 500 A.D. and 1200 A.D., roughly, the Hindus of India were quite active in the broad field of indeterminate analysis as were the Arabs to a lesser extent. The leading mathematicians of that span were Aryabhata, Bhaskara I, Brahmagupta, Mahavira, Aryabhata II and Bhaskara II. The Hindus were very much interested in the remainder theorem and were also capable of putting it to good use. This work done by the Hindus in that large time span was instrumental in keeping the realization of the remainder theorem alive [7, pp. 214-217].

It was only about 100 years after Sun-Tsu that Aryabhata was in his prime as a mathematician. In Aryabhata's work, which is now known as the *Arbhatiya*, he gave a method for the remainder problem that became known as the kuttaka method. Kuttaka was the Hindu's version of the ta-yen rule. The word kuttaka is derived from the root kutt, which means "to pulverize." So the method is referred to as either kuttaka or the pulverizer. This method was the backbone of Hindu indeterminate analysis and is basically the method of continued fractions. (The relationship between the kuttaka and ta-yen methods will be discussed later.) Aryabhata knew the solution method whose general structure is given on page 493. The pulverizer was implemented in that method also [4, pp. 127-139].

Disciples of Aryabhata, such as Bhaskara I and Brahmagupta, reinforced the method of kuttaka by using it effectively, and hence sustaining its popularity, These men were not only mathematicians, but were interested in the remainder problem's application to the stars. That is, the general purpose of their work seemed to be astronomical (in a literal sense) [10, pp. 121-123]. This is why one of the Hindus' greatest advances in this area was the generalization of the conjunct pulverizer, or samslistakuttaka. This took place from the mid-ninth century until the thirteenth century. The conjunctness was useful because it put the problem in a multi-equation form.

$$b_1y_1 = a_1x \pm c_1$$

$$b_2y_2 = a_2x \pm c_2$$

$$b_3y_3 = a_3x \pm c_3$$ [4, pp. 136-137].

The generalization was important because it let the divisors as well as the remainders vary. This gave much more opportunity to use it in a real setting. On account of its important applications in mathematical astronomy, this modified system has received special treatment in the hands of Hindu algebraists from Aryabhata II (950 A.D.) onwards. So, it was this peripheral interest of the Hindus in astronomy which fueled the strengthening of the remainder theorem during that span of time when, for one reason or another, there is a lack of Chinese writing on the subject.

Now we're into the early thirteenth century, and by this time Europe was getting its first taste of the Chinese remainder theorem. The oldest example of an indeterminate problem in Europe appears in the *Propositiones* ad *acuendos juvenes*, which is attributed to Alain (730 A.D. - 804 A.D.). However, this problem had nothing to do with the remainder problem. The first actual appearance of the remainder problem in Europe is from the *Liber* Abaci (1202) of Leonardo Pisano better known as Fibonacci. There are two problems that use the ta-yen rule much like the way Sun-Tsu used it. In fact, like Sun-Tsu, Fibonacci did not give the slightest theoretical or general explanation of his method for the solution of the remainder problem [7, p. 241].

By the middle of the thirteenth century it was finally time for the Chinese to step back onto center stage as the main developers of the theorem which was later named after them. The main reason for this resurgence is the work of Ch'in Chiu-shao. The Shu-shu *Chiu-chang* is the work of Ch'in which is dated 1247 [8, p. 51]. *Shu-shu Chiu-chang* translated is *Nine* Sections of *Mathematics.* This is not to be confused with the much older *Arithmetic* in Nine Sections mentioned earlier. The thirteenth century was a time of terror for China. It suffered attack after attack by the Mongolians and constantly lost territory until, finally, China was overrun and destroyed in 1279 [8, pp. 63-78]. The environment of this era makes Ch'in's advances look even more impressive. During these hard times Ch'in Chiu-shao was writing down the process of solving numerical equations of all degrees. This is also when Ch'in recorded the long awaited generalized version of the ta-yen rule. With this work by Ch'in, he includes and solves ten remainder problems. These problems are worked out at great length, so instead of going through several problems, some will be listed to give the reader an idea of what Ch'in had in mind for his new generalization.

1. To compute the amount of money in the treasuries.
2. Checking old calendars.
3. To erect a dyke.
4. Division of agriculture.
5. Choosing the right size bricks for a building foundation [7, pp. 382-4121.

With the writings of Ch'in Chiu-shao known to all the ta-yen rule has attracted all of the great mathematicians into refining it. It was refined over and over again until T. J. Stieltjes' version (1890) was written. This was ranked the best method by a panel of mathematicians in the early 1970's [7, p. 380]. Stieltjes' version ranked ahead of Ch'in's work, but taking into account the early date of Ch'in's work, it is clear that Ch'in laid the foundation for the application of the ta-yen rule.

The actual listings of the steps in the Chinese method of ta-yen and the Hindus' method of Kuttaka are much too long to include in this paper. This being the case, we would like to tell the reader where they can be found. The list of steps to the ta-yen method by Ch'in Chiu-shao can be found in the writings of Ulrich Libbrecht [7, pp. 328-332]. The method of kuttaka will be found in the same publication [7, pp. 359-366], and in chapter 13 of Datta's book [4]. These sources

should be quite helpful to the curious reader.

One aspect of the kuttaka and ta-yen methods that we discuss is their relationship to each other. For centuries there has been a continuing controversy between mathematical historians over which method was first. That is, they are arguing over which method was derived from the other. Historian A. Wylie, while speaking of the ta-yen rule, states:

> "This appears to be the formula, or something very like it, which was knownto the Hindus under the name of kuttaka, or as it's translated 'Pulverizer,' implying unlimited multiplication which is not so far from the meaning of ta-yen or 'Great Extension.'" [7, pp. 159-1601

Historians of this point of view were essentially saying that we might be giving far too much credit to Ch'in-Chiu shao for his generalization. The most recent argument found on this topic seems to be the most sensible and is backed up by extensive comparisons. This is the argument given in 1973 by Libbrecht in the same book mentioned above. His argument concludes that the two methods are different and he states that "it makes no sense to accept the idea of a historical relationship between the Chinese ta-yen rule and the Indian kuttaka. We will never be sure if one was the product of the other. We know that they were both very instrumental in the development of the Chinese remainder theorem" [7, p. 366].

Before talking about some of the most useful applications of the Chinese remainder theorem we first discuss proving such a theorem. There are several proofs of the theorem mainly because there are several variations of its method. As stated before, Sun-Tsu and others of that very early time did not include a proof either because of their lack of generalization or simply just not seeing its importance. What will follow is the proof of the Chinese remainder theorem as it was stated on page 2 (as given in W. E. Deskins' Abstract Algebra). I will first state a lemma which is referred to in the proof:

Lemma. "There exists an integer x such that $x \equiv a \bmod m$ and

$$x \equiv b \bmod n,$$

where a, b, m, and n are positive integers, if and only if

$$a \equiv b \bmod (m,n).$$

If x and y satisfy the congruences, then

$$x \equiv y \bmod \langle m,n \rangle."$$

'**Proof** of CRT. Proof by induction. "Suppose the proposition is true for the positive integer k and consider the set of $k + 1$ congruences

$$x \equiv a_1 \bmod m_1, \ldots, x \equiv a_{k+1} \text{ mod,, where } (m_i, m_3) = 1 \text{ for } i \neq j.$$

By the induction hypothesis the set of k congruences,

$$x \equiv a_2 m_2, \ldots, x \equiv a_{k+1} \bmod m_{k+1},$$

has a solution r, and the elements of $[r]$ modulo m. Where

$$m \equiv m_2 m_3, \ldots, m_{k+i},$$

are precisely the solutions of the set.

Now we consider

$$x \equiv r \bmod m \text{ and } x \equiv a_1 \bmod m_1,$$

First we see that the integers m and m_1 are relatively prime since a prime divisor of m is necessarily a divisor of exactly one of the k integers $m_2, \ldots, m_{k+1}$. Since these numbers are relatively prime to m_1, we conclude that

$$(m, m_1) = 1.$$

Therefore by (the previous lemma) this pair of congruences has a solution x', and moreover the solutions of the pair are exactly the elements of the congruences class $[x']$ modulo mil,. Then x' is certainly a solution of the $k + 1$ congruences above, and it is easy to see that the set of solutions of these $k + 1$ congruences is exactly the class $[x^t]$ modulo $m_1 m_2 \ldots m$. After all, a solution of these $k + 1$ congruences is certainly a solution of the k congruences from above as well as

$$x \equiv a_1 \bmod m_1.$$

This concludes the induction, and proves the theorem." [5, p. 140-141]

This was an example of a relatively modern proof. However, the interested reader can find different proofs of the Chinese remainder theorem in most abstract algebra books. A particularly interesting proof of Ch'in's ta-yen method, which is a bit more historical in procedure but not as easily followed, can be found in the writing of I-Chen Chang [2].

Having given the background and the content of the Chinese remainder theorem, the paper will conclude with a discussion on applications of the theorem. We can now ask: Why were Sun-Tsu, Aryabhata, Brahmagupta, Bhaskara II, Fibonacci, Ch'in-Chui-shao, and all the others interested in the solution of problems of this kind? As stated earlier, one big reason was an interest in astronomy. You ask: what about astronomy, how can the Chinese remainder theorem be used in astronomy to produce something useful? There are many ways. Perhaps one of the most famous applications is the computation of the Julian period.

Some 400 years ago, in 1593 Joseph Justice Scaliger introduced a period of 7980 years that is known as the Julian period upon which the Julian Day system is based. This system, as it was developed, counts days sequentially from January 4713 B.C., which is defined as Julian day number 0. The question is why or how did Scaliger choose 4713 B.C. to be the base year. This was done by using three divisors and their product (7980) and plugging them into the remainder theorem. What were these divisors? One was 28, the number associated with the solar cycle of the calendar (every 28 years the days of the week occur on the same calendar day). Next was 19, the number of years in the Metonic cycle of lunar phases (every 19 years the phases of the moon occur on approximately the same day of the year). The third divisor is 15 (the number of years in the later Roman Empire's taxation cycle, or indictio). These numbers, when manipulated according to the Chinese remainder theorem, produce 3268 as the nearest base year. However, we need a year in the past on which to base this calendar. So we subtract 7980 to obtain -4712 which is the year 4713 B.C. [3].

There are many other calendar type problems in which this theorem was used. Other types of problems are also numerous. Counting problems such as the one solved on page 495 are a good use for this theorem. As mentioned before, Ch'in-Chiu shao was interested in problems on construction and agriculture. Another problem Ch'in expressed interest in is the transportation problem. This problem was stated and solved in his *Shu-shu-chiu-chang*.

> "A military unit wins a victory. At 5:00 A.M. they send three express messengers to the capitol where they arrive at different times to announce the news. A arrives several days earlier at 5:00 P.M., B arrives

several days later at 2:00 P.M., and C arrives todav at 7:00 A.M. According to their statements, A covered 300 li a day, B 240 li a day, C 180 li a day. Find the number of li from the army to the capitol and the number of days spent by each messenger." [7, p. 401]

The Chinese remainder theorem is, in fact, easily applicable to real life situations. These examples given above are just a few of the many problems this theorem handles. Perhaps the reader can now see why the author finds the Chinese remainder theorem one of the most interesting mathematical concepts of its time. Its simplicity, applicability and uniqueness make this theorem especially attractive. Approximately 1500 years after Sun-Tsu recorded his thoughts, there still exists utmost appreciation for the Chinese remainder theorem.

REFERENCES

1. Arazi, B., *A generalization of the Chinese remainder theorem*, Pacific Journal of Mathematics, 79 (1977), no. 2, 289-296.
2. Chang, I-Chen, *The ancient Chinese pearl in number theory: the Chinese remainder theorem*, International Journal of Mathematical Education and Science Technology, 11 (1980), no. 4, 545-556.
3. Craun, E. D., *The origin of the Julian period: an application of congruences and the Chinese remainder theorem*, American Journal Physical, 49 (1981), no. 7, 658-661.
4. Datta, B., *History of Hindu Mathematics*, Allhabad Law Journal Press, 1938.
5. Deskins, W. E., *Abstract Algebra*, MacMillan Company, 1964.
6. Lee, K. H., *The generalized Chinese remainder theorem*, Bulletin of Malaysian Mathematical Society, 1 (1980), 1-4.
7. Libbrecht, U., *Chinese Mathematics in the Thirteenth Century*, MIT Press, 1973.
8. Mikami, Y., *The Development of Mathematics in China and Japan*, Chelsea Publishing, 1974.
9. Ramankutty, P., *The Chinese remainder theorem*, Journal of Korean Mathematics, 11 (1975), 5-6.
10. van der Waerden, B. L., *Geometry and Algebra in Ancient Civilization*, Springer-Verlag, 1983.
11. Yong, L. L., *A Critical Study of the Yan Hui Suan Fa*, Singapore University Press, 1977.

About the author -

David wrote this paper while a mathematics major and a student in the College of Science and Humanities at Iowa State University in 1986.

About the paper -

The paper was written for a course offered by Dr. J. D. H. Smith.

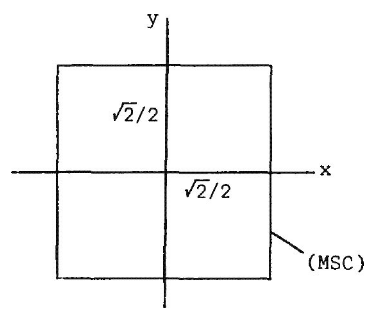
MAXIMUM TRIGONOMETRY

by Richard Bertram
Florida State University

Upon reading the recent paper by Ruth Brisbin and Paul Artola [1] on Taxicab trigonometry, I noticed a strong resemblance between the taxicab norm and the maximum norm. In particular, I found that if one defines the Maximum $\sqrt{2}/2$ Circle (MSC) in one coordinate system and then carries out a 45 coordinate transformation so that the Taxicab Unit Circle (TUC) can be defined in this transformed coordinate system, then those points forming the MSC and those forming the TUC are identical. This observation led me to formulate the following Maximum trigonometry.

Recall that if $x = (x_1, x_2, \ldots, x_n)$ is an element of R^n then $\|x\|_\infty = |x_i|$, $i = 1, 2, \ldots, n$ is the maximum norm of x in R^n. One can use this norm to obtain the distance between any two points $x = (x_1, x_2)$, $y = (y_1, y_2)$ in R^2. That is, $d_\infty(x, y) = \|x-y\|_\infty = |x_i - y_i|_{max}$, $i = 1, 2$.

Consider the points $a = (x, y)$, $a \in A$ which form the $\sqrt{2}/2$ circle given by $\{A \in R^2 : d_p(A, 0) = \sqrt{2}/2\}$ where $0 = (0, 0)$. If the maximum norm is used to measure distance then the MSC is formed and takes the form $\{A \in R^2 : (|x|, |y|)_{max} = \sqrt{2}/2\}$. The graph of the MSC is shown below.



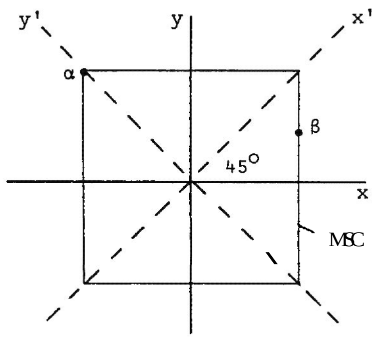
The axes can be rotated 45° so that the following transformation is made

$$x' = \sqrt{2}/2(y + x)$$
$$y' = \sqrt{2}/2(y - x)$$

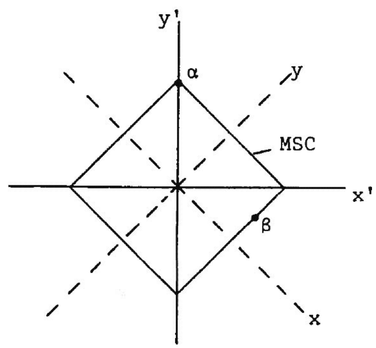
with the inverse transformation

$$x = x'\cos(45^\circ) - y'\sin(45^\circ) = \sqrt{2}/2x' - \sqrt{2}/2y'$$
$$y = x'\sin(45^\circ) + y'\cos(45^\circ) = \sqrt{2}/2x' + \sqrt{2}/2y'.$$

The following graphs depict this transformation.



or



Now, the TUC in the primed coordinate system is defined as $\{A' \in R^2 : |x'| + |y'| = 1\}$ where $a' \in A'$ and $a' = (x', y')$. A couple of examples may convince the reader that those points forming the newly defined TUC are exactly those forming the MSC.

