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PI MU EPSILON JOURNAL

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ON THE REARRANGEMENT OF INFINITE SERIES

Gina Aurello
Seton Hall University

As is known to any student of analysis, the commutative property of finite sums does not always hold for infinite series. That is, while any permutation of the terms of $1 - 1/2 + 1/3 - 114 + 1/5$ will give the same sum, $47/60$, rearrangements of the terms of $1 - 1/2 + 113 - 114 + 115 - \ldots$ can lead to quite different sums. The purpose of this paper is to trace the origin and development of some of the results on rearranging series from Cauchy through Dirichlet and Riemann. Also, we will discuss two theorems, of O. Schlömilch and E. Borel, that are not generally known arid that partially answer the natural questions of how the rearrangement of its terms affects the sum of an infinite series and for which series rearranging terms does not change the sum.

It is commonly agreed [7] that Cauchy was the first to note that the commutative property is not always valid for infinite series. Specifically, he showed that a rearrangement of terms in a convergent infinite series can lead to a divergent series. This result is found in an 1833 paper [2, p. 57]. The example he used was the alternating harmonic series, $\sum_{n=1}^{m} \frac{(-1)^{n+1}}{n}$. He noted that the series is convergent by showing that the sequence of partial sums forms what is now called a Cauchy sequence, that is one in which for a given E there exists N such that $|s_n - s_m| <$ E when n and m are larger than N, where s_k denotes the sum of the first k terms of the series. He then rearranges the terms by grouping together terms from $1/(n+2)$ to $1/(n+2n)$, as

$$1 + \frac{1}{3} - \frac{1}{2} - \frac{1}{4} - \frac{1}{6} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} - \frac{1}{8} - \frac{1}{10} - \frac{1}{12} - \frac{1}{14} - \frac{1}{16}$$
$$- \frac{1}{18} + \frac{1}{11} + \frac{1}{13} + \frac{1}{15} + \frac{1}{17} + \frac{1}{19} + \frac{1}{21} + \frac{1}{23} + \frac{1}{25} + \frac{1}{27} - \ldots .$$

Cauchy said that the sequence of partial sums in the rearrangement cannot converge because it is possible to find arbitrarily large values of m and n so that $|s_m - s_n| > 1/3$. Though Cauchy did not use the integral test, we can see that each block of terms satisfies

$$\sum_{i=1}^{n} \frac{1}{n+2i} \approx \int_0^n \frac{1}{n+2x}\,dx = \frac{1}{2}\ln(n+2x)\Big|_0^n = \frac{1}{2}\ln 3 \approx .55 .$$

According to [7], this is the first example given of a rearrangement of a conditionally convergent series which is divergent.

In Dirichlet's paper about the distribution of primes, in arithmetic progressions [4], in which he proves that there are infinitely many primes of the form $an + b$ for any integers $a > 0$ and b that have greatest common devisor 1, he considers two classes of infinite series. The first class consists of series of real numbers that converge and whose sums are not altered by a rearrangement of terms. He was discussing unconditionally convergent series, which are equivalent to absolutely convergent series for real numbers, although he did not refer to them as such. Also, although Dirichlet did not formally state and prove it, this paper is the source of the following theorem usually attributed to him:

THEOREM. The sum of an absolutely convergent series of real numbers is not altered by a rearrangement of its terms.

The second class consists of series of real numbers which converge but whose sum is dependent on the order of terms. That is, the sum of the series can be changed by rearranging the terms. He made no mention of Cauchy's example given above, but gave examples of convergent series that may be rearranged so as to be divergent or to converge to a different sum. His first example is

$$1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{6}} + \ldots$$

and its rearrangement

$$1 + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{7}} - \frac{1}{\sqrt{4}} + \ldots .$$

The first series converges, while the second diverges. The reason for divergence is that in each group of three terms,

$$\frac{1}{\sqrt{4n+1}} + \frac{1}{\sqrt{4n+3}} - \frac{1}{\sqrt{2n+2}} > \frac{2}{\sqrt{4n+4}} - \frac{1}{\sqrt{2n+2}}$$

$$= \left(1 - \frac{1}{\sqrt{2}}\right)\frac{1}{\sqrt{n+1}}$$

and $\sum_{i=1}^{n} \frac{1 - 1/\sqrt{2}}{\sqrt{n+1}}$ diverges. Dirichlet's second example is

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots$$

and its rearrangement

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \ldots .$$

Though he does not give the sum for either series, Dirichlet notes that they converge to different sums. In fact, the first series converges to ln 2 and the second to (312) ln 2. Dirichlet added to Cauchy's result by showing that a rearrangement of a conditionally convergent series may be convergent to a different sum.

In his paper on representing functions by trigonometric series [5], Riemann noted that understanding the solution to that problem gives additional insight into the convergence of infinite series. He divided convergent series into two categories according to whether the series remains convergent or not when all terms are made positive; that is, absolutely or conditionally convergent series. He noted that in the first category, terms can be displaced and the sum will remain the same, while in the second category the sum can change and the series may even diverge when terms are rearranged. He then described an algorithm for rearranging terms in a conditionally convergent series so that the series converges to an arbitrary sum c: add positive terms until the sum is greater than c, then add negative terms until the sum is less than c, then add positive terms, and so on. As the process is continued, the difference between the sum and c gets closer and closer to zero (because the terms in a conditionally convergent series approach zero) and so the series converges to c. The germ of this idea may have come from a paper by Dirichlet on representing functions by trigonometric series [3]. In that paper, Dirichlet considered the series

$$\sum_{i=1}^{\infty} (-1)^{n+1} k_n, \quad \text{where} \quad k_n = \int_{(n-1)\pi}^{n\pi} \frac{\sin n}{n}\,dn$$

Dirichlet noted that the partial sums of the series oscillated between being greater than or less than $\pi/2$.

O. Schlomilch [6] proved a general theorem on the sum of a rearranged series:

THEOREM. If in the convergent series $s = a_0 - a_1 + a_2 + a_3 - \dots$ the terms are rearranged so that always p positive terms are followed by q negative terms, then the sum of the rearranged series is

$$S = s + \lim_{n\to\infty} \frac{na_n}{2} \ln \frac{p}{q}.$$

Dirichlet's examples can provide two illustrations. For the rearranged series

$$1 + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{7}} - \frac{1}{\sqrt{4}} + \dots,$$

Schlomilch's result becomes

$$S = s + \lim_{n\to\infty} \frac{n(1/\sqrt{n})}{2} \ln \frac{2}{1} = \lim_{n\to\infty} \frac{\sqrt{n}}{2} \ln 2,$$

which diverges to ∞. For the rearranged alternating harmonic series 1 + 113 - 1/2 + 1/5 + 1/7 - 114 + ... the theorem gives

$$S = \ln 2 + \lim_{n\to\infty} \frac{n(1/n)}{2} \ln 2 = \frac{3}{2} \ln 2.$$

The theorem also shows that there are conditions under which the sum of a conditionally convergent alternating series will remain the same for various rearrangements. For example, if $\lim_{n\to\infty} na_n < \infty$ and $p = q$, we have $S = s$. The sum also remains the same if $\lim_{n\to\infty} na_n = 0$. So, in the series $\sum_{n=2}^{\infty} \frac{(-1)''}{n \ln n}$, since $\lim_{n\to\infty} na_n = \lim_{n\to\infty} \frac{n}{n \ln n} = 0$, every rearrangement with p positive terms followed by q negative terms has the same sum. Note that this does not conflict with **Riemann's** algorithm for rearranging a conditionally convergent series to converge to any sum. In **Riemann's** procedure, p and q are not fixed.

