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THE RICHARD V. ANDREE AWARDS

Richard V. Andree, Professor Emeritus of the University of Oklahoma, died on May 8, 1987, at the age of 67.

Professor Andree was a Past-President of Pi Mu Epsilon. He had also served the society as Secretary-Treasurer General and as Editor of the Pi Mu Epsilon Journal.

The Society Council has designated the prizes in the National Student Paper Competition as Richard V. Andree Awards.

First prize winners for 1987-1988 are James E. Georges, California Polytechnic State University, and Annette M. Matthews, Portland State University, for their paper "Maximal Polygons for Equitransitive Periodic Tilings," which appeared in the Fall, 1988, issue of the Journal. The paper was done while the authors were participants in the Research Experiences for Undergraduates program at Oregon State University. James and Annette will share the $200 prize.

Second prize winner is Melanie L. Butt, Middle Tennessee State University, for her paper "Automorphism Groups of Hasse Subgroup Diagrams for Groups of Low Order," which appears in this issue of the Journal. Melanie, who is currently a senior, wrote the paper while she was a junior at Middle Tennessee State University. Melanie presented her paper at the National Meeting of Pi Mu Epsilon at Boulder in August, 1989. Melanie will receive $100.

Third prize winner is Robert A. Coury, University of Washington, for his paper "A Continued Fraction Approach for Factoring Large Numbers," which appears in this issue of the Journal. Robert is a senior at the University of Washington. Robert's paper is a result of research for a talk given at the national meeting in Providence in 1988. Robert will receive $50.

Congratulations James, Annette, Melanie, and Robert.

Two other student-written papers appear in this issue. One is "Energy-Conscious Behavior in Rural Areas: How to Approach a Traffic Light" by Craig Osborn, written while Craig was a senior at Carleton College. The paper is based on a problem presented by Richard Poss of St. Norbert College at the 1987 Annual Pi Mu Epsilon Conference.

The other is Mark Ontkush's "A Closed Formula for Linear Indeterminate Equations in Two Variables." Mark wrote the paper while a senior at the State University of New York at Buffalo. He encountered the formula in a course in discrete mathematics.
AUTOMORPHISM GROUPS OF HASSE SUBGROUP DIAGRAMS
FOR GROUPS OF LOW ORDER

By Melanie L. Butt
Middle Tennessee State University

We begin by reviewing basic group definitions and propositions. A group is a set with a binary operator which is associative, has an identity, and each element has an inverse. An abelian, or commutative, group is one whose operation is also commutative. A subset of a group which also forms a group is called a subgroup.

Proposition 1. If G is a finite group with operation *, and H is a nonempty subset of G, then (H, *) is a subgroup of (G, *) whenever the closure property holds.

More specifically we are interested in Hasse subgroup diagrams. First recall that a poset is a nonempty set P with a relation ≤ on P which is reflexive, antisymmetric, and transitive. A lattice (L, ≤) is a poset with the property that ∀ x, y ∈ P, {x, y} has a least upper bound and a greatest lower bound.

Proposition 2. Let G be a group. Then (L(G), ⊆) is a lattice where

\[ L(G) = \{ H | H \text{ is a subgroup of } G \} \]

and ⊆ is subset inclusion. The greatest lower bound of subgroups H and K is H \cap K. The least upper bound of subgroups H and K is the smallest subgroup of G containing H and K.

We represent lattices of subgroups with subset inclusion by diagrams called Hasse subgroup diagrams. Each subgroup is depicted with a point. Lines are drawn to connect these subgroups according to the following rule: Suppose A and B are subgroups with property A ⊆ B. Then we connect the points with a line and we position B above A. The identity subgroup will be at the bottom of the diagram. We define this subgroup to have height or rank of 0. For subgroups H and K,

\[ \text{rank}(H) = \text{rank}(K) + 1 \]

whenever H is directly above K.

Now we are interested in automorphisms of these diagrams. An automorphism of a Hasse subgroup diagram, H, is a bijection from H to H that preserves or reverses order. Order preserving automorphisms are those with the property that given two elements, x and y, if x ≤ y, then f(x) ≤ f(y). An automorphism is order reversing when x ≤ y implies f(x) ≥ f(y). The identity automorphism is the bijection \[ f: H \rightarrow H \] defined by f(x) = x. The reverse automorphism, if it exists, is the automorphism that turns the Hasse subgroup diagram upside down.

Proposition 3. The set of automorphisms of the Hasse subgroup diagram H forms a group under function composition.

Our goal is to calculate the automorphism groups of the Hasse subgroup diagrams for the groups of low order which are listed in the first column of Table 1.

<table>
<thead>
<tr>
<th>Group</th>
<th>Automorphism Group of Hasse Subgroup Diagram</th>
</tr>
</thead>
<tbody>
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<td>C_1</td>
<td>C_1</td>
</tr>
<tr>
<td>C_2</td>
<td>C_2</td>
</tr>
<tr>
<td>C_3</td>
<td>C_2</td>
</tr>
<tr>
<td>C_4</td>
<td>C_2</td>
</tr>
<tr>
<td>C_5</td>
<td>C_2 × C_2</td>
</tr>
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<td>C_6</td>
<td>C_2</td>
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<tr>
<td>D_6</td>
<td>S_3 × C_2</td>
</tr>
<tr>
<td>C_2 × C_2</td>
<td></td>
</tr>
<tr>
<td>C_4 × C_2</td>
<td></td>
</tr>
<tr>
<td>O</td>
<td>S_3</td>
</tr>
</tbody>
</table>

First we discuss the cyclic groups. The cyclic group with 1 elements, C_1, is the set of the first whole numbers with addition modulo i. Clearly, the automorphism group of the Hasse subgroup diagram of C_1 is C_1 since the only subgroup of C_1 is C_1 itself.

Theorem 1. The automorphism group of the Hasse subgroup diagram of C_i where \( i \in \{2,3,4,5,7,8\} \) is C_2.

Proof. First consider the subgroups we obtain by examining the group tables. Then we find the Hasse subgroup diagrams which are

\[
\begin{align*}
C_2 & \quad \{0,1\} \quad \{0\} \\
C_3 & \quad \{0,1,2\} \quad \{0\} \\
C_4 & \quad \{0,1,2,3\} \quad \{0\} \\
C_5 & \quad \{0,1,2,3,4\} \quad \{0\} \\
C_7 & \quad \{0,1,2,3,4,5,6\} \quad \{0\} \\
C_8 & \quad \{0,1,2,3,4,5,6,7\} \quad \{0,4\} \\
D_4 & \quad \{0\} \\
D_6 & \quad \{0\}
\end{align*}
\]

Because each Hasse subgroup diagram contains only one subgroup at each rank and ≤ is transitive, it follows that the only automorphisms are the identity and the reverse automorphisms. Therefore C_2 is the automorphism group since it is the only group of order two.
Theorem 2. The automorphism group of the Hasse subgroup diagram of $C_6$ is $C_2 \times C_2$.

Proof. The Hasse subgroup diagram of $C_6$ is

Note that the subgroups are labeled by numbers which will be used to refer to the subgroups. By looking at the diagram it is clear that the identity and reverse automorphisms are automorphisms of the Hasse subgroup diagram of $C_6$. Switching subgroups 2 and 3 should also be an automorphism and the function switching subgroups 2 and 3 is order preserving. Let us verify the function switching 2 and 3 is an order preserving automorphism.

Define $f: H \rightarrow H$ by $f(i) = \begin{cases} i & \text{if } i \neq 2,3 \\ 2 & \text{if } i = 3 \\ 3 & \text{if } i = 2 \end{cases}$.

If $i < j$, then $f(i) < f(j) \forall i, j$ is verified by checking

- $1 < 2$ and $f(1) = 1 < 3 = f(2)$,
- $1 < 3$ and $f(1) = 1 < 2 = f(3)$,
- $2 < 4$ and $f(2) = 3 < 4 = f(4)$,
- $3 < 4$ and $f(3) = 2 < 4 = f(4)$.

Now we obtain a fourth automorphism by turning this one upside down. Thus the automorphism groups contain four elements. There are two groups of order four. Since no automorphism has order four, we conclude the automorphism groups is $C_2 \times C_2$.

Another group of low order is $D_4$, the dihedral group with eight elements. The elements of $D_4$ can be thought of as the symmetries of a square. More precisely,

$$D_4 = \langle x, y \mid x^4 = 1, y^2 = 1, (xy)^2 = 1 \rangle;$$

that is, the group generated by two elements, $x$ and $y$, with one element, $x$, of order 4 and the other, $y$, of order 2 which produce the identity when the two elements are multiplied and squared.

Theorem 3. The automorphism group of the Hasse subgroup diagram of $D_4$ is $D_4$.

Proof. First compute the subgroups by inspecting the group table of $D_4$. We obtain the Hasse subgroup diagram

Clearly, the identity will be an automorphism and there will be no reverse automorphism. Automorphisms are obtained by switching the pairs $(2,3)$ or $(5,6)$ or both of them together. Automorphisms are obtained also by switching 7 and 9 along with the pairs $(2,3)$ and $(5,6)$. All of the automorphisms are shown below and will be referred to by their labels.
We provide the details for checking one of the above. The others are similar. Define TL: H → H by

\[ TL(i) = \begin{cases} 
1 & \text{if } i = 1, 4, 8, 10 \\
i + 2 & \text{if } i = 3, 7 \\
i - 3 & \text{if } i = 5, 6 \\
6 & \text{if } i = 2 \\
7 & \text{if } i = 9 
\end{cases} \]

Then we calculate the following:

\[
\begin{align*}
1 < 2 \text{ and } f(1) &= 1 < 6 = f(2), \\
1 < 3 \text{ and } f(1) &= 1 < 5 = f(3), \\
1 < 5 \text{ and } f(1) &= 1 < 2 = f(5), \\
1 < 6 \text{ and } f(1) &= 1 < 3 = f(6), \\
2 < 7 \text{ and } f(2) &= 6 < 9 = f(7), \\
3 < 7 \text{ and } f(3) &= 5 < 9 = f(7), \\
5 < 9 \text{ and } f(5) &= 2 < 7 = f(9), \\
6 < 9 \text{ and } f(6) &= 3 < 7 = f(9), \\
7 < 10 \text{ and } f(7) &= 9 < 10 = f(10), \\
9 < 10 \text{ and } f(9) &= 7 < 10 = f(10).
\end{align*}
\]

To prove the automorphism group is \( D_4 \) we must prove \((xy)^2 = 1\) where \( x \) is an element of order 4 and \( y \) is an element of order 2.

Let \( x = TL \) and \( y = R \). Then \( ((TL)(R))^2 = (TRL)^2 = 1 \). Therefore the automorphism group is \( D_4 \).

The next group is \( S_3 \) where \( S_n \) is the symmetric group on \( n \) objects. \( S_3 \) may be represented as the six symmetries of an equilateral triangle.

**Theorem 4.** The automorphism group of the Hasse subgroup diagram of \( S_3 \) is \( S_4 \times C_2 \).

**Proof.** We compute the subgroups and the Hasse subgroup diagram of \( S_3 \).

\[
\begin{align*}
&\text{(Δ, Δ, Δ, Δ, Δ, Δ, Δ)} \\
&\text{(Δ, Δ)} \\
&\text{(Δ, Δ)} \\
&\text{(Δ, Δ)} \\
&\text{(Δ, Δ, Δ, Δ)} \\
&\text{(Δ)}
\end{align*}
\]

Using the same steps as in the previous proofs, we find there are 24 order preserving automorphisms. There are also 24 order reversing automorphisms. The order preserving automorphisms form the group \( S_4 \) since the four rank 1 subgroups can all be permuted. The reverse automorphism generates the group \( C_2 \). When the reverse automorphism is included with the order preserving automorphisms, 24 new automorphisms are obtained, all order reversing. These also form the group \( S_4 \). Thus the combined automorphism group is \( S_4 \times C_2 \).

**Theorem 5.** The automorphism group of the Hasse subgroup diagram of \( C_2 \times C_2 \) is \( S_3 \times C_2 \).

**Proof.** Consider the subgroups of \( C_2 \times C_2 \); then construct the Hasse subgroup diagram as shown.

\[
\begin{align*}
&\{(0,0), (0,1), (1,0), (1,1)\} \\
&\{(0,0), (0,1)\} \\
&\{(0,0), (1,0)\} \\
&\{(0,0)\}
\end{align*}
\]

By using the same reasoning as in the proof for \( S_3 \), we find the automorphism group is \( S_3 \times C_2 \).

**Theorem 6.** The automorphism group of the Hasse subgroup diagram of \( C_4 \times C_2 \) is \( D_4 \).

**Proof.** Find the subgroups and form the Hasse subgroup diagram for \( C_4 \times C_2 \) as shown.

\[
\begin{align*}
&\{(0,0), (1,0), (2,0), (3,0)\} \\
&\{(0,0), (1,1), (2,0), (3,1)\} \\
&\{(0,0), (2,0)\} \\
&\{(0,0), (1,0)\}
\end{align*}
\]

We find there are 8 automorphisms. By using the definition

\[
D_4 = \left\{ x, y \mid x^4 = 1, y^2 = 1, (xy)^2 = 1 \right\}
\]

we check the automorphisms using

\[
\begin{align*}
&x = 1 \rightarrow 4 \rightarrow 3 \rightarrow 2 \\
y = 2 \rightarrow 3 \rightarrow 4 \rightarrow 5
\end{align*}
\]

We use the same steps as in the previous proofs, we find there are 24 order preserving automorphisms. There are also 24 order reversing automorphisms. The order preserving automorphisms form the group \( S_4 \) since the four rank 1 subgroups can all be permuted. The reverse automorphism generates the group \( C_2 \). When the reverse automorphism is included with the order preserving automorphisms, 24 new automorphisms are obtained, all order reversing. These also form the group \( S_4 \). Thus the combined automorphism group is \( S_4 \times C_2 \).
Then

\[(xy)^2 = \begin{pmatrix}
    8 & 7 & 6 & 5 \\
    7 & 6 & 5 & 4 \\
    6 & 5 & 4 & 3 \\
    5 & 4 & 3 & 2
\end{pmatrix} = \begin{pmatrix}
    5 & 6 & 7 & 8 \\
    6 & 7 & 8 & 1 \\
    7 & 8 & 1 & 2 \\
    8 & 1 & 2 & 3
\end{pmatrix} = 1.
\]

Therefore the automorphism group is \(D_4\).

There are two other groups of order less than or equal to eight. One is the group \(C_2 \times C_2 \times C_2\). The other is the quaternion group, \(Q\), which contains the elements \([\pm 1, \pm i, \pm j, \pm k]\)

and where

\[ij = k = -ji, \quad k = i = -ki, \quad \text{and} \quad ki = j = -ik.\]

Theorem 7. The automorphism group of the Hasse subgroup diagram of \(Q\) is \(S_3\).

Proof. After finding the subgroups of \(Q\) and the Hasse subgroup diagram

\[
\begin{align*}
    \{1, -1, i, -i\} & \quad \{1, -1, j, -j\} & \quad \{1, -1, k, -k\} \\
    1 & \quad 1 & \quad 1 \\
    -1 & \quad -1 & \quad -1
\end{align*}
\]

we find there are 6 order preserving automorphisms and clearly no order reversing automorphisms. The only groups of order 6 are \(C_6\) and \(S_3\). Checking group tables, we find the automorphism group of the Hasse subgroup diagram of \(Q\) is \(S_3\). This result can also be obtained by observing that the automorphisms permute the 3 rank 2 subgroups in all possible ways.