Examples:

$\alpha = (-\sqrt{2}/2, \sqrt{2}/2)$, $d_\infty(\alpha, 0) = |\sqrt{2}/2|$ $\quad\therefore \alpha \in$ MSC

$\alpha = (1, 0)'$, $d'_1(\alpha, 0) = |1| + |0| = |1|$ $\therefore \alpha \in$ TUC

$\beta = (\sqrt{2}/2, 1/2)$, $d_\infty(\beta, 0) = |\sqrt{2}/2|$ $\quad\therefore \beta \in$ MSC

$\beta = ((2 + \sqrt{2})/4, (\sqrt{2} - 2)/4)$, $d'_1(\beta, 0) = |x'| + |y'|$

$= (2 + \sqrt{2})/4 + (2 - \sqrt{2})/4$

$= 1 \quad \therefore \beta \in$ TUC .

If one accepts the statement that those points forming the MSC and those forming the TUC are identical upon transformation then the definitions, tables, and identities developed by Brisbin and Artola for the TUC can be extended to the MSC. If the Diamond sine and cosine

are defined as

$$x' \equiv \cos d\theta' = f_j(\sin\theta', \cos\theta')$$
$$y' \equiv \sin d\theta' = g_j(\sin\theta', \cos\theta')$$

for all x', y' satisfying $|x'| + |y'| = 1$ where $\theta' = \theta - 45$ and f_j, g_j are defined for $j = 1, 2, 3, 4$ in Table 1, then

$$\sqrt{2}/2(y + x) = \cos d(\theta - 45^\circ)$$
$$\sqrt{2}/2(y - x) = \sin d(\theta - 45^\circ).$$

Solving for x and y one obtains

$$x = \sqrt{2}/2(\cos d(\theta - 45^\circ) - \sin d(\theta - 45^\circ))$$
$$y = \sqrt{2}/2(\cos d(\theta - 45^\circ) + \sin d(\theta - 45^\circ)).$$

Now define $x \equiv \cos m(\theta)$, $y \equiv \sin m(\theta)$ for all x, y satisfying $(|x|, |y|)_{max} = \sqrt{2}/2$ then

$$\cos m(\theta) = \sqrt{2}/2(\cos d(\theta - 45^\circ) - \sin d(\theta - 45^\circ))$$
$$\sin m(\theta) = \sqrt{2}/2(\cos d(\theta - 45^\circ) + \sin d(\theta - 45^\circ)).$$

Also,

$$\cos m(-\theta) = \sqrt{2}/2(\cos d(-(\theta + 45^\circ)) - \sin d(-(\theta + 45^\circ)))$$
$$= \sqrt{2}/2(\cos d(\theta + 45^\circ) + \sin d(\theta + 45^\circ))$$
$$\sin m(-\theta) = \sqrt{2}/2(\cos d(-(\theta + 45^\circ)) + \sin d(-(\theta + 45^\circ)))$$
$$= \sqrt{2}/2(\cos d(\theta + 45^\circ) - \sin d(\theta + 45^\circ)).$$

Hence,

$$\cos m(-\theta) = \sin m(\theta + 90^\circ)$$
$$\sin m(-\theta) = \cos m(\theta + 90^\circ).$$

Table 1

quadrant*	$\cos d\theta' = f_j$	$\sin d\theta' = g_j$
1. $(0^\circ \le \theta' \le 90^\circ)$	$\cos\theta'/(\sin\theta' + \cos\theta')$	$\sin\theta'/(\sin\theta' + \cos\theta')$
2. $(90^\circ \le \theta' \le 180^\circ)$	$\cos\theta'/(\sin\theta' - \cos\theta')$	$\sin\theta'/(\sin\theta' - \cos\theta')$
3. $(180^\circ \le \theta' \le 270^\circ)$	$-\cos\theta'/(\sin\theta' + \cos\theta')$	$-\sin\theta'/(\sin\theta' + \cos\theta')$
4. $(270^\circ \le \theta' \le 360^\circ)$	$-\cos\theta'/(\sin\theta' - \sin\theta')$	$-\sin\theta'/(\sin\theta' - \cos\theta')$

As with Euclidean trigonometry, it is important to develop reference angle identities. To do this, sinm and cosm shall be written in terms of θ'.

*from Brisbin and Artola

$$x = \mathrm{cos}m(\theta' + 45^\circ) = \sqrt{2}/2(\mathrm{cos}d\theta' - \mathrm{sin}d\theta')$$
$$y = \mathrm{sin}m(\theta' + 45^\circ) = \sqrt{2}/2(\mathrm{cos}d\theta' + \mathrm{sin}d\theta') .$$

The reference angles are defined as follows:

quadrant	reference angle
1 $(0^\circ \le \theta' \le 90^\circ)$	$\bar{\theta}' = \theta'$
2 $(90^\circ \le \theta' \le 180^\circ)$	$\bar{\theta}' = 180^\circ - \theta'$
3 $(180^\circ \le \theta' \le 270^\circ)$	$\bar{\theta}' = \theta' - 180^\circ$
4 $(270^\circ \le \theta' \le 360^\circ)$	$\bar{\theta}' = 360^\circ - \theta'$.

Brisbin and Artola showed that if θ'_i is a given angle in quadrant i and $\bar{\theta}'_i$ is its associated reference angle, then

$$\mathrm{cos}d\theta'_1 = \mathrm{cos}d\bar{\theta}'_1 \qquad \mathrm{cos}d\theta'_2 = -\mathrm{cos}d\bar{\theta}'_2$$
$$\mathrm{sin}d\theta'_1 = \mathrm{sin}d\bar{\theta}'_1 \qquad \mathrm{sin}d\theta'_2 = \mathrm{sin}d\bar{\theta}'_2$$
$$\mathrm{cos}d\theta'_3 = -\mathrm{cos}d\bar{\theta}'_3 \qquad \mathrm{cos}d\theta'_4 = \mathrm{cos}d\bar{\theta}'_4$$
$$\mathrm{sin}d\theta'_3 = -\mathrm{sin}d\bar{\theta}'_3 \qquad \mathrm{sin}d\theta'_4 = -\mathrm{sin}d\bar{\theta}'_4 .$$

Hence,

$$\begin{aligned}
\mathrm{cos}m(\theta'_2 + 45^\circ) &= \sqrt{2}/2(\mathrm{cos}d\theta'_2 - \mathrm{sin}d\theta'_2)\\
&= \sqrt{2}/2(-\mathrm{cos}d\bar{\theta}'_2 - \mathrm{sin}d\bar{\theta}'_2)\\
&= -\sqrt{2}/2(\mathrm{cos}d\bar{\theta}'_2 + \mathrm{sin}d\bar{\theta}'_2)\\
&= -\mathrm{sin}m(\bar{\theta}'_2 + 45^\circ)\\
\mathrm{sin}m(\theta'_2 + 45^\circ) &= \sqrt{2}/2(\mathrm{cos}d\theta'_2 + \mathrm{sin}d\theta'_2)\\
&= \sqrt{2}/2(-\mathrm{cos}d\bar{\theta}'_2 + \mathrm{sin}d\bar{\theta}'_2)\\
&= -\sqrt{2}/2(\mathrm{cos}d\bar{\theta}'_2 - \mathrm{sin}d\bar{\theta}'_2)\\
&= -\mathrm{cos}m(\bar{\theta}'_2 + 45^\circ)
\end{aligned}$$

and similarly,

$$\mathrm{cos}m(\theta'_1 + 45^\circ) = \sqrt{2}/2(\mathrm{cos}d\bar{\theta}'_1 - \mathrm{sin}d\bar{\theta}'_1) = \mathrm{cos}m(\bar{\theta}'_1 + 45^\circ)$$
$$\mathrm{sin}m(\theta'_1 + 45^\circ) = \sqrt{2}/2(\mathrm{cos}d\bar{\theta}'_1 + \mathrm{sin}d\bar{\theta}'_1) = \mathrm{sin}m(\bar{\theta}'_1 + 45^\circ)$$
$$\mathrm{cos}m(\theta'_3 + 45^\circ) = -\sqrt{2}/2(\mathrm{cos}d\bar{\theta}'_3 - \mathrm{sin}d\bar{\theta}'_3) = -\mathrm{cos}m(\bar{\theta}'_3 + 45^\circ)$$
$$\mathrm{sin}m(\theta'_3 + 45^\circ) = -\sqrt{2}/2(\mathrm{cos}d\bar{\theta}'_3 + \mathrm{sin}d\bar{\theta}'_3) = -\mathrm{sin}m(\bar{\theta}'_3 + 45^\circ)$$
$$\mathrm{cos}m(\theta'_4 + 45^\circ) = \sqrt{2}/2(\mathrm{cos}d\bar{\theta}'_4 + \mathrm{sin}d\bar{\theta}'_4) = \mathrm{sin}m(\bar{\theta}'_4 + 45^\circ)$$
$$\mathrm{sin}m(\theta'_4 + 45^\circ) = \sqrt{2}/2(\mathrm{cos}d\bar{\theta}'_4 - \mathrm{sin}d\bar{\theta}'_4) = \mathrm{cos}m(\bar{\theta}'_4 + 45^\circ) .$$

Now if $\theta' + 45^\circ$ is replaced by θ,

$$\mathrm{cos}m(\theta_1) = \mathrm{cos}m(\bar{\theta}_1)$$
$$\mathrm{sin}m(\theta_1) = \mathrm{sin}m(\bar{\theta}_1)$$
$$\mathrm{sin}m(\theta_2) = -\mathrm{cos}m(\bar{\theta}_2)$$
$$\mathrm{cos}m(\theta_2) = -\mathrm{sin}m(\bar{\theta}_2)$$
$$\mathrm{cos}m(\theta_3) = -\mathrm{cos}m(\bar{\theta}_3)$$
$$\mathrm{sin}m(\theta_3) = -\mathrm{sin}m(\bar{\theta}_3)$$
$$\mathrm{sin}m(\theta_4) = \mathrm{cos}m(\bar{\theta}_4)$$
$$\mathrm{cos}m(\theta_4) = \mathrm{sin}m(\bar{\theta}_4)$$

where

quadrant	reference angle
1 $(45^\circ \le \theta \le 135^\circ)$	$\bar{\theta}_1 = \theta_1$
2 $(135^\circ \le \theta \le 225^\circ)$	$\bar{\theta}_2 = 270^\circ - \theta_2$
3 $(225^\circ \le \theta \le 315^\circ)$	$\bar{\theta}_3 = \theta_3 - 180^\circ$
4 $(315^\circ \le \theta \le 405^\circ)$ or $(-45^\circ \le \theta \le 45^\circ)$	$\bar{\theta}_4 = 450^\circ - \theta_4$.

With the reference angle identities I leave Maximum trigonometry. As a last comment, I emphasize that the Maximum trigonometric functions are written explicitly in terms of the Taxicab trigonometric functions and hence implicitly in terms of the Euclidean trigonometric functions. Therefore, Maximum trigonometric tables can be easily developed using either the well known Euclidean tables or the table presented in the paper by Brisbin and Artola.

REFERENCES

1. Brisbin, Ruth and Artola, Paul, *Taxicab Trigonometry*, Pi Mu Epsilon Journal, No. 11, Spring (1985).

About the author -

Richard Bertram graduated from Florida State University in the Fall of 1985.

About the paper -

This paper was written during Richard's senior year and was motivated by curiosity.

A LOCUS OF POINTS FROM CONVEX COORDINATES

by *James N. Boyd*
St. Christopher's School
Richmond, VA 23226

Introduction. Recently, I came across a problem which must have been designed with convex coordinates [1, 2, 3] in mind:

Problem *1*. A straight line segment $\overline{AB}$ is divided internally at P where P is equally likely to be any point of the segment. What is the probability that

$$\frac{AP}{AB} \cdot \frac{PB}{AB} \leq \frac{3}{16} ?$$

First, let me give a solution and follow that with a more general version of the problem for the line segment. Then I will devote the principal part of this paper to analogous problems for triangles.

Solution. The convex (or barycentric) coordinates of P with respect to A and B are $a_A(P) = \frac{PB}{AB}$ and $a_B(P) = \frac{AP}{AB}$. We know that, for $P \in \overline{AB}$, $\alpha_A(P)$, $\alpha_B(P) \geq 0$ and that $a_A(P) + \alpha_B(P) = 1$.

We can solve $a_A(P) \cdot a_B(P) = a_A(P)(1 - a_A(P)) = 3/16$ to obtain $(a_A(P), \alpha_B(P)) = (1/4, 3/41$ or $(\alpha_A(P), \alpha_B(P)) = (3/4, 1/4)$. Then

$AB \cdot AB \leq 3/16$ for $\alpha_A(P) \in [0, 1/41 \cup [3/4, 1]$.

Therefore, the probability that $\frac{AP}{AB} \cdot \frac{PB}{AB} \leq 3/16$ is

$$\frac{(1/4 - 0) + (1 - 3/41}{1 - 0} = 1/2.$$

Now suppose that $P \in \overline{AB}$ so that

$$\text{(1)} \qquad \frac{AP}{AB} \cdot \frac{PB}{AB} \leq k.$$

Problem *2*. Determine the possible values of k and find the probability that P divides $\overline{AB}$ so that Inequality *(1)* is satisfied.

Solution. Again we let $a_A(P) = \frac{PB}{AB}$ and $a_B(P) = \frac{AP}{AB}$. Since $\alpha_A(P)$, $\alpha_B(P) \geq 0$, we know that $k \geq 0$. We can maximize the product $\alpha_A(P) \cdot \alpha_B(P)$ subject to the constraint $\alpha_A(P) + \alpha_B(P) = 1$ to obtain $k(max) = 1/4$ for $\alpha_A(P) = \alpha_B(P) = 1/2$. As we would expect from symmetry, P for $k(max)$ is the midpoint of the segment. Thus $0 \leq k \leq 1/4$.

Proceeding as in our first problem, we consider $aA(P)(1 - \alpha_A(P)) = k'$. Since the values $\alpha'_A(P) = \frac{1 -}{2}$ and $\alpha''_A(P) = \frac{1 + \sqrt{1 - 4k'}}{2}$ satisfy the equation, the probability that Inequality *(1)* is satisfied must be $[\alpha'_A(P) - 0] + [1 - \alpha''_A(P)] = 1 - \sqrt{1 - 4k'}$.

Convex Coordinates in the Plane. A Euclidean plane is determined by two distinct intersecting lines. If the lines are perpendicular, they can be taken as Cartesian axes, and every point of the plane can be uniquely specified with an ordered pair of real numbers which are called the Cartesian coordinates of the point. Likewise, three noncollinear points, V_1, V_2, V_3, determine a plane, and the points of the plane can also be designated with respect to those three points. Ordered triples of real numbers, $(\alpha_1, a\ , \alpha_3)$ such that $\alpha_1 + a + a = 1$, can be used for that purpose. The three numbers are called the convex (or barycentric) coordinates of a point with respect to V_1, V_2, V_3, in that order.

If the Cartesian coordinates of point P of the plane are (x, y), and those of $V.$ are $(a:_I, y_i)$, $i = 1, 2, 3$, then there exist unique convex coordinates such that

$$\text{(2)} \qquad \begin{aligned} x &= a_1x_1 + a_2x_2 + a_3x_3, \\ y &= a_1y_1 + a_2y_2 + a_3y_3, \text{ and} \\ 1 &= a_1 + a_2 + a_3. \end{aligned}$$

There is one-to-one correspondence between the set of points in the plane and the set of convex coordinates. [4]

The closed triangular region with vertices V_1, V_2, V_3 may be defined by

$$\Delta V_1V_2V_3 = \{(x, y) | x = \sum_{i=1}^{3} \alpha_i x_i,\ y = \sum_{i=1}^{3} \alpha_i y_i,\ \alpha_1, \alpha_2, \alpha_3 \geq 0, \text{ and } \alpha_1 + \alpha_2 + \alpha_3 = 1\}.$$

Although convex coordinates may be unfamiliar to some readers, they can be exceedingly useful in developing properties of triangles, and their study is rewarding. They admit of numerous interpretations; and in different settings, one interpretation may be more helpful than another. [5, 6]

In the introductory problems, one-dimensional convex coordinates were given as ratios of lengths. For our next purposes, I shall interpret two-dimensional convex coordinates in terms of areas. I hope that this interpretation will seem sensible, but I leave it to the reader to seek a justification among the references given at the end of this paper. [1, 7]

Let $P \in \Delta V_1V_2V_3$. Then P determines three subtriangles as shown in Figure 1.