The theorem also shows that if $\lim_{n\to\infty} na_n = \infty$ then with $p \neq q$ a **rearranged** series is divergent. Thus, no rearrangement of the convergent series $\sum_{n=1}^{\infty} (-1)^{n-1}/\sqrt{n}$ with fixedp and q will converge, since $\lim_{n\to\infty} na_n = \lim_{n\to\infty} \sqrt{n} = \infty$.

In **[1]**, E. Borel was concerned with conditions under which a rearrangement of terms in a conditionally convergent series will lead to the same sum. Before stating his theorem, a definition is **necessary.** If $\sum a_n$ is a conditionally convergent series and $\sum b_n$ is a rearrangement of it, then the displacement of a_n is defined by $\alpha_n = |n - m|$, where $a_n = b_m$. The maximum displacement of terms preceding the mth is $A = \max_{n \le m} a$. Borel proved the

THEOREM. Let $\sum a$ be a conditionally convergent series with sum s. Let $\sum b_n$ be a rearrangement of $\sum a_n$ and let $P_n = A \max_{r>n} |a_r|$. If $\lim_{m\to\infty} P_m = 0$ then $\sum b_n = s$.

As an example, Borel used

$$\sum a_n = 1 - \frac{1}{2\ln 2} + \frac{1}{3\ln 3} - \frac{1}{4\ln 4} + \frac{1}{5\ln 5} - \dots$$

with rearrangement

$$\sum b_n = 1 + \frac{1}{3\ln 3} - \frac{1}{2\ln 2} + \frac{1}{5\ln 5} + \frac{1}{7\ln 7} - \dots.$$

The terms of the series are related by $a_{2n} = b_{3n}$, $a_{4n+1} = b_{3n+1}$, and $a_{4n+3} = b_{3n+2}$ for $n \ge 1$. The sequence of displacements is $\{\alpha_n\} = \{0, 1, 1, 2, 1, 3, 2, 4, \dots\}$. It can be shown that the maximum displacement satisfies $\lambda_n < n + 1$. Hence

$$P_n = \lambda_n \max_{r>n} |a_r| < (n+1)\frac{1}{n \ln n}$$

so $\lim_{n\to\infty} P_n = 0$. Hence the two series have the same sum. Note that this result also follows from Schlomilch's theorem. Of course, **Borel's** theorem applies to many series for which Schlomilch's does not.

The theorems of Borel and Schlomilch can be used to generate many other examples that are not found in typical analysis texts.

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Gina *Aurello* prepared this paper, supervised by Professor John J. Saccoman and supported by a grant from the Clare Booth *Luce* Fund, before graduating from *Seton* Hall in 1993. Her mathematical talents are presently employed by the Securities Data Company.

Words and Numbers

In base 16, the digits past 9 are A, B, C, D, E, and F. Thus, some integers when written in base 16 are also English words, as 2766 is ACE and 48879 is BEEF. The problem is, what is the largest integer that, in base-16, is an ordinary English word?

A possible answer, not guaranteed to be right, appears on page 661.

The problem for languages other than English appears to be open.

New Mersenne Prime

On January 12, 1994, David **Slowinski** announced a new largest prime: $2^{859433} - 1$. It was found by a Cray C90 computer; the **Lucas-Lehmer** test for this exponent took 7.2 hours of computer time.

A NUMBER-THEORETIC IDENTITY ARISING FROM BURNSIDE'S ORBIT FORMULA

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What do you think of when you hear the term "group"? Assuming, of course, that you actually think of a mathematical object and not a psychological therapy, you probably think of a set with a binary composition that satisfies the axioms of associativity, identity, and inverses. But this viewpoint alone gives us only an isolated and static picture of the group; what we see amounts to little more than a multiplication table.

The group can reveal more of itself to us when it acts on a set, that is, it behaves as a group of permutations of the set. In this note, we will discuss a useful and elementary theorem concerning groups acting on sets called the *Burnside* orbit formula. This formula has many interesting combinatorial and number-theoretic consequences, one of which we will discuss.

Definition. A group G is said to act on a set X (on the right) just if there is a function $X \times G \to X$ (written $(x, g) \mapsto (x)g$) that satisfies

(1) The identity acts trivially: $(x)e = x$ for all $x \in X$.
(2) The action composes naturally: $x(gh) = (xg)h$ for all $g, h \in G$, $x \in X$.

Equivalently, G acts on X when we have a homomorphism $G \to Sym(X)$, say ϕ. In this way, each $g \in G$ can be considered to be a permutation of the set X by setting $xg := x(g\phi)$.

For instance, the group $Sym(n)$ acts on the set $\{1, \dots, n\}$ in the obvious manner, and so does any subgroup of $Sym(n)$. In particular, the cyclic group Z_n acts on $\{1, \dots, n\}$ by translation (mod n).

A natural question is to ask how elements get **fixed.** We can answer this question from either the viewpoint of G or that of X.

Definition. Given $g \in G$, $Fix(g)$ is the set of elements of X that g fixes. That is, $Fix(g) = \{x \in X \mid xg = x\}$.

Definition. Given $x \in X$, the stabilizer of x, labeled G_x, is the set of group elements of G that fix x. That is, $G_x = \{g \in G \mid xg = x\}$. Note that G_x is in fact a subgroup of G.

Now we turn to combinatorial information that arises when the group is finite. This first theorem, a counting principle, is absolutely fundamental.

THEOREM **(Orbit-Stabilizer Reciprocity).** The size of the orbit of an element equals the number of cosets of the stabilizer of that element, i. e. $|x^G| = [G:G_x]$.

Proof. To prove this, set up a bijection between the elements $\{xg \mid g \in G\}$ of x^G and the cosets $\{G_xg \mid g \in G\}$ by $xg \mapsto G_xg$. It is straightforward to verify that this mapping is well-defined and bijective.

The core result of this note is commonly called **Burnside's** Orbit Formula, although according to P. M. Neumann **[4]**, it antedates **Burnside's** treatment and is more properly attributed to **Cauchy** and **Frobenius.** Theorems have often been named for popularizers, collaborators, and even bystanders; so with that acknowledgement out of the way we will not feel too guilty about opting for name recognition.