My work with automorphism groups was done by inspection of the Hasse subgroup diagrams. Even though some generalizations are easy to state, I do not yet know the theory needed to prove generalizations because I have not yet taken a course in abstract algebra. This also presents a problem when working with \(C_2 \times C_2 \times C_2\) since its Hasse subgroup diagram is more complex.

REFERENCES


Thus, one of \( z + a \) and \( z - a \) must be divisible by \( p \) and the other by \( q \). The factor \( p \) (or \( q \)) can be determined by using the Euclidean algorithm to find the greatest common divisor of \( z + a \) and \( N \) (or \( z - a \) and \( N \)). This method also works if \( N \) has more than two prime factors; one simply reapply the method to the composite factor.

In trying to determine a nontrivial solution to Legendre's congruence, we first find the infinite simple continued fraction expansion of \( \sqrt{N} \). With the sequences \( h_n, k_n \), and \( s_n \) defined as usual for this continued fraction expansion, we have equation (1) which is valid for all \( N \). This reduces to the congruence \( h_n^2 - 1 = (-1)^n s_n \mod N \).

Thus to find a solution to Legendre's congruence, we simply expand \( \sqrt{N} \) until a perfect square \( s_n = z^2 \) is found such that \( n \) is even. Then Legendre's congruence has the solution \( x \equiv h_{n-1}, y = A \mod N \).

If this is not one of the trivial solutions, the prime factors of \( N \) can be found by applying the Euclidean algorithm to determine the greatest common divisor of \( N \) and \( h_{n-1} - A \) of \( N \) and \( h_{n-1} + A \).

**Example.** Let \( N = 7104007 \); then \( \sqrt{N} \) has a period of length 2206. Computing the \( s_n \)'s we find that the first square occurs at \( s_{92} = 2209 \) \(-\) 472. The subscript is even so there is a possibility that we can get a factorization. We have \( h_{92} = 7103960 = 2209 \) \(-\) 47 \( \mod N \). Thus \( h_{92} + 47 \equiv 0 \mod N \), which means that \( N \) divides \( h_{92} + 47 \), and so we end up with a trivial factorization of \( N \).

The next square is \( s_{18} = 841 = 29^2 \). Once again, the subscript is even, so we check to see if our method produces a nontrivial factorization. We have \( h_{15} = 23772920 = 2460899 \mod N \); thus \( h_{15}^2 = s_{16} = h_{16}^2 = 841 = (h_{15} - 29)(h_{15} + 29) \mod N \). 2460870-2460898 (mod N). We now use the Euclidean algorithm to find the greatest common divisor of \( N \) and \( 2460870 \) which is 739, and the greatest common divisor of \( N \) and \( 2460898 \), which is 9613. In fact, it is easy to check that \( N = 739 \times 9613 \).

The program listed at the end of this paper, written in Microsoft's QuickBasic 4.0, will factor numbers up to sixteen digits long. The program runs fairly quickly and factors most numbers in less than a second. It is difficult to predict the time needed to factor a given integer. However, the following table gives some idea of factorization times required for a variety of numbers. The results were obtained by running the compiled program on an IBM AT compatible with an operating speed of 12 MHz and equipped with a math co-processor. The time is in seconds; \( n \) is the subscript for \( s \) that produces a nontrivial factorization; and \( \text{Square} \# \) is the number of squares the program checks until it finds a square that yields a nontrivial factorization. We have

<table>
<thead>
<tr>
<th>Number</th>
<th>Time</th>
<th>( n )</th>
<th>( \text{Square} # )</th>
<th>Factor 1</th>
<th>Factor 2</th>
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</thead>
<tbody>
<tr>
<td>30973</td>
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<td>42</td>
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</table>

**Concluding remarks.** The technique of using continued fractions to produce a factorization is actually an old idea. However, it was not really practical before the advent of fast computers, because of the many steps generally required to produce a square that works.

In 1982 the method was implemented on a 'reasonably fast computer that could be used almost exclusively for factorization' (Riesel [3]). It factored a 35-digit number in about one hour, a 45-digit number in one day, and a 50-digit number in one week. The most difficult number reported was a 56-digit number which was factored after 35 days and resulted in a 23-digit factor and a 33-digit factor.

**REFERENCES**

```
DIM A(-2 TO 7999), P#(2), N#(-2 TO 7999), R#(-2 TO 7999), S#(-2 TO 7999)
CLS
5 PRINT : PRINT : INPUT ID#
  D# = ID#: DP = 0
IF SQR(D#) - INT(SQR(D#)) THEN
  PRINT "NUMBER IS A PERFECT SQUARE"
GOTO 30
IF SQR(D#) = INT(SQR(D#)) THEN
  PRINT "NUMBER IS A PERFECT SQUARE"
GOTO 30
END IF
10 RDA = INT(SQR(D#))
F#(1) = 0: F#(2) = 0: H#(-2) = 0: H#(-1) = 1: R#(0) = 0: RE = 0: REL = 0
S#(0) = 1: M = 0
15 A#(N) = INT(RDA + F#(N(N) / S#(N))): H#(N) = A#(N) * H#(N - 1) + H#(N - 2)
IF H#(N) > D# THEN
  M = INT(H#(N) / D#): H#(N) = H#(N) - M * D#
END IF
15 R#(N + 1) = A#(N) * S#(N) - RUIN: SHIN + 1 = (D# - RUIN + 1 - 2) / SHIN
IF SQR(S#(N + 1)) = INT(SQR(S#(N + 1))) AND (N / 2) <> INT(N / 2) THEN 35
20 IF A#(N) = 2 * A#(0) THEN
  P = N
GOTO 60
ELSE
N = N + 1
END IF
GOTO 15
END IF
25 PRINT "PERIOD IS " ; P
30 PRINT "NOW DOPING WITH A FACTOR OF 5" D# = D# + 1: D# = D# + 5
IF SQR(D#) = INT(SQR(D#)) THEN
  PRINT "NUMBER IS A PERFECT SQUARE"
GOTO 30
END IF
GOTO 10
35 I = N + 1
H# = SQR(S#(1)): F#(1) = H# - 1 - 1 - H#: F#(2) = H# * (1 - 1) * M
IF F#(1) / D# - INT(F#(1) / D#) OR F#(2) / D# = INT(F#(2) / D#) THEN 20
END IF
GOTO 20
IF F#(2) > D# THEN
  A# = F#(2): B# = D#
ELSE
  A# = D#: B# = F#(2)
END IF
45 QA = INT(A# / B#): REH = A# - QA * B#
IF REH = 0 THEN
  F#(2) = REH
GOTO 50
ELSE
  REH = RED: A# = B#: B# = REH
GOTO 45
END IF
50 F#(1) = D# / F#(2)
55 IF F#(1) = 1 OR F#(2) = 1 THEN
  IF DPR = 1 THEN
    DPR = 0: DP = DP + 1
  END IF
GOTO 20
END IF
IF DP <> 0 THEN
  IF F#(1) / 5 = INT(F#(1) / 5) THEN
    F#(1) = F#(1) / 5
  ELSEIF F#(2) / 5 = INT(F#(2) / 5) THEN
    F#(2) = F#(2) / 5
  END IF
  DP = DP - 1: DPR = 1
GOTO 55
END IF
PRINT ID#: "FACTOR INTO": F#(1); "AND": F#(2)
GOTO 5
60 PRINT "PERIOD OF" ; P; "FINISHED WITHOUT SUCCESS"
GOTO 30
```

**The Problem.**

A motorist is driving along a lazy country road when she comes over a hill and sees a red traffic light ahead. She is well acquainted with this road, so she knows how far it is to the intersection. Her car is the new improved friction-free Chevy Slipster, so she can coast at constant speed, that is, without being slowed by friction. Because she is low on gas, however, she is not willing to accelerate before passing the intersection. She wishes to find a strategy that will allow her the highest speed through the intersection, subject to the constraint that she must come to a full stop if the light is red when she arrives.

**Possible Conditions.**

a. She rounds the top of the hill near the light (close enough to pass it some time during the upcoming green cycle) and she knows how long she has until it turns green.

b. She is near the light, but doesn't know how long is left in the red cycle.

c. She is far away, so that there may well be several red/green cycles left before she reaches the intersection (in which case it might make little difference what color the light is when she first sees it).

**Question.** What is the best strategy under each of these conditions?

Case a. of this problem was presented by Dr. Richard Poss of St. Norbert College at the 1987 Annual Pi Mu Epsilon Student Conference. Dr. Mark Krusemeyer of Carleton College suggested cases b. and c. for further investigation. In this paper I will present solutions to cases a. and b; case c is apparently still unsolved.

In the following I assume that the driver will watch the light and discontinue any braking (that is, begin to coast) as soon as the light turns green. In effect, then, our problem is to maximize the "green-light speed," which is defined to be the car's speed at the moment the light changes to green.

![Figure 1](image-url)
Case a. I will propose a strategy and then show that it is the best possible one. Suppose our driver divides the (known) distance remaining by the time she knows she has before the light changes. This will give a velocity \( v_1 \), and she could brake immediately to this velocity (we've assumed she's not far from the light, so we can take the calculated speed to be slower that normal driving speed) and then coast to the light, as shown in Figure 1.

Here \( v_0 \) denotes the car's original velocity and \( t_1 \) is the duration of the red light (from the time it is first seen). Note that the shaded area, given by

\[
A = \int_{t=0}^{t_1} v(t) \, dt,
\]

is the distance to the intersection. In other words, we were already given \( A \) and the duration \( t_1 \) of the light. Now let \( C_1 \) be the curve shown in Figure 1, and assume that \( C_1 \) falls as nearly vertically as possible before leveling off. If we take any other curve \( C_2 \) with the same area \( A \) underneath it, then \( C_2 \) must be above \( C_1 \) somewhere and below \( C_1 \) somewhere. Since \( C_1 \) is (almost!) everywhere horizontal and only nonincreasing functions are allowed, \( C_2 \) must be above \( C_1 \) before it is below \( C_1 \). However, there is then no way for \( C_2 \) to rise to \( v_1 \) at the moment the light turns green. Thus \( C_1 \) shows the best strategy, since it allows the highest green-light velocity.

Case b. Now let \( t_1 \) be the maximum possible time for the light to remain red. We know that the time \( t_0 \) at which the light actually turns green will be somewhere at random between 0 and \( t_1 \). We seek a (velocity) function which:

1. is continuous and nonincreasing from 0 to \( t_1 \);
2. has no more than area \( A \) below it between 0 to \( t_0 \); AND
3. maximizes the average terminal velocity, where the average is taken over all possible values of \( t_0 \) between 0 and \( t_1 \).

Because of restriction 2) and the fact that \( t_0 \) ranges all the way up to \( t_1 \), the area below the function between 0 and \( t_1 \) must not be more than \( A \). Figure 2 shows the graphs of some candidate functions.

Once again, we want to maximize the average green-light speed, averaged over all possible durations \( t_0 \) of the red light between 0 and \( t_1 \). If the car reaches the intersection before the light turns, the green-light speed is obviously zero by the assumption of legality.

Let us proceed as Newton would. To find the average intersection velocity of a given candidate function, divide the interval \([0, t_1]\) into several, say ten, equal subintervals. This gives eleven distinct times at which we will allow the light to turn. We can now average the intersection velocities by adding up the eleven velocities and dividing by 11. The function with the highest average "wins" because it allows the driver to pass the intersection with the highest expected velocity for an arbitrary \( t_0 \).

To increase our accuracy, we could divide the interval into 100 subintervals and average the 101 velocities. This looks familiar — it's integration. In effect, we want the velocity function which has the most area below it on the interval \([0, t_1]\). Since the candidate functions all have the same area \( A \) below them, they are all optimal.

**One caveat:** If at any time her velocity is such that she can coast constantly and reach the intersection at time \( t_1 \), the driver must not slow down any more. If she did, it would cause the area below the function to become less than \( A \). The driver is therefore constrained as follows:

The distance already covered at any time \( t_0 \) is \( \int_{0}^{t_0} v(t) \, dt \). The remaining distance to the intersection is then \( A - \int_{0}^{t_0} v(t) \, dt \). The time left before the light's "deadline" is \( (t_1 - t_0) \). The minimum velocity is the remaining distance divided by the remaining time

\[
v(t_0) \geq v_{\text{min}} = \frac{A - \int_{0}^{t_0} v(t) \, dt}{t_1 - t_0}.
\]

In summary,

1. She must not slow down so much as to prohibit her from reaching the intersection by \( t_1 \).
2. She must watch the light so that she can begin to coast as soon as it turns green, in case it does so before she arrives.
3. Within these limitations, we can now choose any nonincreasing velocity function — that is, any combination of coasting and braking, subject to rule 1 above. A rather surprising result!
A CLOSED FORMULA FOR LINEAR INDETERMINATE EQUATIONS IN TWO VARIABLES

By Mark Ontkush
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This formula requires the necessary following conditions: two integers X and Y, \( X > 1 \) and \( Y > 1 \), and (X, Y) = 1 (X and Y are mutually prime). Given these three conditions, then there exists a number M such that all integers greater than M can be expressed as a sum \( AX + BY = C \), \( C > M \), where A and B are positive integers. The integer M equals \( X(Y - 1) - Y \).

Proof. Take X as the smaller number without any loss of generality. Then, if one divides any number C by X, the result is some integer plus a remainder that is less than X. Thus, there are exactly X - 1 remainders that are possible. However, using a linear combination of X's and Y's, it will be possible to form all of these remainders.

Let the first remainder be represented as such:

\[
R_1 = Y - QX
\]

(1)

where \( Q = \lfloor Y/X \rfloor \). Q is commonly known as the greatest integer function. For example, \( \lfloor 3.01 \rfloor = 3 \), \( \lfloor 4.9 \rfloor = 4 \), and \( \lfloor 5.00 \rfloor = 5 \). Then, given equation (1), the rest of the remainders can be computed as follows:

\[
R_2 = 2Y - 2QX - \lfloor 2R_1/X \rfloor X
\]

or, in general, as

\[
R_n = nY - nQX - \lfloor nR_1/X \rfloor X
\]

Example. Let X = 5 and Y = 7. Then there are X - 1, or 4, remainders, \( R_1 \) through \( R_4 \). They can be computed as follows:

\[
\begin{align*}
R_1 &= 7 - 5 = 2 \\
R_2 &= 2(7) - 2(5) - \lfloor 2(2)/5 \rfloor (5) = 4 \\
R_3 &= 3(7) - 3(5) - \lfloor 3(2)/5 \rfloor (5) = 1 \\
R_4 &= 4(7) - 4(5) - \lfloor 4(2)/5 \rfloor (5) = 3
\end{align*}
\]

Note that all of the integers from 0 to X-1 are expressed here. This is no accident. It has been proven that, for any X and Y, if \( X \equiv 0 \) (mod Y), then the sequence of the remainders modulo Y is a rearrangement of the sequence 1, 2, 3, ..., X-1.

We wish to find the remainder that requires the largest number of \( X \)'s so that we can find a lower bound for the number M. By inspecting the remainders, it is clear that \( R_{X-1} \) will always have the largest number of \( X \)'s.

\[
R_{X-1} = (X - 1) Y - ((X - 1) Q + \lfloor ((X - 1) R_1)/X \rfloor) X.
\]

(2)

The number of \( X \)'s in this equation is (note that square brackets denote greatest integer function):

\[
\begin{align*}
&= (X - 1) Q + \lfloor ((X - 1) R_1)/X \rfloor \\
&= QX - Q + \lfloor ((X - 1)(Y - QX))/X \rfloor \\
&= QX - Q + \lfloor (XY - QX^2 - Y + QX)/X \rfloor \\
&= QX - Q + \lfloor Y - QX - (Y/X) \rfloor \\
&= QX - Q + Y - QX + \lfloor -Q/(Y/X) \rfloor
\end{align*}
\]

since \( Y \) and \( QX \) are both integers and \( \lfloor Y \rfloor = Y \) and \( \lfloor QX \rfloor = QX \).

