PI MU EPSILON JOURNAL

VOLUME 9 FALL 1989 NUMBER 1

CONTENTS

(Continued on inside back cover)

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(Continued on inside back cover)

PI MU EPSILON JOURNAL

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THE RICHARD V. ANDREE AWARDS

*Richard V. Andree, Professor Emeritus of the University of Oklahoma, died on May 8, **1987,** at the age of 67.*

Professor Andree was a Past-President of Pi Mu Epsilon. He had also served the society as Secretary-Treasurer General and as Editor of the Pi Mu Epsilon Journal.

The Society Council has designated the prizes in the National Student Paper Competition as Richard V. Andree Awards.

First prize winners for 1987-1988 are James E. Georges, California Polytechnic State University, and Annette M. Matthews, Portland State University, for their paper "Maximal Polygons for Equitransitive Periodic Tilings," which appeared in the Fall, 1988, issue of the Journal. *The paper was done while the authors were participants in the Research Experiences for Undergraduates program at Oregon State University. James and Annette will share the $200 prize.*

Second prize winner is Melanie L. Butt, Middle Tennessee State University, for her paper "Automorphism Groups of Hasse Subgroup Diagrams for Groups of Low Order," which appears in this issue of the Journal. *Melanie, who is currently a senior, wrote the paper while she was a junior at Middle Tennessee State University. Melanie presented her paper at the National Meeting of Pi Mu Epsilon at Boulder in August, 1989. Melanie will receive $100.*

*Third prize winner is Robert A. **Coury,** University of Washington, for his paper* "A *Continued Fraction Approach for Factoring Large Numbers," which appears in this issue of the* Journal. *Robert is a senior at the University of Washington. Robert's paper is a result of research for a talk given at the national meeting in Providence in 1988. Robert will receive $50.*

Congratulations James, Annette, Melanie, and Robert.

*Two other student-written papers appear in this issue. One is "Energy-Conscious Behavior in Rural Areas: How to Approach a Traffic Light" by Craig Osborn, written while Craig was a senior at **Carleton** College. The paper is based on a problem presented by Richard Poss of St. Norbert College at the 1987 Annual Pi Mu Epsilon Conference.*

The other is Mark Ontkush's "A *Closed Formula for Linear Indeterminate Equations in Two Variables." Mark wrote the paper while a senior at the State University of New York at Buffalo. He encountered the formula in a course in discrete mathematics.*

AUTOMORPHISM GROUPS OF HASSE SUBGROUP DIAGRAMS FOR GROUPS OF LOW ORDER

By Melanie L. Butt
Middle Tennessee State University

We begin by reviewing basic group definitions and propositions. A ***group*** is a set with a binary operator which is associative, has an identity, and each element has an inverse. An ***abelian,*** or ***commutative,*** group is one whose operation is also commutative. A subset of a group which also forms a group is called a ***subgroup.***

Proposition 1. if G is a finite group with operation •, and H is a nonempty subset of G, then (H, •) is a subgroup of (G, •) whenever the closure property holds.

More specifically we are interested in Hasse subgroup diagrams. First recall that a ***poset*** is a nonempty set P with a relation ≤ on P which is reflexive, antisymmetric, and transitive. A ***lattice*** (L, ≤) is a poset with the property that ∀ x,y ∈ P, {x,y} has a least upper bound and a greatest lower bound.

Proposition 2. Let G be a group. Then $(L(G), \subseteq)$ is a lattice where

$$L(G) = \{H \mid H \text{ is a subgroup of } G)$$

and ⊆ is subset inclusion. The greatest lower bound of subgroups H and K is H ∩ K. The least upper bound of subgroups H and K Is the smallest subgroup of G containing H and K.

We represent lattices of subgroups with subset inclusion by diagrams called ***Hasse subgroup diagrams.*** Each subgroup Is depicted with a point. Lines are drawn to connect these subgroups according to the following rule: Suppose A and B are subgroups with property A ⊆ B. Then we connect the points with a line and we position B above A. The identity subgroup will be at the bottom of the diagram. We define this subgroup to have ***height*** or ***rank*** of 0. For subgroups H and K,

$$\text{rank}(H) = \text{rank}(K) + 1$$

whenever H is directly above K.

Now we are interested in automorphisms of these diagrams. An ***automorphism*** of a Hasse subgroup diagram, H, is a bijection from H to H that preserves or reverses order. ***Order preserving*** automorphisms are those with the property that given two elements, x and y, if x ≤ y, then f(x) ≤ f(y). An automorphism is ***order reversing*** when x ≤ y implies f(x) ≥ f(y). The ***identity automorphism*** Is the bijection i: H → H defined by i(x) = x. The ***reverse automorphism,*** if it exists, is the automorphism that turns the Hasse subgroup diagram upside down.

Proposition 3. The set of automorphisms of the Hasse subgroup diagram H forms a group under function composition.

Our goal is to calculate the automorphism groups of the Hasse subgroup diagrams for the groups of low order which are listed in the first column of Table 1.

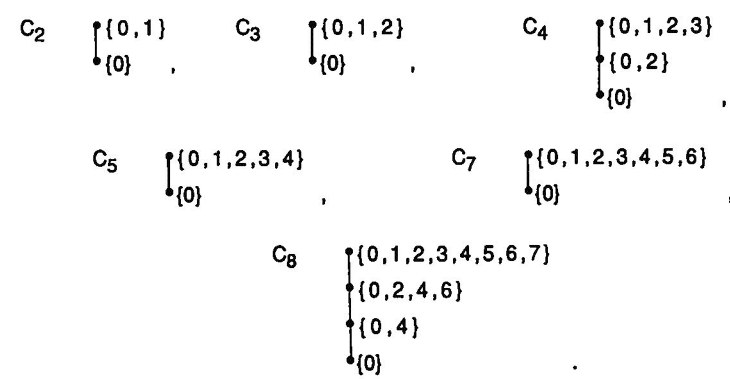
TABLE 1

Group	Automorphism Group of Hasse Subgroup Diagram
C_1	C_1
C_2	C_2
C_3	C_2
C_4	C_2
C_5	C_2
C_6	$C_2 \times C_2$
C_7	C_2
C_8	C_2
D_4	D_4
S_3	$S_4 \times S_2$
$C_2 \times C_2$	$S_3 \times C_2$
$C_4 \times C_2$	D_4
Q	S_3

First we discuss the cyclic groups. The ***cyclic group with I elements.*** C_is Is the set of the first i whole numbers with addition modulo i. Clearly, the automorphism group of the Hasse subgroup diagram of C_1 is C_1 since the only subgroup of C_1 is C_1 Itself.

Theorem 1. The automorphism group of the Hasse subgroup diagram of C_i where i ● {2,3,4,5,7,8} is C_2.

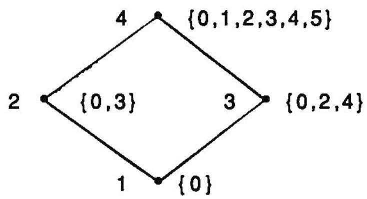
Proof. First consider the subgroups we obtain by examining the group tables. Then we find the Hasse subgroup diagrams which are



Because each Hasse subgroup diagram contains only one subgroup at each rank and ⊆ is transitive, it follows that the only automorphisms are the identity and the reverse automorphisms. Therefore C_2 is the automorphism group since it is the only group of order two.

Theorem 2. The automorphism group of the Hasse subgroup diagram of C_6 Is $C_2 \times C_2$.

Proof. The Hasse subgroup diagram of C_6 is



Note that the subgroups are labeled by numbers which will be used to refer to the subgroups. By looking at the diagram it is clear that the identity and reverse automorphisms are automorphisms of the Hasse subgroup diagram of C_6. Switching subgroups 2 and 3 should also be an automorphism and the function switching subgroups 2 and 3 is order preserving. Let us verify the function switching 2 and 3 is an order preserving automorphism.

$$\text{Define } f\colon H \to H \text{ by } f(i) = \begin{Bmatrix} i & \text{if } i \neq 2,3 \\ 2 & \text{if } i = 3 \\ 3 & \text{if } i = 2 \end{Bmatrix}.$$

If $i < j$, then $f(i) < f(j)$ $\forall$ i,j Is verified by checking

$$\begin{aligned} 1 < 2 &\text{ and } f(l) = 1 < 3 = f(2), \\ 1 < 3 &\text{ and } f(l) = 1 < 2 = f(3), \\ 2 < 4 &\text{ and } f(2) = 3 < 4 = f(4), \\ 3 < 4 &\text{ and } f(3) = 2 < 4 = f(4). \end{aligned}$$

Now we obtain a fourth automorphism by turning this one upside down. Thus the automorphism groups contain four elements. There are two groups of order four. Since no automorphism has order four, we conclude the automorphism groups is $C_2 \times C_2$.

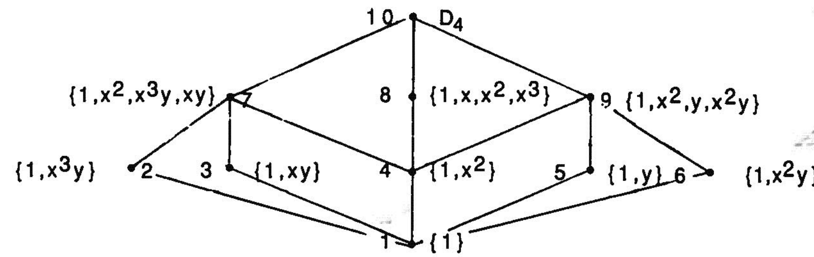
Another groups of low order is D_4, ***the dihedral*** group ***with eight elements.*** The elements of D_4 can be thought of as the symmetries of a square. More precisely,

$$D_4 = \langle x,y \mid x^4 = 1,\ y^2 = 1,\ (xy)^2 = 1 \rangle ;$$

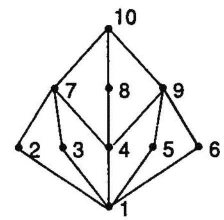
that is, the group generated by two elements, x and **y,** with one element, **x,** of order 4 and the other, **y,** of order 2 which produce the identity when the two elements are multiplied and squared.

Theorem 3. The automorphism group of the Hasse subgroup diagram of D_4 is D_4.

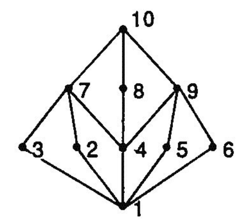
Proof. First compute the subgroups by inspecting the group table of D_4. We obtain the Hasse subgroup diagram



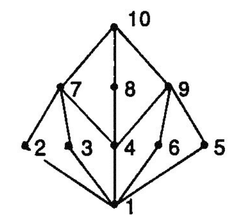
Clearly, the identity will be an automorphism and there will be no reverse automorphism. Automorphisms are obtained by switching the pairs **(2,3)** or **(5,6)** or both of them together. Automorphisms are obtained also by switching 7 and 9 along with the pairs **(2,3)** and **(5,6).** All of the automorphisms are shown below and will be referred to by their labels.



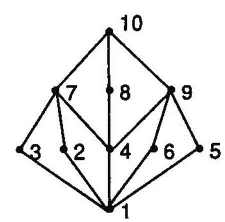
A



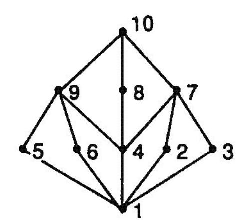
L



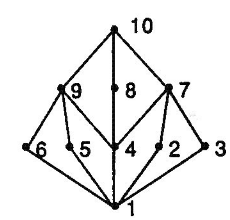
R



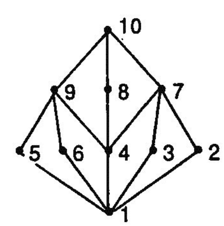
LR



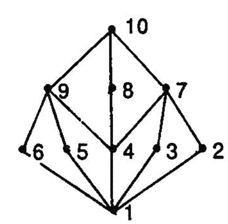
T



TL



TR



TLR

We provide the details for checking one of the above. The others are similar. Define TL: $H \to H$ by

$$TL(i) = \begin{cases} 1 & \text{if } i = 1, 4, 8, 10 \\ i+2 & \text{if } i = 3, 7 \\ i-3 & \text{if } i = 5, 6 \\ 6 & \text{if } i = 2 \\ 7 & \text{if } i = 9 \end{cases} .$$

Then we calculate the following:

$$\begin{aligned}
&1 < 2 \text{ and } f(1) = 1 < 6 = f(2),\\
&1 < 3 \text{ and } f(1) = 1 < 5 = f(3),\\
&1 < 5 \text{ and } f(1) = 1 < 2 = f(5),\\
&1 < 6 \text{ and } f(1) = 1 < 3 = f(6),\\
&2 < 7 \text{ and } f(2) = 6 < 9 = f(7),\\
&3 < 7 \text{ and } f(3) = 5 < 9 = f(7),\\
&5 < 9 \text{ and } f(5) = 2 < 7 = f(9),\\
&6 < 9 \text{ and } f(6) = 3 < 7 = f(9),\\
&7 < 10 \text{ and } f(7) = 9 < 10 = f(10),\\
&9 < 10 \text{ and } f(9) = 7 < 10 = f(10).
\end{aligned}$$

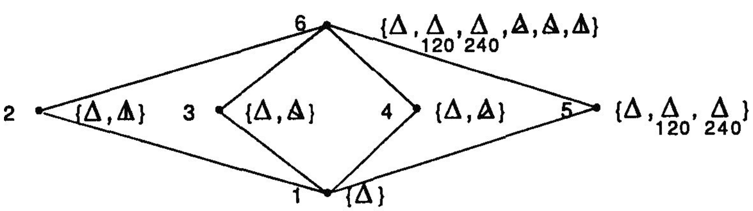
To prove the automorphism group is D_4 we must prove $(xy)^2 = 1$ where x is an element of order 4 and y is an element of order 2.

Let x = TL and y = R. Then $((TL)(R))^2 = (TRL)^2 = 1$. Therefore the automorphism group is D_4.

The next group is S_3 where S_n is the *symmetric group* on *n objects.* S_3 may be represented as the six symmetries of an equilateral triangle.

Theorem 4. The automorphism group of the Hasse subgroup diagram of S_3 is $S_4 \times C_2$.

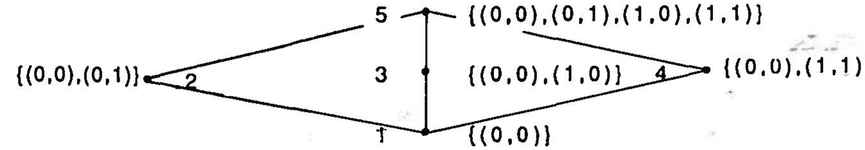
Proof. We compute the subgroups and the Hasse subgroup diagram of S_3.



Using the same steps as in the previous proofs, we find there are 24 order preserving automorphisms. There are also 24 order reversing automorphisms. The order preserving automorphisms form the group S_4 since the four rank 1 subgroups can all be permuted. The reverse automorphism generates the group C_2. When the reverse automorphism is included with the order preserving automorphisms, 24 new automorphisms are obtained, all order reversing. These also form the group S_4. Thus the combined automorphism group is $S_4 \times C_2$.

Theorem 5. The automorphism group of the Hasse subgroup diagram of $C_2 \times C_2$ is $S_3 \times C_2$.

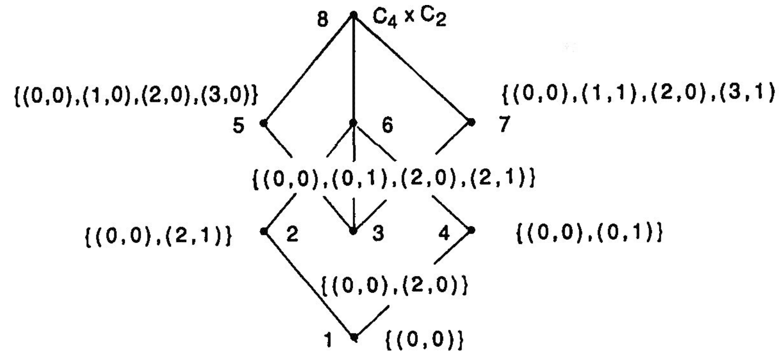
Proof. Consider the subgroups of $C_2 \times C_2$; then construct the Hasse subgroup diagram as shown.



By using the same reasoning as in the proof for S_3, we find the automorphism group is $S_3 \times C_2$.

Theorem 6. The automorphism group of the Hasse subgroup diagram of $C_4 \times C_2$ is D_4.

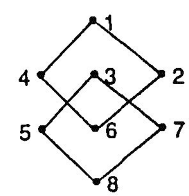
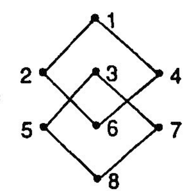
Proof. Find the subgroups and form the Hasse subgroup diagram for $C_4 \times C_2$ as shown.



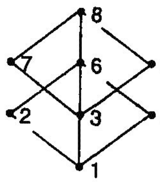
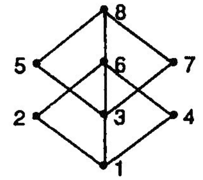
We find there are 8 automorphisms. By using the definition

$$D_4 = \langle\, x,y \mid x^4 = 1,\ y^2 = 1,\ (xy)^2 = 1 \,\rangle$$

we check the automorphisms using

x =  and y = 

Then

$(xy)^2 =$ ( $)^2$ $=$  $= 1.$

Therefore the automorphism group is D_4.

There are two other groups of order less than or equal to eight. One Is the group $C_2 \times C_2 \times C_2$. The other is the quaternion group, Q, which contains the elements

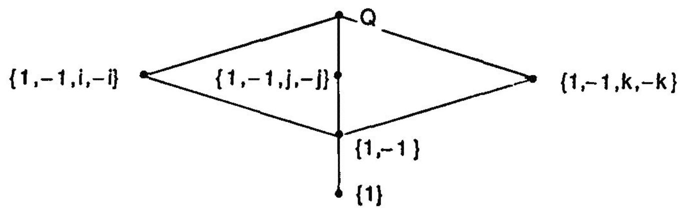
$$\{\pm 1, \pm i, \pm j, \pm k\}$$

and where

$$i^2 = j^2 = k^2 = -1,$$
$$ij = k = -ji,\ jk = i = -kj, \text{ and } ki = j = -ik.$$

Theorem 7. The automorphism group of the Hasse subgroup diagram of Q is S_3.

Proof. After finding the subgroups of Q and the Hasse subgroup diagram



we find there are 6 order preserving automorphisms and clearly no order reversing automorphisms. The only groups of order 6 are C_6 and S_3. Checking group tables, we find the automorphism group of the Hasse subgroup diagram of Q is S_3. This result can also be obtained by observing that the automorphisms permute the 3 rank 2 subgroups in all possible ways.

My work with automorphism groups was done by inspection of the Hasse subgroup diagrams. Even though some generalizations are easy to state, I do not yet know the theory needed to prove generalizations because I have not yet taken a course in abstract algebra. This also presents a problem when working with $C_2 \times C_2 \times C_2$ since its Hasse subgroup diagram is more complex.

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A CONTINUED FRACTION APPROACH FOR FACTORING LARGE NUMBERS

By Robert A. Coury
University of Washington

Introduction. The factoring algorithm described is based on a congruence of Legendre and uses the methods of continued fractions. Legendre's congruence $x^2 \equiv y^2$ (mod N) is an important tool for factoring very large numbers. The equation

$$h_{n-1}^2 - Nk_{n-1}^2 = (-1)^n s_n \tag{1}$$

also plays an important role.

Legendre's congruence is used in several important factoring methods, among them Fermat's, Euler's, Gauss', and Shanks'. These methods differ only in the way in which the solution to $x^2 \equiv y^2$ (mod N) is found.

Background. We begin with some background material on continued fractions. Let $b_0, b_1, b_2, \ldots$ be positive integers. Set $[b_0, b_1] = b_0 + 1/b_1$; $[b_0, b_1, b_2] = b_0 + 1/[b_1, b_2]$; and so on. $[b_0, b_1, \ldots, b_n]$ is called a simple continued fraction.