THEOREM. (Burnside's Orbit **Formula).** The average size of $Fix(g)$ is equal to the number of orbits of G on X. That is,

$$\frac{1}{|G|}\sum_{g\in G} |Fix(g)| = |X/G|.$$

Proof. The proof consists of counting the set $\{(x, g) \mid g \text{ fixes } x\}$ in two ways. First sum over X and get $|Fix(g)|$ for each g to get $|G|$ times the left-hand side.

Next, sum over g to get $|G_x|$ for each x. Now on any one orbit, **Orbit-**Stabilizer Reciprocity says that $|G_x| = |G|/|x^G|$. Since $|G_x|$ appears in the sum exactly $|x^G|$ times, we get a contribution of $|G|$ for each orbit. Thus we get $|G|$ times the right-hand side.

Now we prove the number-theoretic result alluded to in the title. This sum probably looks as if it came completely out of the blue.

THEOREM. Let n be a positive integer and let $\phi(n)$ and $d(n)$ denote the Euler phi-fundion and the number of positive divisors of n (including n), respectively. Then

$$\sum_{\substack{a=0\\(a,n)=1}}^{n-1} (a-1, n) = \phi(n)d(n)$$

As a purely number-theoretic inequality, it does not immediately yield to a natural attack. But it is a fairly easy application of **Burnside's Orbit** Formula, the only trick is to find a group and a set for it to act on.

Proof. The action is surprisingly natural here. Let Z_n be the additive group of integers modulo n. Then the automorphism group $Aut(Z_n)$ is a group of order $\phi(n)$ which consists of the isomorphisms $\{f: x \mapsto xa \mid (a, n) = 1\}$, that is, right multiplications by the relatively prime integers less than n. Let $Aut(Z_n)$ act on the set Z_n.

Now we calculate the group order, the fixed points, and the number of orbits. As we have seen, the group elements of $Aut(Z_n)$ correspond to the integers that are relatively prime to and less than n, so $|Aut(Z_n)| = \phi(n)$.

An automorphism a fixes an element $k \in Z_n$ exactly when $ka \equiv k \pmod n$, that is, when $(a-1)k \equiv 0 \pmod n$. Now, we know from elementary number theory that $bx \equiv c \pmod n$ has solutions if and only if $(b, n) \mid c$, and then exactly (b, n) of them. So we see that the automorphism a has $(a-1, n)$ fixed points.

Finally, we count the number of orbits of $Aut(Z_n)$ on Z_n. Every automorphism (of any group) takes an element of a given order to one of the same order. In the case of Z_n, we show that the orbits are no smaller; that is, $Aut(Z_n)$ is transitive on the element of a given order.

It is clear that $Aut(Z_n)$ is transitive on the generators of Z_n. An element $k \in Z_n$ has order d exactly when $(k, n) = n/d$, so k has the form $(n/d)k'$ where k' is relatively prime to n. So, given two such elements $(n/d)k'$ and $(n/d)k''$, the automorphism $k' \mapsto k''$ takes one to the other. Therefore, the orbits look like $\{k \in Z_n \mid (k, n) = d\}$ and so there are exactly $d(n)$ of them.

Thus

$$\frac{1}{\phi(n)}\sum_{a\in Aut(Z_n)} (a-1, n) = d(n),$$

that is,

$$\sum_{\substack{a=0\\(a,n)=1}}^{n-1} (a-1, n) = d(n)\phi(n).$$

This formula can in fact be thought of as counting the number of subgroups of Z_n. This idea is expanded in **[6].**

There is also a classical theorem of **MacMahon** on the number of ways to construct a necklace of n beads from k colors of beads (the group action of Z_n becomes relevant when we ask that two necklaces be considered the same if we can rotate one into the other). It is proven in several of the references; for an instructive discussion of the proof and how to present it, see [3]. For further reading on Burnside's Orbit Formula (and groups in general), see [2] and [7]. A generalization of the orbit-counting method, Pólya's method of *weights*, is discussed in [1] and [5].

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This paper was prepared while the author was an ***undergraduate*** *at Kansas State University. He would like to thank Professor David Surowski for introducing him to the subject matter of the paper, and to algebra in general. He is presently a second-year graduate student at Princeton, aged 20.*

Chapter Report

Professor Doug Meade, chapter advisor with Professor Hong **Wang,** reports that the SOUTH CAROLINA ALPHA Chapter (University of South Carolina) initiated nine new members last fall. Events planned include a program on careers in mathematics and a "Math Night at the **Movies."**

A NEW METHOD FOR EVALUATING $\sum_{i=1}^{n} i^p$

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The problem of finding a formula for the sum $S_n^p = \sum_{i=1}^{n} i^p$, where p is a positive integer, has been studied by Luthar [2], Schaumberger [3], and many others. Their formulas for S_n^p involve S_n^k for $k = 1, 2, \dots, p-1$. In this note we present a new method, based on the binomial theorem, that requires only half of the values of S for $k < p$.

From the binomial theorem we know that

$$(1) \qquad (m+1)^{2k} = \sum_{i=1}^{2k} \binom{2k}{i} m^i + 1,$$

and that

$$(2) \qquad (m-1)^{2k} = \sum_{i=1}^{2k} \binom{2k}{i} (-1)^i m^i + 1.$$

Subtracting (2) from **(1),** we obtain

$$(m+1)^{2k} - (m-1)^{2k} = 2\sum_{j=1}^{k} \binom{2k}{2j-1} m^{2j-1}.$$

Summing as m goes from 1 to n, we have

$$\sum_{m=1}^{n} [(m+1)^{2k} - (m-1)^{2k}] = 2\sum_{m=1}^{n}\sum_{j=1}^{k} \binom{2k}{2j-1} m^{2j-1}.$$

Interchanging the order of summation and a little algebra gives

$$(n+1)^{2k} + n^{2k} - 1 = 2\sum_{j=1}^{k} \binom{2k}{2j-1} \sum_{m=1}^{n} m^{2j-1}$$

or

$$\sum_{j=1}^{k} \binom{2k}{2j-1} S_n^{2j-1} = \frac{1}{2}[(n+1)^{2k} + n^{2k} - 1].$$

Hence if $p = 2k - 1$ is a positive odd integer, then S_n^p can be found by solving the following triangular system of equations of order k:

$$\frac{1}{2}[(n+1)^2+n^2-1] = \binom{2}{1}S_n^1$$
$$\frac{1}{2}[(n+1)^4+n^4-1] = \binom{4}{1}S_n^1 + \binom{4}{3}S_n^3$$
$$\cdots$$
$$\frac{1}{2}[(n+1)^{2k}+n^{2k}-1] = \binom{2k}{1}S_n^1 + \binom{2k}{3}S_n^3 + \dots + \binom{2k}{2k-1}S_n^{2k-1}.$$

When p is a positive even integer, say $p = 2k$, the above procedure can be applied to the binomial expansions of $(m+1)^{2k+1}$ and $(m-1)^{2k+1}$,

$$(3) \qquad (m+1)^{2k+1} = \sum_{i=1}^{2k+1}\binom{2k+1}{i}m^i + 1$$

and

$$(4) \qquad (m-1)^{2k+1} = \sum_{i=1}^{2k+1}\binom{2k+1}{i}(-1)^{i+1}m^i - 1$$

to get

$$\sum_{m=1}^{n}[(m+1)^{2k+1}-(m-1)^{2k+1}] = 2\sum_{m=1}^{n}\sum_{j=1}^{k}\binom{2k+1}{2j}m^{2j} + 2n.$$

So,

$$\sum_{j=1}^{k}\binom{2k+1}{2j}S_n^{2j} = \frac{1}{2}[(n+1)^{2k+1}+n^{2k+1}-2n-1].$$

Thus S_n^p can be found by solving a triangular system of equations which involve only the previous sums for even powers if p is even and for odd powers if p is odd. In [2] and [3] it was necessary to use all of the previous sums. Thus, the computational effort has been reduced approximately by half.