But \( Q = \lfloor Y/X \rfloor \), and since X and Y are mutually prime,

\[
\lfloor Y/X \rfloor + 1 < (Y/X) < [Y/X],
\]

so

\[
\lfloor Q - (Y/X) \rfloor = [1] = 1
\]

and the number of \( X \)'s in (2) is given by

\[
Y - Q - 1.
\]

(3)

By using (3) in (2), we can solve for \( R_{X-1} \).

\[
R_{X-1} = (X - 1) Y - (Y - Q - 1) X
\]

\[
= XY - Y - (XY - QX - X)
\]

\[
= QX + X - Y
\]

\[
= (Q + 1)X - Y.
\]

If we can discover the number of \( X \)'s required for \( R_{X-2} \) and then add \( R_{X-1} \), we will have \( M \), the largest number that cannot be expressed as \( AX + BY \), where A and B are positive integers. The number of \( X \)'s required for \( R_{X-2} \) is easy: looking at (2) and (3), and remembering \( Y > X \), at most \( Y - Q - 2 \) \( X \)'s will be needed to find this remainder.

\[
M = (Y - Q - 2)X + R_{X-1}
\]

\[
= (Y - Q - 2)X + (Q + 1)X - Y
\]

\[
= XY - QX - 2X + QX + X - Y
\]

\[
= XY - X - Y
\]

\[
= X(Y - 1) - Y
\]

Example. Find M for \( X = 62, Y = 79 \), and show that M cannot be expressed as \( AX + BY = M \), but \( M + 1 \) can be.

\[
M = 62(79 - 1) - 79
\]

\[
= 4757
\]
If we divide 4757 by 62, we get 76 with remainder 45. However, 61(79) - (77)(62) = 45, so 45 is the worst possible remainder. There is no way M can be expressed without using a negative A or B, as

\[
4757 = 76(62) + 61(79) - (77)(62) = 61(79) - 62
\]

M + 1, however, can be expressed as a sum AX + BY. Dividing 4758 by 62, we get 76 with remainder 46. A little experimentation shows that 46 = 10(79) - 12(62). So

\[
4758 = 76(62) + 10(79) - 12(62) = 10(79) + 64(62)
\]

REFERENCES

Award Certificates
Your chapter can make use of the Pi Mu Epsilon Award Certificates available to help you recognize mathematical achievements of your students. Contact Professor Robert Woodside, Secretary-Treasurer.

Matching Prize Fund
If your chapter presents awards for Outstanding Mathematical Papers or for Student Achievement in Mathematics, you may apply to the National Office for an amount equal to that spent by your Chapter up to a maximum of fifty dollars. Contact Professor Robert Woodside, Secretary-Treasurer.

SOME SHORTCUTS FOR FINDING ABSOLUTE EXTREMA

By Subhash C. Saxena
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In the discussion of absolute extrema, most elementary calculus books correctly suggest the following procedure for finding absolute maximum and absolute minimum of a continuous function \( f \) on a closed interval \([a, b]\).

"Find all the critical points of \( f \) on \([a, b]\). Then find the values of \( f \) at each of these points and also at \( a \) and \( b \). The largest of these values gives the absolute maximum and the smallest of these is absolute minimum."

However, in several cases, short cuts may be made to find absolute extrema in various situations. The purpose of this note is to explore some of these short cuts.

In the case of a quadratic function \( ax^2 + bx + c \), it is a well-known fact that:

at \( x = -\frac{b}{2a} \) the quadratic has an absolute minimum if \( a > 0 \) and an absolute maximum if \( a < 0 \). Assuming \( -\frac{b}{2a} \) is in the interior of \([a, b]\), the other absolute extremum occurs at the end-point which is farther from \(-\frac{b}{2a}\).

For a cubic polynomial \( p(x) = ax^3 + bx^2 + cx + d \), it is an easily verifiable fact that it has a relative maximum and a relative minimum if and only if \( p'(x) \) has two distinct real roots \( a \) and \( b \) (which happens when \( b^2 - 3ac > 0 \)). Otherwise it has neither a relative maximum nor a relative minimum.

Graphs of \( y = ax^3 + bx^2 + cx + d \) with \( b^2 - 3ac > 0 \) are shown here:

Assuming \( p'(x) \) has two distinct real roots, say \( a \) and \( b \), with \( a < b \), then \( p'(x) = a(x - a)(x - b) \); where \( a \) and \( b \) are all real.

It is obvious that for \( a > 0 \), \( a \) has a relative maximum and \( b \) has a relative minimum. (For \( a < 0 \), \( a \) has a relative minimum and \( b \) has a relative maximum.)

The main result of this note consists of constructing the largest interval containing \( a \) and \( b \) such that at these critical points the cubic has an absolute maximum
It should also be remembered that the function is monotonic on \((-\infty, a)\) and \((\beta, \infty)\).

As an example, consider

\[ p(x) = \frac{1}{3}x^3 - 4x^2 + 12x - 5 \]

\[ p'(x) = x^2 - 8x + 12. \]

The critical points are \(x = 2\) and \(x = 6\) and they produce a relative maximum and a relative minimum, respectively.

Using our theorem, \(p(x)\) also has an absolute maximum and an absolute minimum, respectively, on any subinterval of \([0, 8]\) containing them. \((0 = \frac{3\alpha - \beta}{2}, 8 = \frac{3\beta - \alpha}{2}).\)

If we are to find an absolute maximum and an absolute minimum on \([1, 9]\), we know that an absolute maximum would occur at \(x = 9\), and an absolute minimum at \(x = 6\) (since \(1 > \frac{3\alpha - \beta}{2}\)).

For a fourth degree polynomial \(p(x)\), we may have one of the following two situations:

Case I. \(p(x)\) has exactly one relative extremum. (This happens when either \(p'(x)\) has only one simple real root, the other two roots being complex or coincident; or where all the three roots of \(p'(x)\) are coincident.)

Case II. \(p(x)\) has exactly three relative extrema (two relative maxima and one relative minimum or two relative minima and one relative maximum).

In Case I, the relative extremum is also absolute extremum of the same type, (i.e., relative maximum is absolute maximum, or the relative minimum is absolute minimum).

In Case II we consider a special and easy situation when three real and distinct roots of \(p'(x)\), say \(\alpha, \beta, \gamma\) with \(\alpha < \beta < \gamma\), are such that

\[ \beta - \alpha = \gamma - \beta = k, \ k > 0. \]

It is then an easy matter to show that the relative extrema at \(x = \alpha\) and \(\gamma\) are the absolute extrema since the function is monotonic on \((-\infty, \alpha)\) and \((\gamma, \infty)\).

and an absolute minimum (not necessarily in that order) in that interval.

If \(a\) and \(\beta\) are critical points of \(p(x)\), then

\[ p'(x) = a(x - \alpha)(x - \beta) = a[x^2 - x(\alpha + \beta) + \alpha\beta], \ a \neq 0. \]

Therefore,

\[ p(x) = a\left[\frac{1}{3}x^3 - \frac{1}{2}x^2(\alpha + \beta) + \alpha\beta x\right] + k. \]

Thus, using elementary algebra,

\[ p(x) - p(\alpha) = a\left[\frac{1}{3}(x^3 - \alpha^3) - \frac{1}{2}(x^2 - \alpha^2)(\alpha + \beta) + \alpha\beta(x - \alpha)\right] \]

\[ = \frac{1}{3}a(x - \alpha)^2\left(x - \frac{3\beta - \alpha}{2}\right). \]

Hence, for \(a > 0\) when \(p(a)\) produces a relative maximum for \(p(x)\),

\[ p(x) - p(\alpha) > 0 \text{ if and only if } x > \frac{3\beta - \alpha}{2}. \]

Interchanging \(a\) and \(\beta\) it follows that

\[ p(x) - p(\beta) = a\left[\frac{1}{3}(x - \beta)^3 - \frac{1}{2}(x^2 - \beta^2)(\alpha + \gamma) + \alpha\gamma(x - \beta)\right] \]

\[ = \frac{1}{3}a(x - \beta)^2\left(x - \frac{3\alpha - \beta}{2}\right). \]

Thus, for \(a > 0\) when \(p(\beta)\) produces a relative minimum for \(p(x)\),

\[ p(x) - p(\beta) < 0 \text{ if and only if } x < \frac{3\alpha - \beta}{2}. \]

For \(a < 0\) when \(p(\alpha)\) is a relative minimum and \(p(\beta)\) is a relative maximum, it can be easily shown that

\[ p(x) < p(\alpha) \text{ if and only if } x > \frac{3\beta - a}{2}, \]

and

\[ p(x) > p(\beta) \text{ if and only if } x < \frac{3a - \beta}{2}, \]

using (1) and (2).

Thus, we have the following result:

**Theorem 1:** Let \(a\) and \(\beta\) be two distinct critical points of a cubic curve. Assuming \(a < \beta\), the largest closed interval containing them and having absolute extrema at \(a\) and \(\beta\) is:

\[ \left[\frac{3a - \beta}{2}, \frac{3\beta - a}{2}\right]. \]

It is interesting to note that the length of this interval is \(2(\beta - a)\) and that each end-point is \(\frac{1}{2}(\beta - a)\) from the nearest critical point.
We have to figure out the largest interval containing \( a, \beta, \) and 7 such that the relative extremum at each of these points is also an absolute extremum.

We have \( a = \beta - k, \gamma = \beta + k, \) and \( \beta \) as distinct roots of \( p'(x). \)

Thus, 
\[
p'(x) = a(x - \beta + k)(x - \beta)(x - \beta - k).
\]

Hence, 
\[
p(x) = \frac{a}{4} (x - \beta)^4 - \frac{a}{2} (x - \beta)^2 + A
\]
\[
= \frac{a}{4} (x - \beta)^2 (x - \beta + \sqrt{2}k)(x - \beta - \sqrt{2}k) + A.
\]

Therefore, 
\[
p(x) - p(\beta) = \frac{a}{4} (x - \beta)^2 (x - (\beta + \sqrt{2}k))(x - (\beta + \sqrt{2}k) - A).
\]

Thus, \( p(x) - p(\beta) \) will have the same sign as \( a \) for \( x < \beta - \sqrt{2}k \) or for \( x > \beta + \sqrt{2}k. \) It will have sign opposite to \( a \) for \( \beta - \sqrt{2}k < x < \beta + \sqrt{2}k. \)

For the sake of convenience, replacing \( \frac{a}{4} \) by \( a \) we have the following result:

**Theorem 2:** For a fourth degree function \( y = a(x - \beta)^4 - 2ak^2(x - \beta)^2 + A, \) \( k > 0, \) the absolute maximum (minimum) occurs if \( a < 0 (a > 0) \) at \( x = \beta \pm k, \) on any interval containing any of these points; and the absolute minimum (maximum) occurs at \( x = \beta \) on any interval containing \( \beta \) which is a subinterval of \( \beta - \sqrt{2}k, \beta + \sqrt{2}k. \)

The interval \( [\beta - \sqrt{2}k, \beta + \sqrt{2}k] \) is the largest interval containing \( a, \beta, \) and 7, such that at each of these points the relative extremum is also an absolute extremum.

I wish to thank Joseph Cicero for his valuable suggestion.

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**CALL FOR NOMINATIONS**

Elections for national officers of the Pi Mu Epsilon Society will be held in the Spring of 1990. The three-year terms of office will begin July 1, 1990.

The committee solicits recommendations for nominees from members. Please submit names and addresses of possible nominees to Milton D. Cox, President, Pi Mu Epsilon, Department of Mathematics and Statistics, Miami University, Oxford, OH 45056.

Additional nominations for officers may be made in accordance with Sections 2 and 3 of Article V. of the Constitution and By-Laws.

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**A QUICK INTRODUCTION TO QUATERNIONS**

By Byron L. McAllister
Montana State University

Viewed strictly as tools, quaternions became nearly obsolete when Gibbs and Heaviside took them apart into the more easily managed vectors and scalars. (For a detailed history, see [1].) On the other hand, there is a certain charm about quaternions that makes them keep coming up. This note concerns some interesting properties of quaternions themselves that are quite elementary, given the experience most of us have today with dot and cross products.

Notation. We may think of a quaternion \( q = a + V \) of a number \( a \) plus a vector \( V. \) The number \( a \) is called the scalar part of \( q \) and \( V \) is called the vector part of \( q. \)

The sum of two quaternions is defined to be the quaternion whose scalar part is the sum of the scalar parts of the two quaternions and whose vector part is the sum of their vector parts. That is
\[
(a + V) + (b + W) = (a + b) + (V + W).
\]

The product of \( q_1 = a + W \) by \( q_2 = b + W \) may be defined in terms of vector dot and cross products. The scalar part of the product \( q_1q_2 \) is \( ab - V \cdot W, \) and the vector part is \( aW + bV + V \times W. \) That is,
\[
(a + V)(b + W) = (ab - V \cdot W) + (aW + bV + V \times W).
\]

Note that the presence of the cross product in the vector part implies that the product is not commutative unless \( V \times W = 0. \)

We shall denote a quaternion generically by \( q, \) its scalar part by \( a, \) and its vector part by \( V. \) Also, we shall use \( r \) to denote the radius of \( q, \) that is, the length of the vector \( V. \) If \( u \) is a vector of unit length pointing in the direction of \( V, \) then we may also denote \( V \) by \( ru. \)

Thus the notation \( a + ru \) is another general notation for \( q. \)

Since a vector has three components, we may regard a quaternion as having four, the fourth being the scalar part. This point of view suggests the notation
\[
q = a + xi + yj + zk.
\]

The square root of the sum of the squares of \( a, x, y, \) and \( z \) will be called the modulus of the quaternion \( q, \) and will be denoted by \( m. \)

Pythagorean quintuples. Suppose that \( a, x, y, \) and \( z \) are integers and consider the quaternion \( q^2 = qq. \) It is easy to show that the modulus of the product of two quaternions is equal to the product of their moduli. Since the squares of the four components of \( q^2 \) add to form the square \( m^2 \) of the modulus \( m \) of \( q^2, \) and since \( m^2 \) is clearly also a positive integer, we generate in this way a sort of "Pythagorean quintuple," that is a set of five positive integers the squares of four of which add to the square of the fifth. This is an analogue of the fact that, in a similar manner, the square of a complex number with integer real and imaginary parts gives us a Pythagorean triple. (The reader will easily find a modification that generates "Pythagorean quadruples."")
An Isomorphism. That there are analogies between quaternions and complex numbers is not surprising since Hamilton invented quaternions as generalized complex numbers. It is quite well-known that the set of quaternions of the form \( a + xi \) is isomorphic to the field of complex numbers, and the same is true if \( i \) is replaced by \( j \) or by \( k \). Perhaps more surprising is the following: Let \( u \) be any unit vector and let \( a \) and \( b \) be two real numbers. Let \( f \) be a mapping from the complex plane into the quaternions defined by the rule

\[
 f(a + bi) = a + bu.
\]

Clearly \( f \) is one-to-one onto its range, and easy calculations show that \( f \) "preserves" addition, multiplication, and multiplication by a real number (scalar). Thus, for any fixed unit vector \( u \), the set of all \( a + bu \) is isomorphic to the complex numbers. (An interesting further inquiry is as to when, for a given set of three perpendicular unit vectors, \( u, v, \) and \( w \), it happens that \( q(a + xi + yj + zk) = a + xu + yv + zw \) is an isomorphism of the quaternions onto themselves.)