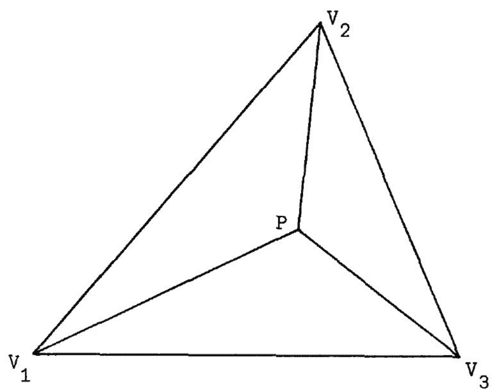


Figure 1

We can define $(\alpha_1, \alpha_2, \alpha_3)$ by

$$\alpha_1 = \frac{\text{Area}\Delta PV_2V_3}{\text{Area}\Delta V_1V_2V_3},\ \alpha_2 = \frac{\text{Area}\Delta PV_1V_3}{\text{Area}\Delta V_1V_2V_3},\ \alpha_3 = \frac{\text{Area}\Delta PV_1V_2}{\text{Area}\Delta V_1V_2V_3}.$$

In three dimensions, convex coordinates can be defined as ratios of volumes of tetrahedra; and in higher dimensional Euclidean spaces, we can say that convex coordinates are ratios of "hypervolumes of $(n+1)$-simplices."

Products of Areas and a Locus of Points. It is easy to show that the convex coordinates of vertex V_i are $\alpha_i = 1$, $\alpha_j = 0$ for $j \neq i$, and that, on the boundary of $\Delta V_1V_2V_3$, at least one of the convex

coordinates must always be zero. So on the boundary, we must have $\alpha_1\alpha_2\alpha_3 = 0$.

By the method of Lagrange multipliers, we can also show that the maximum value of the product $\alpha_1\alpha_2\alpha_3$ for $P \in \Delta V_1V_2V_3$ is 1/27 when $\alpha_1 = \alpha_2 = \alpha_3 = 1/3$. Thus the product of the ratios of the areas of the three subtriangles of $\Delta V_1V_2V_3$ is such that

$$0 \le \alpha_1\alpha_2\alpha_3 \le 1/27.$$

The maximum occurs at the centroid which has convex coordinates (1/3, 1/3, 1/3).

Let us now ask the question: What is the locus of points in $\Delta V_1V_2V_3$ such that $\alpha_1\alpha_2\alpha_3 = k$ where $k \in [0, 1/27]$?

To find the general answer, let us first work a particular problem and then transform the answer to that problem into the desired, general result.

Problem 3. Consider the right triangle whose vertices have Cartesian coordinates as given: V_1: (s, t), V_2: (s, t'), V_3: (s', t).

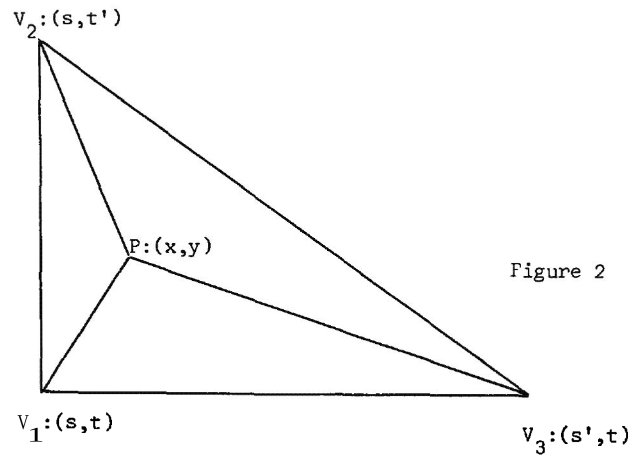


Figure 2

Find a Cartesian equation for the locus of points in the triangle such that $\alpha_1\alpha_2\alpha_3 = k$.

Solution. Let P: (x, y) be a typical point of the locus. With the aid of Equations (2) and Cramer's Rule, we can transform the convex equation $\alpha_1\alpha_2\alpha_3 = k$ into

(3) $\qquad (\text{Const.})F_1(x, y)\cdot F_2(x,y)\cdot F_3(x,y) = k$

where $F_i(x, y) = 0$ is a linear equation for the side of $\Delta V_1V_2V_3$ which stands opposite to vertex V_i. The determinant in the denominator of the Cramer's Rule solution for a_i cannot be zero since it is always twice the area of $\Delta V_1V_2V_3$. The constant factor can be obtained from the requirement that $k = 1/27$ when (x, y) are the Cartesian coordinates of the centroid of the triangle.

The computations leading to Equation (3) are straightforward but not interesting enough in themselves to warrant their inclusion with this discussion. I leave it to the diligent reader to carry them out it he so desires.

Every triangle in the plane can be obtained from $\Delta V_1V_2V_3$ of Problem 3 with the vertex V_1 at the origin of Cartesian coordinates. Suppose that we need to obtain $\Delta V'_1V'_2V'_3$. By means of linear transformations (shears, dilations, stretches), $\Delta V_1V_2V_3$ can be made congruent to the desired triangle. With further linear transformations (rotations and reflections), the intermediate triangle can be given the same orientation in the plane as that of $\Delta V'_1V'_2V'_3$. Then by a translation, our congruent triangle can be taken to the desired location.

Before making use of linear transformations and translations to generalize the result of Equation (3) to any triangle in the plane, we need to establish a bit of matrix notation. Each linear factor $F_i(x, y)$ can be written in the form

$$ax + by + c = (abc)\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}.$$

The translation of $P:(x, y)$ to $P':(x + g, y + h)$ can be effected by the matrix multiplication

$$T\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix}$$

where

$$T = \begin{pmatrix} 1 & 0 & g \\ 0 & 1 & h \\ 0 & 0 & 1 \end{pmatrix}.$$

Noting that

$$T^{-1} = \begin{pmatrix} 1 & 0 & -g \\ 0 & 1 & -h \\ 0 & 0 & 1 \end{pmatrix},$$

we can give the corresponding transformation of the factor $F_i(x, y)$ by

$$(abc)T^{-1}T\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = (abc)T^{-1}\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix}$$

$$= a'x' + b'y' + c' = F'_i(x', y').$$

Under this transformation, $F_i(x, y) = F'_i(x + g, y + h)$.

If a linear transformation which keeps the origin fixed has matrix representation

$$R = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix},$$

then we can write

$$R\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

and

$$R\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{R}} & 0 \\ & & 0 \\ \hline 0 & 0 & 1 \end{array}\right)\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & 0 \\ r_{21} & r_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix}.$$

Clearly

$$R^{-1} = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{R^{-1}}} & 0 \\ & & 0 \\ \hline 0 & 0 & 1 \end{array}\right).$$

The corresponding transformation of the linear factor $F_i(x, y)$ is given by

$$(abc)R^{-1}R\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}.$$

Let T be the matrix product representation of the composition of linear transformations and translation which carries $\Delta V_1V_2V_3$ into $\Delta V'_1V'_2V'_3$. Under the transformation of $\Delta V_1V_2V_3$, the factor $F.(x, y)$ becomes

$$F'.(x', y') = (abc)T^{-1}T\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = a'x' + b'y' + c'.$$

Proposition. Let P denote any point of $\Delta V_1V_2V_3$, and let the convex coordinates of P be $(\alpha_1, \alpha_2, \alpha_3)$. The locus of points in $\Delta V_1V_2V_3$ such that

$$a_1a_2a_3 = k, \; k \in [0, 1/27]$$

has equation

$$(\text{Const.})F_1(x, y)\cdot F_2(x, y)\cdot F_3(x, y) = k$$

where $F_i(x, y) = 0$ is a linear equation for the side of the triangle opposite to vertex V_i, and the constant factor is chosen so that $k = 1/27$ at the centroid of the triangle.

Proof, By starting with a right triangle as previously described, we can transform the triangle into $\Delta V_1V_2V_3$. The corresponding transformation applied to each of the linear factors of Equation (3) yields the desired result.

An Application. Let us conclude with a computation which makes use of our proposition.

Problem 5. Consider the triangle with vertices at the points $V_1:(-1, 0)$ $V_2:(0, 1)$, $V_3:(1, 0)$. Let point P be randomly chosen from the closed triangular region. What is the probability that

$$\frac{\text{Area } \Delta PV_2V_3}{\text{Area } \Delta V_1V_2V_3}\;\frac{\text{Area } \Delta PV_1V_3}{\text{Area } \Delta V_1V_2V_3}\;\frac{\text{Area } \Delta PV_1V_2}{\text{Area } \Delta V_1V_2V_3} \le \frac{1}{54} \; ?$$

Solution. The equation of the sides of the triangle are $y = 0$, $x + y - 1 = 0$, $x - y + 1 = 0$. We need to consider the equation

$$(\text{Const.})y(x + y - 1)(x - y + 1) = k.$$

The centroid of $\Delta V_1V_2V_3$ has coordinates $(\bar{x}, \bar{y}) = (0, 1/3)$. Letting $(\bar{x}, \bar{y}) = (x, y)$ and $k = 1/27$, we can evaluate the constant factor to be $-1/4$. So our equation becomes

(4) $(-1/4)y(x + y - 1)(x - y + 1) = 1/54.$

By Monte Carlo techniques implemented on an Apple IIe computer, the area of the region bounded by the curve defined by Equation (4) was found to be .43. The value .43 was obtained by making 50,000 random choices of points from the rectangle having vertices (-1, 0), (-1, 1), (1, 1), (1, 0) and multiplying the area of the rectangle by the percentage of the points which fell within the region of interest.

The area of $\Delta V_1V_2V_3$ is 1. Therefore the desired probability is $1 - .43 = .57$.

REFERENCES

1. M. Hausner, The Center of Mass and *Affine* Geometry, The American Mathematical Monthly, 69 (1962) 724-37.
2. J. N. Boyd and P. N. Raychowdhury, Convex Coordinates, Probabilities, and the Superposition of States, The College Mathematics Journal, 18 (1987) 186-94.
3. J. N. Boyd and P. N. Raychowdhury, Applications of Extended Convex Coordinates, Mathematical Gazette (to appear).
4. J. N. Bovd and P. N. Raychowdhury, Uniqueness of Convex Coordinates from Physical Considerations, Mathematics and Computer Education, 17 (1983) 113-8.
5. J. N. Boyd, J. S. Cooper, and P. N. Raychowdhury, *Euclidean* Geometry from Classical Mechanics, International Journal of Mathematical Education in Science and Technology (to appear).
6. J. N. Boyd, R. W. Farley, and P. N. Raychowdhury, *Invariance* of Geometric Equations in Convex Coordinates, Pure and Applied Mathematical Sciences (to appear).
7. J. N. Boyd and P. N. Raychowdhury, More Convex Coordinates, Mathematics and Computer Education, 18 (1984) 121-4.

ʃ ʃ

Matching Prize Fund

If your Chapter presents awards for Outstanding Mathematical Papers or for Student Achievement in Mathematics, you may apply to the National Office for an amount equal to that spent by your Chapter up to a maximum of fifty dollars. Contact Professor Robert Woodside, Secretary-Treasurer

A NOTE ON EVALUATING DEFINITE INTEGRALS BY SUBSTITUTION

by Peter A. Lindstrom
North Lake College, Irving, Texas

Beginning calculus students will often evaluate

$$\int_1^3 (3x+2)^2\,dx$$

by expanding the integrand with the Binomial Theorem and then using the Fundamental Theorem of Calculus. But if you tell them to use the Binomial Theorem/Fundamental Theorem of Calculus approach on

$$\text{(1)} \qquad \int_1^3 (3x+2)^{95}\,dx,$$

they will say "there must be an easier way." Two such ways are:

(i.) verify that $(3x+1)^{96}/(3)(96)$ is an antiderivative of the integrand, and so on,

(ii.) use the following theorem that can be found in all calculus texts:

DEFINITE INTEGRAL SUBSTITUTION THEOREM. Let f be a function that is continuous on $[a,b]$, let $u = h(x)$ be a differentiable function on $[a,b]$, and let g be a continuous function such that $f(x)\,dx = g(u)\,du$ or $f(x) = g(h(x))h'(x)$. Then

$$\int_a^b f(x)\,dx = \int_{h(a)}^{h(b)} g(u)\,du.$$

Applying this theorem to (1), where $u = h(x) = (3x+2)$, $(1/3)du = dc$, $g(u) = u^{95}$, $h(1) = 5$, and $h(3) = 11$, then

$$\text{(2)} \qquad \int_1^3 (3x+2)^{95}dx = \int_5^{11} (u)^{95}(1/3)\,du,$$

and so on.

There is still another way to solve this problem; evaluate the definite integral (1) by taking the limit of an appropriate Riemann Sum. At first this approach might seem like a rather poor method to use because the two methods outlined above do such a fine job. Also, by the limit of a Riemann Sum approach, the Binomial Theorem and numerous summation formulas would be needed, along with much paper, time and patience!

The limit of a Riemann Sum approach does have one interesting feature though; it helps to motivate the above theorem and to give a better understanding of the details of the theorem. The purpose of this note is to show how the limit of a Riemann Sum approach yields the same result as shown above by using the theorem.

Consider the following argument. Expressing (1) as the limit of a Riemann Sum by selecting, say, the right end point of a regular partition of n subintervals of [1,3], we obtain

$$\int_1^3 (3x+2)^{95}dx = \lim_{n\to\infty} \sum_{i=1}^{n} \{3(1+i(2/n)) + 2\}^{95}(2/n),$$

$$\text{(3)} \qquad = \lim_{n\to\infty} \sum_{i=1}^{n} \{5 + i(6/n)\}^{95}(2/n).$$

But the quantity $\{5 + i(6/n)\}^{95}$ of the Riemann Sum in (3) suggests another definite integral of, say, $g(u) = u^{95}$ that starts at 5 and is over a closed interval of length 6, again using the right end point of each subinterval of a regular partition. With such an interval of integration being 6 units in length, then (3) can be written as

$$\lim_{n\to\infty} \sum_{i=1}^{n} \{5 + i(6/n)\}^{95}(1/3)(6/n) = \int_5^{11} (u^{95})(1/3)\,du.$$

Hence,

$$\int_1^3 (3x+2)^{95}dx = \int_5^{11} (u^{95})(1/3)\,du,$$

which is the same definite integral obtained by using the Definite Integral Substitution Theorem in (2).

A MULTIPLIER PROBLEM

by *William M. Perel*
Wichita State University

The Problem. Given a single digit, is it always possible to find a natural number such that the product of the number and the single digit will have the same digits as the original number* in the same order, except that the first digit of the product will be the last digit of the original number? The example $4 \times 179487 = 717948$ shows that it is sometimes possible.

In some cases, one must allow the first digit of the number to be the digit *0*. For example, consider $4 \times 076923 = 307692$.

The Solution. Consider

$$x_n \ldots x_2 x_1 h \times m = h x_n \ldots x_2 x_1,$$

which may be written

$$[h + \sum_{i=1}^{n} x_i 10^i] \times m = \sum_{i=1}^{n} x_i 10^{i-1} + h \times 10^n.$$

Let

$$N = \sum_{i=1}^{n} x_i 10^{i-1}, \; M = h + 10N,$$

then

$$(1) \qquad (h + 10N) \times m = N + h \times 10^n,$$

$$10^n \times h = hm + 10Nm - N = hm + N(10m - 1).$$

Next, consider the congruence

$$10^n \equiv m \pmod{10m - 1},$$

which may be solved (by brute force) for $m = 1, 2, 3, \ldots, 9$.

Let $n(m)$ be the smallest (positive) solution for each m and let $y(m)$ satisfy the equation

$$10^{n(m)} = m + y(m)[10m - 1].$$

Equation *(1)* then becomes

$$(2) \qquad [m + y(10m - 1)]h = hm + N(10m - 1)$$

or

$$y(10m - 1) \times h = N(10m - 1)$$

or $\quad N = hy$

and $\quad M = h + 10hy = h(1 + 10y),$

which is the desired answer.

Below, a table of values for $n(m)$ and $y(m)$ is provided for each m, *1* through *9*.

m	$n(m)$	$y(m)$
1	1	1
2	17	5263157894736842
3	27	344827586206896551724137 93
4	5	2564
5	41	20408163265306122448979591836734693877 55
6	57	169491525423728813559322033898305084745762711864406779 6
7	21	144927536231884057 97
8	12	12658227848
9	43	112359550561797752808988764044943820224719

Examples of calculations.