Adding (1) and (2) will also give a general formula for S_n^p when p is even, as adding (3) and (4) will give one when p is odd. These two are slightly different than the above formulas.

It should be noted that there are many efficient and fast methods for solving a triangular system of equations, see for example [1]. However, that is not the concern of this paper.

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Annals of the Applications of Mathematics

Bitter Arctic air, combined with moderate winds, will create dangerous wind chills this weekend. People who become stranded should remain calm. Fear causes blood vessels to constrict, reducing the flow of warm blood to extremities and thus raising the risk of frostbite.

Alcohol causes an increase in the flow of blood to the skin, sometimes leading to a drop in body-core temperature. Symptoms include violent shivering.

For minor shivering, solve a math problem. Research has shown that a calculation as simple as 156 times 64 will temporarily stop the shivers.

(From the New York *Times* news service, printed in the Cincinnati *Enquirer,* January 14, 1994, page A-6.)

GENERALIZED RANDOM WALKS: AREAS AND LENGTHS

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Introduction. The theory of probability is a rich branch of mathematics that draws on a variety of techniques and concepts from calculus, combinatorics, and other areas. The theory of random walks is a rich topic within probability which provides many examples of probabilistic concepts. In this paper, we give an introduction to generalized random walks to illustrate such concepts as arclength, area, average value, variance, martingales, and stopping times. Throughout, there are exercises for further study.

Most of the results in this article were originally developed in an undergraduate research seminar during spring 1993. The seminar was directed by the first author as the faculty sponsor and was completed by the second author as a junior mathematics major.

Random Walk Notation. We consider a "particle" which begins at a particular height j, labeled $(0, j)$, and which can move up one unit with probability p, move down one unit with probability q, or remain fixed with probability $r = 1 - p - q$. Thus, randomness is one-dimensional along the y-axis. However, we shall graph the movement in two dimensions by letting the x-axis denote time,

To define the process formally, we let $\Omega = \{w = \{w_i\}_{i=1}^{\infty} : w_i = -1, 0, \text{ or } 1\}$ be the sample space of all possible paths consisting of downward, constant, and upward steps. The random walk X is a function of time t and path w, denoted $X_t(w)$, which gives the height of the particle at time t as it travels path w. We also assume that the particle makes each movement uniformly through a unit time interval; thus, $X_0(w) = j$ and for integers $n \geq 1$,

$$X_n(w) = j + w_1 + \cdots + w_n, \tag{1}$$

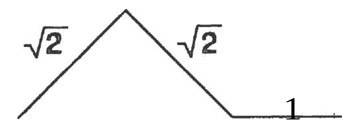
where $w = \{w_i\}_{i=1}^{\infty}$. Since the movement is uniform through time, we obtain a continuous graph by linearly connecting the steps so that for $n \leq t \leq n + 1$,

$$X_t(w) = X_n(w) + (X_{n+1}(w) - X_n(w))(t - n). \tag{2}$$

Stopped Random Walk. If we stop a random walk after a fixed number of steps n, then several calculus and probability concepts can be illustrated, such as arclength, area, average value, and conditional expectation.

First, we can count the numbers of upward, downward, and constant steps, denoted Y_p, Y_q, and Y_r respectively. Each is a binomial random variable and collectively form a **multinomial** distribution. Since $Y_p \sim b(n, p)$, there is an average of np upward steps. Likewise, an average of nq downward steps and nr constant steps would occur.

Second, we can compute the arclength of a particular graph and then the average arclength of all graphs. From the figure, we see that each upward step and each downward step contribute $\sqrt{2}$ to the length while each constant step adds 1 to the length. Thus, for a path w which is stopped after n steps, the arclength is given by



$$\mathcal{L}(w) = \sqrt{2}\cdot(Y_p(w) + Y_q(w)) + 1\cdot Y_r(w). \tag{3}$$

The average (or expected) value of the arclength is

$$E[\mathcal{L}] = \sqrt{2}(E[Y_p] + E[Y_q]) + E[Y_r] = \sqrt{2}n(p + q) + nr. \tag{4}$$

We can also find the variance of the arclength. Let Z count the number of upward or downward steps. Then $Z \sim b(n, p + q)$; hence, $E[Z] = n(p + q)$ and $\mathrm{Var}(Z) = n(p + q)(1 - p - q) = n(p + q)r$. Moreover, we can rewrite $\mathcal{L}$ as

$$\mathcal{L} = \sqrt{2}Z + (n - Z) = (\sqrt{2} - 1)Z + n.$$

Hence,

$$\mathrm{Var}(\mathcal{L}) = \mathrm{Var}((\sqrt{2} - 1)Z + n) = (\sqrt{2} - 1)^2\,\mathrm{Var}(Z) = (\sqrt{2} - 1)^2 n(p + q)r. \tag{5}$$

Next, we consider the height X_i of the particle at each unit step i. For example, X_0 is a constant function equal to the initial height j. For $1 \leq i \leq n$, X_i is a random variable with range $\{j - i, j - i + 1, \ldots, j, \ldots, j + i - 1, j + i\}$. For instance, X_i assumes the value $j + i$ with probability p^i (occurring if and only if the first i steps are upward). We shall derive $E[X_i]$ indirectly by first obtaining the conditional expectation of X_i given X_{i-1}:

$$E[X_i \mid X_{i-1}] = p(X_{i-1} + 1) + q(X_{i-1} - 1) + rX_{i-1} = (p + q + r)X_{i-1} + p - q = X_{i-1} + p - q. \tag{6}$$

From this expression, we see that the collection $\{X_i\}$ is an example of another important probabilistic concept. When p = q, then $E[X_i \mid X_{i-1}] = X_i$, so $\{X_i\}$ is a martingale, or fair game. In other words, on average there is no net gain or loss. For p < q, $E[X_i \mid X_{i-1}] \le X_{i-1}$, and thus $\{X_i\}$ is a supermartingale, or unfair game since there is an expected net loss. For p > q, $\{X_i\}$ is a submartingale.