Inverses. Unlike vectors, the system of quaternions includes multiplicative inverses, and hence supports a concept of division. For any quaternion \( q \), except \( 0 + 0 \), of course, the inverse \( q^{-1} \) of \( q \) is obtainable by subtracting the vector part of \( q \) from the scalar part and then dividing the result by the square of the modulus of \( q \). Analogously to complex numbers, the result of subtracting the vector part of \( q \) from the scalar part is called the conjugate of \( q \). Thus, denoting the conjugate of \( q \) by \( C(q) \), we may write

\[
 q^{-1} = \frac{C(q)}{m^2}.
\]

That \( q^{-1}q = qC(q) = C(q)q = m^2 \) follows directly from \( qC(q) = C(q)q = m^2 \). The vector part of \( q^{-1} \) is seen to be directed oppositely to the vector part of \( q \). Since multiplication is not commutative, quaternion division of \( q_1 \) by \( q_2 \) takes two forms, depending on whether \( q_2^{-1} \) is multiplied on the left or on the right of \( q_1 \). It is amusing to note that this provides "inverses" and hence "division" (two kinds!) for vectors. For, if \( V \) is a vector, we may identify \( V \) with the quaternion \( 0 + V \), so that \( V^{-1} \) is seen to be

\[
 V^{-1} = \frac{-V}{r^2}.
\]

But although \( V^{-1} \) is a vector, i.e., a quaternion with scalar part equal to 0, the (quaternionic) product of a vector with its inverse is not a vector. (It's the scalar 1, of course.) The inverse of \( V \) is directed oppositely to \( V \), and the inverse of a unit vector (i.e., of a quaternion with scalar part 0 and with radius 1) is its negative. If \( V \) and \( W \) are two vectors, we may "divide" \( V \) by \( W \) on the left to produce \( (VxW)/(W.W) \) or on the right to produce \( (WXV)/(W.W) \). Since cross product is anticommutative, the two quotients are negatives of each other.

Square roots. Now let's consider the square roots of a quaternion. Since

\[
 b = p = k^2 = -1,
\]

and since the square of the negative of \( i, j, \) or \( k \) is therefore also \(-1\), it is sometimes said that in the system of quaternions, there are six square roots of \(-1\). Unfortunately, this is somewhat misleading. The truth is that most quaternions have exactly two square roots, given by the formula

\[
 \sqrt{q} = \pm \frac{a + m + V}{(2a + 2m)^{1/2}},
\]

where \( m \) is the modulus of \( q = a + V \). This formula is valid where it makes sense.

Because \( m \geq a \), the formula fails to make sense only if \( a + m = 0 \), and this can only happen if \( V = 0 \) and \( a = 0 \). That is, the formula works unless the vector part of \( q \) is zero and the scalar part is non-positive. To see what happens in that case, consider the following proof of the formula: Think of the vector \( V \) as being given in the form \( bu \), where \( u \) is a unit vector. That is, \( q = a + bu \). If \( V = 0 \), the unit vector \( u \) may be chosen arbitrarily, and \( b = 0 \). (But then be sure to remember the arbitrariness of \( u \).) We've seen that the set of all such \( a + bu \) forms a system isomorphic to the complex numbers. and standard methods (algebraic or geometric) then give us the formula (1) unless \( b = 0 \). Indeed, when \( b = 0 \), if \( a > 0 \) the formula is still valid. In this case the two roots of \( q \) are symmetrically placed on the real axis - i.e., on the axis of scalars. As \( a \) approaches 0, so do both roots, and when \( a \) reaches 0, the roots coalesce to 0. As \( a \) continues its decrease into negative values, we know from our experience with complex numbers, for which the roots become pure imaginary, that for quaternions the two roots must lie on the two rays of the line through 0 and \( u \). That is, a square root is found at a distance \((-a)^{1/2}\) in the \( u \) direction (and another at an equal distance in the opposite direction.) But the arbitrariness of \( u \) means that an entire sphere of such square roots exists. (Thus the estimate of six square roots for \(-1\) is far short of the mark!) Note that when \( V \) is not 0, \( u \) is not arbitrary, so that we don't get the sphere in that case. Similarly, when \( V \) is 0 but a > 0, the vector parts of the roots are 0 and no sphere is obtained.

**Reference**

**OBLIQUE PYTHAGOREAN LATTICE TRIANGLES**

By Stanley Rabinowitz
Westford, Massachusetts 01886

A lattice point is a point in the plane with integer coordinates. A lattice triangle is a triangle whose vertices are lattice points. A Pythagorean triangle is a right triangle with integer sides.

It is obvious that, given any Pythagorean triangle, a congruent copy can be found in the lattice with its legs parallel to the coordinate axes.

**Definition.** A triangle is oblique (or is embedded in an oblique manner), if no side is parallel to one of the coordinate axes.

In general, given a Pythagorean triangle (such as a 3-4-5 triangle), it is not possible to find a congruent copy embedded obliquely in the lattice. The author asked in this journal ([3]) if there is an oblique lattice triangle similar to a 3-4-5 right triangle. A solution was given in [1]. In this note, we will investigate this question in more detail.

A computer search reveals that the smallest oblique lattice triangle similar to a 3-4-5 triangle has vertices at (0, 0), (4, 4), and (7, 1). This triangle is shown in Figure 1.

![Figure 1](image)

Note that the sides of this triangle have lengths $3\sqrt{2}$, $4\sqrt{2}$, and $5\sqrt{2}$. A more interesting question is: Can such a triangle have integral sides? The answer is "yes" as we will see below.

We can find an entire family of lattice triangles similar to the 3-4-5 triangle by considering the three points:

$$
O = (0, 0) \\
B = (4m, 4n) \\
C = (4m + 3n, 4n - 3m)
$$

where $m$ and $n$ are any positive integers. Note that letting $m = 1$ and $n = 1$ yields the triangle previously found by the computer search.

To make the sides of the triangle integral, first make $OB$ integral. To do this, apply the general formula for the sides of a Pythagorean triangle: let $m = r^2 - q^2$ and $n = 2pq$. This yields the 2-parameter solution

$$
O = (0, 0) \\
B = (4p^2 - 4q^2, 8pq) \\
C = (4p^2 - 4q^2 + 6pq, 8pq - 3p^2 + 3q^2)
$$

In some of these, a side may be parallel to one of the axes. It is simple to avoid such a case. For example, choose $p = 2$ and $q = 1$ to get the integral triangle with vertices at $(0, 0)$, $(12, 16)$, and $(24, 7)$. This triangle has sides of lengths 15, 20, and 25. Its sides are 5 times as large as the sides of a 3-4-5 triangle. A computer search reveals that this is the smallest integral triangle similar to a 3-4-5 triangle with no side parallel to an axis.

We now show this can be done in general.

**Theorem 1.** Given a Pythagorean triangle, one can find an oblique Pythagorean lattice triangle similar to the given triangle.

**Proof.** Suppose the given Pythagorean triangle has sides $r$, $s$, and $t$, with $t$ being the length of the hypotenuse. Let $A = (m, n)$. Lay off $r$ copies of $OA$ along ray $OA$ to bring us to the point $B = (rm, rn)$. Erect a perpendicular to $OB$ at $B$ and lay off $s$ copies of $OA$ to bring us to the point $C = (rm - sn, rm + sn)$.

Now let $m = p^2 - q^2$ and $n = 2pq$ to guarantee that $OA$ has integral length. Then we have constructed a Pythagorean lattice triangle $OBC$ similar to the given triangle. Sides $OB$ and $BC$ are clearly not parallel to any axis. $OC$ might be parallel to the $y$-axis. To prevent this, take $p = 4s$ and $q = 1$. Then the sides of the resulting triangle are:

$$
O = (0, 0) \\
B = (16rs^2 - r, 8sr) \\
C = (16rs^2 - r - 8s^2, 8rs + 8s^2)
$$

The line $OC$ cannot be parallel to the $y$-axis, since that would require $16rs^2 = r + 8s^2$ or $s^2 = r(2r - 1) \leq (2r - 1)6(2r - 1) = 1/6$, which cannot be since $s^2$ is a positive integer.

Recall that a Pythagorean triangle is called primitive if its three sides are relatively prime.

The above procedure always produces a non-primitive Pythagorean triangle, since all sides of the triangle formed are divisible by the length of $OA$ and it is clear that $OA > 1$. It is therefore natural to ask if there is a primitive Pythagorean triangle embedded obliquely in the lattice. We answer this question in the negative.

**Theorem 2.** No primitive Pythagorean triangle can be embedded obliquely in the lattice.

**Proof.** Suppose Pythagorean triangle $ABC$ (with right angle at $C$) is embedded obliquely in the lattice. Translate the triangle so that $C$ coincides with the origin. Then perform a rotation through a multiple of $\pi/2$ until ray $CB$ lies in the first quadrant. Point $B$ will not be mapped onto an axis since the triangle is still embedded obliquely (and this property is not affected by the translations or rotations just performed). We may assume that point $A$ has been moved into the second quadrant, for if it moved into the third quadrant, we may perform a reflection about the line $y = x$ to bring it into the second quadrant, leaving $B$ in the first quadrant. Furthermore, we may assume that $B$ lies further from the $x$-axis than $A$, for if $A$ were further from the $x$-axis, we could perform a reflection about the $y$-axis and then relabel points $A$ and $B$. Thus, $AABC$ is situated as shown in Figure 2.
Let $D$ be the foot of the perpendicular from $B$ to the $x$-axis, and let $E$ be the foot of the perpendicular from $A$ to $BD$. Since $B$ was further from the $x$-axis than $A$, point $E$ lies between $B$ and $D$. Also note that since $A$ and $B$ are lattice points, the coordinates of points $A, B, D,$ and $E$ are integers. Quadrilateral $ACEB$ is cyclic since $\angle AEB = \angle AEB = \pi/2$. Thus, $\angle ABC = \angle AEC$. But $AE$ is parallel to $CD$, so $\angle AEC = \angle LECD$. But triangles $ECD$ and $ABC$ are right triangles. Hence they are similar. Let the ratio of similarity be $pq$ with $\gcd(p, q) = 1$. This ratio is rational since it is equal to the ratio of $DE$ to $AC$, both of which are integral. But $AB > BC > CE$, so $AABC$ is strictly larger than $ACDE$, and $q > 1$. Now $CE = (p/q) \cdot AB$, so $CE$ is rational. But $CE^2 = CD^2 + DE^2$, so $CE$ is an integer. If a rational number squared is integral, the rational number must itself be an integer. Hence $CE$ is an integer. Let the lengths of the sides of $AABC$ be $a, b,$ and $c$. Then the lengths of the sides of $\triangle ECD$ are $pa/q, pb/q$, and $pdq$. But these lengths are integers and $p$ and $q$ are relatively prime. So $q \mid a, q \mid b,$ and $q \mid c$. Thus, $q \mid \gcd(a, b, c)$ and consequently, $AABC$ is not primitive.

**Corollary.** The set of diophantine equations

\[
\begin{align*}
    a^2 + b^2 &= r^2 \\
    (b + d)^2 + c^2 &= s^2 \\
    (a + c)^2 + d^2 &= t^2 \\
    r^2 + s^2 &= t^2
\end{align*}
\]

has no solution with $r, s,$ and $t$ being relatively prime.

**Proof.** In the preceding configuration, let point $B$ have coordinates $(c, d)$, let $C$ have coordinates $(-a, b)$, and let $AC = r$, $AB = s$, and $BC = t$. Now the above equations represent the Pythagorean Theorem applied to the various right triangles involved.

Although no oblique lattice triangle congruent to the $3$-$4$-$5$ triangle exists in the planar lattice, what about in the higher dimensions? We conclude this paper with the following surprise: An oblique $3$-$4$-$5$ triangle exists in the integer lattice in $7$-dimensional space! Its vertices are given by the points

\[
\begin{align*}
    O &= (0, 0, 0, 0, 0, 0, 0) \\
    B &= (1, 2, 2, 0, 0, 0, 0) \\
    C &= (0, 0, 0, 0, 2, 2, 2).
\end{align*}
\]

For other easily-stated but unsolved problems concerning lattice points, consult [2].

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**REFERENCES**


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**ON THE COVER OF THE SPRING 1989 ISSUE**

**Editor**

The formulas for the two functions presented on the front and back covers of the Spring 1989 issue are:

Front: \( (\text{abs}(x) + \text{abs}(y))/4 \mod 3 \) 

Back: \( 7 \cdot \log(x^2 + y^2 + 2 \cdot \text{abs}(x \cdot y)) = 0.001 \mod 3 \)

The front and back covers commemorating the 75th Anniversary of the founding of Pi Mu Epsilon were designed and prepared by Professor E. P. Miles, Jr. Florida State University, Tallahassee, Florida, at the FSU Muench Center for Color Graphics, on an INTERCOLOR 2427, DATAVUE, and PRINTACOLOR GP 1024.

Professor Miles presented the J. Sutherland Frame Lecture at the Summer Meeting of Pi Mu Epsilon in Pittsburgh, PA in 1981 on "The Beauties of Mathematics Revealed in Color Block Graphs."
A NOTE ON THE ADDITION FORMULAS FOR SINE

By Arthur Guetter
Hamline University

Many formulas in mathematics, especially in number theory, are derived by evaluating some quantity in two different ways. The purpose of this note is to show how the addition and subtraction formulas for the sine function can be derived by calculating the area of a triangle in two ways. A cursory search of several texts did not reveal the following derivations, though I would doubt if they are new. I will assume in the sequel that $0 < \theta < \pi/2$, $0 < \phi < \pi/2$, and $\phi < \theta$.

I first noticed that these derivations would be possible while grading an assignment which required finding the area of a triangle. Comparing an answer which seemed to be different than mine revealed the double angle formula for sine. We start with an isosceles triangle with the length of the equal sides 1, and the angle between these sides with measure $2\theta$.

![Figure 1a](image1.png)  ![Figure 1b](image2.png)

Each of the two smaller triangles in Figure 1a has area given by $(1/2)hx = (1/2)\cos \theta \sin \theta$, so that twice the area of the triangle is $2\cos \theta \sin \theta$. In Figure 1b, we calculate twice the area of the triangle as $h = \sin 2\theta$. Putting this together gives the double angle formula

$$\sin 2\theta = 2\cos \theta \sin \theta$$

After making this observation, I wondered if I could derive the addition formula for sine in this manner. I needed a triangle with one angle given by $\theta + \phi$, the segment which divides these angles to be an altitude, and one side of length 1.

In Figure 2a, we note that $\cos \phi = \frac{h}{z} = \cos \theta$. Then twice the area of the triangle is

$$xh + yh = h(x + y) = z \cos \theta (x + y) = z \cos \theta (\sin \phi + z \sin \theta) = z (\cos \theta \sin \phi + z \cos \theta \sin \theta) = z (\cos \theta \sin \phi + \cos \phi \sin \theta)$$

![Figure 2a](image3.png)  ![Figure 2b](image4.png)

In Figure 2b, we calculate twice the area as $zh = z \sin (\theta + \phi)$. Equating these areas gives

$$\sin (\theta + \phi) = \cos \theta \sin \phi + \cos \phi \sin \theta,$$

which is of course the addition formula for sine.

We can obtain the subtraction formula for sine in a similar manner. In this case, we use a right triangle with one leg of length one.