Now let N be a positive integer that is not a perfect square. Let a_0 be the greatest integer in $\sqrt{N}$. We compute the series r_n and s_n inductively as follows: if $\alpha_n = (\sqrt{N} + r_n)/s_n$ then $a_n = [\alpha_n]$, $r_{n+1} = a_n s_n - r_n$, and $s_{n+1} = (N - r_{n+1}^2)/s_n$.

Write the rational number $[a_0, a_1, a_2]$ in lowest terms as h_n/k_n (the nth convergent of $\sqrt{N}$). In a certain sense, these convergents represent the best rational approximation of $\sqrt{N}$ (see [1], Chapter 7). The series h_n and k_n can also be defined inductively.

The expression $[a_0, a_1, a_2, \ldots]$ is called the infinite simple continued fraction expansion of $\sqrt{N}$. For each positive value of n, the positive integers h_n, k_n, and s satisfy the following relation: $h_{n-1}^2 - Nk_{n-1}^2 = (-1)^n s_n$.

Finally, if N is not a perfect square, $\sqrt{N}$ has a continued fraction expansion that repeats. The length of the repeating part is called the period.

We now outline how continued fraction expansions are used to factor large numbers.

The method. Suppose N is composite; we assume that N = pq, where p and q are distinct primes. Then Legendre's congruence $x^2 \equiv a^2$ (mod N) has a pair of nontrivial solutions $x \equiv \pm z$ (mod N) in addition to the trivial pair x ≡ Â± (mod N). This fact can be used to factor N.

First, find a nontrivial solution z to $x^2 \equiv a^2$ (mod N). Since $z^2 - a^2 = (z - a)(z + a) \equiv 0$ (mod N), neither z + a nor z – a can be divisible by both p and q: tor example, if z + a were divisible by both p and q, then it would be divisible by N. This would mean that $z \equiv -a$ (mod N), which yields the trivial factorization of N.

Thus, one of $z + a$ and $z - a$ must be divisible by p and the other by q. The factor p (or q) can be determined by using the Euclidean algorithm to find the greatest common divisor of $z + a$ and N (or $z - a$ and N). This method also works if N has more than two prime factors; one simply reapplies the method to the composite factor.

In trying to determine a nontrlvial solution to Legendre's congruence, we first find the infinite simple continued fraction expansion of $\sqrt{N}$. With the sequences h_n, k_n, and s_n defined as usual for this continued fraction expansion, we have equation (1) which is valid for all n. This reduces to the congruence $h_{n-1}^2 \equiv (-1)^n s_n \pmod{N}$.

Thus to find a solution to Legendre's congruence, we simply expand $\sqrt{N}$ until a perfect square $s_n = A^2$ is found such that n is even. Then Legendre's congruence has the solution $x \equiv h_{n-1}$, $y = A \pmod{N}$.

If this is not one of the trivial solutions, the prime factors of N can be found be applying the Euclidean algorithm to determine the greatest common divisor of N and $h_{n-1} - A$ and of N and $h_{n-1} + A$.

Example. Let N = 7104007; then $\sqrt{N}$ has a period of length 2206. Computing the s_n's we find that the first square occurs at s_8 = 2209 = 472. The subscript is even so there is a possibility that we can get a factorization. We have h_7 = 7103960 ≡ −47 (mod N). Thus $h_7 + 47 \equiv 0 \pmod{N}$, which means that N divides $h_7 + 47$, and so we end up with a trivial factorization of N.

The next square is $s_{16} = 841 = 29^2$. Once again, the subscript is even, so we check to see if our method produces a nontrivial factorization. We have $h_{15} = 23772920$ $\equiv 2460899 \pmod{N}$; thus $h_{15}^2 - s_{16} = h_{15}^2 - 841 = (h_{15} - 29)(h_{15} + 29) \equiv$

$2460870 \cdot 2460928 \pmod{N}$. We now use the Euclidean algorithm to find the greatest common divisor of N and 2460870, which is 739, and the greatest common divisor of N and 2460928, which is 9613. In fact, it is easy to check that N = 739·9613.

The program. The program listed at the end of this paper, written in Microsoft's **QuickBasic** 4.0, will factor numbers up to sixteen digits long. The program runs fairly quickly and factors most numbers in less than a second.

It is difficult to predict the time needed to factor a given integer. However, the following table gives some idea of factorization times required for a variety of numbers. The results were obtained by running the compiled program on an IBM AT compatible with an operating speed of 12 MHz and equipped with a math co-processor. The time is in seconds; n is the subscript for s that produces a nontrivial factorization; and Square # is the number of squares the program checks until it finds a square that yields nontrivial factors. The last two columns give the factorization of the number.

The program also includes a routine for 'doping' the number. Doping is a process that multiplies the number to be factored by another number in order to gain a longer period. The reason this is done is that a larger number need not have a long period. For example, a number that is of the form $n^2 + 1$ has a period of length one. It may happen that a number will have a period that is too short to find a square that yields a nontrivial factorization. When this happens, the program multiplies the number to be factored by five and then reapplies the algorithm (remembering at the end to remove the doping constant from the factors obtained). The doping factor does not have to be five; it may be some other suitable number.

Number	Time	n	Square #	Factor 1	Factor 2
30973	.11	42	3	47	659
37913	.06	20	2	31	1223
96571	.11	30	1	269	359
303181	.01	6	1	137	2213
826471	.11	28	2	28499	29
917387	.05	28	1	409	2243
1000009	.05	18	2	293	3413
1597537	.22	82	2	2339	683
2282237	.94	342	11	2753	829
2633383	.50	174	5	7589	347
3237301	.05	4	1	16433	197
3579517	.22	84	3	8543	419
7104007	.05	16	2	739	9613
7322371	.72	260	4	1046053	7
12634801	.01	12	1	45613	277
13237301	.01	6	1	539	24559
14722741	.05	14	1	139	105919
17322371	.01	10	1	1018963	17
739128463	.05	12	1	1373	538331
5231211683	1.32	470	1	15581	335743
9156487871	.11	34	1	85574653	107
12345678971	1.48	522	1	15260419	809
12603664039	.16	68	1	36961	340999
16042282237	.16	46	1	4733633	3389
56789876543	1.21	446	2	35207611	1613

Concluding remarks. The technique of using continued fractions to produce a factorization is actually an old idea. However, it was not really practical before the advent of fast computers, because of the many steps generally required to produce a square that works.

In 1982 the method was implemented on a 'reasonably fast computer that could be used almost exclusively for factorization' (Riesel **[3]**). It factored a 35-digit number in about one hour, a 45-digit number in one day, and a 50-digit number in one week. The most difficult number reported was a 56-digit number which was factored after 35 days and resulted in a 23-digit factor and a 33-digit factor.

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2. C. D. Olds, ***Continued Fractions,*** Random House, New York, 1963.
3. H. Riesel, ***Prime Numbers and Computer Methods for Factorization,*** **Birkhäuser,** Boston, 1985.

```
     DIM A#(-2 TO 7999), F#(2), H#(-2 TO 7999), R#(-2 TO 7999), S#(-2 TO 7999)
     CLS
5    PRINT : PRINT : INPUT ID#
     D# = ID#: DP = 0
     IF D# = 0 THEN END
     IF SQR(D#) = INT(SQR(D#)) THEN
          PRINT "NUMBER IS A PERFECT SQUARE"
          GOTO 30
     END IF
10   RDÃ = INT(SQR(D#))
     F#(1) = 0: F#(2) = 0: H#(-2) = 0: H#(-1) = 1: R#(0) = 0: RE = 0: RE1 = 0
     S#(0) = 1: N = 0
15   A#(N) = INT((RD# + R#(N)) / S#(N)): H#(N) = A#(N) * H#(N - 1) + H#(N - 2)
     IF H#(N) > D# THEN
          M = INT(H#(N) / D#): H#(N) = H#(N) - M • D#
     END IF
     R#(N + 1) = A#(N) • S#(N) - RUIN): SHIN + 1) = (D# - RUIN + 1) ^ 2) / SHIN)
     IF SQR(S#(N + 1)) = INT(SQR(S#(N + 1))) AND (N / 2) <> INT(N / 2) THEN 35
20   IF A#(N) = 2 • A#(0) THEN
          P = N
          GOTO 60
     ELSE
          N = N + 1
          IF N = 7950 THEN
               P = 7950
               GOTO 25
          END IF
          GOTO 15
     END IF
25   PRINT "PERIOD IS "; P
30   PRINT "NOW DOPING WITH A FACTOR OF 5"
     DP = DP + 1: D# = D# * 5
     IF SQR(D#) = INT(SQR(D#)) THEN
          PRINT "NUMBER IS A PERFECT SQUARE"
          GOTO 30
     END IF
     GOTO 10
35   I = N + 1
     M# = SQR(S#(I)): F#(1) = H#(I - 1) - MÃ‡ F#(2) = H#(I - 1) + M#
     IF F#(1) / D# = INT(F#(1) / D#) OR F#(2) / D# = INT(F#(2) / D#) THEN 20
     RE1# = F#(2)
     IF F#(2) > D# THEN
          A# = F#(2): B# = D#
     ELSE
          A# = D#: B# = F#(2)
     END IF
45   QÃ = INT(A# / B#): RE# = All - QÃ • B#
     IF RE# = 0 THEN
        F#(2) = RE1#
        GOTO 50
     ELSE
        RE11 = RED: A# = B#: B# = RE#
        GOTO 45
     END IF
50   F#(1) = D# / F#(2)
55   IF F#(1) = 1 OR F#(2) = 1 THEN
          IF DPR = 1 THEN
               DPR = 0: DP = DP + 1
          END IF
          GOTO 20
     END IF
     IF DP <> 0 THEN
          IF F#(1) / 5 = INT(F#(1) / 5) THEN
               F#(1) = F#(1) / 5
          ELSEIF F#(2) / 5 = INT(F#(2) / 5) THEN
               F#(2) = F#(2) / 5
          END IF
          DP = DP - 1: DPR = 1
          GOTO 55
     END IF
     PRINT ID#; "FACTORS INTO"; F#(1); "AND"; F#(2)
     GOTO 5
60   PRINT "PERIOD OF"; P; "FINISHED WITHOUT SUCCESS"
     GOTO 30
```


ENERGY-CONSCIOUS BEHAVIOR IN RURAL AREAS: HOW TO APPROACH A TRAFFIC LIGHT

By Craig Osborn
Carleton College

The Problem.

A motorist is driving along a lazy country road when she comes over a hill and sees a red traffic light ahead. She is well acquainted with this road, so she knows how far it is to the intersection. Her car is the new improved friction-free Chevy **Slipster,** so she can coast at constant speed, that is, without being slowed by friction. Because she is low on gas, however, she is not willing to accelerate before passing the intersection. She wishes to find a strategy that will allow her the highest speed through the intersection, subject to the constraint that she must come to a full stop if the light is red when she arrives.

Possible Conditions.

a. She rounds the top of the hill near the light (close enough to pass it some time during the upcoming green cycle) and she knows how long she has until it turns green.

b. She is near the light, but doesn't know how long is left in the red cycle.

c. She is far away, so that there may well be several red/green cycles left before she reaches the intersection (in which case it might make little difference what color the light is when she first sees it).

Question. What is the best strategy under each of these conditions?

Case a. of this problem was presented by Dr. Richard Poss of St. Norbert College at the 1987 Annual Pi Mu Epsilon Student Conference. Dr. Mark Krusemeyer of Carleton College suggested cases b. and c. for further investigation. In this paper I will present solutions to cases a. and b.; case c. is apparently still unsolved.

In the following I assume that the driver will watch the light and discontinue any braking (that is, begin to coast) as soon as the light turns green. In effect, then, our problem is to maximize the "green-light speed," which is defined to be the car's speed at the moment the light changes to green.

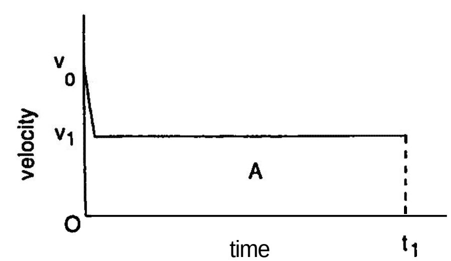

Figure 1

Case a. I will propose a strategy and then show that it is the best possible one. Suppose our driver divides the (known) distance remaining by the time she knows she has before the light changes. This will give a velocity v_1, and she could brake immediately to this velocity (we've assumed she's not far from the light, so we can take the calculated speed to be slower that normal driving speed) and then coast to the light, as shown in Figure 1.

Here v_0 denotes the car's original velocity and t_1 is the duration of the red light (from the time it is first seen). Note that the shaded area, given by

$$A = \int_{t=0}^{t_1} v(t) \quad dt,$$

is the distance to the intersection. In other words, we were already given A and the duration t_1 of the light. Now let C_1 be the curve shown in Figure 1, and assume that C_1 falls as nearly vertically as possible before leveling off. If we take any other curve C_2 with the same area A underneath it, then C_2 must be above C_1 somewhere and below C_1 somewhere. Since C_1 is (almost!) everywhere horizontal and only nonincreasing functions are allowed, C_2 must be above C_1 before it is below C_1. However, there is then no way for C_2 to rise to v_1 at the moment the light turns green. Thus C_1 shows the best strategy, since it allows the highest green-light velocity.

Case b. Now let t_1 be the **<u>maximum</u> possible** time for the **light** to **remain** red. We know that the time t_g at which the light actually turns green will be somewhere at random between 0 and t_1. We seek a (velocity) function which:

1) is continuous and nonincreasing from 0 to t_1;
2) has no more than area A below it between 0 to t_g; AND
3) maximizes the average terminal velocity, where the average is taken over all possible values of t_g between 0 and t_1.

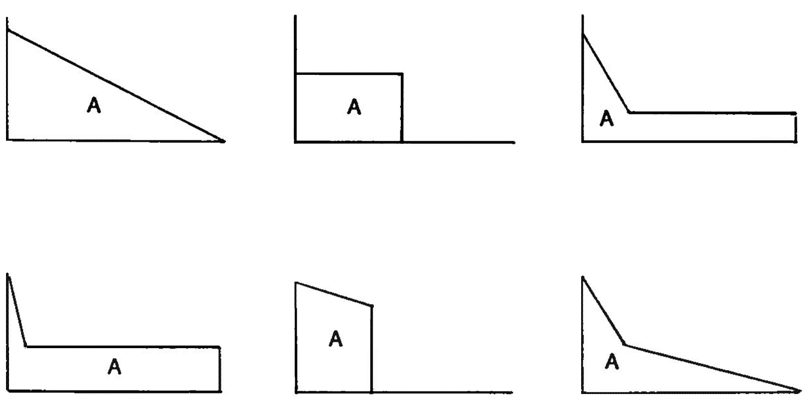

Figure 2

Because of restriction 2) and the fact that t_g ranges all the way up to t_1, the area below the function between 0 and t_1 must not be more than A. Figure 2 shows the graphs of some candidate functions.

Once again, we want to maximize the average green-light speed, averaged over all possible durations t_g of the red light between 0 and t_1. If the car reaches the intersection before the light turns, the green-light speed is obviously zero by the assumption of legality.

Let us proceed as Newton would. To find the average intersection velocity of a given candidate function, divide the interval $[0, t_1]$ into several, say ten, equal subintervals. This gives eleven distinct times at which we will allow the light to turn. We can now average the intersection velocities by adding up the eleven velocities and dividing by 11. The function with the highest average "wins" because it allows the driver to pass the intersection with the highest expected velocity for an arbitrary t_g.

To increase our accuracy, we could divide the interval into 100 subintervals and average the 101 velocities. This looks familiar -- it's integration. In effect, we want the velocity function which has the most area below it on the interval $[0, t_1]$. Since the candidate functions all have the same area A below them, they are **<u>all</u> optimal!**

One caveat: If at any time her velocity is such that she can coast constantly and reach the intersection at time t_1, the driver must not slow down any more. If she did, it would cause the area below the function to become less than A. The driver is therefore constrained as follows:

The distance already covered at any time t_0 is $\int_0^{t_0} v(t)$ dt. The remaining distance to the intersection is then $A - \int_0^{t_0} v(t)$ dt. The time left before the light's "deadline" is $(t_1 - t_0)$. The minimum velocity is the remaining distance divided by the remaining time to deadline: $\dfrac{A - \int_0^{t_0} v(t)\, dt}{t_1 - t_0}$. Thus at any time t_0, we must have

$$v(t_0) \geq v_{min} = \frac{A - \int_0^{t_0} v(t)\, dt}{t_1 - t_0}$$

In summary,

1. She must not slow down so much as to prohibit her from reaching the intersection by t_1.
2. She must watch the light so that she can begin to coast as soon as it turns green, in case it does so before she arrives.
3. Within these limitations, we can now choose any nonincreasing velocity function -- that is, any combination of coasting and braking, subject to rule 1 above. A rather surprising result!

A CLOSED FORMULA FOR LINEAR INDETERMINATE EQUATIONS IN TWO VARIABLES

By Mark *Ontkush*
State University of New York at Buffalo

This formula requires the necessary following conditions: two integers X and **Y,** $X > 1$ and $Y > 1$, and $(X, Y) = 1$ (X and Y are mutually prime). Given these three conditions, then there exists a number M such that all integers greater than M can be expressed as a sum $AX + BY = C$, $C > M$, where A and B are positive integers. The integer M equals $X(Y - 1) - Y$.

Proof. Take X as the smaller number without any loss of generality. Then, if one divides any number C by X, the result is some Integer plus a remalnder that Is less than X. Thus, there are exactly $X - 1$ remalnders that are possible. However, using a linear combination of X's and **Y's,** it will be possible to form all of these remainders.

Let the first remainder be represented as such:

$$R_1 = Y - QX \tag{1}$$

where $Q = [Y/X]$. Q is commonly known as the greatest integer function. For example, $[3.01] = 3$, $[4.9] = 4$, and $[5.00] = 5$. Then, given equation **(1),** the rest of the remainders can be computed as follows:

$$R_2 = 2Y - 2QX - [2R_1/X]X$$

or, in general, as

$$R_n = NY - NQX - [NR_1/X]X$$

Example. Let $X = 5$ and $Y = 7$. Then there are $X - 1$, or 4, remainders, R_1 through R_4. They can be computed as follows:

$$\begin{aligned}
R_1 &= 7 - 5 = 2 \\
R_2 &= 2(7) - 2(5) - [2(2)/5](5) = 4 \\
R_3 &= 3(7) - 3(5) - [3(2)/5](5) = 1 \\
R_4 &= 4(7) - 4(5) - [4(2)/5](5) = 3 .
\end{aligned}$$

Note that all of the integers from 0 to X–1 are expressed here. This is no accident. It has been proven that, for any X and **Y,** if $X \not\equiv 0 \pmod{Y}$, then the sequence of the remainders **modula** Y is a rearrangement of the sequence 1, 2, 3, ..., X–1.

We wish to find the remainder that requires the largest number of X's so that we can find a lower bound for the number M. By inspecting the remainders, it is clear that R_{X-1} will always have the largest number of X's.

$$R_{X-1} = (X - 1)Y - \{(X - 1)Q + [((X - 1)R_1)/X]\}X . \tag{2}$$

The number of X's in this equation is (note that square brackets denote greatest integer function):

$$\begin{aligned}
&(X - 1)Q + [((X - 1)R_1)/X] \\
&= QX - Q + [((X - 1)(Y - QX))/X] \\
&= QX - Q + [(XY - QX^2 - Y + QX)/X] \\
&= QX - Q + [Y - QX - (Y/X) + Q] \\
&= QX - Q + Y - QX + [Q - (Y/X)] ,
\end{aligned}$$

since Y and QX are both integers and $[Y] = Y$ and $[QX] = QX$.