Since $E[X_i \mid X_{i-2}] = E[E[X_i \mid X_{i-1}] \mid X_{i-2}]$, we see that $E[X_i \mid X_{i-2}] = X_{i-2} + 2(p - q)$. Continuing recursively, we obtain $E[X_i \mid X_0] = X_0 + i(p - q) = j + i(p - q)$. But since $E[X_i] = E[E[X_i \mid X_0]]$ ([1], p. 130), X_i has average value

$$E[X_i] = j + i(p - q). \tag{7}$$

Lastly, we can evaluate the signed area under the graph of a stopped path and the average area under all such graphs. The area under path w is simply the definite integral $A(w) = \int_0^n X_t(w)\,dt$. However, since the area is made up of trapezoids and rectangles, we can find it by using the trapezoidal rule: $A = \frac{1}{2}[X_0 + 2X_1 + \dots + 2X_{n-1} + X_n]$. From this expression and (7), we get

$$E[A] = \frac{1}{2}\left[j + 2\sum_{i=1}^{n-1}(j + i(p - q)) + j + n(p - q)\right] = \tag{8}$$

$$j + \sum_{i=1}^{n-1} j + \sum_{i=1}^{n-1} i(p - q) + \frac{n}{2}(p - q) =$$

$$nj + (p - q)\left(\frac{(n-1)n}{2}\right) + \frac{n}{2}(p - q) = nj + \frac{n^2(p - q)}{2}$$

Exercises:

1. How many possible paths are there which are stopped after n steps? Is each one equally likely to occur? What is the probability of obtaining n_1 upward, n_2 downward, and n_3 constant steps? What condition will make each path equally likely to occur?

2. Suppose the particle increases height h with probability p, decreases height k with probability q, or remains constant with probability r = 1 − q − p. Suppose further that each step is uniform throughout a time period of length l. Show that the average arclength after n steps is given by

$$E[\mathcal{L}] = np\sqrt{l^2 + h^2} + nq\sqrt{l^2 + k^2} + nrl.$$

3. Derive a formula for the variance of the arclength under the conditions of exercise 2.

4. Under the conditions of exercise 2, find $E[X_i \mid X_{i-1}]$ for $1 \le i \le$ n. If p = .2 and q = .6, what relationship between h and k will make the process $\{X_i\}$ a martingale? A submartingale? Find also $E[X_i]$.

5. Use the results of exercise 4 to show that the average area, when starting at height j under the conditions of exercise 2, is given by

$$E[A] = l\left[nj + \frac{n^2}{2}(ph - qk)\right].$$

6. Derive a formula for the variance of the area under standard conditions and under the generalized conditions of exercise 2.

Random Stopping Times. Rather than stop a path after a fixed number of steps, we can stop it at a random time which depends on the path w, such as the first time the path behaves a certain way. We shall consider several such stopping times, each of which will be an example of a geometric random variable. For example, we can let $T(w)$ be the first step in which path w does something other than increase. We could also consider the first time for a path to remain constant.

In any case, we shall be careful to stop at an event which is certain to occur with probability 1. It is particularly important to do so if we want to evaluate the area under the curve since the area under the graph of an infinite curve may not exist. Therefore, we shall consider only stopping times T which, as functions of the path w, are almost surely finite-valued. We can then consider the average time to stop $E[T]$, the average arclength after stopping $E[\mathcal{L}]$, the area under the graph of one path $\int_0^{T(w)} X_t(w)\,dt$, and the average area $E[A] = E\left[\int_0^T X_t\,dt\right]$.

First Time to Decrease or Stay Constant. For $0 \le p < 1$, let $T(w)$ be the first step on which path w either decreases or remains constant. Since p < 1, T is a geometric random variable and, since a path will decrease or stay constant on each step with probability q + r = 1 − p, the average time to stop is $E[T] = 1/(q + r)$ ([1] p. 36, p. 551).

Now suppose a path stops on the nth step; i.e., $T(w) = n$. Then the path has made n − 1 upward steps followed by either a downward or constant step. If the last step is downward, then the arclength is $\sqrt{2}\,n$; otherwise, the arclength is $\sqrt{2}(n - 1) + 1$. Since these arclengths occur with probabilities $p^{n-1}q$ and $p^{n-1}r$ respectively, the average arclength is

$$E[\mathcal{L}] = \sum_{n=1}^{\infty} \sqrt{2}\,np^{n-1}q + (\sqrt{2}(n - 1) + 1)p^{n-1}r \tag{9}$$

which simplifies to

$$\frac{\sqrt{2} + r - \sqrt{2}\,r}{1 - p}$$

Next, if a path begins at height j and decreases on the nth step, the area formed is the sum of the areas of two trapezoids and is given by

$$(n-1)\cdot(j+j+n-1)/2+(j+n-1+j+n-2)/2=(n^2+2jn-2)/2.$$

Otherwise, the area is given by

$$(n-1)\cdot(j+j+n-1)/2+(j+n-1)=(n^2+2jn-1)/2.$$

Thus, the average area is

$$(10)\quad E[A]=\sum_{n=1}^{\infty}\left((n^2+2jn-2)/2\right)p^{n-1}q+\left((n^2+2jn-1)/2\right)p^{n-1}r$$

which simplifies to

$$\frac{1+p}{2(1-p)^2}+\frac{j}{1-p}-\frac{2q+r}{2(1-p)}.$$

Exercises

7. Discuss why the first time to decrease or remain constant T is **finite**-valued with probability 1 but is not finite-valued for every path w.

8. The average **value** of the square of the arclength is given by

$$E[\mathcal{L}^2]=\sum_{n=1}^{\infty}(\sqrt{2}n)^2p^{n-1}q+(\sqrt{2}(n-1)+1)^2p^{n-1}r.$$

Find the variance of the arclength: $\mathrm{Var}(\mathcal{L})=E[\mathcal{L}^2]-(E[\mathcal{L}])^2$.