Let $m = 8$. We must then solve the congruence $10^n \equiv 8 \pmod{79}$, which has the solution $n = 12$. Then $y = (10^{12} - 8)/79 = 12658227848$, as is given in the table.

Now suppose $h = 3$. Then $M = 3(1 + 10y) = 3(126582278481) = 379746835443$ and $M \times 8 = 0379746835443 \times 8 = 3037974683544$, so M is the answer.

Suppose $h = 9$. Then $M = 9(1 + 10y) = 9(126582278481) = 1139240506329$ and $M \times 8 = 1139240506329 \times 8 = 9113924050632$, so M is the answer.

* * *

Did you know that - $12345679 \times 9 = 111111111$

$98765432 \times 9 = 888888888$?

A COMMON MISUSE OF "DENOTED"

by I. J. Good
Virginia Polytechnic Institute and State University

Although most students of the humanities can barely believe it, a large majority of scientists have for some years been using such expressions as "the area of this triangle is denoted x" when they mean "the area of this triangle is denoted by x." Editors fail in their duty to protect the language when they allow such expressions to appear in print. The omission of the preposition "by" is certainly a blunder if "to denote" means "to represent" as in the expression "x denotes the area of this triangle." The same blunder is exemplified by "The ball was kicked Tom" when the intended meaning is "The ball was kicked by Tom." But I have never heard even a child use such a construction when all the words are familiar.

After a copy-cat makes the blunder he might rationalize it by saying that "denoted" means "called" or "named"; but he would presumably agree that it means "represents" when he says "x denotes this integral." He might not notice that his meaning changes from "represent" to "call" depending on whether he uses the active or the passive form. This abrupt change of meaning sometimes occurs even within a single sentence! I doubt if there is another verb whose meaning changes in this manner.

Another possible defense is that "denoted x" has, by frequent misusage, become an idiomatic form, that is, "to denote" is the only transitive English verb in which the "by" can be omitted in the passive form. But this defense is weak because idioms, such as "it's me," nearly always involve very common words and even some of these idioms, when printed, can be unacceptable to most literate people, for example, "most all." If "denoted x" is becoming an idiom, let's stamp it out before it's too late.

John Stuart Mill may have originated the misuse in his <u>System of Logic</u>, perhaps because he learned French thoroughly at the age of 14 (Minto and Mitchell, 1911, p. 454). In French it is correct to say "noté X" because "noter" means, among other things, "to brand" (Girard, 1962).

REFERENCES

Girard, D. (1962). <u>The New Cassell's French Dictionary</u> (New York: Funk and Wagnalls).

Minto, William and Mitchell, John Malcolm (1911). "Mill, John Stuart," <u>Encyclopaedia Britannica</u>, 11th edn., Vol. 18, 454-459.

THE IMPROPER INTEGRAL $\int_0^\infty \sin x\,dx$ AS AN ALTERNATING SERIES

by Mohammad K. Azarian
University of Evansville
Evansville, Indiana 47722

Although another way to show that the improper integral $\int_0^\infty \sin x\,dx$ is divergent is by looking at it as an alternating series, I have not found any calculus book that uses this method. Therefore, we show the divergence of $\int^\infty \sin x\,dx$ as an alternating series here, as follows:

$$\int_0^\infty \sin x\,dx = \int^\pi \sin x\,dx + \int_\pi^{2\pi} \sin x\,dx + \dots + \int_{n\pi}^{(n+1)\pi} \sin x\,dx + \dots$$

$$= \sum_{n=0}^{\infty} (-1)^n 2.$$

A similar argument applies to $\int_0^\infty \cos x\,dx$.

LETTER TO THE EDITOR

Dear Editor:

I was intrigued by Schaumberger's proof that $e^\pi > \pi^e$, but I felt that it was flawed by the use of integration and the mean-value theorem. I therefore submit the enclosed note to the Pi Mu Epsilon Journal.

Both Schaumberger and I take it for granted that $\pi > e$. A purist might object that our proofs rely on a calculator to this extent. This objection could be countered by the following argument:

LETTER TO THE EDITOR (*continued*) -

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots < 2 + \frac{1}{2} + \frac{1}{6}\left(1 + \frac{1}{4} + \frac{1}{4^2} + \dots\right)$$

$$= 2.5 + \frac{1}{6}\cdot\frac{1}{1 - 1/4} < 2.722;$$

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right)$$

$$> 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}\right) > 2.723 > e,$$

by hand *calculation*.

Sincerely yours,

R. P. *Boas*
Northwestern University
Evanston, IL 60201

∫ ∫

THE SIGN OF $e^\pi - \pi^e$

by *R. P. Boas*
Northwestern University

The following discussion seems to be even simpler than Schaumberger's [1].

The positivity of $e^\pi - \pi^e$ is equivalent to the inequality

$$\pi \log e > e \log \pi,$$

that is, to

$$(1) \qquad \frac{\pi}{\log \pi} > \frac{e}{\log e}.$$

Now $x/\log x$ increases when $x > e$ (because its derivative is positive).

Since $\pi > e$, we have (1) and hence $e^\pi - \pi^e > 0$.

REFERENCES

1. N. Schaumberger, *Another Approach to* $e^\pi > \pi^e$, this Journal 8 (1986), 251.

PI MU EPSILON PLANS A 75TH BIRTHDAY CELEBRATION

"The Pi Mu Epsilon Fraternity, incorporated on May 25, 1914, under the laws of the State of New York, is celebrating its golden anniversary as a national mathematics honorary fraternity with nearly 100 active chapters in 39 states and the District of Columbia," Thus began Prof. J. S. Frame's article "Fifty Years in the Pi Mu Epsilon Fraternity" in the Spring 1964 issue of the Pi Mu Epsilon Journal.

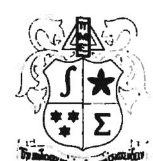
In 1989, the Pi Mu Epsilon Fraternity, Inc. - now the Pi Mu Epsilon, Inc. National Honorary Mathematics Society - will be 75 years old. Councillors and officers have been making plans for appropriate ways of celebrating the birthday.

Professor Frame's updated history of Pi Mu Epsilon will be featured in the Spring 1989 issue of this Journal, along with a complete list of past officers and councillors, winners of the annual paper competitions, presenters of papers at the annual summer meetings, and much more.

The dates and place of the National Pi Mu Epsilon Meeting for the summer of 1989 have not yet been determined. Chapter advisors will be kept informed. All available information will be published in the Fall 1988 issue of this Journal.

∫ ∫

Do you know - here is a copy of the Pi Mu Epsilon shield, what is the motto of the Society? What are the colors? What do the four parts of the shield represent?



Pi Mu Epsilon, Inc.

NATIONAL HONORARY MATHEMATICS SOCIETY

FIFTEENTH ANNUAL
PI MU EPSILON
STUDENT CONFERENCE
MIAMI UNIVERSITY
OXFORD, OHIO

Call for student papers and guests
Friday and Saturday
September 30 and October 1, 1988
Held in conjunction with
The Conference on Recreational Mathematics
featuring
Persi Diaconis, Doris Schattschneider, and Richard Guy

WE INDITE **YOU** TO JOIN US. THERE MILL BE SESSIONS OF THE STUDENT **CONFERENCE** ON **FRIDAY** EUENING **AND SATURDAY AFTERNOON.** FREE **OUERNIGHT LODGING** FOR **ALL** STUDENTS MILL BE **ARRANGED** WITH **MIAMI STUDENTS. EACH STUDENT** SHOULD BRING **A** SLEEPING **BAG. ALL** STUDENT **GUESTS ARE INUITED TO A** FREE **FRIDAY** EUENING **PIZZA PARTY** SUPPER, AND **SPEAKERS WILL** BE **TREATED** TO **A SATURDAY** NOON PICNIC **LUNCH. TALKS MAY** BE ON **ANY** TOPIC **RELATED** TO **MATHEMATICS, STATISTICS** OR COMPUTING. WE WELCOME ITEMS **RANGING** FROM **EXPOSITORY TO RESEARCH, INTERESTING APPLICATIONS, PROBLEMS,** SUMMER EMPLOYMENT, **ETC. PRESENTATION** TIME SHOULD BE **FIFTEEN** OR **THIRTY** MINUTES.

Me need **your title,** presentation time (15 or 30 **min.),** preferred date **(Fri.** or **Sat.)** and a 50 **(approx.)** word abstract by September 22,1900. Please send to Professor Milton **D.** Con, Department of Mathematics and Statistics **Miami University, Oxford, Ohio** 45056.
The Conference on **Recreational** Mathematics begins Friday afternoon, September 30
CONTACT US FOR MORE **DETAILS**

PUZZLE SECTION

Edited by

Joseph D. E. Konhauser

The PUZZLE SECTION is for the enjoyment of those readers who are addicted to working doublecrostics or who find an occasional mathematical puzzle attractive. We consider mathematical puzzles to be problems whose solutions consist of answers immediately recognizable as correct by simple observation and requiring little formal proof. Material submitted and not used here will be sent to the Problem Editor if deemed appropriate for the PROBLEM DEPARTMENT.

Address all proposed puzzles and puzzle solutions to Professor Joseph D. E. Konhauser, Mathematics and Computer Science Department, Macalester College, St. Paul, Minnesota 55105. Deadlines for puzzles appearing in the Fall Issue will be the next February 15, and for the puzzles appearing in the Spring Issue will be the next September 15.

PUZZLES FOR SOLUTION

1. *Proposed by the Editor.*

Determine a rule of formulation for the table and fill in the blanks.

I					
1	2				
1	$\sqrt{2}$	3			
2	1	$\sqrt{5}$	4		
1	$\sqrt{2}$	$\sqrt{2}$		5	
$\sqrt{2}$	1		$\sqrt{2}$		6

2. *Proposed by the Editor.*

The set S consists of five numbers. If pairs of distinct elements of S are added, the following ten sums are obtained:

0, 6, 11, 12, 17, 20, 23, 26, 32, and 37

What are the elements of S?

3. *A Special Case of a Michael Goldberg Proposal.*

Dissect an equilateral triangle into four pieces which can be reassembled to form two equilateral triangles.

4. *A Storage and Retrieval Puzzle.*

Given the set $S = \{a, b, c, d, e\}$ find the shortest sequence of elements of S such that the members of each of the 31 non-empty subsets of S appear as consecutive terms of the sequence at least one time.

5. *Contributed.*

The positive integer 11223 is composite since it is divisible by 9. If we change the 2 in the ten's position to a 1, then the resulting number 11213 is prime. What is the smallest positive composite integer which cannot be changed into a prime by changing exactly one digit?

6. *Contributed.*

Find the smallest positive integer consisting of the ten digits 0 through 9, each used just once, which is divisible by each of the digits 2 through 9.

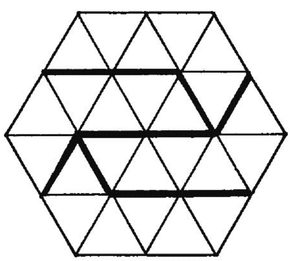
7. *A Special Case of a Problem Posed by P. Erdös and G. Purdy.*

Locate nine points in a plane so that each of the nine points is at a unit distance from exactly four of the others.

* * *

COMMENTS ON PUZZLES 1 - 7, FALL 1987

Of the four respondents to ***Puzzle # 1*** only Jeanette Bickley and Richard I. Hess provided correct partitions of a regular hexagon into four congruent six-sided figures. Their solution, which is the same as that of the proposer, John M. Howell, is reproduced below. A good way to start is to partition the hexagon into 24 congruent equilateral triangles as in the sketch.



There were eight responses to ***Puzzle # 2.*** Seven of these contained the correct arrangement of cards after one shuffle

9 A 4 Q J 7 3 2 10 5 K 8 6

John Schue argued this way:

"The shuffler can be thought of as a permutation σ on the set $\{A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K\}$. Then σ^2 is given by the cycle $\sigma^2 = (A, 10, J, 6, 3, Q, 2, 9, 5, K, 7, 4, 8)$. Since σ^{13} must be the identity permutation, $\sigma = (\sigma^2)^7 = (A, 9, 10, 5, J, K, 6, 7, 3, 4, Q, 8, 2)$. Thus the order of the cards after the first shuffle was $(9, A, 4, Q, J, 7, 3, 2, 10, 5, K, 8, 6)$."

Thirteen correct responses were received for ***Puzzle # 3.*** Most responses consisted of just the correct answer. John Schue wrote

"Let $N = abcd$ be the given number. Then there is a positive integer M such that $N^2 = 10000M + 1000(2ad + 2bc) + 100(2bd + c^2) + 10(2cd) + d^2$. Thus $d^2 \equiv d \pmod{10}$ so that $d = 0, 1, 5$, or 6. If $d = 0$, this leads to $N = 0000$. If $d = 1$, this leads to $N = 0001$. If $d = 5$, this leads to $N = 0625$. Since N is a 4-digit number we must have $d = 6$. Then $12c + 3 \equiv c \pmod{10}$ so $c \equiv -3 \equiv 7$, $12b + 7 \equiv 6 \pmod{10}$ so $b \equiv -7 \equiv 3$, and $12a + 1 \equiv a \pmod{10}$ so $a \equiv -1 \equiv 9$. Thus $N = 9376$ is the unique answer. Note: $N^2 = 87909376$."

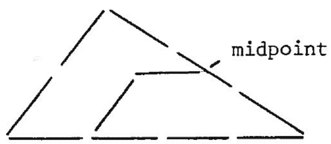
Robert Prielipp and Victor Feser remarked that a number whose square ends with the given number is said to be ***automorphic.*** See the literature for more on such numbers.

Only Richard I. Hess and Emil Slowinski responded to ***Puzzle # 4.*** Their answer to the question "If the side lengths of a convex quadrilateral are positive integers such that each divides the sum of the other three, can the four side lengths be different numbers?" was "NO." An argument given by Michael Clipson, a one-time student of the proposer goes as follows:

"Suppose there is a quadrilateral with side lengths a, b, c and d satisfying the conditions of the problem. Set $S = a + b + c + d$. Since a divides $b + c + d$ as well as itself, $S = k_1a$ for some positive integer k_1. Similarly there exist positive integers k_2, k_3 and k_4 such that $S = k_2b = k_3c = k_4d$. Thus $S = S/k_1 + S/k_2 + S/k_3 + S/k_4 = S(1/k_1 + 1/k_2 + 1/k_3 + 1/k_4)$. But the quadrilateral is convex so each k_i must be larger than 2. Since a, b, c and d are to be different, we have $S \leq S(1/3 + 1/4 + 1/5 + 1/6) = 19S/20$, which is impossible. Hence no such quadrilateral exists."

Puzzle # 5 is a source of embarrassment for the Editor. The question should have read "If the four triangular faces of a tetrahedron have equal *perimeters* must the faces be congruent?" With perimeter in place of area the puzzle is easy. With area, the answer is still "YES" but the proof is not trivial. The Editor first encountered the results while browsing through some old copies of *The American Mathematical Monthly*. In a report on undergraduate mathematics clubs, in the April, 1926, *Monthly*, B. H. Brown discussed both versions of the problem in a note entitled *Bang's Theorem/Isosceles Tetrahedra*. It is this note which served as a basis for Ross Honsberger's expository essay *A Theorem of Bang and the Isosceles Tetrahedron* in his book *Mathematical Gems II* which was published in 1976. The Editor apologizes for making the goof. Only Richard I. Hess and Emil Slowinski responded to the puzzle, as proposed.

Only Emil Slowinski submitted a correct solution to *Puzzle # 6*. Richard I. Hess read too much into the statement of the problem. He claimed correctly that no solution exists if the two interior matches must have their ends end-to-end with the matchsticks which form the triangle. Slowinski's solution follows:



Seven readers submitted answers to *Puzzle # 7*. The answers for the four-rowed array and their finders are given first:

```
6  10  1  8      8  10  1  6      8  10  3  9      9  10  3  8
  4  9  7          2  9  5          2  7  6          1  7  5
    5  2             7  4             5  1             6  2
      3                3                4                4
```

Hess McKeon Oman Prielipp Slowinski	Boulger Oman Prielipp Slowinski	Ashbacher McKeon Oman Prielipp Slowinski	Ashbacher Oman Prielipp

Of course, arrays obtained by reflections in "vertical" lines are also solutions. Solutions for five-rowed arrays and the numbers 1 through 15 are:

```
13  3  15  14  6        6  14  15  3  13
  10  12  1  8            8  1  12  10
    2  11  7                7  11  2
      9  4                    4  9
        5                       5
```

Ashbacher Hess McKeon Oman Prielipp Slowinski	Ashbacher

In addition to the given three-rowed array, Oman and Prielipp submitted three others:

```
2  6  5      6  2  5      6  1  4      4  6  1
  4  1         4  3         5  3         2  5
    3            1            2            3
```

Several respondents reported that they were unable to find any six- or seven-rowed arrays with the specified property. An anonymous contributor, perhaps the proposer, referred to Martin Gardner's *Mathematical Games* column in the April 1977 Scientific American.