In Figure 3, twice the area of the lower triangle is $zh \sin (\theta - \phi)$, twice the area of the whole triangle is $z \sin \theta$, and twice the area of the upper triangle is $x = h \sin \phi$. It follows that

$$zh \sin (\theta - \phi) = z \sin \theta - h \sin \phi = \frac{z \sin \theta \sin \phi}{h} = \cos \phi \sin \theta - \cos \theta \sin \phi.$$  

We have used the relations $1/h = \cos \phi$ and $1/z = \cos \theta$. The last line is the subtraction formula

$$\sin (\theta - \phi) = \cos \phi \sin \theta - \cos \theta \sin \phi.$$  

It is now an easy exercise to extend these formulas to all values of $\theta$ and $\phi$. 

![Figure 3](image5.png)
A FALLACY IN PROBABILITY

By Prem N. Bajaj
The Wichita State University

A card is drawn from a standard well-shuffled deck and put aside. Then a second card is drawn. Let \( Q \) denote the event that the first card is a queen. Let \( K \) denote the event that the second card is a king. We are interested in verifying the identity:

\[
P(K) = P(Q)P(K|Q) + P(Q^c)P(K|Q^c)
\]  

(A)

where \( Q^c \) denotes the event that the first card is not a queen, \( P(K) \) is the probability for the event \( K \), and \( P(K|Q) \) denotes the conditional probability of \( K \) when event \( Q \) has happened, etc.

To compute \( P(K) \), condition it whether the first card is a king or not. If \( K_1 \) denotes the event that the first card is a king, we have

\[
P(K) = P(K_1)P(K|K_1) + P(K_1^c)P(K|K_1^c)
\]

\[= \frac{4}{52} \cdot \frac{3}{51} + \frac{48}{52} \cdot \frac{4}{51} = \frac{4}{52}.
\]  

(i)

Clearly
\[P(K/Q) = \frac{4}{51}.
\]  

(ii)

To find \( P(K/Q^c) \), notice that the first card, which is not a queen, may or may not be a king. Consequently

\[
P(K/Q^c) = \frac{4}{52} \cdot \frac{3}{51} + \frac{48}{52} \cdot \frac{4}{51} = \frac{4}{52} \cdot \frac{47}{51}.
\]  

(iii)

Finally, \( P(Q) = \frac{4}{52} \), \( P(Q^c) = \frac{48}{52} \) together with (i), (ii) and (iii) do not verify the identity (A). What went wrong?

Solution: Computation of \( P(K/Q^c) \) is in error. Indeed, we have

\[
P(K/Q^c) = \frac{P(Q^c|K)P(K/Q^c|K) + P(Q^c|K^c)P(K/Q^c|K^c)}{P(Q^c)}
\]

\[= \frac{P(Q^c|K_1)P(K/Q^c|K_1) + P(Q^c|K_1^c)P(K/Q^c|K_1^c)}{P(Q^c)}.
\]

With this value of \( P(K/Q^c) \), identity (A) is verified to be true.

NOTE ON A WELL-KNOWN LIMIT

By Prem N. Bajaj
The Wichita State University

In the Spring 1989 issue of this journal \[\lim_{n \to \infty} \left( \frac{\sqrt{n}}{n} \right) \] is obtained using the fact

\[\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = R \] implies that \[\lim_{n \to \infty} \sqrt[n]{u_n} = R, u_n > 0. \] (The converse is not true of course.) However the above limit can be obtained using the definition of an integral and the technique of integration by parts. To see this, recall that (with usual notation):

\[
\int_a^b f(x)dx = \lim_{[a]\to 0} \left( \sum_{k=1}^{n} f(x_k)\Delta_k \right)
\]

In particular,

\[
\int_0^1 f(x)dx = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} f\left( \frac{k}{n} \right).
\]

Now let
\[L = \lim_{n \to \infty} \left( \frac{\sqrt{n}}{n} \right),
\]
then

\[
\log L = \lim_{n \to \infty} \left( \frac{1}{n} \sum_{k=1}^{n} \log \frac{k}{n} \right)
\]

\[= \int_0^1 \log x \, dx = -1,
\]

using integration by parts.

Hence, \( L = \frac{1}{e} \).
THE INEQUALITY BETWEEN POWER MEANS VIA COORDINATE GEOMETRY

By Norman Schaumberger
Bronx Community College

The inequality between power means states that if \( r > s \) are nonzero real numbers then for any positive numbers \( a_1, a_2, \ldots, a_n \):

\[
\left( \frac{a_1^r + a_2^r + \cdots + a_n^r}{n} \right)^{1/r} \geq \left( \frac{a_1^s + a_2^s + \cdots + a_n^s}{n} \right)^{1/s}
\]

with equality holding if and only if \( a_1 = a_2 = \ldots = a_n \).

If \( x > 0 \) and \( r > s \geq 1 \) then the graph of \( f(x) = sx^{-s} + \frac{r-s}{x^{s-1}} \) is concave upward and has \( y = rx \) as a tangent line at \((1, r)\). This follows from the fact that

\[
f(1) = r, \quad f'(1) = r \quad \text{and} \quad f''(x) = s(r-s+1)(r-s)x^{-s-1} + (r-s)(-s)x^{-s-1}
\]

is positive.

Hence \( sx^{-s} + \frac{r-s}{x^{s-1}} \geq rx \), or

\[
sx^r + r-s \geq rx^s
\]

with equality if and only if \( x = 1 \).

Let \( P = \left( \frac{a_1^s + a_2^s + \cdots + a_n^s}{n} \right)^{1/s} \) and substitute \( x = \frac{a_i}{P} \) \((i = 1, 2, \ldots, n)\)

successively into (2). Adding gives

\[
\left( \frac{a_1^r + a_2^r + \cdots + a_n^r}{n} \right) + r(n-s) \geq r \left( \frac{a_1^s + a_2^s + \cdots + a_n^s}{n} \right) = r n.
\]

It follows that

\[
\frac{a_1^r + a_2^r + \cdots + a_n^r}{n} \geq \frac{a_1^s + a_2^s + \cdots + a_n^s}{n} \frac{r}{s}
\]

with equality if

and only if \( \frac{a_i}{P} \) equals 1, \((i = 1, 2, \ldots, n)\), or \( a_1 = a_2 = \ldots = a_n \).

Hence, we have proved (1) for the important special case \( r > s \geq 1 \). For example, putting \( r = 2 \) and \( s = 1 \) in (1) gives the familiar arithmetic-quadratic mean inequality:

\[
\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n} \geq \left( \frac{a_1 + a_2 + \cdots + a_n}{n} \right)^2
\]

with equality if and only if \( a_1 = a_2 = \ldots = a_n \).
Dear Editor,

S. Councilman, *Pi Epsilon Journal* 8 (1989), 669-671, suggested a matrix generalization of complex numbers. A different and natural generalization of considerable interest consists of the "skew-circulants" (matrices whose determinants are skew circulants), exemplified in the 4 by 4 case by

\[
A = \begin{b matrix}
    a_0 & a_1 & a_2 & a_3 \\
    -a_3 & a_0 & a_1 & a_2 \\
    -a_2 & -a_3 & a_0 & a_1 \\
    a_1 & a_2 & -a_3 & a_0
\end{b matrix}
\]

A theory of Junctions of such matrices, coiled complicated numbers, is presented by Good, "A simple generalization of complex functions", *Expositiones Mathematicae* 6 (1988), 289-311. In three dimensions, poles of Junctions are replaced by straight lines. Skew circulants are also of interest in the theory of numbers, for example, every prime of the form 8n+1 is equal to a 4 by 4 skew circulant with integer elements, just as in the classic theorem that every prime of the form 4n+1 is of the form \( a_0^2 + a_1^2 \) (Good, *Fibonacci Quarterly* 24, 1986, 47-60; 176-177; Waterhouse, *Fibonacci Quarterly* 26, 1988, 172-177).

Yours sincerely,

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Dear Editor,

The solvers of Problem #663 (page 617, Fall 1988) were too industrious to see the easy methods. The question was to express \( \int_0^{\pi/2} \frac{x}{\sin x} \, dx \) as a series.

(a) \[ \text{Int. } \int_0^\infty \frac{x}{\sin x} \, dx = 2 \int_0^\infty (e^{-x} - e^{-3x} + e^{-5x} - \ldots ) \, dx = 2 \left( \frac{1}{1^2} + \frac{1}{3^2} + \ldots \right). \]

The first step is clear from complex function theory, but can also be (one by first-year calculus methods as follows:

\[
\int_0^{\pi/2} \frac{x}{\sin x} \, dx = 2 \int_0^1 \frac{\tan^{-1} t}{t} \, dt = 2 \int_0^1 \frac{\log t}{1 + t^2} \, dt = 2 \int_0^1 \frac{x}{e^x - e^{-x}} \, dx = \int_0^{\pi/2} \frac{x}{\cosh x} \, dx
\]

by changing variables, then integrating by parts, and then changing variables again in the obvious way.

(b) Another calculation, slightly less elementary, second or third year, is as follows:

Put \( x = \frac{1}{2} \left( \sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} \right) \). and then use the fact that

\[
\int_0^{\pi/2} \frac{\sin (2n+1)x}{\sin x} \, dx = \frac{\pi}{2^n}. \quad \text{The same series expression comes out.}
\]

Of course, you are welcome to use these bits if you think them of any interest, but I would rather not have my name attached. Modern youngsters have a deplorable tendency to look up books and believe what they read instead of working things out for themselves. But now that I have retired I try not to worry about it.

Maine withheld by request
1989 NATIONAL PI MU EPSILON MEETING

The Annual Meeting of the Pi Mu Epsilon National Honorary Mathematics Society was held at the University of Colorado in Boulder August 7 through August 9. The year, 1989, marked the 75th Anniversary of the founding of Pi Mu Epsilon and the 40th Anniversary of the establishment of the Pi Mu Epsilon Journal.

Letters of congratulation and/or certificates were received from the American Mathematical Society, the Governor of New Jersey, President George Bush, the Governor of Colorado, the National Council of Teachers of Mathematics, Kappa Mu Epsilon, The Mathematical Association of America and the Association for Women in Mathematics. Memorabilia, including the original journal of the Mathematics Club of Syracuse University on the founding of Pi Mu Epsilon, were on exhibit in Boulder, courtesy of the Library of Syracuse University.

A generous National Security Agency grant enabled Pi Mu Epsilon to support an increased number of student paper presenters at the meeting. In honor of Pi Mu Epsilon’s 75th Anniversary, the American Mathematical Society announced an annual grant to be administered by Pi Mu Epsilon to further scholarship in undergraduate mathematics. In 1989, part of this grant was used to provide prizes to students whose paper presentations were judged to be of especially high quality by members of the Pi Mu Epsilon Council.

W. H. Freeman and Company Publishers, PWS-Kent Publishing Company and Brooks/Cole Publishing Company provided financial support for the opening reception and a selection of books to further the goals of the Society. Pi Mu Epsilon hosted the Western Hoe Down, the big social event of the joint meeting with The Mathematical Association of America and the American Mathematical Society.

The AMS-MAA-PME Invited Address “The Mathematics of Identification Numbers” was presented by Joseph A. Gallian, University of Minnesota, Duluth. The J. Sutherland Frame Lecturer was Professor Jane Cronin Scanlon, Rutgers University. Her lecture was “Entrainment of Frequency: A Recurring Theme.” A special T-shirt in honor of the Society’s 75th Anniversary was on sale and is still available from the Editor. An ad for the T-shirt appears on page 72.

At the Annual Banquet, $100 awards for excellence in presentation were awarded to the following nine students: Beth-Allyn Eggsens, Chikako Mese, Darrin Frey, William C. Regli, M. Chris Haase, Robert A. Cullen, Stephen J. Smith, Nicholas Ahn, and Michele Pezet. The complete program of 46 student papers follows.

PROGRAM • STUDENT PAPER SESSIONS

**Mathematics & Digital Image Processing**

- Nicholas Ahn
- Illinois Zeta
- Elmhurst College

**Chaotic Linear Transformations on a Toms**

- Joel Atkins
- Indiana Gamma
- Rose-Hulman Institute of Technology

**Hamiltonian and Eulerian Circuits in the Join of Two Connected Graphs**

- Timothy Bohmer
- Ohio Zeta
- University of Dayton

**Solving Diophantine Equations Using Continued Fractions**

- Jim Banoczi
- Ohio Xi
- Youngstown State University

**A Brief Introduction to Fractal Images**

- Mark Boardman, presenter
- David Lewitt
- Nebraska Alpha
- University of Nebraska

**The Hyperbolic Geometry of M. C. Escher**

- Kathleen L. Brigham
- Illinois Epsilon
- Northern Illinois University

**Automorphism Groups of Hasse Subgroup Diagrams for Groups of Low Order**

- Melanie L. Butt
- Tennessee Gamma
- Middle Tennessee State University

**Evolutionary Evaluation of Risk Strategies**

- Elizabeth Clarkson
- Kansas Gamma
- Wichita State University

**Put Up more Wallpaper, It’s Friezeing in Here**

- James E. Collander
- Minnesota Gamma
- Macalester College

**The Classification of Finite Simple Groups**

- Robert A. Cullen
- Wisconsin Alpha
- Marquette University

**Fibonacci Periods mod(m)**

- Keith R. Dean
- Texas Delta
- Stephen F. Austin State University

**A Generalization of Odd and Even Vertices in Graphs. Part I**

- Amy Dykstra
- Michigan Epsilon
- Western Michigan University

**A Phase Aversion method in Geophysics**

- Richard L. Edington
- Texas Delta
- Stephen F. Austin State University

**Change for a Dollar - How Many Ways?**

- Beth-Allyn Eggsens
- Ohio Xi
- Youngstown State University
Is a Transitive Banach Space a Hilbert Space?

mathematics for a Digital Controlling Unit Used in a Forestry Experiment

Plucking a Leaf off a Tree and Other Graphs

An Application of the Rayleigh-Ritz method

Evolutionary Operation

An Elementary Analysis of Conformal mappings of Simply-Connected Domains

An Approximation for the number of Primes between \( k \) and \( k^2 \) when \( k \) is an Integer

A mathematical method for Finding Anisotropy Constants

A Computer Is Worth a Thousand Blackboards

Bounding the Chromatic number of a Graph

How Many Licks Does It Take to Reach the Center of a Tootsie Roll Pop?

Games, Graph Theory, Algorithms, and Kayles

The Determination of the Expected Length of a Coin Toss Game
A Study of Linear Singularly Perturbed Systems

Ahmad Reza Sarhangi
Kansas Gamma
Wichita State University

A Generalization of Odd and Even Vertices in Graphs, Part 2

Michelle Schultz
Michigan Epsilon
Western Michigan University

Computer Go

Stephen J. Smith
Pennsylvania Eta
Dickinson College

A Statistical Soft Drink Taste Test

Wendy R. Smith
South Carolina Gamma
College of Charleston

making "tents" Out of Math - One of Its Practical Uses

Jenny Spence
Wisconsin Delta
St. Norbert College

Resonance: Is It Live, or Is It . . .?

Tim Strnad
Wisconsin Delta
St. Norbert College

Seen Any Good Films Lately? - An Introduction to Some of the notions of Geometric measure Theory

Karen B. Taylor
Kansas Gamma
Wichita State University

Pseudo-Orbit Shadowing on the Unit Interval

Jeffrey Van Eeuwen, presenter
Tim Pennings
Michigan Delta
Hope College

Changes of Address/Inquiries

Subscribers to the Journal should keep the Editor informed of changes in mailing address. Journals are mailed at bulk rate and are not forwarded by the postal system. The cost of sending replacement copies by first class mail is prohibitive.