9. In equations (8) and (9) we used the facts that $\sum_{n=1}^{\infty}p^n=p/(1-p)$, $\sum_{n=1}^{\infty}np^n=p/(1-p)^2$, and $\sum_{n=1}^{\infty}n^2p^n=(p+p^2)/(1-p^2)$. Show that for $-1<p<1$,

$$\sum_{n=1}^{\infty}n^3p^n=(p+4p^2+p^3)/(1-p)^4$$

and

$$\sum_{n=1}^{\infty}n^4p^n=(p+11p^2+11p^3+p^4)/(1-p)^5.$$

10. For $r=0$ and $q=1-p$, use the fact that the variance of the area is given by

$$E[A^2]-(E[A])^2=\sum_{n=1}^{\infty}\left(\frac{n^2+2jn-2}{2}\right)^2p^{n-1}q-(E[A])^2$$

to show that

$$\mathrm{Var}(A)=\frac{p^3+10p^2+9p+12qjp+4q^2j^2p+4qjp^2}{4q^4}$$

11. Under the generalized conditions of exercise 2, again let T be the first time to decrease or remain constant. Show that the average arclength is now given by

$$E[\mathcal{L}]=\frac{\sqrt{h^2+l^2}\,p+\sqrt{l^2+k^2}\,q+lr}{1-p}$$

Verify that when $h=k=l=1$, this **formula** reduces to equation (9). Show also that the average area is now given by

$$E[A]=\frac{hl(1+p)}{2(1-p)^2}+\frac{jl}{1-p}-\frac{l[(h+k)q+hr]}{2(1-p)}.$$

12. For $0\le q<1$, let $T(w)$ be the first step on which path w either increases or remains constant. Show that the average arclength under standard conditions is

$$E[\mathcal{L}]=\frac{\sqrt{2}+r-\sqrt{2}r}{1-q},$$

and the average (signed) area under the curves is

$$E[A]=\frac{-(1+q)}{2(1-q)^2}+\frac{j}{1-q}+\frac{2p+r}{2(1-q)}$$

Derive formulas for $E[\mathcal{L}]$ and $E[A]$ under the generalized conditions of exercise 2.

13. For $0\le r<1$, let $T(w)$ be the first step on which path w either increases or decreases. Show that the average arclength under standard conditions is

$$E[\mathcal{L}]=\frac{\sqrt{2}+r-\sqrt{2}r}{1-r},$$

and the average area under the curves is

$$E[A]=\frac{2j+p-q}{2(1-r)}$$

Derive formulas for $E[\mathcal{L}]$ and $E[A]$ under the generalized conditions of exercise 2.

First Time to Remain Constant. For $0<r\le 1$, we let $T(w)$ be the first step on which path w takes a constant step. Then T is again a geometric random variable and, since a path stays constant with probability r, the average time to stop is $E[T]=1/r$. To compute the average arclength and average area, we examine the case where the first constant step is on the $(n+1)$st step.

There are 2^n such paths since the first n steps may be either up or down but the $(n+1)$st must be constant. Each such path has arclength $\sqrt{2}n+1$; thus, the average **arclength** of this group is $\sqrt{2}n+1$. Moreover, the probability that one from this group occurs is $(1-r)^nr$ (from n **non-constant** steps followed by one constant). Thus, the average arclength of all paths is

$$\text{(11)} \qquad E[\mathcal{L}] = \sum_{n=0}^{\infty} (\text{fin} \ + 1)(1 - r)^n r$$

which simplifies to

$$\frac{\sqrt{2} - \sqrt{2}\,r + r}{r}$$

To compute the average area, we again consider the group of paths with the first constant step being on the (n + 1)st step. The area under such a path w is $A(w) = A_n(w) + X_n(w)$, where $A_n(w)$ is the area under the first n steps and $X_n(w)$ is the constant height between the nth and (n + 1)st steps. The average area of this group is then $E[A_n] + E[X_n]$. But $E[A_n]$ is simply the average area under the graphs of random walks after n steps where the steps are either up with probability p $= p/(p$ + q), down with probability q $= q/(p$ + q), and constant with probability $\bar{r} = 0$. Thus, by (8),

$$\text{(12)} \qquad E[A_n] = nj + \frac{n^2(\bar{p} - \bar{q})}{2}.$$

Likewise, by (7), we have

$$\text{(13)} \qquad E[X_n] = j + n(\bar{p} - \bar{q}).$$

Thus, since the average area of the group is $E[A_n] + E[X_n]$ and one from this group occurs with probability $(1 - r)^n r$, the average area under all graphs is

$$\text{(14)} \qquad \sum_{n=0}^{\infty} \left[nj + \frac{n^2}{2}(\bar{p} - \bar{q}) + j + n(\bar{p} - \bar{q})\right](1 - r)^n r$$

which becomes

$$j + \frac{j(1 - r)}{r} + \frac{\bar{p} - \bar{q}}{2r^2}(-r^2 - r + 2),$$

where p $= p/(p$ + q) and $\bar{q} = q/(p$ + q).

We can similarly derive the average arclength and average area for the first time to decrease. If we consider the group of paths which decrease for the first time on the (n + 1)st step, then the arclength and area formed by one such path w can be written as $\mathcal{L}_n(w) + \sqrt{2}$ and $A_n(w) + X_n(w)$ – 112, where $\mathcal{L}_n$ and A_n are the arclength and area after n steps. We think of these first n steps as paths which either increase with probability $\bar{p} = p/(p$ + r) or remain constant with probability $\bar{r} = r/(p$ + r). Then by (4), (7), and (8), $E[\mathcal{L}_n] = \sqrt{2}\,n\bar{p} + n\bar{r}$, $E[X_n] = j + n\bar{p}$, and $E[A_n] =$ nj $+ n^2\bar{p}/2$. Thus, the average arclength of all paths will be given by

$$E[\mathcal{L}] = \sum_{n=0}^{\infty} (\sqrt{2}\,n\bar{p} + n\bar{r} + \sqrt{2})(1 - q)^n q$$

and the average area under all paths will be given by

$$E[A] = \sum_{n=0}^{\infty} \left(nj + \frac{n^2\bar{p}}{2} + j + n\bar{p} - \frac{1}{2}\right)(1 - q)^n q.$$

The details are left in the following exercises.

Exercises:

14. For $0 < q \le 1$, let $T(w)$ be the first time for path w to decrease. Show that the average arclength and average area are given by

$$E[\mathcal{L}] = \left[\sqrt{2}\left(\frac{p}{p + r}\right) + \frac{r}{p + r}\right]\left[\frac{1 - q}{q}\right] + \sqrt{2},$$

and

$$E[A] = j - \frac{1}{2} + \frac{j(1 - q)}{q} + \frac{\big(p/(p + r)\big)(-q^2 - q + 2)}{2q^2}.$$

15. For $0 < p \le 1$, let $T(w)$ be the first time for path w to increase. Derive formulas for the average arclength and average area formed by the paths.

Reference

1. Rice, J. A., Mathematical Statistics and Data Analysis. Wadsworth & Brooks/Cole, Pacific Grove, California, 1988.

David Neal is an *assistant professor* at Western Kentucky who has done research in stochastic integration and is currently developing a workbook of computer projects to supplement probability and statistics courses. Lon Maynard is a senior mathematics major, expecting child number one, working at two jobs, and preparing for actuarial examination number three.

Here is a solution to the problem (page 646) of what is the largest integer which, in base 16, is an ordinary English word.

What has been done lo a building whose front has been pulled off?, Clearly, it has been *DEFACADED*. In decimal, this is 59855646189.

A NOTE ON A RECURRENCE RELATION

Russell Euler
Northwest Missouri State University

If a and b are constants, standard methods of solving the recurrence relation

(1) $$x_n = ax_{n-1} + b$$

are iteration, mathematical induction, or by using its characteristic equation. The purpose of this note is to show an alternative method of solution.