List of respondents: Charles Ashbacher *(3,7)*, Jeanette Bickley *(1)*, William Boulger *(2,3,7)*, Mark Evans *(3,6)*, Victor G. Feser *(3)*, Robert C. Gebhardt *(3)*, Richard I. Hess *(1,2,3,4,5,6,7)*, John M. Howell *(3)*, James Maccaline *(2,3)*, Scott M. McKeon *(2,3,7)*, Thomas M. Mitchell *(3)*, John Oman *(7)*, Robert Prielipp *(3,7)*, John Schue *(2,3)*, Emil Slowinski *(2,3,4,5,6,7)*, and Michael Taylor *(2,3)*.

Solution to Mathacrostic No. 25. (See Fall 1987 Issue).

Words:

A. grammar	L. paraselene	V. isonemal fabrics
B. figure eight	M. offspring	W. Catalan
C. rhadamanthine	N. lacunae	X. the future
D. accidence	O. one-track	Y. unmitigated
E. next friend	P. get off the earth	Z. robot retailing
F. Captain Video	Q. immersion	a. encaenia
G. impedance match	R. complex	b. bastinado
H. sagitta	S. at loggerheads	c. ocotillo
I. absorption laws	T. light-handed	d. otherworldly
J. trammel	U. paradigm	e. kite
K. ogham		

Quotation: *... the vertical blackboard is ... the ... prime medium of pedagogic expression. No amount of explanation accompanying a complete figure on a page can match the information transmitted while creating the same figure at the board, talking all the while. The ease of adding and subtracting detail, of correcting errors and amending diagrams are familiar to all math teachers.*

Solved by: Jeanette Bickley, Webster Groves High School, MO; Charles R. Diminnie, St. Bonaventure University, NY; Victor G. Feser, University of Mary, Bismarck, ND; Robert Forsberg, Lexington, MA; Dr. Theodor Kaufman, Brooklyn, NY; Charlotte Maines, Caldwell, NJ; Stephanie Sloyan, Georgian Court College, Lakewood, NJ; Michael J. Taylor, Indianapolis Power and Light Co., IN; Jeffrey Weeks and Nadia Marano, Ithaca College, NY; and Barbara Zeeberg, Denver, CO.

* * *

A First-class Puzzle Revisited - Posthaste, give the next two terms in the sequence: 2, 3, 2, 3, 4, 5, 6, 8, 10, 13, 15, 18, 20, ...

Mathacrostic No. 26

Proposed by Joseph D. E. Konhauser

The 229 letters to be entered in the numbered spaces in the grid will be identical to those in the 29 keyed *Words* at the matching numbers. The key numbers have been entered in the diagram to assist in constructing the solution. When completed, the initial letters of the *Words* will give the name(s) of the author(s) and the title of a book; the completed grid will be a quotation from that book.

The solution to *Mathacrostic No. 25* is given elsewhere in the PUZZLE SECTION.

1 A	2 I	3 W		4 E	5 U	6 M		7 H	8 G	9 K	10 P	11 C	12 S	13 F
14 A		15 M	16 F	17 I	18 H	19 E	20 L	21 K		22 V		23 S	24 b	25 H
26 N	27 I	28 R	29 P	30 M		31 C	32 O	33 R	34 G		35 U	36 c		37 W
38 D	39 Z	40 R	41 I		42 Y	43 Z	44 b		45 N	46 M	47 A	48 H	49 a	50 c
51 Q		52 U	53 C	54 J	55 D	56 O	57 a	58 N	59 P		60 R	61 b	62 B	
63 W	64 O	65 X	66 Y	67 J	68 E		69 N	70 c		71 C	72 N		73 W	
74 O	75 c	76 I	77 S	78 H	79 F	80 X	81 N		82 G	83 X		84 T	85 W	86 Z
	87 E	88 V	89 Q	90 M	91 B	92 K		93 W	94 X	95 F	96 O	97 D	98 K	
99 O	100 H	101 S		102 N	103 Y	104 F	105 G	106 H	107 a		108 O	109 U		110 C
111 V	112 b		113 M	114 D	115 Q	116 B	117 F	118 S	119 E		120 A	121 O	122 Q	123 N
124 C	125 R	126 Z		127 c	128 R	129 W		130 B	131 J	132 L	133 I	134 N	135 H	136 V
137 Y	138 T	139 K	140 G		141 c	142 P	143 F		144 Z	145 N	146 R	147 U	148 V	149 T
150 X	151 O	152 c	153 b	154 Y	155 S	156 B		157 H	158 V		159 Y	160 J	161 P	162 R
163 M	164 E	165 L		166 U	167 X	168 K	169 O	170 D	171 N	172 Z	173 Q	174 B	175 a	
176 J	177 X	178 H	179 P	180 S	181 V	182 I		183 C		184 X	185 b	186 W	187 U	188 M
189 H	190 E	191 L	192 J		193 W	194 M	195 I	196 Y		197 c	198 L		199 V	200 W
201 I		202 K	203 U	204 G		205 S	206 T	207 Z	208 c		209 b	210 S	211 H	212 V
213 E	214 R		215 G	216 Y	217 L		218 A	219 P	220 S	221 Y		222 N	223 X	224 b
225 E	226 a	227 L	228 O	229 V										

Definitions	*Words*
A. a projection in a building	218 47 120 1 14
B. comprehends (2 wds.)	156 91 174 116 130 62
C. a poset each two elements of which have a glb and a lub	124 183 31 110 71 53 11
D. to make an angle	97 170 55 38 114
E. a musical instrument that sounds by the vibration of its constituent material	190 119 225 19 87 68 213 4 164
F. formerly, in Ireland, a tenant renting a small farm under the rack-rent system	95 104 79 117 16 143 13
G. full of intellectual complications	105 215 82 204 34 8 140
H. an idealized, frictionless machine (2 wds.)	189 178 18 211 157 135 78 48 7 25 100 106
I. dark nebula in the Constellation Orion	41 76 17 201 27 2 133 195 182
J. infinity; the endless (Hindu mythology)	54 192 160 131 67 176
K. arbiters	9 98 92 21 168 139 202
L. pressed together	165 217 20 227 132 198 191
M. common sense (2 wds.)	15 46 113 194 30 163 6 188 90
N. 'henge-type monument north of Stonehenge believed to have preceded Stonehenge (2 wds.)	222 26 171 102 58 45 81 134 69 145 123 72
O. form of truth table for representing Boolean or logical expressions so that the manner in which they can be simplified is made apparent (2 wds.)	96 121 56 228 99 108 74 32 64 151 169
P. an edge of a graph whose removal separates the graph	179 59 161 142 10 219 29
Q. bright, glossy, lustrous	122 173 89 115 51
R. one of the first to recognize Karl Weierstrass' mathematical talent (1798 - 1652)	40 162 146 125 28 214 33 60 128
S. rectification	210 101 205 23 220 180 77 118 155 12
T. hard problems	138 206 84 149
U. a type of magic lantern that projects the image of an opaque object onto a screen	5 187 35 52 166 203 109 147
V. a virtually defunct star (2 wds.)	136 111 88 229 181 212 199 22 148 158
W. what a set S is said to be with respect to the point P if the intersection of S with each ray emanating from P is a segment or is the entire ray	63 193 200 37 186 85 73 93 3 129
X. a three-pointed figure in embroidery (2 wds.)	223 80 65 177 184 83 167 94 150
Y. a winding, tending, or twisting	137 159 196 66 103 221 154 216 42
Z. his achievement was a completely successful set of definitions for a calculus of ratios (408 - 355 B.C.)	86 39 126 43 172 144 207
a. a short vertical flat spring with a bob at the top used for measuring slight horizontal vibrations	107 57 49 226 175
b. horn-shaped	224 61 209 24 185 44 112 153
c. a magnesium metasilicate, grayish-white to brown, occurring massive and in orthorhombic crystals	50 152 36 141 127 70 197 208 75

* * *

Editor's Note – Mathacrostic No. 26 was prepared by Carl E. Gadow on a Sun 3/50 Workstation using Metaform Professional by Intran Corporation.

PROBLEM DEPARTMENT

Edited by Clayton W. Dodge
University of Maine

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

All communications should be addressed to C. W. Dodge, Math. Dept., University of Maine, Orono, ME 04469. Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly identified with name and address. Solutions to problems in this issue should be mailed by December 15, 1988.

Problems for Solution

665. *Proposed by John M. Howell, Littlerock, California.*

An international committee translated the *EINS-ZWEI-DREI* problem, Problem *626* [Fall *1986*, Fall 1987]. Even though the languages are all mixed up, there is a base ten solution in which *O*, *D*, and *T*, the initial letters of *ONE*, *DOS*, and *TRE*, are in increasing arithmetic progression. Find that solution.

$$\begin{array}{r} \text{ONE} \\ \text{DOS} \\ +\ \text{TRE} \\ \hline \text{SEX} \end{array}$$

666. *Proposed by John M. Howell, Littlerock, California.*

Five dice are rolled to form a "poker hand." Find the probabilities of the hands: no matches, one pair, two pairs, three of a kind, four of a kind, five of a kind, full house, and straight.

667. *Proposed by John M. Howell, Littlerock, California.*

Each special die has one face with 1 spot, two faces with 2 spots each, and three faces with 3 spots each. Find the probability of tossing a sum of *8* with four special dice.

668. *Proposed by R. S. Luthar, University of Wisconsin Center, Janesville, Wisconsin.*

Evaluate the integral

$$\int_0^{\pi/2} \frac{2\cos^3 x}{2 - \sin 2x}\, dx.$$

669. *Proposed by Mohammad K. Azarian, University of Evansville, Evansville, Indiana.*

Let n be any positive integer. Show that $\prod_{k=n}^{2n} (k! + 1)$ has a factor greater than $\prod_{k=n}^{2n} k$.

670. *Proposed by Peter A. Lindstrom, North Lake College, Irving, Texas.*

If $F_n = 2^{2^n} + 1$ is the nth Fermat number, find all values of n so that F_n and $F_n - 2$ are twin primes.

671. *Proposed by J. S. Frame, Michigan State University, East Lansing, Michigan.*

Find all sequences of $2k + 1$ consecutive integers $a, a + 1, \ldots, a + 2k$ such that the sum of the squares of the first $k + 1$ of these integers is equal to the sum of the squares of the last k. That is, find a formula for $a = a_k$ as a function of k. For example, $a_1 = 3$ since $3^2 + 4^2 = 5^2$.

672. *Proposed by Barry Brunson, Western Kentucky University, Bowling Green, Kentucky.*

Find a series representation for

$$\int_0^1 x^x\, dx.$$

673. *Proposed by Stanley Rabinowitz, Alliant Computer Systems, Littleton, Massachusetts.*

Let *AB* be an edge of a regular tesseract (a four-dimensional cube) and let C be the tesseract's vertex that is furthest from *A*. Find the measure of angle *ACB*.

674. *Proposed by Russell Euler, Northwest Missouri State University, Maryville, Missouri.*

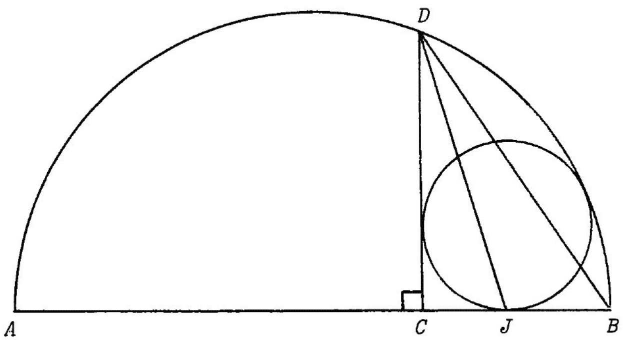
Find necessary and sufficient conditions for the arithmetic mean of the roots of the polynomial equation

$$a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n = 0$$

to be equal to the geometric mean of the roots.

675. *Proposed by John H. Scott, Macalester College, Saint Paul, Minnesota.*

Erect a semicircle on segment AB as diameter. From point D on the semicircle drop a perpendicular to point C on AB. Draw a circle tangent to CB at J and tangent to the semicircle and to segment CD. Prove that angles CDJ and JDB have equal measures. See the figure below.



676. *Proposed by John M. Howell, Littlerock, California.*

Show that, for k > 0,

$$\sum_{i=1}^{n}\left[\prod_{j=0}^{k}(i+j)\right]^{-1} = \frac{1}{k\cdot k!} - \frac{1}{k}\left[\prod_{j=1}^{k}(n+j)\right]^{-1}.$$

677. *Proposed by Jack Garfunkel, Flushing, New York.*

If A, B, and C are the angles of a triangle? then show that

$$\frac{\cos A\cos B\cos C}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} \le \frac{\sqrt{3}}{9}.$$

Solutions

639. [Spring 19871 *Proposed by Charles W. Trigg, San Diego, California.*

Find the smallest SLICE the KNIFE can cut from the *CAKE* if

$$CAKE + KNIFE = \text{SLICE}.$$

Solution by James Wei Qiang Li, James Madison High School, Brooklyn, New York.

Since we must have E + E = E (mod 10), then E = 0. From the hundreds column we get that A = 9. It follows that I is carried into each column after the tens column. Since we are looking for the smallest SLICE, we first find the smallest value for S. We have K + I = S, so K < 8. Also $K + F$ > 10, so K > 2. From the thousands column we have C + N = L + 9 with L > 0 and N < 9, so C > 1.

If K = 3, then C = 1, which is not possible. If K = 4, then F = 8 and C = 2, so N < 8, again producing an impossibility. If K = 5, then F = 7 or 8, so C = 2 or 3 and N = 8 or 7, producing L = 1. The resulting $SLICE$s are 61I20 and 61I30. The smallest I available is 3 in the former case and 2 in the latter. Thus the answer is the latter, and the complete addition is

$$\begin{array}{r} 3950 \\ \underline{57280} \\ 61230 \end{array}.$$

Also solved by **CHARLES ASHBACHER**, *Mount Mercy College, Cedar Rapids, IA,* **JAMES E. CAMPBELL**, *Indiana University at Bloomington,* **MARK EVANS**, *Louisville, KY,* **VICTOR G. FESER**, *University of Mary, Bismarck, ND,* **ROBERT C. GEBHARDT**, *Hopatcong, NJ,* **RICHARD I. HESS**, *Rancho Palos Verdes, CA,* **JOHN H. SCOTT**, *Macalester College, Saint Paul, MN, and* **KENNETH M. WILKE**, *Topeka, KS. Both* **FRANK P. BATTLES**, *Massachusetts Maritime Academy, Buzzards Bay, and the* **PROPOSER** *obtained the near solution* 2950 + 58370 = 61320. *Wilke commented that "The smallest* SLICE *comes from the largest CAKE by using the smallest KNIFE."*

***640. [Spring 1987]** *Proposed by John M. Howell, Littlerock, California.*

Find the largest value of $S(n)$ and the limit of $S(n)$ as $n \to \infty$ if

$$S(n) = \sum_{x=1}^{n-1} \frac{n}{\binom{n}{x}}.$$

Composite of solutions by Richard I. Hess, Rancho Palos Verdes, California, and Al Terego, Malden, Massachusetts.

We have that $S(1)$ is undefined, $S(2)$ = 1, $S(3)$ = 2, $S(4)$ = 2 2/3, $S(5)$ = 3, $S(6)$ = 3 1/10, $S(7)$ = 3 1/15, $S(8)$ = 2 34/35, and $S(9)$ = 2 6/7.

For large n,

$$S(n) = \frac{n}{n} + \frac{2n}{n(n-1)} + \frac{6n}{n(n-1)(n-2)} + \dots + \frac{2n}{n(n-1)} + \frac{n}{n}$$

$$= 1 + \frac{2}{n-1} + \frac{6}{(n-1)(n-2)} + \frac{24}{(n-1)(n-2)(n-3)} + \dots + 1.$$

Hence $\lim_{n\to\infty} S(n) = 2$ since all terms but the first and last drop out.