Inquiries about certificates, pins, posters, matching prize funds, support for regional meetings, and travel support for national meetings should be directed to the Secretary-Treasurer, Robert M. Woodside, Department of Mathematics, East Carolina University, Greenville, NC 27858. 919-757-6414.
Dear Professor Poiani:

On behalf of the Association for Women in Mathematics, I extend warm congratulations to Pi Mu Epsilon on the occasion of its 75th Anniversary. It is our hope that, through its role as a national honor society promoting research and scholarship in mathematics, Pi Mu Epsilon will encourage more undergraduate women to continue in mathematics, and to go on to successful careers in the mathematical sciences.

Sincerely,

Bill P. Mestrov
President

Association for Women in Mathematics

Office Address: Box 178, Wellesley College, Wellesley, Massachusetts 02181
Telephone: 617-285-0220 Tel. 2403

July 6, 1989

The great German mathematician Carl Friedrich Gauss called mathematics the "queen of the sciences" -- an apt description for this field of knowledge that has, from the very beginning of civilization, been one of man's ablest tools in understanding and working in the world around him. Medicine, engineering, space exploration -- the great feats accomplished in these and so many other fields would be impossible without mathematics.

For 75 years, your society has encouraged and furthered excellence in mathematics. In so doing, you have not only enriched the scholarly pursuits of your members but also touched the lives of all, because we all depend on the fruits of applied mathematics in our everyday lives.

I salute you for your efforts and achievements, and wish you an enjoyable celebration and every future success. God bless you.

JPM/cc
4. Proposed by the Editor.

(A timely variation on a familiar theme.) Find a law of formation for the 5 x 5 array

\[
\begin{array}{|c|c|c|c|c|}
\hline
164 & 244 & 306 & 128 & 448 \\
\hline
268 & 348 & 410 & 232 & 552 \\
\hline
387 & 467 & 529 & 351 & 671 \\
\hline
425 & 505 & 567 & 389 & 709 \\
\hline
276 & 356 & 418 & 240 & 560 \\
\hline
\end{array}
\]

5. Proposed by the Editor.

Label the sixteen vertices of the "cube within a cube" so that the twenty-four quadrilateral faces have equal vertex sums.

6. Proposed by the Editor.

By making cuts along its diagonals, a square can be dissected into four pieces which can be reassembled to form two congruent squares. By making cuts along the line segments joining the midpoints of opposite sides, the square can be dissected into four congruent squares. By cutting a square along the four line segments joining vertices to midpoints of opposite sides, the square can be dissected into nine pieces which can be reassembled to form two congruent squares. Dissect a square into a "small number" of pieces which can be reassembled to form three congruent squares.

7. Proposed by the Editor.

In a certain mathematics journal, seven puzzles were proposed. In response, for each puzzle the Editor received two correct solutions. In all, 14 solutions were submitted by 7 different readers, two solutions from each. Is it possible to publish the readers' solutions so that exactly one from each of the seven contributors will appear?

COMMENTS ON PUZZLES 1⁻7, SPRING 1989

Responses to Puzzle #1 were either 101! = 1111000 in base 2 or 010! = 3628800 in base 10. In Puzzle #2, several readers recognized the old puzzle of drawing a continuous path of four line segments through a 3x3 array of points without passing through any of the nine points more than one time. The secret is to "overshoot" the 2 and the 4. For Puzzle #3, the nine responses were quite varied. The most succinct was RICHARD HESS: "These are the integers expressible in base 3 using only ones and zeros." In Puzzle #4, the shortest solution for going from ONE to TWO was VICTOR FESER's ONE - ORE - ORT - OAT - TAT - TOT - TOO - TWO. Nineteen readers responded to the matching problem in Puzzle #5 and were in complete agreement (1 - comb, 2 - pen, 3 - key, 4 - book). The solution to Puzzle #6 is not unique. One solution is to arrange the numbers 1 through 15 in three rows 1, 2, 11, 12, 14; 8, 9, 10, 7, 6; 15, 1, 3, 5, 4. In all solutions, row sums are 40 and column sums 24. ROBERT PRIELIPP pointed out that Puzzle #7 had appeared as Problem 73 in the January 1970 issue of the Journal of Recreational Mathematics. The longest chain consists of six isosceles triangles with degrees 124°, 28°, 28°, 28°, 76°, 76°, 76°, 52°, 52°, 52°, 64°, 64°, 64°, 58°, 58° and 58°, 61°, 61°.

SOLVERS: Charles Aschbacher (1, 3, 5, 6), Amy Bohacheck (5, 6, 7), Margaret Boles (5), William Bougler (1, 3, 4, 5, 6, 7), Matthew Broadhead (2, 3, 4, 5, 6, 7), William Chau (1, 3, 5, 6, 7), Chris Conrad (6, 7), Anna Contadino (5), Victor Feser (1, 4, 5), Robert C. Gephart (5), I. J. Good (4), Richard J. Hess (1, 2, 3, 5, 6, 7), Donna Hiendst (3, 6), Jon Lange (7), Bro. Howard Lohrey, SM. (2, 5, 6), Thomas Mitchell (5), Donald B. Onnen (1, 2, 3, 4, 5, 6, 7), Robert Prieplipp (4, 7), Emil Slowniak (1, 3, 4, 5, 6, 7), Michael Taylor (5, 6), Katharine Vance (5), Tian-Yih Wang (5, 6) and Yvonne Zhou (1, 5, 6).

ERRATA

William Chau and Thomas Mitchell pointed out the omission of a square root symbol on page 679 of the Spring 1989 issue in the discussion of the solution to Puzzle #3 in the Fall 1988 issue.

Solution to Mathacrostic No. 28 (Spring 1989)

WORDS

A. Wythoff's Nim
B. Penrose Tiles
C. Offshoot
D. Unpolished
E. Neusis
F. Dehydrated Elephant
G. Swivel Joint
H. Time Reversal
I. One-time Pad
J. Norm
K. Ectotone
L. Lettover
M. Axiom of Choice
N. Benford's Law
Q. Yang-Mills Gauge Field
P. Relativity
Q. Itself
R. Neurile
S. Trapdoor
T. Hilbert's Hotel
U. Slingshot Effect
V. Outlier
W. Flowsnake
X. Race
Y. Eotvos
Z. Aeolian
a. Sphinx
b. Ophiuride
T. Necker Cube

AUTHOR AND TITLE: W. POUNDSTONE LABYRINTHS OF REASON

QUOTATION: There is a subversive joy in seeing logic tumble like a house of cards. All the well-known paradoxes of confirmation theory and epistemology were conceived more or less in the spirit of intellectual play. In few other fields is it possible for the interested nonexpert to sample so much of the true flavor of the field and have fun doing it.

SOLVERS: JEANETTE BICKLEY, St. Louis Community College at Meramec, MO; J. KEVIN COLLIGAN, National Security Agency; CHARLES R. DIMINNIE, St. Bonaventure University, NY; ROBERT FORSBERG, Lexington, MA; MICHELE HEBBERG, Herman, MN; JOAN AND DICK JORDAN, Indianapolis, IN; DR. THEODORE KAUFMAN, Brooklyn, NY; HENRY S. LIEBERMAN, Waban, MA; CHARLOTTE MAINES, Rochester, NY; DON PFAFF, University of Nevada-Reno; STEPHANIE SLOYAN, Georgian Court College, Lakewood, NJ; MICHAEL TAYLOR, Indianapolis Power and Light, Co., IN; and BARBARA ZEEBERG, Denver, CO.

Mathacrostic No. 29

Proposed by Joseph D. E. Konhauser

The 239 letters to be entered in the numbered spaces in the grid will be identical to those in the 25 keyed words at the matching numbers. The key numbers have been entered in the grid to assist in constructing the solution. When completed, the initial letters of the Words will give the names(s) of the author(s) and the title of a book; the completed grid will be a quotation from that book.
## Definitions

A. **Wood inlay which flourished in Italy during the Renaissance**

B. **One of order n gives rise to n-1 mutually orthogonal Latin squares (2 wds.)**

C. **A device consisting of balls of equal mass on strings of equal length to illustrate elastic impact (2 wds.)**

D. **A multi-layered structure which simulates chaotic folding (2 wds.)**

E. **John________ pseudonym under which mathematician, Eric Temple Bell, wrote science fiction**

F. **A period or state of decline (2 wds.)**

G. **The pivoted swinging bar to which the traces of a harness are fastened and by which a vehicle or implement is drawn**

H. **A small, stemless aquatic plant of the mustard family having slender, sharp-pointed leaves and minute white flowers**

I. **Geometry, a picturesque but inaccurate description of the intrinsic topology of a surface (sometimes comp.)**

J. **A very small amount (3 wds.; or 2 wds., one comp.)**

K. **A movement in art and literature, 1918-1922, intended to outrage and offend by flouting traditional aesthetic standards and social mores**

L. **Complete (comp.)**

M. **Something that is seen or intuited**

N. **The three concepts whose unity is symbolized by the triple pentagon emblem of the Berlin Philharmonic (3 wds.)**

O. **James Lovelock's theory that the earth, its oceans and atmosphere, and all living things are parts of one great organism**

P. **The upper integral of the characteristic function of a point set P on an interval (a,b) (2 wds.)**

Q. **To become apparent**

R. **Compact, connected, and locally connected metric spaces (2 wds.)**

S. **"Books are the____ of men."
Mark Twain (2 wds.)**

T. **Dodecahedron-based game sold to a London toymaker for 25 £ in 1859 by Sir William Rowan Hamilton (3 wds.)**

### Words

| 144 | 60 | 14 | 226 | 167 | 49 | 205 | 150 |
| 81 | 159 | 52 | 35 | 121 | 169 | 113 | 170 | 61 | 255 | 39 |
| 108 | 166 | 197 | 9 | 87 | 45 | 185 | 69 | 129 | 209 | 173 |
| 238 | 191 |
| 141 | 156 | 48 | 131 | 239 | 3 | 208 | 17 | 56 | 72 | 195 |
| 106 | 123 | 53 | 181 |

### Clue Answers

| 18 | 138 | 96 | 204 | 42 | 26 | 192 | 176 | 6 |
| 180 | 177 | 25 | 82 | 79 | 162 | 8 | 95 | 233 | 154 | 29 |
| 88 | 207232 | 143 | 174 | 114 | 125 |
| 109 | 236 | 139 | 102 | 70 | 177 | 66 |
| 92 | 88 | 161 | 213 | 231 | 59 | 200 | 33 | 196 | 101 |
PROBLEM DEPARTMENT

Edited by Clayton W. Dodge
University of Maine

This department welcomes problems believed to be new and at a level appropriate for the readers of this Journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

All communications should be addressed to C. W. Dodge, Math. Dept., University of Maine, Orono, ME 04469. Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly identified with name and address. Solutions to problems in this issue should be mailed by July 1, 1990.

PROBLEMS FOR SOLUTION

704. Proposed by the late Charles W. Trigg, San Diego, California.
Find the least HEAT necessary to BOIL the H2O:

\[ \text{HEAT} + \text{HHO} = \text{BOIL} \]

705. Proposed by the late Charles W. Trigg, San Diego, California.
In this "Ovis" group, the EWES and every LAMB are in prime condition. Find the two solutions:

\[ \text{RAM} + \text{EWES} + \text{LAMB} + \text{IAMB} = \text{SHEEP}. \]

706. Proposed by John Dulbec, Ohio Xi Chapter, Youngstown State University, Youngstown, Ohio.
This alphametic is too "compact" to have a unique solution. If, however, one CECHs for primality, then there is just one conclusion:

\[ \text{STONE} + \text{CECH} = \text{LECAR}. \]

707. Proposed by Murray S. Khanji, University of Alberta, Edmonton, Alberta, Canada.
From a point R taken on any circular arc PQ of less than a quadrant, two segments are drawn, one to an extremity P of the arc and the other RS perpendicular to the chord PQ of the arc and terminated by it. Determine the maximum of the sum PR + RS of the lengths of these two segments. This problem without solution is given in Todhunter's Trigonometry.

Find a Mascheroni construction (a construction using only compasses -- no straightedge allowed) for the orthic triangle of an acute \( \triangle ABC \).

If \( a, b, \) and \( c \) are the lengths of the sides of a triangle and if \( K \) and \( P \) are the area and perimeter, respectively, then prove that

\[ a^2b^2 + b^2c^2 + c^2a^2 \geq 12K^2 + \frac{a^4}{108} \]

with equality if and only if the triangle is equilateral.

710. Proposed by Thomas E. Moore, Bridgewater State College, Bridgewater, Massachusetts.
Under what conditions on the positive integers \( a \) and \( b \) will the sides of a nondegenerate triangle be formed by

a) \( a, \) \( b, \) and \( \gcd(a,b)? \)
b) \( a, b, \) and \( \text{lcm}(a,b)? \)

711. Proposed by James N. Boyd, St. Christopher's School, Richmond, Virginia.
A pentagon is constructed with five segments of lengths 1, 1, 1, 1, and \( w \). Find \( w \) so that the pentagon will have the greatest area.

A cube 4 inches on a side is painted. Then it is cut into 64 one-inch cubes. A cube is chosen at random and tossed. Find the probability that none of the five faces that are showing is painted.

713. Proposed by R. S. Luthar, University of Wisconsin Center, Janesville, Wisconsin.
Evaluate

\[ \int_{\pi/30}^{\pi/60} \tan5x \tan3x \tan2x \, dx . \]
714. Proposed by Sam Pearsall, Loyola Marymount University, Los Angeles, California.
A flea crawls at the constant rate $r = 1$ foot per minute along a uniformly stretched elastic band, starting at one end. The band is initially $L = 1$ yard in length and is instantaneously and uniformly stretched $L = 1$ yard at the end of each minute while the flea maintains his grip on the band at the instant of each stretch. It is well known that the flea will reach the other end of the band in under 11 minutes. Find all lengths $L$ such that the flea will reach the other end of the band in finite time.

715. Proposed by Christopher Stuart, New Mexico State University, University Park, New Mexico.
Euler's constant $\gamma$ is defined by the equation
\[
\gamma = \lim_{N \to \infty} \left( \sum_{k=1}^{N} \frac{1}{k} - \ln N \right)
\]
Show that
\[
\gamma = \sum_{k=2}^{\infty} \sum_{j=1}^{\infty} \frac{(-1)^k}{kj^k}.
\]

It is known that, for $x, y, z > 0$,
\[
\sqrt{xy} + \sqrt{yz} + \sqrt{zx} \leq 3\sqrt[3]{xyz} + xy + xz.
\]
Prove the "other side" of this inequality, namely,
\[
\sqrt{xy} + \sqrt{yz} + \sqrt{zx} \geq 3\sqrt{\frac{xyz}{x+y+z}}.
\]

717. Proposed by Russell Euler, Northwest Missouri State University, Maryville, Missouri.
Find all positive integers $n$ for which
\[
\sum_{k=1}^{n} (-1)^{k-1} \frac{n}{k} \text{ is an integer}.
\]

718. Proposed by David Petry, Eugene, Oregon.
Prove or find a counterexample: If $a, b, c, p$ are integers such that $0 \leq a < b < c \leq 2p + 1$, then $a^p + b^p \leq c^p$.