But $Q = [Y/X]$, and since X and Y are mutually prime,

$$[Y/X] + 1 < (Y/X) < [Y/X],$$

so

$$[Q - (Y/X)] = [1] = 1,$$

and the number of X's in (2) is given by

$$Y - Q - 1 . \tag{3}$$

By using (3) in **(2),** we can solve for R_{X-1}.

$$\begin{aligned}
R_{X-1} &= (X - 1)Y - (Y - Q - 1)X \\
&= XY - Y - (XY - QX - X) \\
&= QX + X - Y \\
&= (Q + 1)X - Y .
\end{aligned}$$

If we can discover the number of X's required for R_{X-2} and then add R_{X-1}, we will have **M,** the largest number that cannot be expressed as $AX + BY$, where A and B are positive integers. The number of X's required for R_{X-2} is easy: looking at (2) and **(3),** and remembering $Y > X$, at most $(Y - Q - 2)$ X's will be needed to find this remalnder.

$$\begin{aligned}
M &= (Y - Q - 2)X + R_{X-1} \\
&= (Y - Q - 2)X + (Q + 1)X - Y \\
&= XY - QX - 2X + QX + X - Y \\
&= XY - X - Y \\
&= X(Y - 1) - Y
\end{aligned}$$

Example. Find M for $X = 62$, $Y = 79$, and show that M cannot be expressed as **$AX + BY = M$, but $M + 1$ can be.**

$$\begin{aligned}
M &= 62(79 - 1) - 79 \\
&= 4757
\end{aligned}$$

If we divide 4757 by 62, we get 76 wlth remainder 45. However, 61(79) − (77)(62) = 45, so 45 is the worst possible remainder. There is no way M can be expressed without using a negative A or B, as

$$4757 = 76(62) + 61(79) - (77)(62)$$
$$= 61(79) - 62$$

M + 1, however, can be expressed as a sum AX + BY. Dividing 4758 by 62, we get 76 with remainder 46. A little experimentation shows that 46 = 10(79) − 12(62). So

$$4758 = 76(62) + 10(79) - 12(62)$$
$$= 10(79) + 64(62) .$$

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Your chapter can make use of the Pi Mu Epsilon Award Certificates available to help you recognize mathematical achievements of your students. Contact Professor Robert Woodside, Secretary-Treasurer.

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SOME SHORTCUTS FOR FINDING ABSOLUTE EXTREMA

By Subhash C. Saxena
University of South Carolina — Coastal

In the discussion of absolute extrema, most elementary calculus books correctly suggest the following procedure for finding absolute maximum and absolute minimum of a continuous function f on a closed interval $[\mu, \upsilon]$.

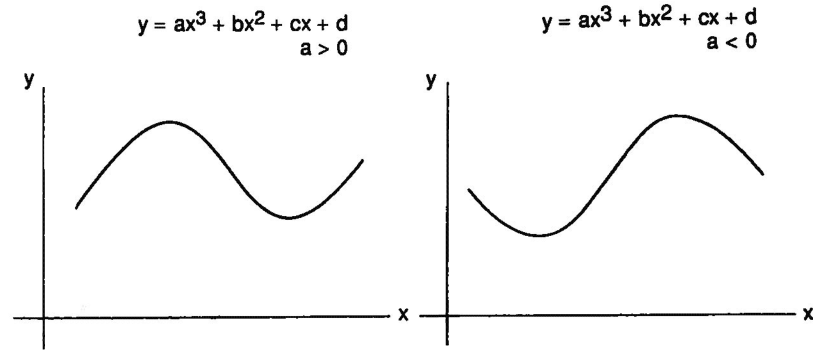
"Find all the critical points of f on $[\mu, u]$. Then find the values of f at each of these points and also at μ and υ. The largest of these values gives the absolute maximum and the smallest of these is absolute minimum."

However, in several cases, short cuts may be made to find absolute extrema in various situations. The purpose of this note is to explore some of these short cuts.

In the case of a quadratic function ax2 + bx + c, it is a well-known fact that: at $x = -\frac{b}{2a}$ the quadratic has an absolute minimum if a > 0 and an absolute maximum if a < 0. Assuming $-\frac{b}{2a}$ is in the interior of $[\mu, \upsilon]$, the other absolute extremum occurs at the end-point which is farther from $-\frac{b}{2a}$.

For a cubic polynomial $p(x) = ax^3 + bx^2 + cx + d$, it is an easily verifiable fact that it has a relative maximum and a relative minimum if and only if p'(x) has two distinct real roots a and β (which happens when $b^2 - 3ac > 0$). Otherwise it has neither a relative maximum nor a relative minimum.

Graphs of y = ax3 + bx^2 + cx + d, $b^2 - 3ac > 0$ are shown here:



Assuming p'(x) has two distinct real roots, say a and β with a < β, then $p'(x) = a(x - \alpha)(x - \beta)$; where a, a, and β are all real.

It is obvious that for a > 0, a has a relative maximum and β has a relative minimum. (For a < 0, a has a relative minimum and β has a relative maximum.).

The main result of this note consists of constructing the largest interval containing a and β such that at these critical points the cubic has an absolute maximum

It should also be remembered that the function Is monotonic on $(-\infty, a)$ and (β, ∞).

As an example, consider

$$p(x) = \frac{1}{3}x^3 - 4x^2 + 12x - 5$$
$$p'(x) = x^2 - 8x + 12.$$

The critical points are $x = 2$ and $x = 6$ and they produce a relative maximum and a relative minimum, respectively.

Using our theorem, $p(x)$ also has an absolute maximum and an absolute minimum, respectively, on any subinterval of $[0, 8]$ containing them. $\left(0 = \frac{3\alpha - \beta}{2},\ 8 = \frac{3\beta - \alpha}{2}\right)$.

If we are to find an absolute maximum and an absolute minimum on $[1, 9]$, we know that an absolute maximum would occur at $x = 9$, and an absolute minimum at $x = 6$ (since $1 > \frac{3\alpha - \beta}{2}$).

For a fourth degree polynomial $p(x)$, we may have one of the following two situations:

Case **I.** $p(x)$ has exactly one relative extremum. (This happens when either $p'(x)$ has only one simple real root, the other two roots being complex or coincident; or where all the three roots of $p'(x)$ are coincident.)

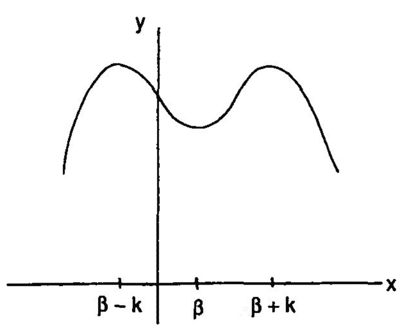
Case **II.** $p(x)$ has exactly three relative extrema (two relative maxima and one relative minimum or two relative minima and one relative maximum).

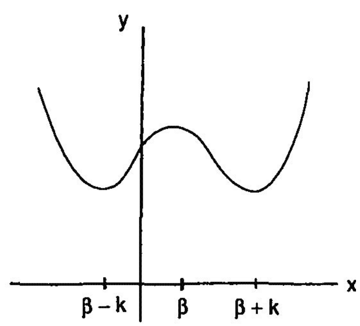
In Case I, the relative extremum is also absolute extremum of the same type, (i.e. relative maximum is absolute maximum, or the relative minimum is absolute minimum).

In Case II we consider a special and easy situation when three real and distinct roots of $p'(x)$, say a, β, and γ with $a < \beta < 7$, are such that

$$\beta - \alpha = \gamma - \beta = k,\ k > 0.$$

It is then an easy matter to show that the relative extrema at $x = a$ and γ are the absolute extrema since the function is monotonic on $(-\infty, a)$ and (γ, ∞).





and an absolute minimum (not necessarily in that order) in that interval.

If a and β are critical points of $p(x)$, then

$$p'(x) = a(x - \alpha)(x - \beta) = a[x^2 - x(\alpha + \beta) + \alpha\beta],\ a \neq 0.$$

Therefore,

$$p(x) = a\left[\frac{1}{3}x^3 - \frac{1}{2}x^2(\alpha + \beta) + \alpha\beta x\right] + k.$$

Thus, using elementary algebra,

$$\begin{aligned} p(x) - p(\alpha) &= a\left[\frac{1}{3}(x^3 - \alpha^3) - \frac{1}{2}(x^2 - \alpha^2)(\alpha + \beta) + \alpha\beta(x - \alpha)\right] \\ &= \frac{1}{3}a(x - \alpha)^2\left(x - \frac{3\beta - \alpha}{2}\right). \end{aligned} \quad (1)$$

Hence, for $a > 0$ when a produces a relative **maximum** for $p(x)$,

$$p(x) - p(\alpha) > 0 \text{ if and only if } x > \frac{3\beta - \alpha}{2}.$$

Interchanging a and β it follows that

$$p(x) - p(\beta) = \frac{1}{3}a(x - \beta)^2\left(x - \frac{3\alpha - \beta}{2}\right). \quad (2)$$

Thus, for $a > 0$ when β produces a relative minimum for $p(x)$,

$$p(x) - p(\beta) < 0 \text{ if and only if } x < \frac{3\alpha - \beta}{2}.$$

For $a < 0$ when $p(\alpha)$ is a relative minimum and $p(\beta)$ is a relative maximum, it can be easily shown that

$$p(x) < p(\alpha) \text{ if and only if } x > \frac{3\beta - a}{2},$$

and

$$p(x) > p(\beta) \text{ if and only if } x < \frac{3a - \beta}{2},$$

using (1) and (2).

Thus, we have the following result:

Theorem 1: Let a and β be two distinct critical points of a cubic curve. Assuming $a < \beta$, the largest closed interval containing them and having absolute extrema at a and β is:

$$\left[\frac{3\alpha - \beta}{2}, \frac{3\beta - \alpha}{2}\right].$$

It is interesting to note that the length of this interval Is $2(\beta - a)$ and that each end-point is $\frac{1}{2}(\beta - a)$ from the nearest critical point.

We have to figure out the largest Interval containing α, β, and γ such that the relative extremum at each of these points Is also an absolute extremum.

We have $\alpha = \beta - k$, $\gamma = \beta + k$, and β as distinct roots of $p'(x)$.

Thus,

$$p'(x) = a(x - \beta + k)(x - \beta)(x - \beta - k).$$

Hence,

$$\begin{aligned} p(x) &= \frac{a}{4}(x-\beta)^4 - \frac{a}{2}k^2(x-\beta)^2 + \lambda \\ &= \frac{a}{4}(x-\beta)^2(x-\beta+\sqrt{2}\,k)(x-\beta-\sqrt{2}\,k) + \lambda \end{aligned}$$

Therefore,

$$p(x) - p(\beta) = \frac{a}{4}(x-\beta)^2[x - (\beta - \sqrt{2}\,k)][(x - (\beta + \sqrt{2}\,k)].$$

Thus, $p(x) - p(\beta)$ will have the same sign as a for $x < \beta - \sqrt{2}\,k$ or for $x > \beta + \sqrt{2}\,k$. It will have sign opposite to a for $\beta - \sqrt{2}\,k < x < \beta + \sqrt{2}\,k$.

For the sake of convenience, replacing $\frac{a}{4}$ by a we have the following result:

Theorem 2. For a fourth degree function $y = a(x - \beta)^4 - 2ak^2(x - \beta)^2 + \lambda$, $k > 0$, the absolute maximum (minimum) occurs if $a < 0$ $(a > 0)$ at $x = \beta \pm k$, on any interval containing any of these points; and the absolute minimum (maximum) occurs at $x = \beta$ on any interval containing β which is a subinterval of $[\beta - \sqrt{2}\,k, \beta + \sqrt{2}\,k]$.

The interval $[\beta - \sqrt{2}\,k, \beta + \sqrt{2}\,k]$ is the largest interval containing α, β, and γ, such that at each of these points the relative extremum Is also an absolute extremum.

I wish to thank Joseph Cicero for his valuable suggestion.

CALL FOR NOMINATIONS

Elections for national officers of the Pi Mu Epsilon Society will be held in the Spring of 1990. The three-year terms of office will begin July 1, 1990.

The committee solicits recommendations for nominees from the membership. Please submit names and addresses of possible nominees to Milton D. Cox, Past-President, Pi Mu Epsilon, Department of Mathematics and Statistics, Miami University, Oxford, OH 45056.

Additional nominations for officers may be made in accordance with Sections 2. and 3. of Article V. of the Constitution and By-Laws.

A QUICK INTRODUCTION TO QUATERNIONS

By Byron L. *McAllister*
Montana State University

Viewed strictly as tools, quaternions became nearly obsolete when Gibbs and Heaviside took them apart into the more easily managed vectors and scalars. (For a detailed history, see [1].) On the other hand, there is a certain charm about quaternlons that makes them keep coming up. This note concerns some interesting properties of quaternions themselves that are quite elementary, given the experience most of us have today with dot and cross products.

Notation. We may think of a quaternion q as a formal sum a + V of a number a plus a vector V. The number a is called the scalar part of q and V is called the vector part of q. The sum of two quaternlons is defined to be the quaternion whose scalar part Is the sum of the scalar parts of the two quaternions and whose vector part is the sum of their vector parts. That is

$$(a + V) + (b + W) = (a + b) + (V + W).$$

The product of $q_1 = a + W$ by $q_2 = b + W$ may be defined in terms of vector dot and cross products. The scalar part of the product q_1q_2 is $ab - V\cdot W$, and the vector part is $aW + bV + V\times W$. That is,

$$(a + V)(b + W) = (ab - V\cdot W) + (aW + bV + V\times W).$$

Note that the presence of the cross product In the vector part implies that the product is not commutative unless $V\times W$ Is 0.

We shall denote a quaternion generically by **q**, its scalar part by a, and its vector part by V. Also, we shall use r to denote the radius of **q**, that is, the length of the vector V. If u is a vector of unit length pointing In the direction of V, then we may also denote V by **r u**. Thus the notation a + **r u** is another general notation for q.

Since a vector has three components, we may regard a quaternion as having four, the fourth being the scalar part. This point of view suggests the notation

$$q = a + xi + yj + zk.$$

The square root of the sum of the squares of a, x, y, and z will be called the modulus of the quaternion **q**, and will be denoted by m.

Pythagorean quintuples. Suppose that a, x, y and z are integers and consider the quaternion $q^2 = qq$. It is easy to show that the modulus of the product of two quaternions is equal to the product of their moduli. Since the squares of the four components of q^2 add to form the square m^4 of the modulus m^2 of q^2, and since m^2 is clearly also a positive integer, we generate in this way a sort of "Pythagorean quintuple," that is a set of five positive integers the squares of four of which add to the square of the fifth. This is an analogue of the fact that, in a similar manner, the square of a complex number with integer real and imaginary parts gives us a Pythagorean triple. (The reader will easily find a modification that generates "Pythagorean quadruples.")

An Isomorphism. That there are analogies between quaternions and complex numbers is not surprising since Hamilton invented quaternions as generalized complex numbers. It is quite well-known that the set of quaternions of the form a + xi is isomorphic to the field of complex numbers, and the same is true if **i** is replaced by j or by k. Perhaps more surprising is the following: Let **u** be any unit vector and let a and b be two real numbers. Let f be a mapping from the complex plane into the quaternions defined by the rule

$$f(a + bi) = a + bu.$$

Clearly f is one-to-one onto its range, and easy calculations show that f "preserves" addition, multiplication, and multiplication by a real number (scalar). Thus, for any fixed unit vector **u**, the set of all a + *b***u** is isomorphic to the complex numbers. (An interesting further inquiry is as to when, for a given set of three perpendicular unit vectors, **u**, **v**, and **w**, it happens that g(a + xi + yj + zk) = a + xu + yv + zw is an isomorphism of the quaternions onto themselves.)

Inverses. Unlike vectors, the system of quaternions includes multiplicative inverses, and hence supports a concept of division. For any quaternion **q**, except 0 + 0, of course, the inverse q^{-1} of q is obtainable by subtracting the vector part of q from the scalar part and then dividing the result by the square of the modulus of q. Analogously to complex numbers, the result of subtracting the vector part of q from the scalar part Is called the conjugate of q. Thus, denoting the conjugate of q by **C(q)**, we may write

$$q^{-1} = \frac{C(q)}{m^2}.$$

That $q^{-1}q = qq^{-1} = 1$ follows directly from $qC(q) = C(q)q = m^2$. The vector part of q^{-1} is seen to be directed oppositely to the vector part of q. Since multiplication is not commutative, **quaternion** division of q_1 by q_2 takes two forms, depending on whether q_2^{-1} is multiplied on the left or on the right of q_1. It is amusing to note that this provides "inverses" and hence "division" (two kindsl) for vectors. For, if V is a vector, we may identify V with the quaternion 0 + V, so that V^{-1} is seen to be

$$V^{-1} = \frac{-V}{r^2}.$$

But although V^{-1} is a vector, **i.e.,** a quaternion with scalar part equal to 0, the (quaternionic) product of a vector with its inverse is not a vector. (It's the scalar 1, of course.) The inverse of V is directed oppositely to V, and the inverse of a unit vector (**i.e.,** of a quaternion with scalar part 0 and with radius 1) is its negative. If V and W are two vectors, we may "divide" V by W on the left to produce **(VxW)/(W·W)** or on the right to produce **(WxV)/(W·W).** Since cross product is anticommutative, the two quotients are negatives of each other.

Square roots. Now let's consider the square roots of a quaternion. Since

$$i^2 = j^2 = k^2 = -1,$$

and since the square of the negative of **i, j,** or k is therefore also –1, it is sometimes said that in the system of quaternions, there are six square roots of –1. Unfortunately, this is somewhat misleading. The truth is that most quaternions have exactly two square roots, given by the formula

$$\mathrm{sqrt}\ (q) = \pm \frac{a + m + V}{(2a + 2m)^{1/2}}, \qquad (1)$$

where *m* is the modulus of q = a + V. This formula is valid where it makes sense.

Because $m \geq a$, the formula fails to make sense only if a + *m* = 0, and this can only happen if V = 0 and a a 0. That is, the formula works unless the vector part of **q is** zero and the scalar part is non-positive. To see what happens in that case, consider **the** following proof of the formula: Think of the vector V as being given In the form ***bu***, where **u** is a unit vector. That is, q = a + *b***u**. If V = 0, the unit vector **u** may be chosen arbitrarily, and b is 0. (But then **be sure** to remember the arbitrariness of u.) We've seen that the set of all such a + *b***u** forms a system isomorphic to the complex numbers. and standard methods (algebraic or geometric) then give us the formula (1) unless *b* = 0. Indeed, when b = 0, if a > 0 the formula is still valid. In this case the two roots of q are symmetrically placed on the real axis — **i.e.,** on the axis of scalars. As a approaches 0, so do both roots, and when a reaches 0, the roots coalesce to 0. As a continues its decrease into negative values, we know from our experience with complex numbers, for which the roots become pure imaginary, that for quaternions the two roots must lie on the two rays of the line through 0 and **u.** That is, a square root is found at a distance $(-a)^{1/2}$ in the **u** direction (and another at an equal distance in the opposite direction.) But the arbitrariness of **u** means that an entire sphere of such square roots exists. (Thus the estimate of six square roots for –1 is far short of the mark!) Note that when V is not 0, **u** is not arbitrary, so that we don't get the sphere in that case. Similarly, when V is 0 but a > 0, the vector parts of the roots are 0 and no sphere is obtained.

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[1] Michael J. Crowe, A *History* of Vector Analysis, N.Y., Dover Publications, 1985 (originally published by University of Notre Dame Press, 1967).

OBLIQUE PYTHAGOREAN LATTICE TRIANGLES

By Stanley *Rabinowitz*
Westford, Massachusetts 01886

A lattice point is a point in the plane with integer coordinates. A lattice triangle is a triangle whose vertices are lattice points. A Pythagorean triangle is a right triangle with integer sides.

It is obvious that, given any Pythagorean triangle, a congruent copy can be found in the lattice with its legs parallel to the coordinate axes.

Definition. ***A*** triangle is oblique (or is embedded in an oblique manner), if no side is parallel to one of the coordinate axes.