We now show that $\{S(n)\}$ is monotone decreasing for $n > 8$. For this purpose we write

$$S(2n) = \frac{2n}{(2n)!}[1!(2n-1)! + 2!(2n-2)! + \dots + n!n! + \dots + (2n-1)!1!]$$

$$= \frac{2\cdot 2n}{(2n)!}[1!(2n-1)! + 2!(2n-2)! + \dots + (n-1)!(n+1)!] + \frac{2n}{(2n)!}n!n!$$

$$= 2\left[1 + \frac{2!}{2n-1} + \frac{3!}{(2n-1)(2n-2)} + \frac{4!}{(2n-1)(2n-2)(2n-3)} + \dots + \frac{(n-1)!}{(n+1)(n+2)\dots(2n-1)}\right] + \frac{2n\cdot n!}{(n+1)(n+2)\dots(2n)}$$

and similarly

$$S(2n+1) = 2\left[1 + \frac{2!}{2n} + \frac{3!}{2n(2n-1)} + \frac{4!}{2n(2n-1)(2n-2)} + \dots + \frac{(n-1)!}{(n+2)(n+3)\dots(2n-1)}\right] + \frac{2n\cdot n!}{(n+2)(n+3)\dots(2n)}.$$

Next we subtract corresponding terms to get that

$$S(2n+1) - S(2n) = 2\left[0 + 2!\left(\frac{1}{2n} - \frac{1}{2n-1}\right) + \frac{3!}{2n-1}\left(\frac{1}{2n} - \frac{1}{2n-2}\right) + \dots + \frac{(n-1)!}{(n+3)(n+4)\dots(2n-1)}\left(\frac{1}{2n} - \frac{1}{n+2}\right)\right] + \frac{2\cdot n!}{(n+2)(n+3)\dots(2n)}\left(1 - \frac{n}{n+1}\right)$$

$$= 2\left[\frac{-2!}{2n(2n-1)} + \frac{-2\cdot 3!}{(2n-2)(2n-1)(2n)} + \dots + \frac{-(n-2)(n-1)!}{(n+2)(n+3)\dots(2n)}\right] + \frac{2\cdot n!}{(n+1)(n+2)\dots(2n)}$$

In this last expression all the terms in the brackets are negative. Now we obviously will have that $S(2n+1) < S(2n)$ if we can show that

$$\frac{2\cdot 2!}{2n(2n-1)} \geq \frac{2\cdot n!}{(n+1)(n+2)\dots(2n)}.$$

The argument leading to this inequality makes sense only for $n > 2$ since $S(6)$, for example, has only five terms. Thus we take $n > 2$ and rewrite the inequality in the equivalent form

$$P(n):\quad 2(n+1)(n+2)\dots(2n-2) \geq n!.$$

Now $P(3)$ states that $(2)(4) > 3!$, which is true. By mathematical induction we assume that $n > 2$ and that $P(n)$ is true and we take

$$2(n+2)(n+3)\dots(2n) > 2(n+2)(n+3)\dots(2n)\cdot\frac{(n+1)^2}{(2n-1)(2n)}$$

$$= 2(n+1)(n+2)\dots(2n-2)\cdot(n+1)$$

$$> n!\cdot(n+1)$$

$$= (n+1)!,$$

so $P(n+1)$ is true whenever $P(n)$ is true and $n > 2$.

Similarly we obtain that

$$S(2n) - S(2n-1) = \frac{2}{(2n-1)!}\left[0 - 2(2n-3)! - 2\cdot 3!(2n-4)! - \dots - (n-2)n!(n-1)!\right] + \frac{n!n!}{(2n-1)!},$$

and then prove that, for $n > 4$,

$$\frac{2}{(2n-1)!}\cdot 2(2n-3)! \geq \frac{n!n!}{(2n-1)!}.$$

This completes the proof that $\{S(n)\}$ is monotone decreasing for $n > 5$. Therefore $S(6) = 3\ 1/10$ is the largest $S(n)$.

Also solved by GEORGE P. EVANOVICH, *Saint Peters College, Jersey City, NJ,* MARK EVANS, *Louisville, KY,* JOHN H. SCOTT, *Macalester College, Saint Paul, MN, and the* PROPOSER. *All also-solvers obtained the correct limit and the largest* $S(n)$, *but none proved that* $S(6)$ *was indeed greatest. Scott commented that "*$S(n)$ *approaches 2 very, very slowly, but infinity is very, very, very large."*

***641.** *[Spring 1987] Proposed by Paul A. McKlueen, Raleigh, North Carolina.*

Let $f_1 = f_2 = 1$ and $f_{k+2} = f_k + f_{k+1}$ for $k > 0$ define the Fibonacci sequence. It is known that $f_k f_{k+1} = \sum_{i=1}^{k} f_i^2$ for any positive integer k. Find a similar formula for the generalized Fibonacci sequence g_k, where g_1 through g_n are given and for $k > 0$,

$$g_{k+n} = \sum_{i=1}^{n} g_{k+i-1}.$$

I. *Solution by Mark Evans, Louisville, Kentucky.*

Since we have that

$$\prod_{i=k+1}^{k+n} g_i = \left[\prod_{i=k+1}^{k+n-1} g_i\right]\left[\sum_{i=k}^{k+n-1} g_i\right]$$

$$= \left[\prod_{i=k+1}^{k+n-1} g_i\right]\left[g_k + \sum_{i=k+1}^{k+n-1} g_i\right]$$

$$= \prod_{i=k}^{k+n-1} g_i + \left[\prod_{i=k+1}^{k+n-1} g_i\right]\left[\sum_{i=k+1}^{k+n-1} g_i\right],$$

then it is readily shown by mathematical induction that

$$\prod_{i=k+1}^{k+n} g_i = \sum_{j=0}^{k}\left[\prod_{i=j+1}^{j+n-1} g_i\right]\left[\sum_{i=j+1}^{j+n-1} g_i\right] + \prod_{i=1}^{n} g_i - \left[\prod_{i=1}^{n-1} g_i\right]\left[\sum_{i=1}^{n-1} g_i\right].$$

II. *Solution by Ai? Terego, Malden, Massachusetts.*

When m is a positive integer, we have that

$$g_{m+n+1} = g_{m+1} + g_{m+2} + \cdots + g_{m+n}$$

$$= (g_m + g_{m+1} + \cdots + g_{m+n-1}) + g_{m+n} - g_m$$

$$= 2g_{m+n} - g_m.$$

Hence

$$g_{m+n}g_{m+n+1} = g_{m+n}(2g_{m+n} - g_m) = 2g_{m+n}^2 - g_{m+n}g_m$$

$$= 2g_{m+n}^2 - g_m(2g_{m+n-1} - g_{m-1})$$

$$= 2g_{m+n}^2 - 2g_m g_{m+n-1} + g_m g_{m-1},$$

so we can prove by mathematical induction that, when m and k are positive integers,

$$g_{m+k(n+1)}g_{m+k(n+1)+1} = g_m g_{m+1} + \sum_{j=1}^{k}\left(2g_{m+j(n+1)}^2 - 2g_{m+1+(j-1)(n+1)}g_{m-1+j(n+1)}\right).$$

642. [Spring 1987] *Proposed by Dmitry P. Mavlo, Moscow, U.S.S.R.*

Let $\pi, \mu, \varepsilon > 0$. Prove that

$$(1 + \pi\mu\varepsilon)\left(\frac{1}{\pi(1+\mu)} + \frac{1}{\mu(1+\varepsilon)} + \frac{1}{\varepsilon(1+\pi)}\right) \geq 3,$$

with equality if and only if $\pi = \mu = \varepsilon = 1$.

Solution by P. H. Diananda, Singapore.

Multiply the stated inequality by

$$\pi\mu\varepsilon(1+\pi)(1+\mu)(1+\varepsilon)(1+\pi\mu\varepsilon)$$

to get

$$(\pi^2\mu^2\varepsilon^2\,\Sigma\,\pi + \Sigma\,\pi\mu^2) + (\pi\mu\varepsilon\,\Sigma\,\pi\mu^2 + \Sigma\,\mu\varepsilon) \geq 2\pi\mu\varepsilon\,\Sigma\,\pi\mu + 2\pi\mu\varepsilon\,\Sigma\,\pi.$$

This is true since

$$\pi^2\mu^2\varepsilon^2\pi + \pi\mu^2 \geq 2\pi\mu\varepsilon\pi\mu \quad\text{and}\quad \pi\mu\varepsilon\pi\mu^2 + \mu\varepsilon \geq 2\pi\mu\varepsilon\mu,$$

and so on, with equality if and only if

$$\pi^2\mu^2\varepsilon^2\pi = \pi\mu^2 \quad\text{and}\quad \mu\pi\varepsilon\pi\mu^2 = \mu\varepsilon,$$

and so on, which are equivalent to $\pi = \mu = \varepsilon = 1$.

Also solved by RALPH KING, *St. Bonaventure University,* NY, *and the* PROPOSER.

643. [Spring 1987] *Proposed by M. S. Klamkin, University of, Alberta, Edmonton, Alberta, Canada.*

If $a, b, c, d > 0$, prove that

$$\left(a/b\right)^{2a-b+d}\left(b/c\right)^{2b-c+a}\left(c/d\right)^{2c-d+b}\left(d/a\right)^{2d-a+c} \geq 1.$$

I. *Solution by Mark Evans, Louisville, Kentucky.*

We have that

(1) $$\left(a/b\right)^{2a-b+d}\left(b/c\right)^{2b-c+a}\left(c/d\right)^{2c-d+b}\left(d/a\right)^{2d-a+c}$$

$$(2) \qquad = \frac{a^{3a}}{a^{b+c+d}} \cdot \frac{b^{3b}}{b^{c+d+a}} \cdot \frac{c^{3c}}{c^{d+a+b}} \cdot \frac{d^{3d}}{d^{a+b+c}}$$

$$(3) \qquad = \left(\frac{a}{b}\right)^{a-b}\left(\frac{a}{c}\right)^{a-c}\left(\frac{a}{d}\right)^{a-d}\left(\frac{b}{c}\right)^{b-c}\left(\frac{b}{d}\right)^{b-d}\left(\frac{c}{d}\right)^{c-d}.$$

From Expression (2) we see that the equation is symmetrical in a, b, c, and d. Hence we may let $a \geq b \geq c \geq d > 0$ without loss of generality. Now in Expression (3) all the fractions are greater than or equal to 1 and all the exponents are nonnegative, so Expression (3) is greater than or equal to 1 and the proof is complete.

II. *Solution by Richard I. Hess, Rancho Palos Verdes, California.*

The given inequality can be rewritten in the form

$$F(a,b,c,d) = a^{3a-b-c-d}b^{3b-c-d-a}c^{3c-d-a-b}d^{3d-a-b-c} \geq 1.$$

We let $f(a,b,c,d) = \ln F$ and search for a minimum of f. Thus

$$\frac{\partial f}{\partial a} = \frac{1}{a}(3a - b - c - d) + 3\ln a - \ln b - \ln c - \ln d$$

and

$$\frac{\partial f}{\partial d} = \frac{1}{d}(3d - a - b - c) + 3\ln d - \ln a - \ln b - \ln c.$$

Without loss of generality we take $a \leq b \leq c \leq d$ and we find that $\partial f/\partial a \leq 0$ and $\partial f/\partial d \geq 0$. Hence we can increase a and decrease d to reduce f. This can be done until $a = b$ and $c = d$. Finally consider

$$g(a,d) = f(a,a,d,d) = 4(a - d)(\ln a - \ln d).$$

We see that $g(a,d) > 0$ when $a \neq d$ and that $g(a,a) = 0$. Hence g is minimized when $a = b = c = d$, so $F(a,b,c,d) \geq 1$ with equality if and only if $a = b = c = d$.

III. *Solution by the Proposer.*

The inequality can be rewritten as

$$a^a b^b c^c d^d \geq (abcd)^{(a+b+c+d)/4}.$$

More generally, if $a_i > 0$, then

$$(1) \qquad \prod_{i=1}^{n} (a_i)^{a_i} \geq \left[\prod_{i=1}^{n} a_i\right]^{(a_1+a_2+\cdots+a_n)/n}.$$

Since $y = x\log x$ is concave upwards for $x > 0$, then

$$(2) \qquad \frac{1}{n}\sum_{i=1}^{n} a_i \log a_i \geq \left[\frac{1}{n}\sum_{i=1}^{n} a_i\right]\log\left[\frac{1}{n}\sum_{i=1}^{n} a_i\right].$$

Then, by the arithmetic-geometric mean inequality,

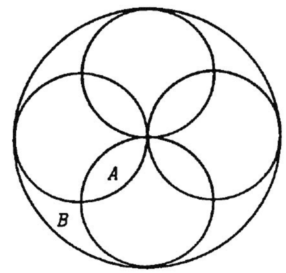
$$(3) \qquad \log\left[\frac{1}{n}\sum_{i=1}^{n} a_i\right] \geq \log \prod_{i=1}^{n} a_i^{1/n}.$$

Finally, statements (2) and (3) imply (1).

Also solved by JAMES E. CAMPBELL, *Indiana University at Bloomington.*

644. [Spring 1987] *Proposed by Richard I. Hess, Rancho Palos Verdes, California.*

In the figure below prove that regions A and B have equal areas.



Solution by Judith P. Kahn, James Madison High School, Brooklyn, New York.

The large circle has an area equal to the sum of the areas of the four small circles plus $4B$ minus $4A$ because the A regions have been counted twice. If r is the radius of the small circles, then $2r$ is the radius of the large circle, and we have

$$\pi(2r)^2 = 4(\pi r^2) + 4B - 4A,$$

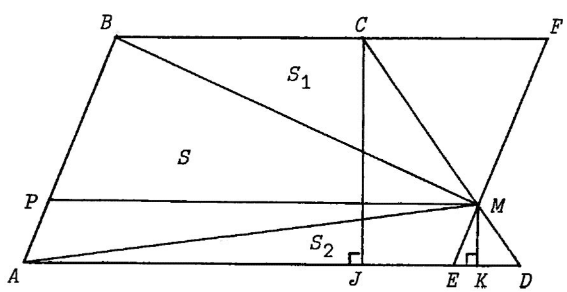
so that $A = B$.

Also solved by STEVE ASHER, *McNeil Pharmaceutical, Spiking House, PA,* CHARLES ASHBACHER, *Mount Mercy College, Cedar Rapids, IA,* SEUNG-JIN BANG, *Seoul, Korea,* FRANK P. BATTLES, *Massachusetts Maritime Academy, Buzzards Bay,* JAMES E. CAMPBELL, *Indiana State University at Bloomington,* RUSSELL EULER, Northwest Missouri State University, Maryville, GEORGE P. EVANOVICH, Saint Peters College, Jersey City, NJ, MARK EVANS, Louisville, KY, VICTOR G. FESER, University of Mary, Bismarck, ND, ROBERT C. GEBHARDT, Hopatcong, NJ, JACK GARFUNKEL, Flushing, NY, RICHARD A. GIBBS, Fort Lewis College, Durango, CO, JOHN P. HOLCOMB, JR., St. Bonaventure University, NY, XIAN SHAN HUI, James Madison High School, Brooklyn, NY, BRUCE KING, Western Connecticut State University, Danbury, RALPH E. KING, St. Bonaventure University, NY, JAMES LI, James Madison High School, Brooklyn, NY, HENRY S. LIEBERMAN, Waban, MA, BOB PRIELIPP, University of Wisconsin-Oshkosh, GEORGE W. RAINEY, Los Angeles, CA, JOHN H. SCOTT, Macalester College, Saint Paul, MN, WADE H. SHERARD, Furman University, Greenville, SC, TIMOTHY SIPKA, Anderson College, IN, STEPHANIE SLOYAN, Georgian Court College, Lakewood, NJ, JIM TODSEN, Iowa State University, Ames, KENNETH M. WILKE, Topeka, KS, and the PROPOSER (2 solutions). Campbell commented that this problem should have been in the Puzzle Section. That may well be true and, if so, then I do apologize for stealing my colleague's material, but there are two reasons why it seems appropriate hum. First, our readers certainly have responded with solutions, and second, almost no one saw the immediate relation that since the area of the large circle equals that of the four small circles, thin $A = B$ immediately.