SOLUTIONS

Find all solutions to this base ten multiplication alphametric in honor of my Soviet mathematician and theoretical physicist pen pal who also is a regular contributor to this department:

\[ \text{DMITRI} = P \cdot \text{MAVLO}. \]

Solution by Alan Wayne, Holiday, Florida.
Because a BASIC program to solve this problem on my small computer takes more than 500 hours to run if no power surges occur, I have resorted to a "by hand" search. It took only about 50 hours, with the following five steps.
1. For $P = 2$ to 9, for $D = 1$ to 9, and for $M = 1$ to 9, $M$ is the greatest integer in $(10D + M)/P$. This determines 32 ordered triples $(P, D, M)$.
2. The product of $P$ and $O$ ends in $I$. This determines 44 ordered triples $(P, O, I)$.
3. Combining the previous results, omitting duplicated digits, we find 99 ordered quintuples $(P, D, M, O, I)$.
4. Each of these quintuples is examined for values of $V$; then for values of $T$; and finally, if need be, for the three remaining values possible for $L$.
5. Two solutions result: 32695 and 356426. Also solved by CHARLES ASHBACHER, Mount Mercy College, Cedar Rapids, IA, MARK EVANS (partial solution), Louisville, KY; ROBERT C. GEBHARDT, Hopatcong, NJ; RICHARD L. HESS, Rancho Palos Verdes, CA; L. J. UPTON, Mississauga, Ontario, Canada; LIEN VUONG, Texas A & M University, College Station, and the PROPOSER.


a) Prove this inequality for positive real numbers $U, S, A$, dedicated to 100 years of American mathematics, as evidenced by the 100th anniversary of the American Mathematical Society:
\[
\frac{U}{(1 + U)(1 + S)} + \frac{S}{(1 + S)(1 + A)} + \frac{A}{(1 + A)(1 + U)} \geq \frac{3\text{USA}}{(1 + \text{USA})^2},
\]
with equality if and only if $U = S = A = 1$.

b) Which inequality, if either, is more general, the USA inequality of part (a) or the $\pi \mu \varepsilon$ inequality of Problem 642 [Spring 1987, Spring 19881?
\[
(1 + \pi \mu \varepsilon) \cdot \left( \frac{1}{\pi(1 + \mu)} + \frac{1}{\mu(1 + \varepsilon)} + \frac{1}{\varepsilon(1 + \pi)} \right) \geq 3
\]
for positive numbers $\pi, \mu, \varepsilon$, with equality if and only if $\pi = \mu = \varepsilon = 1$?

Solution by the proposer.

a) We use the notation $\Sigma U = U + S + A$ and $\Sigma US = US + SA + AU$. First we prove the lemma:

\[
(1) \quad XU + \Sigma US \geq 6\text{USA} \quad \text{with equality if and only if } U = S = A = 1.
\]
By the AM-GM inequality we have \( U + SA > 2\sqrt{USA} \) and two similar inequalities, establishing the inequality of the lemma. Equality occurs when and only when \( U = SA \) and similarly \( S = AU \) and \( A = US \), which are true if and only if \( U = S = A = 1 \).

Next we prove another lemma.

\[
2(1 - USA + U^2S^2A^2) \geq \sqrt{USA}(1 + USA)
\]

with equality iff \( USA = 1 \).

Let \( t = \sqrt{USA} \). Then inequality (2) is equivalent to this chain of inequalities:

\[
2 - 2t^2 + 2t^4 \geq t(1 + t^2),
\]

\[
2 - t - 2t^2 - t^3 + 2t^4 \geq 0,
\]

\[
(t - 1)^2(2t^2 + 3t + 2) \geq 0.
\]

This last inequality is true since the quadratic factor has no real roots and is therefore always positive. Furthermore, equality holds only when the first factor is zero: when \( t = 1 \). Hence inequality (2) is established.

Now we prove the main theorem. Multiply both sides of the proposed inequality by the nonzero expression

\[
2(1 + U)(1 + S)(1 + A)(1 + USA)^2
\]

to get the equivalent inequality

\[
2(1 + USA)^2 (XU + SUM) \geq 6USA(1 + XU + SUM + USA),
\]

which reduces to

\[
2(1 - USA + U^2S^2A^2))(XU + SUM) \geq 6USA(1 + USA).
\]

This inequality is seen to be just the result of multiplying the inequalities (1) and (2) of the two lemmas side for side, establishing the theorem.

b) The USA inequality is more general. (Naturally in real life the prosperity of Pi Mu Epsilon should follow from the prosperity of the country.) To prove this assertion we rewrite both inequalities in the "unified" notation: \( U = x, S = y, A = z, \pi = x, \mu = y, \) and \( e = y: \)

\[
\frac{x}{1 + x} + \frac{y}{1 + y} + \frac{z}{1 + z} \geq \frac{3xyz}{(1 + x)(1 + y)(1 + z)}
\]

and

\[
(1 + xz)(1 + yz)(1 + z) \geq 3.
\]

We must show that the latter inequality follows from the former. To that end we shall rewrite each inequality to have the same left side. We get

\[
[ \frac{1 + xyz}{(1 + x)(1 + y)(1 + z)} ]^2 \geq \frac{3xyz}{(1 + x)(1 + y)(1 + z)}
\]

and

\[
[ \frac{1}{x(1 + x)} + \frac{1}{y(1 + y)} + \frac{1}{z(1 + z)} ] \geq 3.
\]

We must show that the right side of inequality (3) is greater than or equal to the right side of (4), which is equivalent to

\[
\frac{xyz}{(1 + x)(1 + y)(1 + z)} \geq \frac{3}{(1 + x)(1 + y)(1 + z)}
\]

We must show that the right side of inequality (3) is greater than or equal to the right side of (4), which is equivalent to

\[
\frac{xyz}{(1 + x)(1 + y)(1 + z)} \geq \frac{3}{(1 + x)(1 + y)(1 + z)}
\]

The substitution

\[
d = \frac{x}{1 + x}, \quad e = \frac{y}{1 + y}, \quad f = \frac{z}{1 + z}
\]

changes (5) into

\[
\left[ \sqrt{\frac{de}{e}} + \sqrt{\frac{ef}{d}} + \sqrt{\frac{d}{e}} \right]^2 \geq 3(d + e + f),
\]

which is equivalent to the following chain of inequalities:

\[
(de + ef + f^2) \geq 3def(d + e + f),
\]

\[
d^2e^2 - 2de^2f + f^2e^2 + (e^2f^2 - 2de^2f + d^2e^2) + (e^2f^2 - 2de^2f + d^2e^2) \geq 0,
\]

\[
d^2(e - f)^2 + e^2(f - d)^2 + f^2(d - e)^2 \geq 0.
\]

This last inequality is obviously true for any \( d, e, f \) in the reals and hence for any positive \( x, y, z \).


A regular heptagon (seven-sided polygon) is randomly placed far from an observer. Find the probability that the observer can see four sides of the heptagon.
Solution by Richard I. Hess, Rancho Palos Verdes, California.

If any odd-sided polygon (with n sides) is placed far from the observer, then the probability that he can see more (or less) than half the sides approaches 1/2 as the distance increases. To prove this statement, consider that another observer placed diametrically opposite the first one will see the complementary number of sides (for all but a finite number of positions). As the polygon is rotated through one revolution, then, each sees (n + 1)/2 sides just as often as the other. Hence the probability is 1/2.

Also solved by CHARLES ASHBACHER, Mount Mercy College, Cedar Rapids, IA, WILLIAM CHAU, Eggersville, NY, RICHARD DUNLAP (2 solutions), Georgia Tech, Atlanta, GREGORY F. MARTIN, University of North Florida, Jacksonville, PROBLEM SOLVING GROUP, University of Arizona, Tucson, and the PROPOSER.

681. [Fall 1988] Proposed by R. S. Luthar, University of Wisconsin Center, Janesville, Wisconsin.

Professor E. P. B. Umbugio is in the midst of writing his thirteen-volume treatise on analytic geometry. He would like to use the following theorem in Volume 9, but is having difficulty with it. Help the poor old professor by supplying a proof for him.

For I = 1, 2, ..., n, let $P_i$ represent the plane

$$\frac{x_i}{a_i} + \frac{y_i}{b_i} + 1 = 0 \quad \text{where} \quad 3a_i b_1 + 3b_i c_1 + 3c_i a_1 = a_i b_i c_i.$$

Then the intersection of all the planes is nonempty.

Solution by William Chau, Eggersville, New York.

The intersection of all the planes contains at least the point (3,3,3) since the given condition is equivalent to

$$\frac{3}{c_1} + \frac{3}{a_1} + \frac{3}{b_1} = 1.$$

It is clear that if the coefficients (3,3,3) in the given condition are replaced by the three numbers (p,q,r), then the intersection of the three planes will be the point (p,q,r). The problem can also be extended into hyperspace quite readily.

Solution by William Chau, Eggersville, New York.


Find all ordered pairs of nonzero integers a and b with b prime such that

$$a^3 - b^3 = a.$$

I. Solution by Alan Wayne, Holiday, Florida.

The given relation is equivalent to

$$(a - 1)a(a + 1) = b^3.$$  

The left member, being the product of three consecutive integers, contains both 2 and 3 as factors. Hence 6 divides $b^3$, so 6 divides b, as that b cannot be a prime. Therefore there is no solution.

Dropping the requirement that b be prime, the following result is easily proved by applying Descartes' Rule of Signs to the polynomial

$$P(x) = x^3 - x - b^3.$$  

The product of the three consecutive integers x - 1, x, and x + 1 is the cube of an integer b if and only if

$$(a,b) \in \{(-1,0),(0,0),(1,0)\}.$$  

II. Solution by Francis C. Leary, Saint Bonaventure University, New York.

There are no positive integral solutions even if b is not assumed prime. The given equation is equivalent to

$$a^3 - b^3 = a.$$  

Since the left side is even, then so is b. Let b = 2n for some nonzero integer n. Then $a$ must be a root of the polynomial

$$P(x) = x^3 - x - 8n^3.$$  

The discriminant of this polynomial is $D = 4 - 1728n^3$, which is clearly negative if n is a nonzero integer. Thus the polynomial has exactly one real root.

If n > 0, then $P(2n) = -2n < 0$ and $P(2n + 1) = 4n(3n + 1) > 0$. By the intermediate value theorem, $P(x) = 0$ for some x such that $2n < x < 2n + 1$. This x is the unique real root of $P(x) = 0$ and is clearly not an integer. A similar argument holds if n < 0. Thus the only integral solutions are the trivial ones $(a,b) = (1,0), (0,0),$ or $(-1,0)$.

Also solved by CHARLES ASHBACHER, Mount Mercy College, Cedar Rapids, IA, SEUNG-JIN BANG, Seoul, Korea, FRANK P. BATTLES and LAURA L. KELLEHER (2 solutions), Massachusetts Maritime Academy, Buzzards Bay, JAMES F. BURKE, Illinois Benedictine

a) Given three concentric circles, construct an isosceles right triangle so that its vertices lie on one circle.

b) Is the construction always possible?


Let the three circles be centered at the origin of the Cartesian plane and have radii 1, r, and s with r ≤ s, and let the right angle vertex C of right triangle ABC lie at the point where the circle of radius 1 crosses the x-axis. Let vertices A and B lie on the circles of radii r and s respectively. Let the sides of the triangle opposite vertices A, B, and C have lengths a, b, and c. See Figure 1. (This figure covers all cases except that where the circle on which the right angle vertex lies degenerates to a point, in which case the other two circles must coincide and the solution is clear.) Then we see that

\[
A = (1 + b \cos \theta, b \sin \theta) \quad \text{and} \quad B = (1 - a \sin \theta, a \cos \theta),
\]

where \( \theta \) is the angle of inclination of side b. Since A and B lie on circles of radii r and s, we have

\[
(1 + b \cos \theta)^2 + (b \sin \theta)^2 = r^2
\]

and

\[
(1 - a \sin \theta)^2 + (a \cos \theta)^2 = s^2.
\]

Since triangle ABC is isosceles, then a = b and these equations reduce to

\[
2a \cos \theta = 1 - a^2 + r^2 \quad \text{and} \quad 2a \sin \theta = 1 + a^2 - s^2.
\]

Now square both sides of both equations and then add to obtain the quartic in a),

\[
2a^4 - 2(r^2 + s^2) a^2 + (r^2 - 1)^2 + (s^2 - 1)^2 = 0.
\]

Now triangles can be constructed for those values of r and s which yield real roots of (1), in which case those roots have the form \( u \sqrt{v} \), where u and v may or may not be equal. Thus there are at most two solution triangles and their legs are the positive real roots of (1). Considering equation (1) as a quadratic in \( a^2 \), there will be two real roots when its discriminant

\[
D = -4(r^4 + s^4 - 2r^2 s^2 - 4r^2 - 4s^2 + 4) > 0.
\]

Let \( x = r^2 \) and \( y = s^2 \) and graph \( D = 0 \) in this new xy-plane. We get a parabola in the first quadrant, as shown in Figure 2. The solution set for the construction problem is the region inside the parabola and above the line \( y = x \) (so that \( r \leq s \)). That is, any \( r \) and \( s \) such that the point \( (r^2, s^2) \) lies in that region will permit the desired construction, and only those points. So the construction is not always possible. If the point lies on the parabola or if \( r = s = 1 \), there is just one solution triangle; if it is inside and not the point \((1,1)\), then there are two distinct solutions.

When a solution exists, all required operations can be performed with ruler and compass.
b) Let the radii of the three circles be \(a\), \(b\), and \(c\) where \(0 < a < b < c\).

Then the circles \(s\) and \(t'\) of part (a) will intersect if the appropriate following condition is satisfied. If the right angle vertex lies on

- circle (a), then we must have \(c - b \leq a\sqrt{2} \leq c + b\);
- circle (b), then we must have \(c - a \leq b\sqrt{2} \leq c + a\);
- circle (c), then we must have \(b - a \leq c\sqrt{2} \leq b + a\).

These conditions can be rewritten. Thus, if the right angle vertex lies on

- circle (a), then we must have \(b \geq c - a\sqrt{2}\);
- circle (b), then we must have \(a \leq c - b\sqrt{2}\);
- circle (c), then we must have \(a + b \geq c\sqrt{2}\).

**Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA.**

684. [Fall 1988] Proposed by Dmitry S. Mavlo, Moscow, 11 S. S. R.

This problem is dedicated to Paul Erdos on his 75th birthday. Erdos and Hans Debrunner published (EL Math. 11 (1956) 20) the following theorem: Let \(D, E, F\) be points on the interiors of sides \(BC, CA, AB\) of triangle \(ABC\). Then the area \([DEF]\) of triangle \(DEF\) cannot be less than the smallest of the three other triangles formed:

\[ [DEF] \geq \min([AEF], [CDE], [BFD]). \]

a) Prove this generalization of the Erdos-Debrunner inequality: Assuming the configuration of the Erdos-Debrunner inequality, for some fixed real number \(\alpha^*\), if \(-\infty < \alpha \leq \alpha^*\), then

\[ [DEF] \geq M(\alpha), \text{ where } M(\alpha) = \left( \frac{[AEF]^\alpha + [CDE]^\alpha + [BFD]^\alpha}{3} \right)^{1/\alpha} \]

is the power mean of order \(\alpha\) of the three positive areas \([AEF], [CDE], \text{ and } [BFD]\).

b) Determine the maximum value of \(\alpha^*\) for which the inequality holds.

c) Find all cases where equality holds.

d) Prove that, for \(\alpha = -1\), the inequality of part (a) is equivalent to the \(\pi \mu \varepsilon\) inequality referred to in Problem 679(b) above.