In general, given a Pythagorean triangle (such as a 3-4-5 triangle), it is not possible to find a congruent copy embedded obliquely in the lattice. The author asked in this journal ([3]) if there is an oblique lattice triangle similar to a 3-4-5 right triangle. A solution was given in [1]. In this note, we will investigate this question in more detail.

A computer search reveals that the smallest oblique lattice triangle similar to a 3-4-5 triangle has vertices at (0, 0), (4, 4), and (7, 1). This triangle is shown in Figure 1.

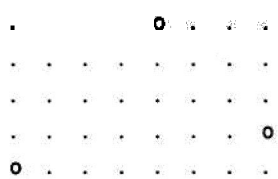

Figure 1

Note that the sides of this triangle have lengths $3\sqrt{2}$, $4\sqrt{2}$, and $5\sqrt{2}$. A more interesting question is: Can such a triangle have integral sides? The answer is "yes" as we will see below.

We can find an entire family of lattice triangles similar to the 3-4-5 triangle by considering the three points:

$$\begin{aligned} O &= (0, 0) \\ B &= (4m, 4n) \\ C &= (4m + 3n, 4n - 3m) \end{aligned}$$

where m and n are any positive integers. Note that letting m = 1 and n = 1 yields the triangle previously found by the computer search.

To make the sides of the triangle integral, first make OB integral. To do this, apply the general formula for the sides of a Pythagorean triangle: let $m = p^2 - q^2$ and n = 2pq. This yields the 2-parameter solution

$$\begin{aligned} O &= (0, 0) \\ B &= (4p^2 - 4q^2, 8pq) \\ C &= (4p^2 - 4q^2 + 6pq, 8pq - 3p^2 + 3q^2) \,. \end{aligned}$$

In some of these, a side may be parallel to one of the axes. It is simple to avoid such a case. For example, choose p = 2 and q = 1 to get the integral triangle with-vertices at (0, 0), (12, 16), and (24, 7). This triangle has sides of lengths 15, 20, and 25. Its sides are 5 times as large as the sides of a 3-4-5 triangle. A computer search reveals that this is the smallest integral triangle similar to a 3-4-5 triangle with no side parallel to an axis.

We now show this can be done in general.

Theorem 1. Given a Pythagorean Triangle, one can find an oblique Pythagorean lattice triangle similar to the given triangle.

Proof. Suppose the given Pythagorean triangle has sides r, s, and t, with t being the length of the hypotenuse. Let A = (m, n). Lay off r copies of OA along ray OA to bring us to the point $B = (rm, rn)$. Erect a perpendicular to OB at B and lay off s copies of OA to bring us to the point $C = (rm - sn, rn + sn)$.

Now let $m = p^2 - q^2$ and n = 2pq to guarantee that OA has integral length. Then we have constructed a Pythagorean lattice triangle OBC similar to the given triangle. Sides OB and BC are clearly not parallel to any axis. OC might be parallel to the y-axis. To prevent this, take p = 4s and $q = 1$. Then the sides of the resulting triangle are:

$$\begin{aligned} O &= (0, 0) \\ B &= (16rs^2 - r, 8sr) \\ C &= (16rs^2 - r - 8s^2, 8rs + 8s^2) \,. \end{aligned}$$

The line OC cannot be parallel to the y-axis, since that would require $16rs^2 = r + 8s^2$ or $s^2 = r/8(2r - 1) \le (2r - 1)/8(2r - 1) = 1/8$, which cannot be since s^2 is a positive integer.

Recall that a Pythagorean triangle is called primitive if its three sides are relatively prime.

The above procedure always produces a non-primitive Pythagorean triangle, since all sides of the triangle formed are divisible by the length of OA and it is clear that $OA > 1$. It is therefore natural to ask if there is a primitive Pythagorean triangle embedded obliquely in the lattice. We answer this question in the negative.

Theorem 2. No primitive Pythagorean triangle can be embedded obliquely in the lattice.

Proof. Suppose Pythagorean triangle ABC (with right angle at C) is embedded obliquely in the lattice. Translate the triangle so that C coincides with the origin. Then perform a rotation through a multiple of $\pi/2$ until ray CB lies in the first quadrant. Point B will not be mapped onto an axis since the triangle is still embedded obliquely (and this property is not affected by the translations or rotations just performed). We may assume that point A has been moved into the second quadrant, for if It moved into the third quadrant, we may perform a reflection about the line y = x to bring it into the second quadrant, leaving B in the first quadrant. Furthermore, we may assume that B lies further from the x-axis than A, for if A were further from the x-axis, we could perform a reflection about the y-axis and then relabel points A and B. Thus, $\triangle ABC$ is situated as shown in Figure 2.

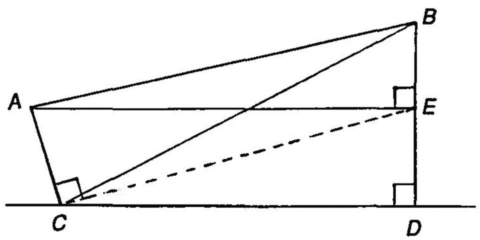

Figure *2*

Let ***D*** be the foot of the perpendicular from ***B*** to the x-axis, and let ***E*** be the foot of the perpendicular from ***A*** to ***BD.*** Since ***B*** was further from the x-axis than ***A,*** point ***E*** lies between ***B*** and ***D.*** Also note that since ***A*** and ***B*** are lattice points, the coordinates of points ***A, B, D,*** and ***E*** are integers. Quadrilateral ***ACEB*** is cyclic since $\angle ACB = \angle AEB = \pi/2$. Thus, $\angle ABC = \angle AEC$. But $AE \parallel CD$ implies that $\angle AEC =$ ***LECD.*** Thus $\angle ABC =$ ***LECD.*** But triangles ***ECD*** and ***ABC*** are **rlght** triangles. Hence they are similar. Let the ratio of similarity be ***plq*** with $\gcd(p, q) = l$. This ratio is rational since it is equal to the ratio of ***DE*** to ***AC,*** both of which are integral. But $AB > BC > CE$, so ***AABC*** is strictly larger than ***ACDE,*** and so $q > 1$. Now $CE = (p/q) \cdot AB$, so ***CE*** is rational. But $CE^2 = CD^2 + DE^2$, so CE^2 is an integer. If a rational number squared is integral, the rational number must itself be an integer. Hence ***CE*** is an integer. Let the lengths of the sides of ***AABC*** be ***a, b,*** and ***c.*** Then the lengths of the sides of ΔECD are ***pa/q, pb/q,*** and ***pdq.*** But these lengths are integers and ***p*** and ***q*** are relatively prime. So $q \mid a$, $q \mid b$, and $q \mid c$. Thus, $q \mid \gcd(a, b, c)$ and consequently, ***AABC*** is not primitive.

Corollary. The set of diophantine equations

$$a^2 + b^2 = r^2$$
$$(b + d)^2 + c^2 = s^2$$
$$(a + c)^2 + d^2 = t^2$$
$$r^2 + s^2 = t^2$$

has no solution with ***r, s,*** and ***t*** being relatively prime.

Proof. In the preceding configuration, let point ***B*** have coordinates ***(c, d),*** let C have coordinates ***(-a, b +*** eft and let $AC = r$, $AB = s$, and $BC = t$. Now the above equations represent the Pythagorean Theorem applied to the various right triangles involved.

Although no oblique lattice triangle congruent to the ***3-4-5*** triangle exists in the planar lattice, what about in the higher dimensions? We conclude this paper with the following surprise: An oblique ***3-4-5*** triangle exists in the integer lattice in **7-** dimensional **spacel** Its vertices are given by the points

$$O = (0, 0, 0, 0, 0, 0, 0)$$
$$B = (1, 2, 2, 0, 0, 0, 0)$$
$$C = (0, 0, 0, 2, 2, 2, 2) .$$

For other easily-stated but unsolved problems concerning lattice points, consult [2].

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[1] Charles **R.** Diminnie, Richard **I.** Hess, and John **Putz,** "Solution to Problem **581",** ***Pi Mu Epsilon Journal. 8 (1985) 194.***

[2] J. Hammer, ***Unsolved Problems Concerning Lattice Points,*** Research Notes in Mathematics. No. ***15.*** **Pitman.** London: ***1978.***

[3] Stanley **Rabinowitz,** "Problem **581",** ***Pi Mu Epsilon Journal. 8 (1984) 43.***

ON THE COVER OF THE SPRING ***1989*** ISSUE

Editor

The formulas for the two functions presented on the front and back covers of the Spring ***1989*** issue are:

Front: $(abs(x) + abs(y))/4 \pmod 3$

Back: $7 \cdot \log(x^2 + y^2 + 2 \cdot abs(x \cdot y) + 0.001) \pmod 3$

The front and back covers commemorating the 75th Anniversary of the founding of Pi Mu Epsilon were designed and prepared by Professor E. P. **Mlles,** Jr., Florida State University, Tallahassee, Florida, at the FSU Muench Center for Color Graphics, on a INTERCOLOR ***2427,*** DATAVUE, and PRINTACOLOR GP ***1024.***

Professor Miles presented the J. Sutherland Frame Lecture at the Summer Meeting of Pi Mu Epsilon in Pittsburgh, PA in ***1981*** on **"The Beauties of Mathematics Revealed in Color Block Graphs."**

A NOTE ON THE ADDITION FORMULAS FOR SINE

By ***Arthur Guetter***
Hamline University

Many formulas in mathematics, especially in number theory, are derived by evaluating some quantity in two different ways. The purpose of this note is to show how the addition and subtraction formulas for the sine function can be derived by calculating the area of a triangle in two ways. A cursory search of several texts did not reveal the following derivations, though I would doubt if they are new. I will assume in the sequel that $0 < 0 < \pi/2$, $0 < \phi$ c $\pi/2$, and $\phi < 0$.

1 first noticed that these derivations would be possible while grading an assignment which required finding the area of a triangle. Comparing an answer which seemed to be different than mine revealed the double angle formula for sine. We start with an isosceles triangle with the length of the equal sides 1, and the angle between these sides with measure 2θ.

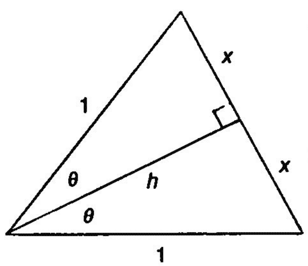


Figure **1a**

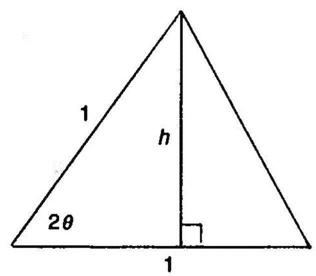


Figure **1b**

Each of the two smaller triangles in Figure **1a** has area given by $(1/2)hx =$ $(1/2)\cos\theta \sin 0$, so that twice the area of the triangle is $2\cos\theta \sin 0$. In Figure **1b,** we calculate twice the area of the triangle as $h = \sin 20$. Putting this together gives the double angle formula

$$\sin 20 = 2\cos 0 \sin\theta$$

After making this observation, I wondered if I could derive the ***addition*** formula for sine in this manner. I needed a triangle with one angle given by $\theta + \phi$, the segment which divides these angles to be an altitude, and one side of length 1.

In Figure **2a,** we note that $\cos\phi = h = z\cos 0$. Then twice the area of the triangle is

$$\begin{aligned} xh + yh &= h(x+y) \\ &= z\cos\theta(x+y) \\ &= z\cos\theta(\sin\phi + z\sin\theta) \\ &= z(\cos\theta\sin\phi + z\cos\theta\sin\theta) \\ &= z(\cos\theta\sin\phi + \cos\phi\sin\theta) \end{aligned}$$

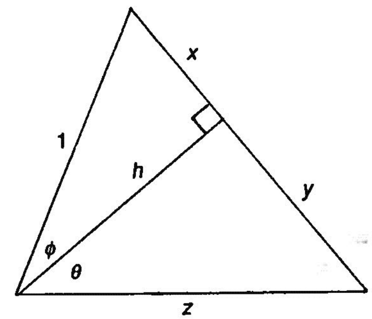


Figure 2a

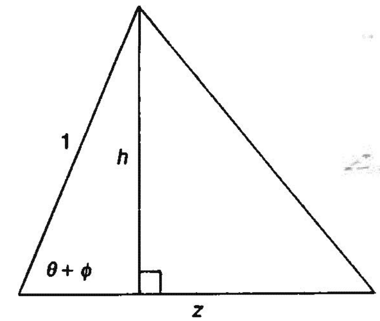


Figure 2b

In Figure **2b,** we calculate twice the area as $zh = z\sin(\theta+\phi)$. Equating these areas gives

$$\sin(\theta+\phi) = \cos 0 \sin\phi + \cos\phi \sin 0,$$

which is of course the addition formula for sine.

We can obtain the subtraction formula for sine in a similar manner. In this case, we use a right triangle with one leg of length one.

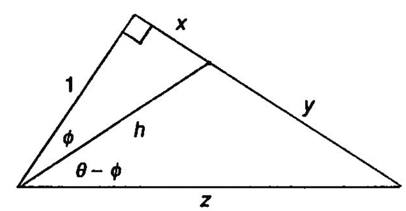


Figure 3

In Figure 3, twice the area of the lower triangle is $zh \sin(0 - \text{ip})$, twice the area of the whole triangle is $z \sin 0$, and twice the area of the upper triangle is $x = h\sin\phi$. It follows that

$$\begin{aligned} zh\sin(\theta-\phi) &= z\sin 0 - h\sin\phi \\ \sin(\theta-\phi) &= \frac{\sin 0}{h} - \frac{\sin\phi}{z} \\ &= \cos\phi\sin\theta - \cos\theta\sin\phi. \end{aligned}$$

We have used the relations $1/h = \cos\phi$ and $1/z = \cos 0$. The last line is the subtraction formula

$$\sin(\theta-\phi) = \cos\phi\sin 0 - \cos\theta\sin\phi.$$

It is now an easy exercise to extend these formulas to all values of 0 and ϕ.

A FALLACY IN PROBABILITY

By Prem N. Bajaj
The Wichita State University

A card is drawn from a standard well-shuffled deck and put aside. Then a second card is drawn. Let **Q** denote the event that the first card is a queen. Let K denote the event that the second card is a king. We are interested in verifying the identity:

$$P(K) = P(Q)P(K/Q) + P(Q^c)P(K/Q^c) \qquad \text{(A)}$$

where Q^c denotes the event that the first card is not a queen, $P(K)$ is the probability for the event K, and $P(K/Q)$ denotes the conditional probability of K when event **Q** has happened, etc.

To compute $P(K)$, condition it whether the first card is a king or not. If K_1 denotes the event that the first card is a king, we have

$$P(K) = P(K_1)P(K/K_1) + P(K_1^c)P(K/K_1^c)$$

$$= \frac{4}{52}\cdot\frac{3}{51} + \frac{48}{52}\cdot\frac{4}{51} = \frac{4}{52}. \qquad \text{(i)}$$

Clearly $$P(K/Q) = \frac{4}{51}. \qquad \text{(ii)}$$

To find $P(K/Q^c)$, notice that the first card, which is not a queen, may or may not be a king. Consequently

$$P(K/Q^c) = \frac{4}{52}\cdot\frac{3}{51} + \frac{44}{52}\cdot\frac{4}{51}$$

$$= \frac{4}{52}\cdot\frac{47}{51}. \qquad \text{(iii)}$$

Finally, $P(Q) = \frac{4}{52}$, $P(Q^c) = \frac{48}{52}$ together with (i), (ii) and (iii) do ***not*** verify the identity (A). What went wrong?

Solution: Computation of $P(K/Q^c)$ is in error. Indeed, we have

$$P(K/Q^c) = \frac{P(Q^cK)}{P(Q^c)}$$

$$= \frac{P(Q^cK_1)P(K/Q^cK_1) + P(Q^cK_1^c)P(K/Q^cK_1^c)}{P(Q^c)},$$

$$= \frac{\frac{4}{52}\,\frac{3}{51} + \frac{44}{52}\,\frac{4}{51}}{\frac{48}{52}} = \frac{4\cdot 47}{48\cdot 51}. \qquad \text{(iv)}$$

With this value of $P(K/Q^c)$, identity (A) is verified to be true.

NOTE ON A WELL-KNOWN LIMIT

By Prem N. Bajaj
The Wichita State University

In the Spring **1989** issue of this journal $\lim_{n\to\infty}\left(\frac{\sqrt[n]{n!}}{n}\right)$ Is obtained using the fact that $\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = R$ implies that $\lim_{n\to\infty}\sqrt[n]{u_n} = R$, $u_n > 0$. (The converse is not true of course.)

However the above limit can be obtained using the definition of an integral and the technique of integration by parts. To see this, recall that (with usual notation):

$$\int_a^b f(x)\,dx = \lim_{\|\Delta\|\to 0}\left(\sum_{k=1}^{n} f(\xi_k)\Delta_k\right)$$

In particular,

$$\int_0^1 f(x)\,dx = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right).$$

Now let
$$L = \lim_{n\to\infty}\left(\frac{\sqrt[n]{n!}}{n}\right),$$

then
$$\log L = \lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n}\log\frac{k}{n}\right)$$

$$= \int_0^1 \log x\,dx = -1,$$

using integration by parts.

Hence, $L = \frac{1}{e}$.

THE INEQUALITY BETWEEN POWER MEANS VIA COORDINATE GEOMETRY

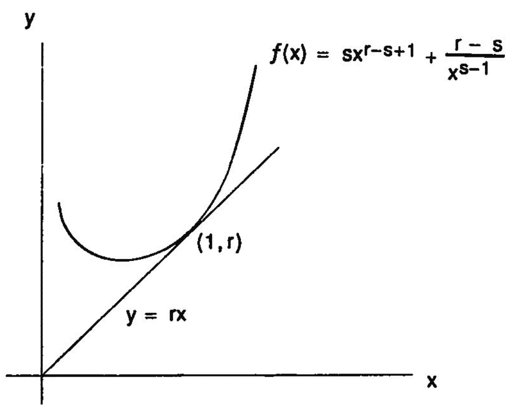
By Norman Schaumberger
Bronx Community College

The inequality between power means states that if $r > s$ are nonzero real numbers then for any positive numbers $a_1, a_2, \ldots, a_n$:

$$\left(\frac{a_1^r + a_2^r + \ldots + a_n^r}{n}\right)^{\frac{1}{r}} \geq \left(\frac{a_1^s + a_2^s + \ldots + a_n^s}{n}\right)^{\frac{1}{r}} \qquad (1)$$

with equality holding if and only if $a_1 = a_2 = \ldots = a_n$.

If $x > 0$ and $r > s \geq 1$ then the graph of $f(x) = sx^{r-s+1} + \frac{r-s}{x^{s-1}}$ is concave upward and has $y = rx$ as a tangent line at $(1, r)$. This follows from the fact that $f(1) = r$, $f'(1) = r$ and $f''(x) = s(r-s+1)(r-s)x^{r-s+1} + (r-s)(-s+1)(-s)x^{-s-1}$ is positive.



Hence $sx^{r-s+1} + \frac{r-s}{x^{s-1}} \geq rx$, or

$$sx^r + r - s \geq rx^s \qquad (2)$$

with equality if and only if $x = 1$.

Let $P = \left(\frac{a_1^s + a_2^s + \ldots + a_n^s}{n}\right)^{\frac{1}{s}}$ and substitute $x = \frac{a_i}{P}$ $(i = 1, 2, \ldots, n)$ successively into (2). Adding gives

$$s\left(\frac{a_1^r + a_2^r + \ldots + a_n^r}{P^r}\right) + rn - sn \geq r\left(\frac{a_1^s + a_2^s + \ldots + a_n^s}{P^s}\right) = rn.$$

It follows that $\frac{a_1^r + a_2^r + \ldots a_n^r}{n} \geq P^r = \left(\frac{a_1^s + a_2^s + \ldots + a_n^s}{n}\right)^{\frac{r}{s}}$ with equality if and only if $\frac{a_i}{P}$ equals 1, $(i = 1, 2, \ldots, n)$, or $a_1 = a_2 = \ldots = a_n$.