645. [Spring 1987] *Proposed by Dmitry P. Mavlo, Moscow, U.S.S.R.*

Let M be an arbitrary point on segment CD of trapezoid $ABCD$ having sides AD and BC parallel. Let S, S_1, and S_2 be the areas of triangles ABM, BCM, and ADM, respectively. Prove that

$$S \geq 2\min(S_1, S_2)$$



I. *Solution by Mark Evans, Louisville, Kentucky.*

Label the trapezoid so that $AD > BC$, as shown in the figure on the previous page. Take P on AB so that MP is parallel to AD and BC, and let J and K be the feet of the perpendiculars dropped from C and M to side AD. Now

$$\text{area}(ABCD) = (CJ)(AD + BC)/2 = S + S_1 + S_2,$$

$$S = (MK)(PM)/2 + (PM)(CJ - MK)/2 = (CJ)(PM)/2,$$

and

$$S_1 = (BC)(CJ - MK)/2.$$

If $PM \geq (AD + BC)/2$, then

$$S \geq (CJ)(AD + BC)/4 \geq \text{area}(ABCD)/2,$$

so $S \geq S_1 + S_2 \geq 2\min(S_1, S_2)$.

If, on the other hand, $PM < (AD + BC)/2$, then $MK > (CJ)/2$ and

$$S_1 = (CJ - MK)(BC)/2 < (CJ - (CJ)/2)(BC)/2 = (CJ)(BC)/4.$$

Since also $PM > BC$, then $(CJ)(PM)/2 > 2((CJ)(BC)/4)$ and finally

$$S > 2S_1 \geq 2\min(S_1, S_2).$$

II. *Solution by Al T. Tood, Triangle, California.*

Assume $AD \geq BC$ and draw EF through M parallel to AB and cutting AD at E and BC at F, as shown in the figure. If $CM \geq MD$, then $\text{area}(ABCD) \leq \text{area}(AEFB)$ since $\text{area}(DEM) \leq \text{area}(CFM)$. Then

$$S = \text{area}(AEB) = \text{area}(AEFB)/2 \geq \text{area}(ABCD)/2.$$

Since now $S \geq \text{area}(ABCD)/2$, then also

$$\text{area}(ABCD)/2 \geq S_1 + S_2 \geq 2\min(S_1, S_2).$$

If $CM < MD$, then

$$S = \text{area}(ABE) = \text{area}(BEF) > 2S_1 \geq 2\min(S_1, S_2).$$

Also solved by GEORGE P. EVANOVICH, Saint Peters College, Jersey City, NJ, JACK GARFUNKEL, Flushing, NY, RICHARD A. GIBBS, Fort Lewis College, Durango, CO, RICHARD I. HESS, Rancho Palos Verdes, CA, JAMES LI, James Madison High School, Brooklyn, NY, HENRY S. LIEBERMAN, Waban, MA, JOHN H. SCOTT, *Macalester College, Saint Paul,* MN, KENNETH M. WILKE, Topeka, KS, and the PROPOSER.

646. [Spring 1987] *Proposed by Dick Field, Santa Monica, California.*

Find the smallest k for which there is only one k-digit palindrome that is the square of an integer.

Solution by Kenneth M. Wilke, Topeka, Kansas.

If $k = 1$, then 1, 4, and 9 are palindrome squares. For $k = 2n + 1$, both $(10^n + 1)^2$ and $[2(10^n + 1)]^2$ produce the respective k-digit palindrome squares

$$100\ldots020\ldots001 \qquad \text{and} \qquad 400\ldots080\ldots004,$$

where each of the four groups contains $n - 1$ zeros. Hence the desired value of k cannot be odd.

If $k = 2n$ for some positive integer n, then the palindrome square (PS) must be divisible by 11 and hence by 121. Since $121 > 100$, then $k > 2$. Checking the values of $121a^2 = (11a)^2$ for $3 \le a \le 9$, no PS occurs, so $k \neq 4$.

Now suppose $k = 6$. Since the PS must end in 1, 4, 5, 6, or 9, its root must end in 1 or 9, 2 or 8, 5, 4 or 6, or 3 or 7, respectively, and we let its root be $11a$ since it must be divisible by 11. Using these criteria we find that: if the PS ends in 1, then $316 < 11a \le 447$, and 319, 341, and 429 need to be tested; if the PS ends in 4, then $632 < 11a \le 707$, and 638 and 682 need testing; if the PS ends in 6, then $774 < 11a \le 836$, and 814 and 836 need testing; and if the PS ends in 9, then $948 < 11a \le 999$, and 957 needs to be tested.

Of these possibilities, only 836 = 698896 produces a PS. Since this is the sole solution having 6 digits, then $k = 6$ is the desired result.

Also solved by CHARLES ASHBACHER, *Mount Mercy College, Cedar Rapids,* IA, FRANK P. BATTLES and LAURA L. KELLEHER, *Massachusetts Maritime Academy, Buzzards Bay,* VICTOR G. FESER, *University of Mary, Bismarck,* ND, RICHARD I. HESS, *Rancho Palos Verdes,* CA, THOMAS M. MITCHELL, *Southern Illinois University at Carbondale,* JOHN H. SCOTT, *Macalester College, Saint Paul,* MN, WADE H. SHERARD, *Furman University, Greenville,* SC, and the PROPOSER.

647. [Spring 1987] *Proposed by Robert C. Gebhardt, Hopatcong, New Jersey.*

For each positive integer n find the earliest row of Pascal's triangle in which the first n terms have the property that each term after the first is an integral multiple of its predecessor.

Solution by Cheung Kwai Hung, Brooklyn, New York.

We use the notation

$$C(m,r) = \binom{m}{r} = \frac{m!}{(m-r)!\,r!}$$

for what we shall call the $(r + 1)$st coefficient in the mth row of Pascal's triangle. Then

$$\frac{C(m,r)}{C(m,r-1)} = \frac{(m-r+1)!\,(r-1)!}{m!} \cdot \frac{m!}{(m-r)!\,r!} = \frac{m-r+1}{r}$$

For this ratio to be an integer, we need to have $m + 1$ divisible by r. Therefore $m + 1 \equiv 0 \pmod{i}$ for $i = 2, 3, 4, \ldots, n - 1$, so the smallest value for $m + 1$ is the least common multiple of 2, 3, ..., $n - 1$. That is, for $n > 3$,

$$m = \operatorname{lcm}(2, 3, 4, \ldots, n - 1) - 1.$$

By examination of Pascal's triangle for $n \le 3$ and by the above formula for $n > 3$, we have the following table:

n:	1	2	3	4	5	6	7	8	...
m:	0	1	3	5	11	59	59	419	.

Also solved by FRANK P. BATTLES and LAURA L. KELLEHER, *Massachusetts Maritime Academy, Buzzards Bay,* JAMES E. CAMPBELL, *Indiana University at Bloomington,* MARK EVANS, *Louisville,* KY, RICHARD I. HESS, *Rancho Palos Verdes,* CA, HENRY S. LIEBERMAN, *Waban,* MA, KENNETH M. WILKE, *Topeka,* KS, and the PROPOSER.

648. [Spring 1987] *Proposed by Jack Garfunkel, Flushing, New York.*

If A, B, C are the angles of a triangle ABC, prove that

$$\Pi \cos\left(\frac{A}{2}\right) \le \frac{\sqrt{3}}{6} \Sigma \cos^2\left(\frac{A}{2}\right).$$

Solution by the Proposer.

Item 5.5 of page 49 of O. Bottema et al, *Geometric Inequalities*, states that

$$s \le \frac{\sqrt{3}}{3}(4R + r).$$

The we have

$$\Pi \cos\frac{A}{2} = \frac{s}{4R} \le \frac{\sqrt{3}}{12R}(3R + R + r)$$

$$= \frac{\sqrt{3}}{6}\left(\frac{3}{2} + \frac{1}{2}\frac{R + r}{R}\right)$$

$$= \frac{\sqrt{3}}{6}\,\Sigma\,\frac{1 + \cos A}{2}$$

$$= \frac{\sqrt{3}}{6}\,\Sigma\,\cos^2\left(\frac{A}{2}\right).$$

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, BOB PRIELIPP, University of Wisconsin-Oshkosh, and JOHN H. SCOTT, Macalester College, Saint Paul, MN.

649. *[Spring 1987] Proposed by Edward J. Arismendi, Jr., California State University, Long Beach, California.*

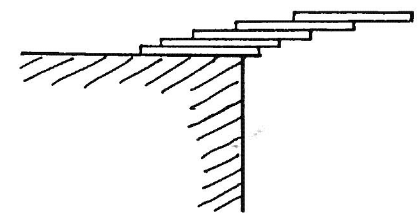
How far beyond the edge of a table can a deck of cards be stacked without the pile falling off the table?

Solution by the Proposer.

Since the mass m of a card is uniformly distributed, the center of of mass of the card is at its geometrical centroid. Hence one card will (just) balance if it extends 1/2 its length over the edge of the table or over the edge of the next card below. Thus place the first card extending 1/2 its length over the edge of a second card. The center of mass of the two cards is midway between their respective centers, 1/4 of the way from the edge of the second card (and 3/4 of the way along the first card). So place these two cards so that their center of mass is at the edge of a third card. The center of mass of the three cards is $(2\cdot 0 + (1/2)\cdot 1)/3 = 1/6$ of the way from the edge of the third card, and so on. Hence the top card of n cards can protrude

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \ldots + \frac{1}{2n}$$

card lengths over the edge of the table. See the following figure.



Also solved by RUSSELL EULER, Northwest Missouri State University, Maryville, VICTOR G. FESER, University of Mary, Bismarck, ND, and RICHARD I. HESS, Rancho Palos Verdes, CA. Hess noted that the protrusion is 2.269022.. *for a deck of* 52 *cards.*

Editor's note.: this delightful problem first appeared in the Pi Mu Epsilon Journal as problem #52 in Volume 1, No. 8, April 1953.

650. *[Spring 1987] Proposed by Richard I. Hess, Rancho Palos Verdes, California.*

The 1980 Wimbleton final between Borg and McEnroe involved a tie-break game that went to 18–16. Given that the server has a 70% chance of winning the point, what is the probability that the two players reach a 16–16 tie in a tiebreak game? (In a tiebreak game the first player serves one point. Thereafter players alternate serving 2 points each. The first player reaching 7 or more points with an advantage of 2 or more points wins the game.)

Solution by James E. Campbell, Indiana University at Bloomington.

In order to go beyond 7 points, there must be a 6–6 tie at the end of the first 12 serves. The probability of this occurring, which we call $P(6\text{-}6)$, is derived as follows. The probability of the receiver winning each of the first 12 serves is $(0.3)^{12}$ and would result in a 6–6 tie. If the players exchange exactly x points, the score is still 6–6 and there are $C(6,x)^2$ such exchanges, each with probability $(0.3)^{12-2x}(0.7)^{2x}$ of occurring. Thus

$$P(6\text{-}6) = (0.3)^{12} + 6^2(0.3)^{10}(0.7)^2 + 15^2(0.3)^8(0.7)^4 + \ldots$$
$$+ 6^2(0.3)^2(0.7)^{10} + (0.7)^{12} = 0.24838.$$

Then, to make sure that neither player gains a two-point advantage, each of the following pairs of points must be split. Since the serve switches within each pair, then either the server or the receiver must win both points. The probability of this happening is

$$(0.3)^2 + (0.7)^2 = 0.58.$$

This must continue for 10 pairs of points, so we have

$$P(16\text{-}16) = P(6\text{-}6)\cdot(0.58)^{10} \approx (0.24838)(0.0043080) = 0.0010700.$$

Also solved by CHARLES ASHBACHER, *Mount Mercy College, Cedar Rapids, IA,* MARK EVANS, *Louisville, KY, and the* PROPOSER, *who pointed out that if the server had no advantage, then the result would be* 0.000220299.

651. [Spring 1987] *Proposed by Al Terego, Malden, Massachusetts.*

Professor E. P. Umbugio has recently been strutting around because he hit upon the solution of the fourth degree equation which results when the radicals are eliminated from the equation

$$x = (x - 1/x)^{1/2} + (1 - 1/x)^{1/2}.$$

Deflate the professor by solving it using nothing higher than quadratic equations. [From Robinson's *Mathematical Recreations*, 1851.]

I. Solution by Kenneth M. Wilke, Topeka, Kansas.

Let $A = x - 1/x$ and $B = 1 - 1/x$. Then the given equation becomes

$$(1) \qquad x = \sqrt{A} + \sqrt{B}.$$

Hence

$$\frac{1}{x} = \frac{\sqrt{A} - \sqrt{B}}{A - B} = \frac{\sqrt{A} - \sqrt{B}}{x - 1}$$

and

$$(2) \qquad 1 - \frac{1}{x} = \frac{x - 1}{x} = \sqrt{A} - \sqrt{B}.$$

Adding equations (1) and (2) yields

$$2\sqrt{A} = x + 1 - \frac{1}{x} = A + 1,$$

so $(\sqrt{A} - 1)^2 = 0$ and thus $A = 1$. Now we have

$$x - \frac{1}{x} = A = 1, \qquad \text{so} \qquad x^2 - x - 1 = 0,$$

which has the roots

$$x = \frac{1 + \sqrt{5}}{2} \qquad \text{and} \qquad x = \frac{1 - \sqrt{5}}{2}.$$

If $x = \frac{1 + \sqrt{5}}{2}$, then $A = 1$ and $B = \frac{3 - \sqrt{5}}{2} = \left(\frac{\sqrt{5} - 1}{2}\right)^2$, so that

$$\sqrt{A} + \sqrt{B} = 1 + \frac{\sqrt{5} - 1}{2} = \frac{1 + \sqrt{5}}{2} = x,$$

so this value is a root of the given equation.

If $x = \frac{1 - \sqrt{5}}{2}$, then $A = 1$ and $B = \frac{3 + \sqrt{5}}{2} = \left(\frac{\sqrt{5} + 1}{2}\right)^2$, so that

$$\sqrt{A} + \sqrt{B} = 1 + \frac{\sqrt{5} + 1}{2} = \frac{3 + \sqrt{5}}{2} \neq x,$$

so $x = \frac{1 + \sqrt{5}}{2}$, the golden ratio, is the unique root.

II. Solution by Richard I. Hess, Rancho Palos Verdes, California.

Square both sides of the given equation, multiply through by x and then isolate the radical to get

$$x^3 - x^2 - x + 2 = 2(x^3 - x^2 - x + 1)^{1/2}.$$

Let $u = x^3 - x^2 - x + 2$. Then the equation reduces to

$$u^2 = 4(u - 1) \qquad \text{or} \qquad u^2 - 4u + 4 = 0,$$

which has the single root $u = 2$. Now

$$2 = u = x^3 - x^2 - x + 2,$$

so

$$x(x^2 - x - 1) = 0.$$

Since $x = 0$ is clearly not a root of the original equation, we solve the quadratic to get that

$$x = \frac{1 \pm \sqrt{5}}{2}.$$

We readily check that only the plus sign applies to the given equation.

III. Pseudosolution by John H. Scott, Macalester College, Saint Paul, Minnesota.

Since it appears that the right value for x will simplify both radicals so that they may be added, let us find the value for x that satisfies the equation $(x - 1/x)^{1/2} = 1$. We square and get the quadratic equation

$$x^2 - x - 1 = 0,$$

whose roots are $x = (1 \pm \sqrt{5})/2$. Clearly the positive sign, giving the golden ratio, has to satisfy Professor Umbugio's problem.

Also solved by **SEUNG-JIN BANG**, *Seoul, Korea,* **RUSSELL EULER**, *Northwest Missouri State University, Maryville,* **GEORGE P. EVANOVICH**, *Saint Peters College, Jersey City, NJ,* **JACK GARFUNKEL**, *Flushing, NY,* **ROBERT C. GEBHARDT**, *Hopatcong, NJ,* **RALPH E. KING**, *St. Bonaventure University, NY,* **WADE H. SHERARD**, *Furman University, Greenville, SC, and the.* **PROPOSER.**

Editorial comment. Bang used the. substitution $t = x - 1/x$, proceding as in Solution 1. The proposer used the method of Solution 11. Sherard squared twice, obtaining the. fourth degree polynomial and then asked what must be true for it to factor into two quadratic polynomials with integral coefficients. He. wrote

$$x^4 - 2x^3 - x^2 + 2x + 1 = (x^2 + ax + b)(x^2 + cx + d),$$

multiplied out the. right hide and equated like coefficients, obtaining $bd = 1$ among other conditions. He chose $b = d = -1$ and then found that the quartic factored into $(x^2 - x - 1)^2$. All the other solvers obtained the. quartic the. same way and then gave Sherard's factorization without any explanation of the reasoning. It is not clear that this technique is "using nothing higher than quadratic equations." Of come, one might well raise the same question regarding Solution 11 above..