To avoid trivialities, assume that a $\neq$ 0 or 1. The motivation for the method given below comes from the observation that $x_n - x_{n-1} = b$ can be easily solved by summation.

Since a $\neq$ 0, (1) can be written

$$\frac{x_n}{a^n} = \frac{x_{n-1}}{a^{n-1}} + \frac{b}{a^n}.$$

This is equivalent to

(2) $$\frac{x_i}{a^i} - \frac{x_{i-1}}{a^{i-1}} = \frac{b}{a^i}.$$

When both sides of (2) arc summed from $i = 1$ to $i = n$, the left-hand side telescopes to give

$$\frac{x_n}{a^n} - x_0 = \sum_{i=1}^{n} \frac{b}{a^i}.$$

The series on the right-hand side is geometric, so

(3) $$x_n = a^n x_0 + \frac{b(a^n - 1)}{a - 1}$$

Finally, note that even if the restrictions on a arc removed, (3) still gives the solution to (1). For instance, if a = 0, (3) gives $x_n = b$. If a = 1 in (1), applying **L'Hôpital's** Rule to (3) gives $x_n = x_0 + nb$.

The same idea can be used to solve

(4) $$x_{n+1} = ax_n + bx_{n-1}.$$

Let r and s be the roots of the characteristic equation $x^2 = ax + b$, divide both sides of (4) by r^{n+1} and let $u_n = x_n/r^n$, and we get

$$u_{n+1} - u_n = \frac{s}{r}(u_n - u_{n-1}).$$

Let $v_n = u_{n-1} - u_n$, and the solution follows quickly.

In addition to writing articles, Russell Euler is an avid problem-solver and has been successful in passing his enthusiasm on to some of his students. *His* Ph. D. research was directed by *the* late *Y. L.* Luke.

Why slope is *m*

In the last issue of the Journal, the question of why the slope of a line is always denoted m was raised. There were some answers.

Professor RICHARD POSS (St. Norbert College) says that he tells his students that *m* is for marginal, as in marginal revenue and marginal cost, and that he has yet to have anyone disagree with him.

Professor JAMES CHEW (North Carolina A. & T. State University) points out that

> The Indonesian word miring means slanted or crooked, as anyone with an Indonesian-English dictionary may verify. Hence if a picture on a wall is miring, someone usually comes along and straightens it, i. e., zeros out its slope.

More seriously, Professor MARK KRUSEMEYER (Carleton College) suggests

> that m might be short for a French word such as *montée.* I was hoping to find confirmation by looking in old French calculus texts, but the only one in the library is Jordan's Cours *d'analyse* (2me. ed., 1893). There I can find the equation of a line only once (on admittedly casual paging through) in (more or less) slope-intercept form. That happens on p. 188, where the slope is called m (actually, there arc two of them, called m and m_0), but it is referred to as the "coefficient angulaire".

Though French is a more likely source than Indonesian for m, the definitive solution to the m-mystery has yet to be found. Professor Chew thought that we should not make a mountain out of a molehill.

Another Application of a Familiar Maximum

Norman Schaumberger
Hofstra University

Because of the interest in demonstrating the well-known **result** $e^{\pi} > \pi^{e}$ it is now a standard exercise in elementary calculus to show that if x > 0, then $f(x) = x^{1/x}$ has an absolute maximum at e. Thus, for any positive real number x, $e^{1/e}$ a $x^{1/x}$ or

$$(1) \qquad e^{x} \geq x^{e}$$

with equality if and only if x = e.

We will use (1) to obtain the possibly surprising formula

$$(2) \qquad \lim_{n \to \infty} \frac{(n!)^{1/n}}{n} = \frac{1}{e}.$$

Thus, one of the many formulas that connect e and n! can be resented early in a first course in calculus. The usual proof of (2) uses Stirling's formula.

Putting $x = \left(\frac{k+1}{k}\right)e$ and then $x = \left(\frac{k}{(k+1)}\right)e$ in (1) gives

$$e^{((k+1)/k)e} > \left[\left(\frac{k+1}{k}\right)e\right]^{e} \quad \text{and} \quad e^{(k/(k+1))e} > \left[\left(\frac{k}{k+1}\right)e\right]^{e}.$$

These can be written as

$$e^{((k+1)/k)-1} > \frac{k+1}{k} \quad \text{and} \quad e^{(k/(k+1))-1} > \frac{k}{k+1}$$

which give

$$\left(\frac{k+1}{k}\right)^{k+1} > e > \left(\frac{k+1}{k}\right)^{k}.$$

Letting k = 1, 2, ..., n and multiplying, we get

$$(3) \qquad \frac{(n+1)^{n+1}}{n!} > e^{n} > \frac{(n+1)^{n}}{n!}$$

or

$$\frac{1}{e}\left(\frac{n+1}{n}\right)(n+1)^{(1/n)} > \frac{(n!)^{1/n}}{n} > \frac{1}{e}\left(\frac{n+1}{n}\right).$$

As $n \to \infty$, $(n+1)/n \to$ I, $(n+1)^{(1/n)} \to 1$, and we have (2).

It should be noted that (3) also gives a familiar estimate for n!,

$$(n+1)\left(\frac{n+1}{e}\right)^{n} > n! > \left(\frac{n+1}{e}\right)^{n}.$$

Norman Schaumberger, recently retired from Bronx Community College of CUNY, is presently an adjunct professor at Hofstra. He is a regular contributor to the problem section of several journals and, as in this paper, he continues to find non-standard approaches to familiar topics.

Chapter Reports

Professor John Petro, corresponding secretary for the MICHIGAN **EPSILON** Chapter (Western Michigan University), reports an ususually full round of activities for the chapter. There was a pizza party, a book sale netting $500 to help support the activities of the chapter, and a sale of Pi Mu Epsilon t-shirts. Thirty-one new members were intitated. In addition to a series of speakers (including Sheldon **Axler,** Peter **Hilton,** and Jean Pedersen), the chapter sponsored a program for high school students that included presentations on mathematical topics as well as information on careers involving mathematics, problem solving, and a closing talk on mathematics and poetry by Professor Arthur White.

The **GEORGIA BETA** Chapter gives book awards to students majoring in applied mathematics who achieve a grade-point average of at least 3.7 in all mathematics courses taken. Last year, faculty correspondent James M. Osborn reports, two awards were given. Awardees get to choose the book they want.

THE NASTINESS OF MATHEMATICIANS

N. L. Mackenzie
Seattle, Washington

The editor of this ***Journal*** [2] issued an appeal for anecdotes about mathematicians "illustrating their human qualities." If he meant their ***admirable*** human qualities, I doubt that he had much of a response. The reason is that anecdotes about mathematicians are almost always about their smartness, their absent-mindedness, or their nastiness.