Hence, we have proved (1) for the important special case $r > s \geq 1$. For example, putting $r = 2$ and $s = 1$ in (1) gives the familiar arithmetic-quadratic mean inequality:

$$\frac{a_1^2 + a_2^2 + \ldots + a_n^2}{n} \geq \left(\frac{a_1 + a_2 + \ldots + a_n}{n}\right)^2$$

with equality if and only if $a_1 = a_2 = \ldots = a_n$

LETTERS TO THE EDITOR

Dear *Editor,*

Samuel Councilman, Pi *flu* ***Epsilon*** Journal 8 **(1989), 669-671, suggested** a matrix generalization of complex numbers. A different and natural **generalization** of **considerable** interest consists of the **"skew-circulices"** (matrices whose determinants are skew **circulants), exemplified** in the 4 **by** 4 case by

$$\mathcal{A} = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 \\ -a_3 & a_0 & a_1 & a_2 \\ -a_2 & -a_3 & a_0 & a_1 \\ -a_1 & -a_2 & -a_3 & a_0 \end{bmatrix}$$

A theory of Junctions of such matrices, coiled ***complicated numbers,*** is presented by Good, "A **simple generalization** of complex functions", ***Expositiones Mathematicae*** 6 **(1988), 289-311.** In **three dimensions,** poles of Junctions are replaced by straight Cines. **Shew circulants** are also of interest **in** the theory of numbers, for **example,** every prime of the form **8n** + 1 is equal to a 4 by 4 **skew circulant** with integer elements, just **as** in the **classic** theorem that **every** prime of the form 4n + 1 is of the form $a_0^2 + a_1^2$ **(a** 2 by 2 skew **circulant)** (Good, ***Fibonacci Quarterly*** 24. **1986, 47-60, 176-177; Waterhouse, *Fibonacci Quarterly*** 26, 1988, **172-177).**

Yours sincerely,

1. 'J. Good
University Distinguished. Professor of Statistics
Adjunct Professor of Philosophy
Virginia **Polytechnic** Institute and State University
Blacksburg, VA 24061

Dear Editor,

The solvers of Problem **#663 (page** 617, Fall 1988) were too industrious to see the easy **methods.** The question was to express $\int_0^{\pi/2} \frac{x}{\sin x}\,dx$ as a series.

$$\text{(a)}\quad \text{Int.} = \int_0^{\infty} \frac{x}{\cosh x}\,dx = 2\int_0^{\infty} x(e^{-x} - e^{-3x} + e^{-5x} - \ldots)\,dx = 2\left(1 - \frac{1}{3^2} + \frac{1}{5^2} - \ldots\right).$$

The first step is clear from **complex** function theory, but can also be ((one by first-year **calculus** methods **as** follows:

$$\int_0^{\pi/2} \frac{x}{\sin x}\,dx = 2\int_0^{1} \frac{\tan^{-1} t}{t}\,dt = -2\int_0^{1} \frac{\log t}{1+t^2}\,dt = -2\int_{-\infty}^{0} \frac{x}{e^x + e^{-x}}\,dx = \int_0^{\infty} \frac{x}{\cosh x}\,dx$$

by **changing variables,** then integrating by parts, and then changing **variables again** in the obvious way.

(b) ***Another calculation, slightly*** less elementary, second or third year, is **as** follows:

Put $x = \frac{4}{\pi}\left(\sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{32} - \right.$. and then use the fact that

$\int_0^{\pi/2} \frac{\sin(2n+1)x}{\sin x}\,dx =$. The same series expression **comes** out.

Of course, you are **welcome** to **use** these bits **if** you *think* them of any interest, but 1 would rather not have my name **attached.** Modern youngsters have a **deplorable** tendency to look up **books** and believe what they read **instead** of working things out for themselves. But now that 1 have retired I try not to worry about it.

Maine **witheld** by request

1989 NATIONAL PI MU EPSILON MEETING

The Annual Meeting of the Pi Mu Epsilon National Honorary Mathematics Society was held at the University of Colorado in Boulder August 7 through August 9. The year, 1989, marked the 75th Anniversary of the founding of Pi Mu Epsilon and the 40th Anniversary of the establishment of the Pi *Mu Epsilon Journal.*

Letters of congratulation and/or certificates were received from the American Mathematical Society, the Governor of New Jersey, President George Bush, the Governor of Colorado, the National Council of Teachers of Mathematics, Kappa Mu Epsilon, The Mathematical Association of America and the Association for Women in Mathematics.

Memorabilia, including the original journal of the Mathematics Club of Syracuse University on the founding of Pi Mu Epsilon, were on exhibit in Boulder, courtesy of the Library of Syracuse University.

A generous National Security Agency grant enabled Pi Mu Epsilon to support an increased number of student paper presenters at the meeting.

In honor of Pi Mu Epsilon's 75th Anniversary, the American Mathematical Society announced an annual grant to be administered by Pi Mu Epsilon to further scholarship in undergraduate mathematics. In 1989, part of this grant was used to provide prizes to students whose paper presentations were judged to be of especially high quality by members of the Pi Mu Epsilon Council.

W. H. Freeman and Company Publishers, PWS-Kent Publishing Company and Brooks/Cole Publishing Company provided financial support for the opening reception and a selection of books to further the goals of the Society.

Pi Mu Epsilon hosted the Western Hoe Down, the big social event of the joint meeting with The Mathematical Association of America and the American Mathematical Society.

The AMS-MAA-PME Invited Address "The Mathematics of Identification Numbers" was presented by Joseph A. Gallian, University of Minnesota, Duluth.

The J. Sutherland Frame Lecturer was Professor Jane Cronin Scanlon, Rutgers University. Her lecture was "Entrainment of Frequency: A Recurring Theme."

A special T-shirt in honor of the Society's 75th Anniversary was on sale and is still available from the Editor. An ad for the T-shirt appears on page 72.

At the Annual Banquet, $100 awards for excellence in presentation were awarded to the following nine students: Beth-Allyn Eggens, Chikako Mese, Darrin Frey, William C. Regli, M. Chris Haase, Robert A. Cullen, Stephen J. Smith, Nicholas Ahn, and Michele Pezet. The complete program of 46 student papers follows.

PROGRAM - STUDENT PAPER SESSIONS

mathematics G Digital Image Processing	*Nicholas Ahn* *Illinois Iota* *Elmhurst College*
Chaotic Linear Transformations on a Toms	*Joel Atkins* *Indiana Gomma* *Rose-Hutman Institute of Technology*

Hamiltonian and Eulerian Circuits in the Join of Two Connected Graphs	*Timothy Bahmer* Ohio Zeta *University of Dayton*
Solving Diophantine Equations Using Continued Fractions	*Jim Banoczi* Ohio Xi *Youngstown State University*
A Brief Introduction to Fractal Images	*Mark Boardman, presenter* *David. Leavitt* *Nebraska Alpha* *University of Nebraska*
The Hyperbolic Geometry of M. C. Escher	*Kathleen L. Brigham* *Illinois Epsilon* *Northern Illinois University*
Automorphism Groups of Hasse Subgroup Diagrams for Groups of Low Order	*Melanie L. Butt* *Tennessee Gamma* *Middle Tennessee State University*
Evolutionary Evaluation of Risk Strategies	*Elizabeth Clarkson* *Kansas Gommu* *Wichita State University*
Put Up more Wallpaper, It's Friezeing in Here	*James Ellis Colliander* *Minnesota Gamma* *Macalester College*
The Classification of Finite Simple Groups	*Robert A. Cullen* *Wisconsin Alpha* *Marquette University*
Fibonacci Periods mod(m)	Keith *R. Dean* *Texas Delta* *Stephen F. Austin State University*
A Generalization of Odd and Even Vertices in Graphs. Part I	*Amy Dykstra* *Michigan Epsilon* *Western Michigan University*
A Phase Assort method in Geophysics	*Richard L. Edington* *Texas Delta* *Stephen F. Austin State University*
Change for a Dollar - How Many Ways?	*Beth-Allyn Eggens* Ohio *Xi* *Youngstown State University*

The Computer as Catalyst	Shari J. Feldman Pennsylvania Rho Dickinson College
Rpplications of Difference Tables in number Theory	Joseph E. Fields Maryland Gamma University of Maryland, Baltimore County
Chaos Theory	James A. FitzSimmons Ohio Theta Xavier University
Conjugations in Inverse Semigroups	Darrin Frey Nebraska Alpha University of Nebraska
The Determination of the Expected Length of a Coin Toss Game	Francis Fung Kansas Beta Kansas State University
Fractals: A New Geometry	Mary Anne Gallagher New Jersey Epsilon Saint Peter's College
The Relationship between a Graph and its Line Graph	Colleen Galligher Ohio Zeta University of Dayton
Fixed Points, Compactness, and Existence Theorems for Differential Equations	Paul Glezen Arkansas Alpha University of Arkansas
A Proposed Secondary mathematics Curriculum for the 1990's	Kevin. Groothuis Michigan Alpha Michigan State University
Elliptic Curves: Theory and Rpplication	M. Chris Haase Ohio Alpha Ohio State University
The Domination number and Uniquely Domatic Graphs	Sheri Jordan Arkansas Beta Hendrix College
Singularly Perturbed Systems (numerical methods for)	Khaled Kahlouni Texas Nu University of Houston - Downtown

Is a Transitive Banach Space a Hilbert Space?	Shinko Kojima Tennessee Alpha Memphis State University
mathematics for a Digital Controlling Unit Used in a Forestry Experiment	Paul E. Lewis Texas Delta Stephen F. Austin State University
Plucking a Leaf off a Tree and Other Graphs	Chikako Mese Ohio Zeta University oJ Dayton
An Rpplication of the Rayleigh-Ritz method	J. Greer Milam Alabama Gamma Samford University
Evolutionary Operation	Pam Miller Ohio Nu University of Akron
An Elementary Analysis of Conformal mappings of Simply-Connected Domains	Jeffrey Osikiewicz Ohio Xi Youngstown State University
An Rpproximation for the number of Primes between k and k^2 when k is an Integer	Randall Osteen Florida Theta University of Central Florida
A mathematical method for Finding Anisotropy Constants	Brad, P a d Ohio Delta Miami University
A Computer Is Worth a Thousand Blackboards	Michele Pezet Michigan Gamma Andrews University
Bounding the Chromatic number of a Graph	Marla Prenger Ohio Zeta University of Dayton
Hour Many Licks Does It Take to Reach the Center of A Tootsie Roll Pop?	Henry W d t Ramsey South Carolina Gamma College oJ Charleston
Games. Graph Theory, Algorithms, and Kayles	William C. Regli Pennsylvania Xi St. Joseph's University

A Study of Linear Singularly Perturbed Syetems	Gholam Reza Sarhangi Kansas Gamma Wichita State University
A Generalization of Odd and Even Vertices in Graphs, Part 2	Michelle Schultz Michigan Epsilon Western Michigan University
Computer Go	Stephen J. Smith Pennsylvania Rho Dickinson College
A Statistical Soft Drink Taste Test	Wendy R. Smith South Carolina Gamma College of Charleston
making "tents" Out of Math - One of Its Practical Uses	Jenny Spence Wisconsin Delta St. Norbert College
Resonance: Is It Live, or Is It . . . ?	Tim Strnad Wisconsin Delta St. Norbert College
Seen Any Good Films Lately? - An Introduction to Some of the notions of Geometric measure Theory	Karen H. Taylor Kansas Gamma Wichita State University
Pseudo-Orbit Shadowing on the Unit Interval	Jeffrey Van Eeuwen, presenter Tim Pennings Michigan Delta Hope College

CHANGES OF ADDRESS/INQUIRIES

Subscribers to the Journal should keep the Editor informed of changes in mailing address. Journals are mailed at bulk rate and are not forwarded by the postal system. The cost of sending replacement copies by first class mail is prohibitive.

Inquiries about certificates, pins, posters, matching prize funds, support for regional meetings, and travel support for national meetings should be directed to the Secretary-Treasurer, Robert M. Woodside, Department of Mathematics, East Carolina University. Greenville, NC 27858. 919-757-6414.

STATE OF COLORADO

EXECUTIVE CHAMBERS
136 State Capitol
Denver, Colorado 80203-1792
Phone (303) 866-2471

Roy Romer
Governor

August 9, 1989

National Conference
Pi Mu Epsilon
University of Colorado
Boulder, Colorado

Greetings:

It gives me pleasure to welcome you to Pi Mu Epsilon's annual conference co-sponsored by the American Mathematical Society and the Mathematical Association of America.

On behalf of the people of Colorado, I extend congratulations to Pi Mu Epsilon on the occasion of its 75th anniversary. Mathematical proficiency is vital in our advancing society. It is encouraging to know that Pi Mu Epsilon, the AMS and the MAA promote mathematic and scholarly development.

Best wishes for a successful and memorable gathering.

Sincerely,

Roy Romer

Roy Romer
Governor

STATE OF NEW JERSEY
OFFICE OF THE GOVERNOR
CN-001
TRENTON
08625

THOMAS H. KEAN
GOVERNOR

August 9, 1989

Dear Friends:

On behalf of the State of New Jersey, I am pleased to extend warm greetings and congratulations to you as you celebrate the 75th Anniversary of Pi Mu Epsilon.

Pi Mu Epsilon has been dedicated to the promotion of scholarly activity in mathematics throughout its long history. Your society has influenced the lives of mathematicians across the country, and you should be proud of your success in encouraging our youth to discover the important role mathematics plays in our world today and will play in the future.

Again, congratulations, and best wishes for additional success.

Sincerely,

Tom Kean

Thomas H. Kean
Governor

American Mathematical Society

Undergraduate mathematics is the foundation upon which future mathematicians build experience and knowledge. It is vital for the future of our discipline that undergraduate mathematics flourish. The Council of the American Mathematical Society congratulates Pi Mu Epsilon for its seventy-five years of nurturing excellence in undergraduate mathematics and it offers best wishes for the future.

August 1989

William Browder
President

Robert M. Fossum
Secretary

NATIONAL COUNCIL OF
Teachers of Mathematics

Scottsdale School District
3811 North 44th St.
Phoenix, AZ 85018
June 7, 1989

Dr. Eileen L. Poiani
President
Saint Peter's College
2641 Kennedy Boulevard
Jersey City, NJ 07306

Dear President Poiani:

Congratulations to Pi Mu Epsilon on the society's 75th anniversary. On behalf of the National Council of Teachers of Mathematics, I send you and your members greetings as you celebrate this significant event.

The promotion or scholarly activity in mathematics among students and staff is a lofty goal that contributes to the advancement of mathematics in general. Students are encouraged, motivated, and challenged to prepare for the opportunity to present their papers and be recognized in the academic community.

Our very best wishes for your continued success in the pursuit of excellence in mathematics. Happy anniversary!

Sincerely,

Shirley Frye
President, NCTM

ms
cc: Dr. James Gates, Executive Director, NCTM

Association for Women in Mathematics

Office Address: Box 178, Wellesley College, Wellesley, Massachusetts 02181
Telephone: 617-235-0320 Ext. 2643

July 6, 1989

Professor Eileen L. Poiani
Saint Peter's College
2641 Kennedy Boulevard
Jersey City, NJ 07306

Dear Professor Poiani:

On behalf of the Association for Women in Mathematics. I extend warm congratulations to Pi Mu Epsilon on the occasion of its 75th Anniversary. It is our hope that, through its role as a national honor society promoting research and scholarship in mathematics. Pi Mu Epsilon will encourage more undergraduate women to continue in mathematics, and to go onto successful careers in the mathematical sciences.

Sincerely,

Jill P. Mesirov
President

JPM/cc

THE WHITE HOUSE
WASHINGTON

June 27, 1989

It is a pleasure to extend warmest greetings to the members of Pi Mu Epsilon as vou celebrate your 75th anniversary.

The great German mathematician Carl **Friedrich** Gauss called mathematics the "queen of the sciences" -- an **apt** description for this **field** of knowledge that has, from the very beginning of civilization, been one of man's ablest tools in understanding and **working** in the world around him. Medicine, engineering. space exploration -- the great feats accomplished in these and so many other fields would be impossible without mathematics.

For 75 years, your society has **encouraged** and furthered excellence in mathematics. In so doing. you have not only enriched the scholarly pursuits of your members but also touched the lives of all. because we all depend on the fruits of applied mathematics in our everyday **lives.**

I salute you for your efforts and achievements, and wish you an enjoyable celebration and every future success. God bless you.

Gy Bush

The MATHEMATICAL ASSOCIATION of AMERICA

Lida K. Barrett President

July 26, 1989

Eileen L. Poiani, President
Pi Mu Epsilon, Inc.
National Honorary Mathematics Society
Saint Peters College
2641 Kennedy Boulevard
Jersey City, New Jersey 07306

Dear Dr. Poiani:

Congratulations to Pi Mu Epsilon on its 75th Anniversary celebration. I am pleased that Pi Mu Epsilon is meeting again this summer as it has each summer since 1952 with the American Math Society and the Mathematical Association at the joint summer meetings. The tradition of having students from across the country present papers on mathematic related topics adds a dimension to our meetings and gives us a hope for and perspective on the future of the mathematics profession.

I recall with considerable pleasure an outcome of a Pi Mu Epsilon meeting in the late 70's. I asked the young woman student representing the University of Tennessee to look up a young man representing Armstrong State, who I had met when I spoke to the Pi Mu Epsilon banquet there. Not only did they get acquainted at the Pi Mu Epsilon meeting, but he came to Tennessee to study. They have been happily married for a number of years and have three delightful children.

Congratulations to Pi Mu Epsilon and its 90,000 members on its 75th anniversary. The Mathematical Association of America looks forward to sharing its summer meetings with Pi Mu Epsilon in the years ahead.

Sincerely,

Lida K. Barrett

LKB/wbl

Kappa Mu Epsilon

MATHEMATICS HONOR SOCIETY

HAROLD L. THOMAS PRESIDENT
DEPARTMENT OF MATHEMATICS
PITTSBURG STATE UNIVERSITY
PITTSBURG KANSAS 66762

ARNOLD D. HAMMEL PRESIDENT ELECT
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BENEDICTINE COLLEGE
ATCHISON KANSAS 66002

MARY S. ELICK, HISTORIAN
DEPARTMENT OF MATHEMATICS
MISSOURI SOUTHERN STATE COLLEGE
JOPLIN MISSOURI 64801

MEMBER OF THE ASSOCIATION OF COLLEGE HONOR SOCIETIES

July 5, 1989

Professor Eileen L. Poiani
President, Pi Mu Epsilon
Saint Peter's College
2641 Kennedy Boulevard
Jersey City, NJ 07306

Dear Professor Poiani:

On behalf of the members OE Kappa Mu Epsilon, I am pleased to extend greetings dnd congratulations to the members of Pi Mu Epsilon on the occasion of your Diamond Jubilee celebration to be held at uw University of Colorado in August. We share with you the pride and satisfaction of promoting and recognizing scholarship in mathematics.

The heritage of Pi Mu Epsilon gives evidence that members serve as an inspiration to others in improving their proficiency in mathematics and provide high goals for them to emulate. Again, our heartiest congratulations in honor of Pi Mu Epsilon's 75th Anniversary.

Sincerely yours,

Harold L. Thomas
President

jk

PUZZLE SECTION

Edited by Joseph D. E. Konhauser
Macalester College

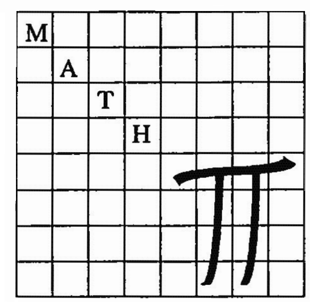
The PUZZLE SECTION is for the enjoyment of those readers who are addicted to working doublecrostics or who find an occasional mathematical puzzle or word puzzle attractive. We consider mathematical puzzles to be problems whose solutions consist of answers immediately recognizable as correct by simple observation and requiring little formal proof. Material submitted and not used here will be sent to the Problem Editor if deemed suitable for the PROBLEM DEPARTMENT.