In the May 7977 issue of Eureka (now Crux Mathematicorum) 1 wrote a biography of Euclide Paracelso Bombasto Umbugio (pages 118 - 125), in which his date of birth mu, given as probably April 1, 1900. Several months later 1 received a letter from Eureka's editor Léo Sauvé with a copy of a letter to him from Charles Trigg. Trigg stated, "While reading The Pentagon, 17 (Spring 1958), p. 106, I was startled to see the following: '174. Proposed by the. Editor ... ' " Then our problem 651, including the. reference to Robinson's Mathematical Recreations of 1851, was stated just as given here. Trigg continued, "This seems to move. Umbugio back to the middle of the. 19th century, at least." Since then I have located this same problem in the. April 1955 issue of the Elementary Problem Department of The American Mathematical Monthly as Problem E1161. Here the professor's name. mu, given as E. P. B. Umbugio:

Trigg's letter appeared to cast great doubt on the validity of my painstaking research into the life of the. great Umbugio and was a terrible blow to my ego. I spent many sleepless nights poring over ancient manuscripts to resolve this discrepancy, and after ten full years I am happy at last to be able to report its final resolution.

It certainly is doubtful that anyone born about 1900 would be proposing problems in 1851, but one. might compare this situation to the case of J. S. Bach's thirteenth son P. D. Q. Each, as reported to the world by Peter Schikele. There is, however, a far more logical explanation. Evidently E. P. Umbugio, Living in 1851, was the grand-father of our E. P. 8. Umbugio. And it is most reasonable that E. P. 8. would have access to his famous ancestor's files and would tun across his more difficult problems and recognize them as worthy of proposing anew. Following his usual form, of course E. P. B. Umbugio would propose them under his own name, rather than waste Apace and confuse people by giving credit to his sources.

1988 NATIONAL PI MU EPSILON MEETING

The National Meeting of the Pi Mu Epsilon National Honorary Mathematics Society will be held August 8 - 11, 1988, in Providence, Rhode Island, in conjunction with the 100th Anniversary of American Mathematics and the Centennial Celebration of the American Mathematical Society.

The main events of the celebration will be at the Omni Biltmore Hotel in downtown Providence.

Lodging for Pi Mu Epsilon participants will be available at very reasonable rates in Brown University housing. A free shuttle service to and from the Brown University campus will operate at regular intervals.

The J. Sutherland Frame Lecturer will be Professor Doris Schnattschneider, Moravian College.

GLEANINGS FROM CHAPTER REPORTS

ARKANSAS BETA (Hendrix College). Several students attended the Arkansas-Oklahoma MAA Meeting at East Central University in Ada, Oklahoma in March. *Jodie Novak* presented a paper on the Poincare Conjecture. The Annual Hendrix-Rhodes-Sewanee Undergraduate Mathematics Symposium was held in Sewanee in April. *Marla Beggs, Jodie Novak, Carol Parker* and *Luann Phillips* presented papers. Guest speakers during the school year included *Dr. William Jaco*, Oklahoma State University, *Professor Paul Fjelstad*, St. Olaf College, *Professor Matthew Gould*, Vanderbilt University, and *Dr. Christine Stevens*, Arkansas State University in Jonesboro.

At the Honors Convocation, the McHenry-Lane Freshman Math Award was shared by *Mike McClure and Allen McNight.* The Hogan Senior Math Award was given to *Jodie Novak.* The Phillip Parker Undergraduate Research Award was given to *Carol Parker.*

MASSACHUSETTS GAMMA (Bridgewater State College). The chapter members met regularly for problem-solving sessions with faculty advisor, *Thomas E. Moore.* Several members attended the Northeast Section Meeting of the MAA at Worcester Polytechnic Institute in November.

On April 20, 1987, *Professor Murray Abramson* died. *Professor Abramson* had been a member of the Mathematics and Computer Science Department for 26 years and was instrumental in the creation of the Pi Mu Epsilon chapter at Bridgewater.

MINNESOTA GAMMA (Macalester College). Fall activities included a showing of Nova's "A Mathematical Mystery Tour," a picnic at Minnehaha Falls, a lecture by *Sue Johnson* on "Mathematics and Statistics in Industry," and an address by *Professor Tom Sibley*, St. John's University, on "Fractals - Math in the Clouds."

Spring activities included a Career Night at a faculty home featuring recent Macalester graduates in a variety of professions, including high school teaching, actuarial science, banking and investment, and industry.

Mohsen Mazaheri spoke at the Annual Pi Mu Epsilon Conference at St. John's University on "A Sum that Jacques Bernoulli could not Evaluate." *Mohsen* later presented his paper at the Spring Meetings of the North Central Section of the MAA at the University of Minnesota in April.

Winners of the Ezra Camp Awards were *Karen Brasel* and *Charlene Barnes.*

Invited speaker at the Annual Installation Ceremonies was *Professor Underwood Dudley*, DePauw University. His topic was "Number Mysticism."

A T-shirt sale and an end-of-classes picnic completed the school year.

MINNESOTA ZETA (Saint Mary's College). *Dr. Andrea Birch*, Professor of Philosophy at Saint Mary's College, presented a historical survey of the roots of analysis. Student talks included *Julie Gustafson* on "What Infinite Matrices Can Do," *Lisa Janikowski* on "Can this Polynomial be Factored?," *Timothy Gryl* on "Iterated Binomial Coefficients," *Karl Patzner* on "A Discrete Look at 1 + 2 + ... + N," *Steve Anderle* on "Facility Location Problems," *Tim Gryl* on "Commutative Rings," *Jon Strauss* on "The Wedderburn-Artin Structure Theorem," and *Karl Patzner* on "Dead Man's Solitaire." In March, Saint Mary's alum *Jerry Karel* spoke on Expert Systems at 3M, where he is employed.

NEW YORK OMEGA (Saint Bonaventure University). The chapter's problem solving sessions have been well-enough received that they are on the verge of becoming part of the curriculum. Sponsored lectures during the academic year included "Why Galaxies have Spiral Arms" by *Professor Guy Johnson*, Syracuse University, "Cryptography" by *Professor Steven Andrianoff*, St. Bonaventure University, and "Transfinite Cardinals" by *Professor Charles Diminnie*, St. Bonaventure University.

The Annual Pi Mu Epsilon Award was presented to *Lori Leccese. Lisa Anderson* and *Ronald Ginnetti* received honorable mention.

NORTH CAROLINA LAMBDA (Wake Forest University). A strong program of seminars and lectures highlighted the year's activities. *Professor Herta T. Freitag*, Hollins College, spoke on "A Pair of Rabbits and a Beauty Contest - Is That All?." *Stun Thomas*, Wake Forest student, reported on his summer experiences at NASA. Wake Forest professor *Richard Carmichael* led a seminar on the delta function. *Sherry Leonard Wagstaff*, Integon Life Insurance Company, discussed actuarial science. *Professor Richard Carmichael* was seminar leader for the topic "Representation of Species." *John Baxley*, Wake Forest University, was leader for a seminar on "Some Challenging Problems." Wake Forest student *Talmadge Rogers* lectured on "Recurrence Formulas to Compute Sums of Integer Powers." Winston-Salem State University students *Sandria Kerr* and *James Gary* spoke on "Fractals." Invited Speaker *Carl Pomerance*, University of Georgia, lectured on "Public Key Cryptography and How to Crack It." *Professor Charles Swartz*, New Mexico State University and University of North Carolina, Chapel Hill, discussed "The Fundamental Theorem of Calculus." *Professor Billy Bryant of, Vanderbilt University* talked on "Some Theorems I Have Loved." Wake Forest students *Salman Azhar* and *Tim Hendrix* lectured on "The Prime Number Theorem" and "Higher Order Derivatives in Multidimensional Calculus," respectively.

The seminar and lecture program was punctuated by ice cream and pizza socials, a picnic, and a report on "Math Education in India" by *Ram S. Pathak* of Banaras Hindu University.

OHIO NU (The University of Akron). At its annual induction banquet these prizes and honors were announced: One-year memberships in the American Mathematical Society were awarded to *Laura B. Humphreys, Steven E. Linhart, John. Patterson, and Robert M. Streharsky.* The Samuel Selby Scholarships were awarded, in the indicated amounts, to *Robert M. Streharsky* ($150), *Steven E. Linhart* ($125), and *Laura B. Humphreys* ($100).

OHIO OMICRON (Mount Union College). *Jim Kirklin* presented a paper at the Thirteenth Annual Pi Mu Epsilon Conference at the University of California in Berkeley in August on "The Effect of Einstein's Theory of Relativity on Interstellar Navigation." *Mark Harker*, former Mount Union student and Pi Mu Epsilon officer, spoke of his work with Arthur Anderson, Inc. *Dr. Michael Zwilling*, faculty member, summarized the work on his dissertation.

OHIO ZETA (The University of Dayton). Twelve student talks were presented during the school year and were the principal activities of the regular meetings. Also, *Dr. John W. McCloskey*, department chairman, spoke on "An Intuitive Approach to Combinatorics." Several chapter members and faculty attended the annual Pi Mu Epsilon Conference at Miami University in Oxford, Ohio, in September. Four students presented talks. *Mary Beth Anderson* spoke on "An Application of Euler's Method to Initial Value Problems for Second Order Ordinary Differential Equations." *LoW. Bohn* discussed "The Busy Beaver Problem." *Rafe Donahue* presented "A Discrete Analogue of the Laplace Transform." *Mark Liatti* gave a talk entitled "The Mad Hatter."

Six students presented papers at the Ohio Section of the MAA Spring Meeting at Ohio University in Athens, Ohio, in April. *Julie Anderson* presented a talk entitled "Is the Set of Positive Divisors of an Integer a Boolean Algebra?" *LoW. Bohn* talked on "The Busy Beaver Problem." *Jeff Diller* discussed "The Isoperimetric Inequality." *Rafe Donahue* spoke on "Finite Calculus, Difference Equations, and the Development of a Transform Method for Finding Solutions." *Mark Liatti* gave a talk entitled "Don't Let it Bug You," and *Margie Mascolino* asked "What's the Difference?" *Jeff, Rafe, Mark* and *Margie* won awards for best presentations.

At the chapter's annual banquet, *Greg Scanlan* was awarded the Sophomore Class Award for Excellence.

PENNSYLVANIA NU (Edinboro University). At the Fall Initiation, *Ate. Etling* spoke on "Generalized Inverses." At the Spring Initiation, *Dr. Nair* lectured on "Artificial Intelligence." At the final meeting of the school year, chapter president *John Humid* presented a talk on "Least Mean Squares." Social events during the school year featured pizza parties.

SOUTH CAROLINA GAMMA (College of Charleston). Highlights of the year included a talk given by *Ph. Frank Guess* from the University of South Carolina on "System Reliability Theory." *Dr. Jim Reneke* from Clemson University visited the campus and discussed Clemson University's latest research project, "Distributed Systems." Departmental math exams (with solutions) from previous semesters were sold by chapter members, netting over $1000 for the chapter.

In the Spring, the chapter sponsored a trip to the Southeastern regional conference of the Mathematical Association of America held at Furman University in Greenville, South Carolina.

The College of Charleston hosted the annual Math Meet in which high school students from four states competed. Chapter members assisted with the administration of competitive events.

TENNESSEE GAMMA (Middle Tennessee State University). During the Fall and Spring semesters, activities included regular monthly meetings, a club-sponsored sophomore calculus contest, and co-sponsorship of a regional junior high mathematics contest. Guest speakers at the monthly meetings included: *Dr. David Cook*, University of Mississippi, who spoke on "Life on a Moebius Band," *Uh. David Sutherland*, who gave a presentation on error detecting/correcting codes, *Mr. Mike Pinter*, who spoke on chromatic polynomials, *Mrs. Lora Clark*, who discussed Riemann's hypothesis, *MA. Dovie Kimmins*, who talked on geometric probability, and *Dr. George Beers*, who gave a presentation on circular inversions.

The winners of the sophomore calculus contest were: first place - *Franklin Mason*, and second place - *Wesley Thompson, III*. The prizes in the amounts $100 and $25, respectively, were awarded at the MTSU Awards Banquet in April.

TEXAS LAMBDA (University of Texas at Austin). Chapter-sponsored lectures included *Bruce Palka's* annual talk on graduate programs in mathematics and *Mike Starbird's* talk on paradoxes in mathematics.

Representatives from the CIA and LBJ School offered recruiting talks and fielded questions from interested math majors.

Informal events during the year included a reception for new members and a picnic.

WISCONSIN DELTA (St. Norbert College). In August, *Michele Kwaterski, Mike Sir* and *Steve Van Lieshout* attended the Pi Mu Epsilon National Conference in Berkeley, held in conjunction with the International Congress of Mathematicians. *Steve* presented a paper entitled "Counting Rectangles in a Multirectangular Region."

In September, *Randy Crozier, Cathi Reinardy, Michele Kwaterski, Summer Quimby* and *Chris Steffanus* attended the Regional Conference at Miami University in Oxford, Ohio, where *Cathi* presented a paper.

In April, *Summer Quimby, Mary Lefeber, Shelly Braatz* and *Annette Lewis* attended the Pi Mu Epsilon Conference at St. John's University in Collegeville, Minnesota, where *Summer Quimby* gave a paper entitled "A Bit of Checking and Correcting."

Also, in April, *Cathi* and *Michele* spoke at the meeting of the Wisconsin Section of the MAA.

Talks during the academic year included "The Mandelbrot Set" by *Dr. John Frohliger*, St. Norbert College, "Communication Satellite Systems, an Application of Operations Research" by *Dr. Charles Reilly*, Ohio State University, "Polish Notation, Parentheses, and Data Base" by *Jim Blahnik* of St. Nobert's Computer Science Department and student *Brain Mohgan*. *Dr. V. Frederick Rickey*, Bowling Green State University, spoke on "The Reliance of Mathematics," *Orville Elseth*, Oconto Falls High School, spoke: on "Four Years of Unified Math," and *Mrs. Bonnie Berken*, St. Norbert College, spoke "On the Status of Statistics."

In October, the chapter hosted a Regional Pi Mu Epsilon Conference which had a total attendance of sixty-five with fourteen student papers.

ΠΜΕ

St. Norbert College
Third Annual
Pi Mu Epsilon
Regional Conference

November 4-5, 1988

Featured Speaker: Phil Straffin
Beloit College

Sponsored by the St. Norbert College Chapter of Π Μ Ε in Conjunction with the St. Norbert Math Club Σ Ν Δ

For information contact:
Rick Poss, St. Norbert College
De Pere, WI 54115
(414) 337-3198

ΣΝΔ

Editor's Note

The Pi Mu Epsilon Journal was founded in 1949 and is dedicated to undergraduate and beginning graduate students interested in mathematics. Submitted articles, announcements and contributions to the Puzzle Section and Problem Department of the Journal should be directed toward this group.

Undergraduate and beginning graduate students are urged to submit papers to the Journal for consideration and possible publication. Student papers are given top priority.

Expository articles by professionals in all areas of mathematics are especially welcome.

Triumph of the Jewelers Art

YOUR BADGE — a triumph of skilled and highly trained Balfour craftsmen is a steadfast and dynamic symbol in a changing world.

Official Badge
Official one piece key
Official one piece key-pin
Official three-piece key
Official three-piece key-pin

WRITE FOR INSIGNIA PRICE LIST.

An Authorized Jeweler to Pi Mu Epsilon

L.G. Balfour Company
ATTLEBORO MASSACHUSETTS

IN CANADA L. G. BALFOUR COMPANY, LTD. MONTREAL AND TORONTO

PI MU EPSILON JOURNAL PRICES

PAID IN ADVANCE ORDERS:

Members:	\$ 8.00 for 2 years \$20.00 for 5 years
Non-Members:	\$12.00 for 2 years \$30.00 for 5 years
Libraries:	\$30.00 for 5 years (same as non-members)
Foreign:	\$15.00 for 2 years (surface mail)
Back Issues	\$ 4.00 per issue
Complete volume	\$30.00 (5 years, 10 issues)
All issues	\$210.00 (7 complete back volumes plus current volume subscription)