---

**Solution of the proposer.**

Let points \(D, E, F\) divide sides \(CB, AC, BA\) in the ratios \(\mu, \varepsilon, \pi\), respectively. Then

- \([AEF] = \frac{\varepsilon}{(\varepsilon + 1)(\pi + 1)} [ABC]\),
- \([BFD] = \frac{\pi}{(\pi + 1)(\mu + 1)} [ABC]\),
- \([CDE] = \frac{\mu}{(\mu + 1)(\varepsilon + 1)} [ABC]\),

and hence

\[ [DEF] = \frac{1 - \frac{\varepsilon}{(\varepsilon + 1)(\pi + 1)} - \frac{\pi}{(\pi + 1)(\mu + 1)} - \frac{\mu}{(\mu + 1)(\varepsilon + 1)}}{[AEF]^k + [CDE]^k + [BFD]^k} [ABC] \]

which becomes, when the above substitutions are made,

\[ [DEF]^k \geq \frac{3}{(\varepsilon + 1)(\pi + 1)(\mu + 1)} \]

and finally

\[ \left(1 - \frac{1}{\varepsilon(\mu + 1)} - \frac{1}{\pi(\varepsilon + 1)} - \frac{1}{\mu(\pi + 1)}\right)^k \geq \frac{3}{(1 + \pi \mu \varepsilon)^k}. \]

Thus we have proved the equivalence of inequality (1) for all \(k\) such that \(k^* \leq k < \infty\) for some \(k^*\) and the inequality of part (a).

We have also proved part \(d\), for if \(k = 1\), inequality (1) is equivalent to the \(\pi \mu \varepsilon\) inequality. Since the \(\pi \mu \varepsilon\) inequality is true, we have also proved the inequality of
part (a) for $a = -1$.

Now define

$$F(\pi, \mu, \varepsilon) = \left[ \frac{1}{\varepsilon (\mu + 1)} \right]^k + \left[ \frac{1}{\mu (\pi + 1)} \right]^k + \left[ \frac{1}{\pi (\varepsilon + 1)} \right]^k - \frac{3}{(1 + \pi \mu \varepsilon)^k}. $$

It is straightforward but tedious to set the three partial derivatives $\partial F/\partial \pi$, $\partial F/\partial \mu$, $\partial F/\partial \varepsilon$ equal to zero and solve simultaneously to get that $\pi = \mu = \varepsilon = 1$. Next we form all second order partial derivatives and evaluate them at $(1, 1, 1)$ to get

$$F_{11} = F_{22} = F_{33} = \frac{1}{2}k(k + 1),$$

$$F_{12} = F_{23} = F_{31} = F_{21} = F_{32} = F_{13} = \frac{1}{4}k(3 - k).$$

By the Sylvester theorem the function $F$ will have the point $(1, 1, 1)$ as a minimum if and only if the following three inequalities hold at the point $(1, 1, 1)$:

$$A_1 = F_{11} = \frac{1}{2}k(k + 1) > 0,$$
$$A_2 = \begin{vmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{vmatrix} > 0,$$
$$A_3 = \begin{vmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{vmatrix} > 0.$$

To that end we calculate that

$$\frac{1}{2}k(k + 1) > 0,$$
$$\frac{3}{16}k^2 - \frac{1}{3}(k + 5) > 0,$$
$$\frac{9}{8}k^3 (k - \frac{1}{3})^2 > 0,$$

which are all true if and only if $k > 113$. That is, for all positive $\pi, \mu, \varepsilon$ and $k \geq 113$, we have

$$F(\pi, \mu, \varepsilon) \geq F(1, 1, 1) = 0.$$

Since $k = -a$, we have shown that the original inequality of part (a) holds for $a \leq -113$. That is, we have proved part (a) and also we have shown that $a' = -113$ is the value that satisfies part (b). Additionally, we have seen that equality holds if and only if $\pi = \mu = \varepsilon = 1$, that is, when points $D, E, F$ are the midpoints of the sides of triangle $ABC$.

Editorial note. The proposer's details of the work summarized in the last two paragraphs will be furnished by the problems editor upon request.

685. [Fall 1988] Proposed by R. S. Luther, University of Wisconsin Center, Janesville, Wisconsin.

In any triangle $ABC$ with $C < 45^\circ$ and given any other angle $D$ with $0^\circ < D < 45^\circ$, prove that

$$b \cos D - c \cos (A - D) < a.$$

Solution by Bob Priest, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin.

More generally, we shall show that if $ABC$ is a triangle and $D$ is any angle with $0^\circ \leq D \leq 180^\circ$, then $b \cos D - c \cos A \cos D < a$.

Since $0^\circ < C < 180^\circ$ and $0^\circ \leq \cos D < 1$, it follows that $a \cos C \cos D < a$. Hence

$$(a \cos C + c \cos A) \cos D - c \cos A \cos D < a,$$

making $b \cos D - c \cos A \cos D$ because $b = a \cos C + c \cos A$.


Determine the matrix $[A^3 - A^2 + I]^{-1}$ where $A$ is an $n$ by $n$ matrix such that $A^5 + A = 5nI$ and $I$ is the identity matrix.

Solution by the proposer.

The number $5n$ can be replaced by any number except $-1$, say $m - 1$. Then

$$ml = A^5 + A + I = [A^2 + A + I][A^3 - A^2 + I],$$

so


For the stated problem, then, we have that

$$[A^3 - A^2 + I]^{-1} = [A^2 + A + I]^{-1} (5n + 1).$$


687. [Fall 1988] Proposed by Basil Rene, Burnside, South Australia.

For positive reals $x$ and $y$, prove the "quaint little inequality,"
4xy \leq (x + y)(xy + 1).

I. Solution by Bob Pridipp, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin. 

The required inequality is equivalent to

\[
\frac{x^2 y + xy^2}{4} + x + y \geq xy = 4\sqrt{x^4 y^4}
\]

which follows immediately from the arithmetic mean-geometric mean inequality with n = 4. Equality holds if and only if

\[
x^2 y = xy^2 = x = y, \quad \text{that is,} \quad x = y = 1.
\]

II. Solution by George P. Evansovich, Saint Peter's College, Jersey City, New Jersey. 

Let a, b, x, y be positive real numbers. By the AM-GM inequality,

\[ab + xy \geq 2\sqrt{abxy}\] and \[ax + by \geq 2\sqrt{abxy}.

Multiply together the two inequalities to get the more general inequality

\[(ab + xy)(ax + by) \geq 4abxy.
\]

Now set a = b = 1 to get the desired inequality.


We have

\[0 \leq (x - 1)^2 + y(x - 1)^2\]

\[= x^2 y - 2y + 1 + y(x^2 - 2x + 1)\]

\[= (x + y)(x + 1) - 4xy.
\]

Equality holds for \(x = y = 1\).

IV. Solution by St. Olaf Problem Solving Class, St. Olaf College, Northfield, Minnesota. 

Since \(x\) and \(y\) are positive, then \(x + \frac{1}{x} \geq 2\) and \(y + \frac{1}{y} \geq 2\). Consequently,

\[x + \frac{1}{x} + y + \frac{1}{y} \geq 4,
\]

so then

\[x^2 y + xy^2 + x = (x + y)(xy + 1) \geq 4xy.
\]

Another solution by RICHARD A. GIBBS, Fort Lewis College, Durango, CO, RICHARD I. HESS, Rancho Palos Verdes, CA, DAVID INY, Westinghouse Electric Corporation, Baltimore, MD, RICHARD DUNLAP, Georgia Tech, Atlanta, RICHARD A. GIBBS, Fort Lewis College, Durango, CO, RICHARD I. HESS, Rancho Palos Verdes, CA, DAVID INY, Westinghouse Electric Corporation, Baltimore, MD, PROBLEM SOLVING GROUP, University of Arizona, Tucson, and the PROPOSER.


A row of \(n\) chairs is to be occupied by \(n\) boys and girls taken from a group of more than \(n\) boys and more than \(n\) girls. If the boys do not want to sit next to one another, in how many ways can the children occupy the chairs? 

Solution by John Putz, Alma College, Alma, Michigan. 

Let \(f(n)\) denote the number of ways of seating \(n\) children. Assuming that the first \(n - 1\) chairs have been filled satisfactorily, the \(n\)th chair can certainly be filled by a girl. Therefore

\[f(n) = f(n - 1) + f(n - 2),
\]

for \(n > 2\). That is, \(f(n)\) is the \((n + 2)\)nd Fibonacci number \(F_n\), where,

\[F_1 = F_2 = 1 \quad \text{and} \quad F_n = F_{n+1} + F_{n+2} \quad \text{for} \quad n > 2.
\]

Also solved by WILLIAM CHAU, Eggertsville, NY; RICHARD DUNLAP, Georgia Tech, Atlanta, RICHARD A. GIBBS, Fort Lewis College, Durango, CO, RICHARD I. HESS, Rancho Palos Verdes, CA, DAVID INY, Westinghouse Electric Corporation, Baltimore, MD, PROBLEM SOLVING GROUP, University of Arizona, Tucson, and the PROPOSER.

Gibbs commented that this problem is a "fairly well-known result," citing a problem sheet he used several years ago. Hess asked about the solution if the boys and girls are distinguishable. Indeed, Dunlap provided a solution for this "more difficult problem."


Show that for any three infinite sequences of natural numbers

\[a_1, a_2, a_3, \ldots \quad b_1, b_2, b_3, \ldots \quad c_1, c_2, c_3, \ldots \]

there can be found numbers \(p\) and \(q\) such that \(a_p > a_q, b_p > b_q,\) and \(c_p > c_q.\)
Solution 5 Chris Long, Rutgers University, New Brunswick, New Jersey.

We prove the stronger result: If
\[ \{x_{11}, x_{12}, \ldots \}, \{x_{21}, x_{22}, \ldots \}, \ldots, \{x_{n1}, x_{n2}, \ldots \} \]

are infinite sequences of natural numbers, then there exist infinitely many pairs of numbers \( p, q \) with \( p < q \) such that
\[ x_{pk} \leq x_{qk} \quad \text{for} \quad 1 \leq k \leq n. \]

We prove the theorem by mathematical induction. For \( n = 1 \), let
\[ p = \inf \{ |x_{11} = \inf \{x_{11}, x_{12}, \ldots \} \}. \]

Then clearly \( p, q \) is such a pair of numbers for all \( q > p \).

Assume that the statement is true for \( 1, 2, \ldots, n-1 \).

Define
\[ \delta (1) = \inf \{ |x_{n1} = \inf \{x_{11}, x_{12}, \ldots \} \}, \]

and recursively define
\[ \delta (m) = \inf \{ |x_{n1} = \inf \{x_{n(m-1)+1}, x_{n(m-1)+2}, \ldots \} \}. \]

Consider the subsequences
\[ \{x_{1\delta (1)}, x_{1\delta (2)}, \ldots \}, \ldots, \{x_{n1, n-1\delta (1)}, x_{n1, n-1\delta (2)}, \ldots \}. \]

By the inductive assumption there are infinitely many pairs of numbers \( \delta (p), \delta (q) \) with \( \delta (p) < \delta (q) \) such that
\[ x_{k\delta (p)} \leq x_{k\delta (q)} \quad \text{for} \quad 1 \leq k \leq n-1. \]

We finish the inductive step by noting that the sequence
\[ \{x_{n\delta (1)}, x_{n\delta (2)}, \ldots \} \]

is nondecreasing by construction, so we also have that
\[ x_{n\delta (p)} \leq x_{n\delta (q)}. \]

Solution by the proposer.

We show that five circles of radius \( r = 0.3261606 \) will cover the square. Consider the figure. Let the diagonals of each of the four corner rectangles be \( 2r \) and let the circumradius of the isosceles triangle in the remaining rectangle be \( r \). Then we must have, using the notation of the figure,
\[ x^2 = 4r^2 - \frac{1}{4} \quad y^2 = 2x - \frac{3}{4} \quad \text{and} \quad r = \frac{1}{2} (1 - x) + \frac{(1 - 2y)^2}{8(1 - x)}. \]

The last equation is from the isosceles triangle. By calculator we find that \( r = 0.3261606, x = 0.4189546, \) and \( y = 0.2964947 \). The sketch in the figure shows that these five circles cover the square.

Now suppose a solution where one of the five circles lies inside the square and each of the other circles covers a vertex and each edge has a point covered by two circles. These four edge points and the four vertices partition the perimeter into eight segments whose lengths total 4 units. The sum of the squares of these segments is not less than \( 8 \cdot (1/2)^2 \) since the midpoint of an edge minimizes the sum of the squares on it. Hence at least one circle covers a segment (hypotenuse of a right triangle) of length at least \( 1/\sqrt{2} \). The radius of that circle is at least half that value, namely 0.35.

The only other possibility is for the fifth circle to cover a portion of one of the sides. This is the solution we have given above.

Solutions were also submitted by Richard I. Hess, Rancho Palos Verdes, CA, and Liem Vuong, Texas A & M University, College Station. Both solutions assumed one circle lying inside the square to produce a radius of \( \sqrt{2}/4 = 0.353553 \). The proposer, who is now at Westinghouse Electric Corp., Baltimore, MD, also gave a solution for six covering circles, proving that \( \sqrt{65}/16 \) is the minimum radius.

CORRECTIONS

Bob Prielipp pointed out a misplaced exponent in the solution to Problem 674 on page 694 of the Spring 1989 issue. The line

A unit square is covered by five circles of equal radius. Find the minimum necessary radius. (See Problem 507, Fall 1982).
IN MEMORIAM

Charles W. Trigg

Born February 7, 1898, he started his career as a chemist and during World War I invented an instant coffee soluble in cold water. In the next 10 years he published nearly 200 articles, notes and editorials on coffee, tea and spices. He began teaching chemistry in 1927. From 1938-43 he taught mathematics and physics at Los Angeles City College.

From 1943-46, serving to Lt. Commander in the U. S. Naval Reserve, Charles earned his wings as a navigator and taught celestial navigation.

In 1946 he returned to Los Angeles City College as Coordinator of Instruction, was promoted to full professor, and in 1955 became Dean of Instruction until his retirement in 1963.

In the ensuing 26 years he proposed hundreds of problems, submitted thousands of solutions, and wrote more than 500 articles, book reviews, and other items in mathematics. The LACC Engineering Department presented him with a diploma awarding him the degree of P.D.P.F. (Polyhedra Doctor in Paper Folding) for his careful cardboard-and-rubber-band geometric models, many of which hung in his office at his San Diego retirement home.

The late Léo Sauvè, editor of Crux Mathematicorum, conveyed upon him the title of "prince of digit delvers," but later demoted him to "count of digit delvers." That still left him with a D.D., Charles said.

Humor enlivens any serious study and Charles was a master at mathematical humor. Several of the editor's pseudonyms used in this department were suggested by Trigg, including S. E. Ducer, M. T. Kopf, Pauvre Fish, Bro. Kenarch, and Titus Canby. Nathan Altshiller Court commended him for endowing his contributions with "a quality which is rare, namely wit." The dedication of Howard Eves' 1988 book, Return to Mathematical Circles reads, "To Charles W. Trigg, the wittiest and cleverest of us all."

Charles W. Trigg died June 28, 1989. He was a delightful mathematician and problemist and a dear friend. We dedicate to his memory this issue of the Problem Department, which in his honor contains two extra of his digit-delving proposals, problems 704 and 705.

(-a_1/a_0)/n^n = ((-1)a_n/a_0)^{1/n}

In the Spring 1988 issue the solution to Problem 642 on page 539 has an error. William Chau discovered that the multiplication factor given there should not contain (1 + \pi\mu\epsilon); it should be only

\pi\mu\epsilon(1 + \pi)(1 + \mu)(1 + \epsilon).

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5. expository papers are actively encouraged
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The Editor and the author(s) appreciate you help. Please be frank with your comments and suggestions.

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Joseph D. E. Konhauser
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Smith College, Northampton, MA

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