Address all proposed puzzles and puzzle solutions to Professor Joseph D. E. Konhauser, Mathematics and Computer Science Department, Macalester College, St. Paul, MN 55105. Deadlines for puzzles appearing in the Fall Issue will be the next March 15, and for the puzzles in the Spring Issue will be the next September 15.

PUZZLES FOR SOLUTION

1. *Proposed by Jeanette Bickley, St. Louis, MO.*

Cut the 8x8 square into four congruent pieces such that each piece has one of the four letters (M, A, T, H) and each has a piece of Π.



2. *Proposed by the Editor.*

A right circular cone of slant height s and generating angle α is "rolled" on a plane so that the apex V remains fixed. How many times will the cone revolve about its axis if the cone is rolled through a complete circle about V?

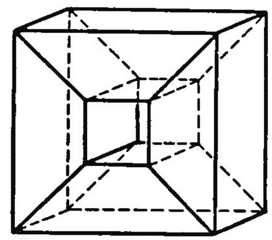
3. *Proposed by the Editor.*

How many positive integers have base ten representations consisting of distinct digits (0 through 9)? By way of example, 7, 13, and 123 are integers to be counted; 11, 122 and 200 are integers not to be counted.

4. ***Proposed by the Editor.***

(A timely variation on a familiar theme.) Find a law of formation for the 5 x 5 array

164	244	306	128	448
268	348	410	232	552
387	467	529	351	671
425	505	567	389	709
276	356	418	240	560



5. ***Proposed by the Editor.***

Label the sixteen vertices of the "cube within a cube" so that the twenty-four quadrilateral faces have equal vertex sums.

6. ***Proposed by the Editor.***

By making cuts along its diagonals, a square can be dissected into four pieces which can be reassembled to form two congruent squares. By making cuts along the line segments joining the midpoints of opposite sides, the square can by dissected into four congruent squares. By cutting a square along the four line segments joining vertices to midpoints of opposite sides, the square can de dissected into nine pieces which can be reassembled to form five congruent squares. Dissect a square into a "small number" of pieces which can be reassembled to form three congruent squares.

7. ***Proposed by the Editor.***

In a certain mathematics journal, seven puzzles were proposed. In response, for each puzzle the Editor received two correct solutions. In all, 14 solutions were submitted by 7 different readers, two solutions from each. Is it possible to publish the readers' solutions so that exactly one from each of the seven contributors will appear?

COMMENTS ON PUZZLES 1 - 7, SPRING 1989

Responses to Puzzle #1 were either 101! = 1111000 in base 2 or 010! = 3628800 in base 10. In Puzzle #2, several readers recognized the old puzzle of drawing a continuous path of four line segments through a 3x3 array of points without passing through any of the nine points more than one time. The secret is to "overshoot" the 2 and the 4. For Puzzle #3, the nine responses were quite varied. The most succinct was RICHARD I. HESS' "These are the integers expressible in base 3 using only ones and zeros." In Puzzle #4, the shortest solution for going from ONE to TWO was VICTOR FESER's ONE - ORE - ORT - OAT - TAT -TOT - TOO - TWO. Nineteen readers responded to the matching problem in Puzzle #5 and were in complete agreement (1 - comb, 2 - pen, 3 - key, 4 - book). The solution to Puzzle #6 is not unique. One solution is to arrange the numbers 1 through 15 in three rows 1, 2, 11, 12, 14; 8, 9, 10, 7, 6; 15, 13, 3, 5, 4. In all solutions, row sums are 40 and column sums 24. ROBERT PRIELIPP pointed out that Puzzle #7 had appeared as Problem 73 in the January 1970 issue of the *Journal of Recreational Mathematics.* The longest chain consists of six isosceles triangles with degrees 124°, 28°, 28°; 28°, 76°, 76°; 76°, 52°, 52°; 52°, 64°, 64°; 64°, 58°, 58° and 58°, 61°, 61°.

SOLVERS: **Charles Aschbacher** (1, 3, 5, 6), Amy Bohachek (5, 6, 7), Margaret **Boles** (5), William **Boulger** (1, 3, 4, 5, 6, 7), Matthew Broadhead (2, 3, 4, 5, 6, 7), **William Chau** (1, 3, 5, 6, 7), Chris Conrad (5, 6, 7), Anna Contadino (5), Victor **Feser** (1, 4, 5), Robert C. Gephardt (5), I. J. Good (4), Richard I. Hess (1, 2, 3, 5, 6, 7), Donna **Hiestand** (3, 6), Jon **Lange** (7), Bro. Howard Lohrey, S.M. (2, 5, 6), Thomas **Mitchell** (5), **Donald B. Onnen** (1, 2, 3, 4, 5, 6, 7), Robert Prielipp (4, 7), **Emil** Slowinski (1, 3, 4, 5, 6, 7), Michael Taylor (5, 6), Katharine Vance (5), **Tian-Yih** Wang (5, 6) and Yvonne Zhou (1, 5, 6).

ERRATA

William Chau and Thomas Mitchell pointed out the omission of a square root symbol on page 679 of the Spring 1989 issue in the discussion of the solution to Puzzle #3 in the Fall 1988 issue.

Solution to Mathacrostic No. 28 (Spring 1989)

WORDS

A. **Wythoff's** Nim	K. **Ecotone**	U. Slingshot Effect
B. **Penrose** Tiles	L. Leftover	V. Outlier
C. Offshoot	M. Axiom of Choice	W. Flowsnake
D. Unpolished	N. **Benford's** Law	X. **Race**
E. Neusis	O. Yang-Mills Gauge Field	Y. Eotvos
F. Dehydrated Elephant	P. Relativity	Z. Aeolian
G. Swivel Joint	Q. Itself	a. Sphinx
H. Time Reversal	R. **Neurite**	b. Ophiuride
I. One-time Pad	S. Trapdoor	c. **Necker Cube**
J. Norm	T. **Hilbert's** Hotel	

AUTHOR AND TITLE: W. POUNDSTONE **LABYRINTHS** OF REASON

QUOTATION: There is a subversive joy in seeing logic tumble like a house of cards. All the well-known paradoxes of confirmation theory and epistemology were conceived more or less in the spirit of intellectual play. In few other fields is it possible for the interested **nonexpert** to sample so much of the true flavor of the field and have fun doing it.

SOLVERS: JEANETTE BICKLEY, St. Louis Community College at **Meramec,** MO; J. KEVIN COLLIGAN, National Security Agency; CHARLES R. DIMINNIE, St. Bonaventure University, NY; ROBERT **FORSBERG,** Lexington, MA; MICHELE **HEIBERG,** Herman, MN; JOAN AND DICK JORDAN, Indianapolis, IN; DR. **THEODOR** KAUFMAN, Brooklyn, NY; HENRY S. LIEBERMAN, Waban, MA; CHARLOTTE **MAINES,** Rochester, NY; DON PFAFF, University of Nevada-Reno; STEPHANIE SLOYAN, Georgian Court College, **Lakewood,** NJ; MICHAEL TAYLOR, Indianapolis Power and **Light,** Co., IN; and BARBARA **ZEEBERG,** Denver, CO.

Mathacrostic No. 29

Proposed by Joseph D. E. Konhauser

The 239 letters to be entered in the numbered spaces in the grid will be identical to those in the 25 keyed words at the matching numbers. The key numbers have been entered in the diagram to assist in constructing the solution. When completed, the initial letters of the Words will give the **names(s)** of the **author(s)** and the title of a **book;** the completed grid will be a quotation from that book.

Definitions	Words
A. Wood inlay which flourished in Italy during the Renaissance	144 60 10 226 167 49 205 130
B. One of order n gives rise to n - 1 mufually orthogonal Latin squares (2 wds.)	81 159 52 35 121 189 113 170 61 235 39
C. A device consisting of balls of equal mass on strings of equal length **to** illustrate elastic impact (2 wds.)	108 166 197 9 87 45 185 69 129 209 173 238 191
D. A multi-layered structure which simulates chaotic folding (2 **wds.)**	141 156 48 131 239 3 208 17 56 72 195 106 123 53 181
E. John ___, pseudonym under which mathematician, Eric Temple Bell, wrote science fiction	193 93 41 184 5
F. A period or state of decline (2 **wds.)**	32 237 187 94 84 78 146
G. The pivoted swinging bar to which the traces of a harness are fastened and by which a vehicle or implement is drawn	153 20 73 168 112 222 2 116 180 203 97
H. A small, **sfemless** aquatic plant of the mustard family having slender, sharp-pointed leaves and minute white flowers	140 224 188 132 70 212 50
I. ___ geometry, a picturesque but inaccurate description of the intrinsic topology of a **surface** (sometimes comp.)	134 128 105 145 152 55 68 38 124 103 206
J. A very small amount **(3** wds.; or 2 **wds.,** one **comp.)**	43 137 149 80 13 229 62 211 165 51 76 122 151
K. A movement in art and **literature, 1918-1922,** intended to outrage and offend by flouting traditional aesfhetic standards and social mores	100 171 136 79
L. Complete **(comp.)**	133 201 15 57 36 147 164 228 46
M. Something that is seen or intuited	183 155 218 199 23
N. The three concepts whose unity is symbolized by fhe triple pentagon emblem of the Berlin Philharmonic (3 wds.)	89 163 223 58 75 31 25 14 230 142 178 44 7
O. James Lovelock's theory that the earth, its oceans and atmosphere, and all **living** things are **parts** of one great organism	215 219 186 21
P. The upper integral of the characteristic function of a point set P on an **interval (a,b)** (2 wds.)	22 54 27 217 83 90 158 71 37 107 119 148
Q. To become apparent	12 67 1 172
R. Compact, connected, and locally **connected** metric spaces (2 wds.)	47 77 126 220 111 85 210 99 34 234 175 169 104
S. "Books are the ___ **of** men." Mark Twain (2 **wds.)**	202 120 24 4 64 30 115 162 40 157 179 214 16 194 127 227
T. Dodecahedron-based game sold to a London **toymaker** for 25 **£** in 1859 by Sir William Rowan **Hamilton** (3 **wds.)**	98 118 11 63 216 74 160 198 110 225 28 91 135 221
U. Standard, touchstone, criterion	18 138 96 204 42 26 192 176 6
V. Winner of **the 1989 World Computer Chess** Championship (2 wds.)	190 117 65 82 150 162 6 95 233 154 29
W. **Inadequate** for or incapable of bringing about an ambitious project	86 207 232 143 174 114 125
X. Any business venture, operation, or **product** that is a dependable source of inwme or profit (2 **wds.)**	109 236 139 102 19 177 66
Y. An advocate of **the** interpretation of **myths** as traditional accounts of historical persons and events	92 88 161 213 231 59 200 33 196 101

1 Q	2 G		3 D	4 S	5 E	6 U		7 N	8 V	9 C		10 A	11 T
	12 Q	13 J	14 N	15 L	16 S	17 D	18 U		19 X	20 G	21 O	22 P	23 M
	24 S	25 N	26 U		27 P	28 T		29 V	30 S	31 N	32 F		33 Y
34 R		35 B	36 L		37 P	38 I	39 B		40 S	41 E	42 U	43 J	44 N
45 C	46 L		47 R	48 D	49 A	50 H		51 J	52 B		53 D	54 P	55 I
	56 D	57 L	58 N	59 Y		60 A	61 B	62 J	63 T	64 S	65 V		66 X
67 Q	68 I		69 C	70 H	71 P	72 D	73 G	74 T	75 N	76 J	77 R	78 F	
79 K		80 J	81 B	82 V	83 P	84 F	85 R	86 W	87 C	88 Y	89 N		90 P
91 T	92 Y	93 E	94 F	95 V	96 U	97 G		98 T	99 R	100 K		101 Y	102 X
103 I		104 R	105 I	106 D	107 P	108 C	109 X	110 T		111 R	112 G		113 B
114 W	115 S	116 G	117 V	118 T	119 P		120 S	121 B		122 J	123 D	124 I	
125 W	126 R	127 S	128 I	129 C	130 A	131 D		132 H	133 L	134 I	135 T	136 K	
137 J	138 U	139 X		140 H	141 D	142 N	143 W	144 A	145 I	146 F	147 L		148 P
149 J		150 V	151 J	152 I		153 G	154 V	155 M	156 D	157 S		158 P	159 B
	160 T	161 Y	162 S		163 N	164 L	165 J	166 C	167 A	168 G	169 R	170 B	
171 K	172 Q	173 C		174 W	175 R	176 U	177 X	178 N	179 S	180 G	181 D	182 V	183 M
184 E	185 C	186 O	187 F	188 H	189 B		190 V	191 C	192 U	193 E	194 S	195 D	196 Y
	197 C	198 T	199 M		200 Y	201 L	202 S	203 G	204 U		205 A	206 I	
207 W	208 D	209 C	210 R	211 J		212 H	213 Y	214 S	215 O	216 T	217 P	218 M	
219 O	220 R	221 T		222 G	223 N	224 H		225 T	226 A	227 S		228 L	229 J
230 N	231 Y	232 W	233 V	234 R	235 B	236 X	237 F	238 C	239 D				

PROBLEM DEPARTMENT

Edited by Clayton W. Dodge
University of Maine

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

All communications should be addressed to C. W. Dodge, Math. Dept., University of Maine, Orono, ME 04469. Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly identified with name and address. Solutions to problems in this issue should be mailed by July 1, 1990.

PROBLEMS FOR SOLUTION

704. *Proposed by the late Charles W. Trigg, San Diego, California.*

Find the least HEAT necessary to BOIL the H_2O:

$$\text{HEAT} + \text{HHO} = \text{BOIL}$$

705. *Proposed by the late Charles W. Trigg, San Diego, California.*

In this "Ovis" group, the EWES and every LAMB are in prime condition. Find the two solutions:

$$\text{RAM} + \text{EWES} + \text{LAMB} + \text{IAMB} = \text{SHEEP}.$$

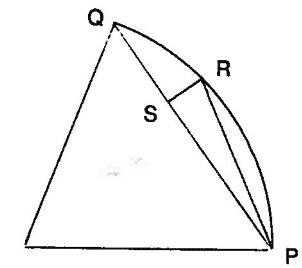
706. *Proposed by John Dalbec, Ohio Xi Chapter, Youngstown State University, Youngstown, Ohio.*

This alphametric is too "compact" to have a unique solution. If, however, one CECHs for primality, then there is just one conclusion:

$$\text{STONE} + \text{CECH} = \text{LECAR}.$$

707. *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.*

From a point R taken on any circular arc PQ of less than a quadrant, two segments are drawn, one to an extremity P of the arc and the other RS perpendicular to the chord PQ of the arc and terminated by it. Determine the maximum of the sum PR + RS of the lengths of these two segments. This problem without solution is given in Todhunter's *Trigonometry*.



708. *Proposed by Jack Garfunkel, Flushing, New York.*

Find a Mascheroni construction (a construction using only compasses -- no straightedge allowed) for the orthic triangle of an acute triangle ABC.

709. *Proposed by Norman Schaumberger, Bronx Community College, Bronx, New York.*

If a, b, and c are the lengths of the sides of a triangle and if K and P are the area and perimeter, respectively, then prove that

$$a^2b^2 + b^2c^2 + c^2a^2 \ge 12K^2 + \frac{P^4}{108}$$

with equality if and only if the triangle is equilateral.

710. *Proposed by Thomas E. Moore, Bridgewater State College, Bridgewater, Massachusetts.*

Under what conditions on the positive integers a and b will the sides of a nondegenerate triangle be formed by

a) a, b, and gcd(a,b)?
b) a, b, and lcm[a,b]?

711. *Proposed by James N. Boyd, St. Christopher's School, Richmond, Virginia.*

A pentagon is constructed with five segments of lengths 1, 1, 1, 1, and w. Find w so that the pentagon will have the greatest area.

712. *Proposed by Robert C. Gebhardt, Hopatcong, New Jersey.*

A cube 4 inches on a side is painted. Then it is cut into 64 one-inch cubes. A cube is chosen at random and tossed. Find the probability that none of the five faces that are showing is painted.

713. *Proposed by R. S. Luthar, University of Wisconsin Center, Janesville, Wisconsin.*

Evaluate

$$\int_{\pi/60}^{\pi/30} \tan 5x \tan 3x \tan 2x \, dx .$$

714. *Proposed by Sam Pearsall, Loyola Marymount University, Los Angeles, California.*

A flea crawls at the constant rate r = 1 foot per minute along a uniformly stretched elastic band, starting at one end. The band is initially L = 1 yard in length and is instantaneously and uniformly stretched L = 1 yard at the end of each minute while the flea maintains his grip on the band at the instant of each stretch. It is well known that the flea will reach the other end of the band in under 11 minutes. Find all lengths L such that the flea will reach the other end of the band in finite time.

715. *Proposed by Christopher Stuart, New Mexico State University, University Park, New Mexico.*

Euler's constant γ is defined by the equation

$$\gamma = \lim_{N\to\infty}\left(\sum_{k=1}^{N}\frac{1}{k} - \ln N\right)$$

Show that

$$\gamma = \sum_{k=2}^{\infty}\sum_{j=1}^{\infty}\frac{(-1)^k}{kj^k}.$$

716. *Proposed by Jack Garfunkel, Flushing, New York.*

It is known that, for x, y, z > 0,

$$\sqrt{xy} + \sqrt{yz} + \sqrt{zx} \le \sqrt{3}\sqrt{xy + yz + zx}\ .$$

Prove the "other side" of this inequality, namely,

$$\sqrt{xy} + \sqrt{yz} + \sqrt{zx} \ge 3\sqrt{3}\sqrt{\frac{xyz}{x+y+z}}.$$

717. *Proposed by Russell Euler, Northwest Missouri State University, Maryville, Missouri.*

Find all positive integers n for which

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\frac{1}{k}$$

is an integer,

718. *Proposed by David Petry, Eugene, Oregon.*

Prove or find a counterexample: If a, b, c, p are integers such that $0 \le a < b < c \le 2p + 1$, then $a^p + b^p \le c^p$.

SOLUTIONS

678. [Fall 1988] *Proposed by Brian Conrad, Centereach High School, Centereach, New York.*

Find all solutions to this base ten multiplication alphametric in honor of my Soviet mathematician and theoretical physicist pen pal who also is a regular contributor to thls department:

DMITRI = P • MAVLO.

Solution by Alan Wayne, Holiday, Florida.

Because a BASIC program to solve this problem on my small computer takes more than 500 hours to run if no power surges occur, I have resorted to a "by hand" search. It took only about 50 hours, with the following five steps.

1. For P = 2 to 9, for D = 1 to 9, and for M = 1 to 9, M is the greatest integer in (10D + M)/P. This determines 32 ordered triples (P, D, M).
2. The product of P and O ends in I. This determines 44 ordered trlples (P, O, I).
3. Combining the previous results, omitting duplicated digits, we find 99 ordered quintuples (P, D, M, O, I).
4. Each of these pentuples is examined for values of V; then for values of T; and finally, if need be, for the three remaining values possible for L.
5. Two solutions result:

130780 = 4 • 32695 and 356426 = 7 • 50918.

Also solved by CHARLES ASHBACHER, *Mount Mercy College, Cedar Rapids, IA,* MARK EVANS *(partial solution), Louisville, KY,* ROBERT C. GEBHARDT, *Hopatcong, NJ,* RICHARD I. HESS, *Rancho Palos Verdes, CA,* L. J. UPTON, *Mississauga, Ontario, Canada,* LIEN VUONG, *Texas A + M University, College Station, and the* PROPOSER.

679. [Fall 1988] *Proposed by Dmitry P. Mavlo, Moscow, U.S.S.R.*

a) Prove this inequality for positive real numbers U, S, and A, dedicated to 100 years of American mathematics, as evidenced by the 100th anniversary of the American Mathematical Society:

$$\frac{U}{(1+U)(1+S)} + \frac{S}{(1+S)(1+A)} + \frac{A}{(1+A)(1+U)} \ge \frac{3USA}{(1+USA)^2},$$

with equality if and only if U = S = A = 1.

b) Which inequality, if either, is more general, the USA inequality of part (a) or the $\pi\mu\varepsilon$ inequality of Problem 642 [Spring 1987, Spring 19881:

$$(1 + \pi\mu\varepsilon)\cdot\left(\frac{1}{\pi(1+\mu)} + \frac{1}{\mu(1+\varepsilon)} + \frac{1}{\varepsilon(1+\pi)}\right) \ge 3$$

for positive numbers π, μ, and ε, with equality if and only if $\pi = \mu = \varepsilon = 1$?

Solution by the proposer.

a) We use the notation $\Sigma U = U + S + A$ and $\Sigma US = US + SA + AU$. First we prove the lemma

(1) $\qquad XU + \Sigma US \ge 6\sqrt{USA}$ with equality iff U = S = A = 1.

By the AM-GM inequality we have $U + SA \geq 2\sqrt{USA}$ and two similar inequalities, establishing the inequality of the lemma. Equality occurs when and only when $U = SA$ and similarly $S = AU$ and $A = US$, which are true if and only if $U = S = A = 1$.

Next we prove another lemma

$$(2) \qquad 2(1 - USA + U^2S^2A^2) \geq \sqrt{USA}(1 + USA)$$

with equality iff $USA = 1$.

Let $t = \sqrt{USA}$. Then inequality (2) is equivalent to this chain of inequalities:

$$2 - 2t^2 + 2t^4 \geq t(1 + t^2),$$

$$2 - t - 2t^2 - t^3 + 2t^4 \geq 0,$$

$$(t - 1)^2(2t^2 + 3t + 2) \geq 0.$$

This last inequality is true since the quadratic factor has no real roots and is therefore always positive. Furthermore, equality holds only when the first factor is zero: when $t = 1$. Hence inequality (2) is established.

Now we prove the main theorem. Multiply both sides of the proposed inequality by the nonzero expression

$$2(1 + U)(1 + S)(1 + A)(1 + USA)^2$$

to get the equivalent inequality

$$2(1 + USA)^2(XU + \Sigma US) \geq 6USA(1 + XU + \Sigma US + USA),$$

which reduces to

$$2(1 - USA + U^2S^2A^2)(XU + \Sigma US) \geq 6USA(1 + USA).$$

This inequality is seen to be just the result of multiplying the inequalities (1) and (2) of the two lemmas side for side, establishing the theorem.

b) The USA inequality is more general. (Naturally in real life the prosperity of Pi Mu Epsilon should follow from the prosperity of the country.) To prove this assertion we rewrite both inequalities in the "unified" notation: $U = x$, $S = y$, $A = z$, $\pi = x$, $\mu = z$, and $\varepsilon = y$:

$$\frac{x}{(1+x)(1+y)} + \frac{y}{(1+y)(1+z)} + \frac{z}{(1+z)(1+x)} \geq \frac{3xyz}{(1+xyz)^2}$$

and

$$(1 + xzy)\left[\frac{1}{x(1+z)} + \frac{1}{z(1+y)} + \frac{1}{y(1+x)}\right] \geq 3.$$

We must show that the latter inequality follows from the former. To that end we shall rewrite each inequality to have the same left side. We get

$$(3) \qquad \left[\frac{1+xyz}{(1+x)(1+y)(1+z)}\right]^2 \geq \frac{3xyz}{\left[\frac{x}{(1+x)(1+y)} + \frac{y}{(1+y)(1+z)} + \frac{z}{(1+z)(1+x)}\right](1+x)^2(1+y)^2(1+z)^2}$$

and

$$(4) \qquad \left[\frac{1+xyz}{(1+x)(1+y)(1+z)}\right]^2 \geq \left[\frac{3}{\left(\frac{1}{x(1+z)} + \frac{1}{z(1+y)} + \frac{1}{y(1+x)}\right)(1+x)(1+y)(1+z)}\right]^2$$

We must show that the right side of inequality (3) is greater than or equal to the right side of (4), which is equivalent to

$$(5) \qquad \left[\frac{\sqrt{xyz}}{x(1+z)} + \frac{\sqrt{xyz}}{z(1+y)} + \frac{\sqrt{xyz}}{y(1+x)}\right]^2 \geq 3\left[\frac{x}{(1+x)(1+y)} + \frac{}{} + \frac{z}{(1+z)(1+x)}\right].$$

The substitution

$$d = \frac{x}{(1+x)(1+y)}, \quad e = \frac{y}{(1+y)(1+z)}, \quad f = \frac{z}{(1+z)(1+x)}$$

changes (5) into

$$\left[\sqrt{\frac{de}{f}} + \sqrt{\frac{ef}{d}} + \sqrt{\frac{fd}{e}}\right]^2 \geq 3(d + e + f),$$

which is equivalent to the following chain of inequalities:

$$(de + ef + fd)^2 \geq 3def(d + e + f),$$

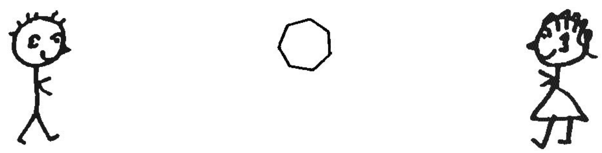
$$(d^2e^2 - 2d^2ef + f^2d^2) + (e^2f^2 - 2de^2f + d^2e^2) + (f^2d^2 - 2def^2 + e^2f^2) \geq 0,$$

$$d^2(e - f)^2 + e^2(f - d)^2 + f^2(d - e)^2 \geq 0.$$

This last inequality is obviously true for any d, e, f in the reals and hence for any positive x, y, z.

680. [Fall 1988] *Proposed by Robert C. Gebhardt, Hopatcong, New Jersey.*

A regular heptagon (seven-sided polygon) is randomly placed far from an observer. Find the probability that the observer can see four sides of the heptagon.



Solution by Richard I. Hess, Rancho Palos Verdes, California.

If any odd-sided polygon (with n sides) is placed far from the observer, then the probability that he can see more (or less) then half the sides approaches 1/2 as the distance increases. To prove this statement, consider that another observer placed diametrically opposite the first one will see the complementary number of sides (for all but a finite number of positions). As the polygon is rotated through one revolution, then, each sees $(n + 1)/2$ sides just as often as the other. Hence the probability is 1/2.

Also solved by CHARLES ASHBACHER, ***Mount Mercy College, Cedar Rapids, IA,*** WILLIAM CHAU, *Eggertsville, NY,* RICHARD DUNLAP *(2 solutions),* ***Georgia Tech, Atlanta,*** GREGORY F. MARTIN, ***University of North Florida, Jacksonville,*** PROBLEM SOLVING GROUP, ***University of Arizona, Tucson,*** and the PROPOSER.

681. [Fall 1988] ***Proposed by R. S. Luthar, University of Wisconsin Center, Janesville, Wisconsin.***

Professor E. P. B. Umbugio is in the midst of writing his thirteen-volume treatise on analytic geometry. He would like to use the following theorem in Volume 9, but is having difficulty with it. Help the poor old professor by supplying a proof for him.

For $i = 1, 2, \ldots, n$, let P_i represent the plane

$$\frac{x}{a_i} + \frac{y}{b_i} + \frac{z}{c_i} = 1 \quad \text{where} \quad 3a_ib_i + 3b_ic_i + 3c_ia_i = a_ib_ic_i.$$

Then the intersection of all the planes is nonempty.

Solution by William Chau, Eggertsville, New York.

The intersection of all the planes contains at least the point (3,3,3) since the given condition is equivalent to

$$\frac{3}{c_i} + \frac{3}{a_i} + \frac{3}{b_i} = 1.$$

It is clear that if the coefficients (3,3,3) in the given condition are replaced by the three numbers (p,q,r), then the intersection of the three planes will be the point (p,q,r). The problem can also be extended into hyperspace quite readily.

Also solved by RICHARD DUNLAP, *Georgia Tech, Atlanta,* RUSSELL EULER, ***Northwest Missouri State University, Maryville,*** RICHARD I. HESS, *Rancho Palos Verdes, CA,* DON PFAFF, ***University of Nevada, Reno,*** MIKE PINTER, *Belmont College, Nashville, TN,*

WADE H. SHERARD, ***Furman University, Greenville, SC,*** ALAN WAYNE, ***Holiday, FL,*** and the PROPOSER.

682. [Fall 1988] ***Proposed by Brian Conrad, Centereach High School, Centereach, New York.***

Find all ordered pairs of nonzero integers a and b with b prime such that

$$a^3 - b^3 = a.$$

I. ***Solution by Alan Wayne, Holiday, Florida.***

The given relation is equivalent to

$$(a - 1)a(a + 1) = b^3.$$

The left member, being the product of three consecutive integers, contains both 2 and 3 as factors. Hence 6 divides b^3, so 6 divides b, so that b cannot be a prime. Therefore there is no solution.

Dropping the requirement that b be prime, the following result is easily proved by applying Descartes' Rule of Signs to the polynomial

$$P(x) = x^3 - x - b^3$$

The product of the three consecutive integers x - 1, x, and x + 1 is the cube of an integer b if and only if

$$(a,b) \in \{(-1,0),(0,0),(1,0)\}.$$

II. ***Solution by Francis C. Leary, Saint Bonaventure University, New York.***

There are no positive integral solutions even if b is not assumed prime. The given equation is equivalent to

$$a^3 - a = b^3.$$

Since the left side is even. then so is b. Let b = 2n for some nonzero integer n. Then a must be a root of the polynomial

$$p(x) = x^3 - x - 8n^3.$$

The discriminant of this polynomial is $D = 4 - 1728n^3$, which is clearly negative if n is a nonzero integer. Thus the polynomial has exactly one real root.

If $n > 0$, then $p(2n) = -2n < 0$ and $P(2n + 1) = 4n(3n + 1) > 0$. By the intermediate value theorem, $p(x) = 0$ for some x such that $2n < x < 2n + 1$. This x is the unique real root of $p(x) = 0$ and is clearly not an integer. A similar argument holds if $n < 0$. Thus the only integral solutions are the trivial ones $(a,b) = (1,0), (0,0)$, or $(-1,0)$.

Also solved by CHARLES ASHBACHER, ***Mount Mercy College, Cedar Rapids, IA,*** SEUNG-JIN BANG, *Seoul, Korea,* FRANK P. BATTLES and LAURA L. KELLEHER *(2 solutions),* ***Massachusetts Maritime Academy, Buzzards Bay,*** JAMES F. BURKE, ***Illinois Benedictine***

College, Lisle, WILLIAM CHAU, *Eggertsville, NY,* RICHARD DUNLAP, *Georgia Tech, Atlanta,* GEORGE P. EVANOVICH, *Saint Peter's College, Jersey City, NJ,* ROBERT C. GEBHARDT, *Hopatcong, NJ,* STEPHEN I. GENDLER, *Clarion University of Pennsylvania,* RICHARD A. GIBBS, *Fort Lewis College, Durango, CO,* RICHARD I. HESS, *Rancho Palos Verdes, CA,* JUDITH P. KHAN, *James Madison High School, Brooklyn, NY,* CARL LIBIS, *Granada Hills, CA,* CHRIS LONG, *Rutgers University, New Brunswick, NJ,* OXFORD RUNNING CLUB, *University of Mississippi, University,* DON PFAFF, *University of Nevada, Reno,* MIKE PINTER, *Belmont College, Nashville, TN,* BOB PRIELIPP, *University of Wisconsin-Oshkosh,* PROBLEM SOLVING GROUP (*2 solutions), University of Arizona, Tucson,* JOHN PUTZ, *Alma College, MI,* ST. OLAF PROBLEM SOLVING CLASS, *St. Olaf College, Northfield, MN,* WADE H. SHERARD, *Furman University, Greenville, SC, and* the PROPOSER.

Two solvers asked if the problem was correctly stated. It was. Prielipp found the theorem "The product of three consecutive natural numbers cannot be a power with exponent greater than 1 of a natural number" in Sierpinski, *Elementary Theory of Numbers,* Hafner Publishing Co., New York, 1964, page 68.

"683. [Fall 1988] *Proposed by Jack Garfunkel, Flushing, New York.*

a) Given three concentric circles, construct an isosceles right triangle so that its vertices lie one on each circle.

b) Is the construction always possible?

I. *Solution by William H. Peirce, Stonington, Connecticut.*

Let the three circles be centered at the origin of the Cartesian plane and have radii 1, r, and s with $r \le s$, and let the right angle vertex C of right triangle ABC lie at the point where the circle of radius 1 crosses the x-axis. Let vertices A and B lie on the circles of radii r and s respectively. Let the sides of the triangle opposite vertices A, B, and C have lengths a, b, and c. See Figure 1. (This figure covers all cases except that where the circle on which the right angle vertex lies degenerates to a point, in which case the other two circles must coincide and the solution is clear.) Then we see that

$$A = (1 + b\cos\theta,\ b\sin\theta) \quad\text{and}\quad B = (1 - a\sin\theta,\ a\cos\theta),$$

where θ is the angle of inclination of side b. Since A and B lie on circles of radii r and s, we have

$$(1 + b\cos\theta)^2 + (b\sin\theta)^2 = r^2$$

and

$$(1 - a\sin\theta)^2 + (a\cos\theta)^2 = s^2 .$$

Since triangle ABC is isosceles, then a = b and these equations reduce to

$$2a\cos\theta = -1 - a^2 + r^2 \quad\text{and}\quad 2a\sin\theta = 1 + a^2 - s^2 .$$

Now square both sides of both equations and then add to obtain the quartic in a),

$$(1) \qquad 2a^4 - 2(r^2 + s^2)a^2 + (r^2 - 1)^2 + (s^2 - 1)^2 = 0.$$

Now triangles can be constructed for those values of r and s which yield real roots of (1), in which case those roots have the form $\pm u, \pm v$, where u and v may or may not be

equal. Thus there are at most two solution triangles and their legs are the positive real roots of (1). Considering equation (1) as a quadratic in a^2, there will be two real roots when its discriminant

$$D = -4(r^4 + s^4 - 2r^2 s^2 - 4r^2 - 4s^2 + 4) > 0.$$

Let $x = r^2$ and $y = s^2$ and graph D = 0 in this new xy-plane. We get a parabola in the first quadrant, as shown in Figure 2. The solution set for the construction problem is the region inside the parabola and above the line y = x (so that $r \le s$). That is, any r and s such that the point (r^2, s^2) lies in that region will permit the desired construction, and only those points. So the construction is not always possible. If the point lies on the parabola or if r = s = 1, there is just one solution triangle; if it is inside and not the point (1,1), then there are two distinct solutions.

When a solution exists, all required operations can be performed with ruler and compass.

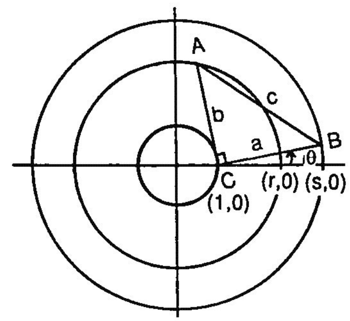


Figure 1

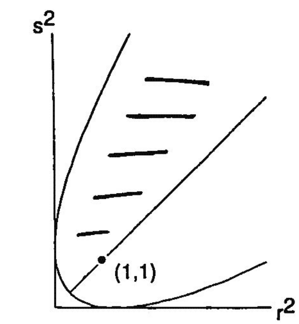
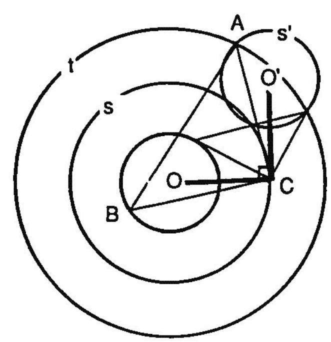


Figure 2

II. *Solution by Bro. Kenarch, Bologna, Italy.*

a) Pick the right angle vertex C on one of the three circles. Rotate the common center O and one of the other two circles s and t, say s, about C through a right angle, either clockwise or counterclockwise, to O' and s'. If circles s' and t intersect, then any such point of intersection is a vertex, say A, of the desired isosceles right triangle. The third vertex B is the preimage of A under the stated rotation. If s' and t intersect in two points, then there are two essentially distinct solutions; if one point, then one solution. The entire figure can be reflected in the line OC to produce other solution(s), which we do not consider as being distinct from the first solution(s).



b) Let the radii of the three circles be a, b, and c where

$$0 < a < b < c.$$

Then the circles s and t' of part (a) will intersect if the appropriate following condition is satisfied. If the right angle vertex lies on

circle (a), then we must have $c - b \le a\sqrt{2} \le c + b$;
circle (b), then we must have $c - a \le b\sqrt{2} \le c + a$;
circle (c), then we must have $b - a \le c\sqrt{2} \le b + a$.

These conditions can be rewritten. Thus, if the right angle vertex lies on

circle (a), then we must have $b \ge c - a\sqrt{2}$;
circle (b), then we must have $a \ge \left|c - b\sqrt{2}\right|$;
circle (c), then we must have $a + b \ge c\sqrt{2}$.

Also solved by RICHARD I. HESS, *Rancho Palos Verdes, CA.*

684. [Fall 1988] *Proposed by Dmitry S. Mavlo, Moscow, U. S. S. R.*

This problem is dedicated to Paul Erdos on his 75th birthday. Erdös and Hans Debrunner published (*El. Math.* 11(1956)20) the following theorem: Let D, E, F be points on the interiors of sides BC, CA, AB of triangle ABC. Then the area [DEF] of triangle DEF cannot be less than the smallest of the three other triangles formed:

$$[DEF] \ge \min\{[AEF], [CDE], [BFD]\}.$$

a) Prove this generalization of the Erdos-Debrunner inequality: Assuming the configuration of the Erdos-Debrunner inequality, for some fixed real number α^*, if $-\infty < \alpha \le \alpha^*$, then

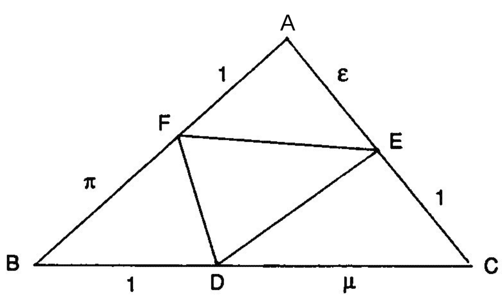
$$[DEF] \ge M^{(\alpha)}, \text{ where } M^{(\alpha)} = \left[\frac{[AEF]^{\alpha} + [CDE]^{\alpha} + [BFD]^{\alpha}}{3}\right]^{1/\alpha}$$

is the power mean of order α of the three positive areas [AEF], [CDE], and [BFD].

b) Determine the maximum value of α^* for which the inequality holds.

c) Find all cases where equality holds.

d) Prove that, for $\alpha = -1$, the inequality of part (a) is equivalent to the $\pi\mu\varepsilon$ inequality referred to in Problem 679(b) above.



Solution 5 the proposer.

Let points D, E, F divide sides CB, AC, BA in the ratios μ, ε, π, respectively. Then

$$[AEF] = \frac{\varepsilon}{(\varepsilon+1)(\pi+1)}[ABC], \quad [BFD] = \frac{\pi}{(\pi+1)(\mu+1)}[ABC],$$

$$[CDE] = \frac{\mu}{(\mu+1)(\varepsilon+1)}[ABC],$$

and hence

$$[DEF] = \left[1 - \frac{\varepsilon}{(\varepsilon+1)(\pi+1)} - \frac{\pi}{(\pi+1)(\mu+1)} - \frac{\mu}{(\mu+1)(\varepsilon+1)}\right][ABC]$$

$$= \frac{(\pi\mu\varepsilon+1)}{(\varepsilon+1)(\pi+1)(\mu+1)}[ABC].$$

In the inequality of part (a) we let $k = -\alpha$ to get

$$[DEF]^k \ge \frac{3}{\dfrac{1}{[AEF]^k} + \dfrac{1}{[CDE]^k} + \dfrac{1}{[BFD]^k}},$$

which becomes, when the above substitutions are made,

$$\frac{(\pi\mu\varepsilon+1)^k}{[(\varepsilon+1)(\pi+1)(\mu+1)]^k} \ge \frac{3}{\left[\dfrac{(\varepsilon+1)(\pi+1)}{\varepsilon}\right]^k + \left[\dfrac{(\pi+1)(\mu+1)}{\pi}\right]^k + \left[\dfrac{(\mu+1)(\varepsilon+1)}{\mu}\right]^k}$$

and finally

$$(1) \qquad \left[\frac{1}{\varepsilon(\mu+1)}\right]^k + \left[\frac{1}{\pi(\varepsilon+1)}\right]^k + \left[\frac{1}{\mu(\pi+1)}\right]^k \ge \frac{3}{(1+\pi\mu\varepsilon)^k}.$$

Thus we have proved the equivalence of inequality (1) for all k such that $k^* \le k < \infty$ for some k^* and the inequality of part (a).

We have also proved part (d), for if $k = 1$, inequality (1) is equivalent to the $\pi\mu\varepsilon$ inequality. Since the $\pi\mu\varepsilon$ inequality is true, we have also proved the inequality of

part (a) for $a = -1$.
Now define

$$F(\pi,\mu,\varepsilon) = \left[\frac{1}{\varepsilon(\mu + 1)}\right]^k + \left[\frac{1}{\pi(\varepsilon + 1)}\right]^k + \left[\frac{1}{\mu(\pi + 1)}\right]^k - \frac{3}{(1 + \pi\mu\varepsilon)^k}.$$

It is straightforward but tedious to set the three first partial derivatives $\partial F/\partial\pi$, $\partial F/\partial\mu$, $\partial F/\partial\varepsilon$ equal to zero and solve simultaneously to get that $\pi = \mu = \varepsilon = 1$. Next we form all second order partial derivatives and evaluate them at $(1,1,1)$ to get

$$F_{11} = F_{22} = F_{33} = \frac{1}{2}k(k + 1)$$
$$F_{12} = F_{23} = F_{31} = F_{21} = F_{32} = F_{13} = \frac{1}{4}k(3 - k).$$

By the Sylvester theorem the function F will have the point $(1,1,1)$ as a minimum if and only if the following three inequalities hold at the point $(1,1,1)$:

$$A_1 = F_{11} = \frac{1}{2}k(k + 1) > 0, \qquad A_2 = \begin{vmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{vmatrix} > 0,$$

and

$$A_3 = \begin{vmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{vmatrix} > 0.$$

To that end we calculate that

$$\frac{1}{2}k(k + 1) > 0, \quad \frac{3}{16}k^2(k - \frac{1}{3})(k + 5) > 0, \quad \text{and} \quad \frac{9}{8}k^3(k - \frac{1}{3})^2 > 0,$$

which are all true if and only if $k > 113$. That is, for all positive π, μ, and ε and $k \geq 113$, we have

$$F(\pi,\mu,\varepsilon) \geq F(1,1,1) = 0.$$

Since $k = -a$, we have shown that the original inequality of part (a) holds for $a \leq -113$. That is, we have proved part (a) and also we have shown that $a' = -113$ is the value that satisfies part (b). Additionally, we have seen that equality holds if and only if $\pi = \mu = e = 1$, that is, when points D, E, F are the midpoints of the sides of triangle ABC.

Editorial note. The proposer's details of the work summarized in the last two paragraphs will be furnished by the problems editor upon request.

685. [Fall 1988] *Troposed by R. S. Luthar, University of 'Wisconsin Center, Janesville, 'Wisconsin.*

In any triangle ABC with $C < 45°$ and given any other angle D with $0° < D < 45°$, prove that

$$b\cos D - c\cos(A - D) < a.$$

Solution By Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, 'Wisconsin.

More generally, we shall show that if ABC is a triangle and D is any angle with $0° \leq D \leq 180°$, then $b\cos D - c\cos A\cos D < a$.

Since $0° < C < 180°$ and $0° \leq D \leq 180°$, then $-1 < \cos C\cos D < 1$. It follows that $a\cos C\cos D < a$. Hence

$$(a\cos C + c\cos A)\cos D - c\cos A\cos D < a,$$

making $b\cos D - c\cos A\cos D$ cabecause $b = a\cos C + c\cos A$.

Also solved by SEUNG-JIN BANG, *Seoul, Korea,* WILLIAM CHAU, *Eggertsville, NY,* JACK GARFUNKEL, *Flushing, NY,* RICHARD I. HESS, *Rancho Tolos Verdes, CA,* RALPH E. KING, *St. Bonaventure University, NY, and the* PROPOSER.

686. [Fall 1988] *Troposed By Murray S. Klamkin, University of Alberta, 'Edmonton, Alberta, Canada.*

Determine the matrix $[A^3 - A^2 + I]^{-1}$ where A is an n by n matrix such that $A^5 + A = 5nI$ and I is the identity matrix.

Solution By the proposer.

The number 5n can be replaced by any number except -1, say $m - 1$. Then

$$mI = A^5 + A + I = [A^2 + A + I][A^3 - A^2 + I],$$

so

$$[A^3 - A^2 + I]^{-1} = [A^2 + A + I]\ An.$$

For the stated problem, then, we have that

$$[A^3 - A^2 + I]^{-1} = [A^2 + A + I]\ /(5n + 1).$$

Also solved by SEUNG-JIN BANG, *Seoul, Korea,* WILLIAM CHAU, *Eggertsville, NY,* JOHN CORTESE, *Reading, MA,* RICHARD DUNLAP, *Georgia Tech, Atlanta,* RICHARD A. GIBBS, *Fort Lewis College, Durango, CO,* RICHARD I. HESS, *Rancho Tolos Denies, CA,* CHRIS LONG, *Rutgers University, New Brunswick, NJ,* MASSACHUSETTS GAMMA, *Bridgewater State College,* DON PFAFF, *University of Nevada, Reno, and* BOB PRIELIPP, *University of Wisconsin-Oshkosh.*

687. [Fall 1988] *'Proposed by 'Basil Rennie, Burnside, South Australia.*

For positive reals x and y, prove the "quaint little inequality,"

$$4xy \le (x + y)(xy + 1).$$

I. *Solution by Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin.*
The required inequality is equivalent to

$$\frac{x^2y + xy^2 + x + y}{4} \ge xy = \sqrt[4]{x^4y^4}$$

which follows immediately from the arithmetic mean-geometric mean inequality with n = 4. Equality holds if and only if

$$x^2y = xy^2 = x = y, \quad \text{that is,} \quad x = y = 1.$$

II. *Solution by George P. Evanovich, Saint Peter's College, Jersey City, New Jersey.*
Let a, b, x, y be positive real numbers. By the AM-GM inequality,

$$ab + xy \ge 2\sqrt{abxy} \quad \text{and} \quad ax + by \ge 2\sqrt{axby}.$$

Multiply together the two inequalities to get the more general inequality

$$(ab + xy)(ax + by) \ge 4abxy.$$

Now set a = b = 1 to get the desired inequality.

III. *Solution by Seung-Jin Bang, Seoul, Korea.*
We have

$$0 \le x(y - 1)^2 + y(x - 1)^2$$

$$= x(y^2 - 2y + 1) + y(x^2 - 2x + 1)$$

$$= (x + y)(xy + 1) - 4xy .$$

Equality holds for x = y = 1.

IV. *Solution by St. Olaf Problem Solving Class, St. Olaf College, Northfield, Minnesota.*
Since x and y are positive, then $x + 1/x \ge 2$ and $y + 1/y \ge 2$. Consequently,

$$x + \frac{1}{x} + y + \frac{1}{y} \ge 4,$$

so then

$$x^2y + y + xy^2 + x = (x + y)(xy + 1) \ge 4xy.$$

Also solved by JOHN T. ANNULIS, *University of Arkansas-Monticello,* FRANK P. BATTLES, *Massachusetts Maritime Academy, Buzzards Bay,* WILLIAM CHAU, *Eggertsville, NY,* DAVID DEL SESTO, *North Scituate, RI,* RICHARD DUNLAP, *Georgia Tech, Atlanta,* RUSSELL EULER, *Northwest Missouri State University, Maryville,* JACK GARFUNKEL, *Flushing, NY,* ROBERT C. GEBHARDT, *Hopatcong, NJ,* RICHARD A. GIBBS, *Fort Lewis College, Durango, CO,* RICHARD I. HESS, *Rancho Palos Verdes, CA,* DAVID INY, *Westinghouse Electric Corporation, Baltimore, MD,* JUDITH P. KHAN, *James Madison High School, Brooklyn, NY,* RALPH E. KING, *St. Bonaventure University, NY,* CARL LIBIS, *Granada Hills, CA,* CHRIS LONG, *Rutgers University, New Brunswick, NJ,* W. MOSER, *McGill University, Montreal, Canada,* YOSHINOBU MURAYOSHI, *Portland, OR,* DON PFAFF, *University of Nevada, Reno,* MIKE PINTER, *Belmont College, Nashville, TN,* PROBLEM SOLVING GROUP (2 *solutions), University of Arizona, Tucson,* JOHN PUTZ, *Alma College, MI,* ALAN WAYNE, *Holiday, FL, and the* PROPOSER.

688. [Fall 1988] *Proposed by Willie Yong, Singapore, Republic of Singapore.*
A row of n chairs is to be occupied by n boys and girls taken from a group of more than n boys and more than n girls. If the boys do not want to sit next to one another, in how many ways can the children occupy the chairs? (This problem is taken from the *Malaysian Math. Bulletin.*)

Solution by John Putz, Alma College, Alma, Michigan.
Let f(n) denote the number of ways of seating n children. Assuming that the first n − 1 chairs have been filled satisfactorily, the nth chair can certainly be filled by a girl. So the number of arrangements in which a girl fills the nth chair is equal to f(n − 1), the number of ways the first n − 1 chairs can be filled. The nth chair can be filled with a boy only if the (n − 1)st chair has been filled with a girl, which can be done in f(n − 2) ways. Therefore

$$f(n) = f(n - 1) + f(n - 2),$$

a Fibonacci sequence! Since f(1) = 2 and f(2) = 3, specifically bg, gg, gb, then we have that

$$f(1) = 2, \quad f(2) = 3, \quad \text{and} \quad f(n) = f(n - 1) + f(n - 2)$$

for n > 2. That is, f(n) is the (n + 2)nd Fibonacci number F_n, where,

$$F_1 = F_2 = 1 \quad \text{and} \quad F_n = F_{n+1} + F_{n+2} \quad \text{for } n > 2.$$

Also solved by WILLIAM CHAU, *Eggertsville, NY,* RICHARD DUNLAP, *Georgia Tech, Atlanta,* RICHARD A. GIBBS, *Fort Lewis College, Durango, CO,* RICHARD I. HESS, *Rancho Palos Verdes, CA,* DAVID INY, *Westinghouse Electric Corporation, Baltimore, MD,* PROBLEM SOLVING GROUP, *University of Arizona, Tucson, and the* PROPOSER.

Gibbs commented that this problem is a "fairly well-known result," citing a problem sheet he used several years ago. Hess asked about the solution if the boys and girls are distinguishable. Indeed, Dunlap provided a solution for this "more difficult problem."

*689. [Fall 1988] *Proposed by Willie Yong, Singapore, Republic of Singapore.*

Show that for any three infinite sequences of natural numbers

$$a_1, a_2, a_3, \ldots, \qquad b_1, b_2, b_3, \ldots, \qquad c_1, c_2, c_3, \ldots$$

there can be found numbers p and q such that $a_p > a_q$, $b_p > b_q$, and $c_p > c_q$.

Solution 5 Chris Long, Rutgers University, New Brunswick, New Jersey.

We prove the stronger result: If

$$\{x_{11}, x_{12}, \dots\}, \{x_{21}, x_{22}, \dots\}, \dots, \{x_{n1}, x_{n2}, \dots\}$$

are infinite sequences of natural numbers, then there exist infinitely many pairs of numbers p, q with $p < q$ such that

$$x_{pk} \le x_{qk} \quad \text{for} \quad 1 \le k \le n.$$

We prove the theorem by mathematical induction. For $n = 1$, let

$$p = \inf\{t \mid x_{1t} = \inf\{x_{11}, x_{12}, \dots\}\}.$$

Then clearly p, q is such a pair of numbers for all $q > p$.

Assume that the statement is true for $1, 2, \dots, n - 1$.
Define

$$\delta(1) = \inf\{t \mid x_{nt} = \inf\{x_{11}, x_{12}, \dots\}\},$$

and recursively define

$$\delta(m) = \inf\{t \mid x_{nt} = \inf\{x_{n(\delta(m-1)+1)}, x_{n(\delta(m-1)+2)}, \dots\}\}.$$

Consider the subsequences

$$\{x_{1\delta(1)}, x_{1\delta(2)}, \dots\}, \dots, \{x_{(n-1)\delta(1)}, x_{(n-1)\delta(2)}, \dots\}.$$

By the inductive assumption there are infinitely many pairs of numbers $\delta(p)$, $\delta(q)$ with $\delta(p) < \delta(q)$ such that

$$x_{k\delta(p)} \le x_{k\delta(q)} \quad \text{for} \quad 1 \le k \le n-1$$

We finish the inductive step by noting that the sequence

$$\{x_{n\delta(1)}, x_{n\delta(2)}, \dots\}$$

is nondecreasing by construction, so we also have that

$$x_{k\delta(p)} \le x_{k\delta(q)}.$$

690. [Fall **1988**] *Proposed by David Iny, Rensselaer Polytechnic Institute, Troy, New York.*

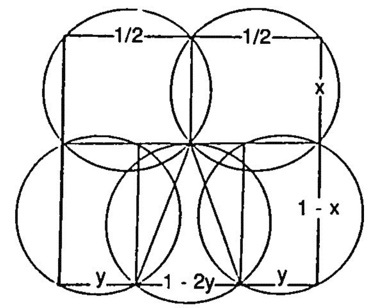
A unit square is covered by five circles of equal radius. Find the minimum necessary radius. (See Problem 507, Fall 1982).

Solution by the proposer.

We show that five circles of radius $r = 0.3261606$ will cover the square. Consider the figure. Let the diagonals of each of the four corner rectangles be $2r$ and let the circumradius of the isosceles triangle in the remaining rectangle be r. Then we must have, using the notation of the figure,

$$x^2 = 4r^2 - \frac{1}{4}, \quad y^2 = 2x - \frac{3}{4}, \quad \text{and} \quad r = \frac{1}{2}(1 - x) + \frac{(1 - 2y)^2}{8(1 - x)}$$

The last equation is from the isosceles triangle. By calculator we find that $r \approx 0.3261606$, $x = 0.4189546$, and $y = 0.2964947$. The sketch in the figure shows that these five circles cover the square.



Now suppose a solution where one of the five circles lies inside the square and each of the other circles covers a vertex and each edge has a point covered by two circles. These four edge points and the four vertices partition the perimeter into eight segments whose lengths total 4 units. The sum of the squares of these segments is not less than $8 \cdot (1/2)^2$ since the midpoint of an edge minimizes the sum of the squares on it. Hence at least one circle covers a segment (hypotenuse of a right triangle) of length at least $1/\sqrt{2}$. The radius of that circle is at least half that value, namely 0.35.

The only other possibility is for the fifth circle to cover a portion of one of the sides. This is the solution we have given above.

Solutions were also submitted by RICHARD I. HESS, *Rancho Palos Verdes, CA, and* LIEN VUONG, *Texas A & M University, College Station.* Both solutions assumed one circle lying inside the square to produce a radius of $\sqrt{2}/4 = 0.353553$. The proposer, who is now at Westinghouse Electric Corp., Baltimore, MD, also gave a solution for six covering circles, proving that $\sqrt{65}/16$ is the minimum radius.

CORRECTIONS

Bob Prielipp pointed out a misplaced exponent in the solution to Problem 674 on page 694 of the Spring 1989 issue. The line

$$(-a_1/a_0)/n^n = ((-1)a_n/a_0)^{1/n}$$

should read

$$(-a_1/a_0)/n = ((-1)^n a_n/a_0)^{1/n}.$$

In the Spring **1988** issue the solution to Problem **642** on page **539** has an error. William Chau discovered that the multiplication factor given there should not contain **$(1 + \pi\mu\varepsilon)$;** it should be only

$$\pi\mu\varepsilon(1 + \pi)(1 + \mu)(1 + \varepsilon).$$

IN MEMORIAM

Charles W. Trigg

Born February **7, 1898,** he started his career as a chemist and during World War **I** invented an instant coffee soluble in cold water. In the next **10** years he published nearly **200** articles, notes and editorials on coffee, tea and spices. He began teaching chemistry in **1927.** From **1938-43** he taught mathematics and physics at Los Angeles City College.

From **1943-46,** serving to Lt. Commander in the U. S. Naval Reserve, Charles earned his wings as a navigator and taught celestial navigation.

In **1946** he returned to Los Angeles City College as Coordinator of Instruction, was promoted to full professor, and in **1955** became Dean of Instruction until his retirement in **1963.**

In the ensuing **26** years he proposed hundreds of problems, submitted thousands of solutions, and wrote more than **500** articles, book reviews, and other items in mathematics. The LACC Engineering Department presented him with a diploma awarding him the degree of P.D.P.F. (Polyhedra Doctor in Paper Folding) for his careful cardboard-and-rubber-band geometric models, many of which hung in **his** office at his San **Diego** retirement home.

The late **Léo Sauvé,** editor of Crux **Mathematicorum,** conveyed upon him the title of "prince of digit delvers," but later demoted him to "count of digit delvers." That still left him with a D.D., Charles said.

Humor enlivens any serious study and Charles was a master at mathematical humor. Several of the editor's pseudonyms used in this department were suggested by Trigg, including S. E. Ducer, M. T. Kopf, Pauvre Fish, Bro. **Kenarch,** and Titus **Canby.** Nathan **Altshiller** Court commended him for endowing his contributions with "a quality which Is rare, namely wit." The dedication of Howard Eves' **1988** book, Return to Mathematical Circles reads, "To Charles W. Trigg, the wittiest and cleverest of us all."

Charles W. Trigg died June **28, 1989.** He was a delightful mathematician and problemist and a dear friend. We dedicate to his memory this issue of the Problem Department, which in his honor contains two extra of his digit-delving proposals, problems **704** and **705**

Editor's Note

The Pi Mu Epsilon Journal ***was founded in 1949 and is dedicated to undergraduate and beginning graduate students interested in mathematics. Submitted articles, announcements and contributions to the Puzzle Section and Problem Department of the Journal should be directed toward this group.***

Undergraduate and beginning graduate students are urged to submit papers to the Journal for consideration and possible publication. Student papers are given top priority. Expository articles by professionals in all areas of mathematics are especially welcome. A copy of the Guidelines for Referees ***follows this note.***

Each year, the National Student Paper Competition awards prizes of $200, $100, and $50, provided that at least five student papers have been submitted to the Editor. All students who have not yet received a Master's Degree, or higher, are eligible for these awards. Awards for 1987-1988 are announced on the first page of this issue.

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2. most readers of the ***Pi Mu Epsilon Journal*** are undergraduates; the paper should be directed to them
3. with rare exceptions, the paper should be of general interest
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5. expository papers are actively encouraged
6. the ***Journal*** does not necessarily expect the same quality of exposition from an undergraduate author as it does from more experienced authors
7. stylistic comments and changes are welcomed and encouraged
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The Editor and the **author(s)** appreciate you help. Please be frank with your comments and suggestions.

If for some reason you find that your schedule does not permit you to referee the enclosed paper within four to six weeks, please return it to the Editor.

Joseph D. E. Konhauser
Editor, ***Pi Mu Epsilon Journal***

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