Everyone knows that mathematicians are smart. The United States Employment Service once tested people in various occupations for intelligence, manual dexterity, motor coordination, and so on. The highest score for intelligence, as measured by IQ score, was 143 for mathematician, above dentist (131), accountant (118), waitress (80), and all the others. That mathematicians are absent-minded is a corollary of the extraordinary power of concentration that the discipline demands. That they are nasty is more difficult to explain, though not more difficult to document.

Anecdotes about smartness go all the way back to Thales, the first mathematician. Remember the one about his cornering the olive-oil market? About telling the woman where to find her lost wash? The one about the mule? Even if you remember the last one, it is so good that it deserves retelling. Thales, the story goes, had a salt mine; mules were used to haul the salt to town for sale. The path to town crossed a stream; one day a mule fell in the stream, the water dissolved most of the salt, and the load was much lighter. The clever mule thereafter fell in the stream on every trip. How would you try to cure the mule of this bad habit? By yelling, by beatings, or by moral suasion? Mules, anecdotes tell us, are stubborn and those methods would not be likely to succeed. Thales, being a mathematician and thus much smarter than the mule, filled the mule's bags on the next trip not with salt but with ... sponges.

Search for anecdotes testifying to the warmth, generosity, or humanity of mathematicians and of course you will find a few. However, for every one you find, you will find two, or ten, testifying to their smartness, absent-mindedness, or nastiness. What are the anecdotes about Newton?

> Once, having dismounted from his horse to lead him up a hill, the horse slipped his head out of the bridle; but Newton, oblivious, never discovered it till, on reaching a tollgate at the top of the hill, he turned to remount and perceived that the bridle he had in his

> hand had no horse attached to it.

Another is

> On getting out of bed in the morning, he has been discovered to sit on his bedside for hours without dressing himself, utterly absorbed in thought.

These, and many others along the same lines can be found in [9]. I will spare you further anecdotes of Newton forgetting to eat, of Archimedes failing to notice that there was a battle going on around him, of Wiener forgetting where he lived and even forgetting his name, ... The list of absent-minded mathematicians could go on and on.

There is nothing bad in being smart and absent-mindedness can be endearing, but nastiness has no redeeming quality. That mathematicians tend towards the nasty is not just my opinion. Here is what Howard Eves (who, by the way, does not have a nasty bone in his body: when I write about the nastiness of mathematicians, it is only of a ***tendency*** towards nastiness, and one by no means to be found in every mathematician) has to say in his valuable and entertaining book, ***In Mathematical Circles*** [4, p. 85-86]:

> Having associated from early years with two particular classes of scholars-botanists (or, more widely, nature lovers) and mathematicians—I came to notice, and through the years have confirmed, a striking general difference between the two classes. The botanists are usually the most pleasant sort of people to be with; they radiate gentle modesty, are open-minded, enjoy each other's company, are kind in their professional comments about one another, and are found interesting by their non-botanical friends. The mathematicians, on the other hand, are too often unpleasant to be with; they frequently exude self-importance, are professionally opinionated, tend to bicker and quarrel among themselves and say unkind things about one another, take an almost gleeful pleasure in unearthing an error in another's work, and are quite often boring to their nonmathematical acquaintances.
>
> The unpleasant features of the mathematical group are noticeable even among some of the more gifted high school students of the subject, become sharper among the college graduate students of mathematics, and often attain an undignified aspect among college instructors and professors of mathematics.

How true! How easy to find examples! Here is David Hilbert in action at a meeting of the mathematics seminar in Gottingen, as reported by

Constance Reid [10, p. 1691. The meeting room had two doors: one to the hall and another which led to the mathematics reading room. A visitor was lecturing on a topic in differential equations when

> . he [Hilbert] interrupted the speaker with, "My dear colleague, I am very much afraid that you do not know what a differential equation is." Stunned and humiliated, the man turned instantly and left the meeting, going into the next room, which was the reading room. "You really shouldn't have done that," everyone scolded Hilbert. "But he doesn't know what a differential equation is," Hilbert insisted. "Now, you see, he has gone to the reading room to look it up!"

That this discreditable anecdote should appear in a sympathetic biography is testimony both to Hilbert's nastiness and to the lack of better anecdotes.

An article, "Mathematical Anecdotes" [6], provides more evidence. Its purpose was certainly not to discredit the anecdotees — Bergman, Besicovich, Gödel, Lefshetz, and Wiener: its author says

> ... the mathematicians described here are among the gods of twentieth-century mathematics. ... The enormous scholarly reputation of these men sometimes cause their humanity to be forgotten. ... In telling stories about them we bring them back to life and celebrate their careers.

What do we find about Bergman?

> Bergman had always felt that the value of his ideas was not sufficiently appreciated. ... [At a conference] I sat next to him at most of the principal lectures. In each of these, he listened carefully for the phrase "and in 1922 Stefan Bergman invented the kernel function." Bergman would then dutifully record this fact in his notes—and nothing more. I must have seen him do this twenty times during the three-week conference.

That is one anecdote. Another is

> On another occasion a young mathematician gave Bergman a manuscript he had just written. Bergman read it and said "I like your result. Let's make it a joint paper, and I'll write the next one."

Then there is the story about how Bergman was going to leave his own wedding reception to discuss mathematics, until he was threatened with losing his job if he did. There also is

> Once he phoned a student, at the student's home number, at 2:00 a. m. and said "Are you in the library? I want you to look something up for me!"

Then there was his student who got married and went on his honeymoon, by bus. Bergman went along.

> The student protested that the trip was to be part of his honeymoon, and that he could not talk mathematics on the bus. Bergman promised to behave. When the bus took off, Bergman was at the back of the bus and, just to be safe, Bergman's student took a window seat near the front with his wife in the adjacent aisle seat. But after about ten minutes Bergman got a great idea, wandered up the aisle, leaned across the scowling bride, and began to discuss mathematics. It wasn't long before the wife was in the back of the bus and Bergman next to his student—and so it remained for the rest of the bus trip!

For this he should be remembered fondly?

As was pointed out in [2], mathematicians tend to have dull non-mathematical lives. Perhaps the dullness is part of the reason that they tend to be nasty. And nasty they can be. In the obituary of an eminent mathematician that appeared in [8] we read

> Some of his former students recollect his sense of humor.

There are only two examples in the obituary. One is his arranging to appear twice in a composite photograph by running from one end of the group to the other between shots—funny enough. But the other is

> . a former student who felt himself a novice in teaching sought advice from [him] to which the latter gave the terse reply: "Always start writing in the upper left-hand comer of the blackboard."

Are you laughing? Remember, this is the best that the author could find to illustrate the deceased's warmth and sense of humor. The obituary contains two other human touches. One was his habit of opening classroom windows whatever the weather was like (*he* was moving around, lie wouldn't get cold) and the other was his habit of throwing into the wastebasket at the end of each class the piece of chalk he had been using.

> Besides, during the run of a lecture he was occasionally known to have used an inattentive student as a target for the chalk.

I am certain that the author was not indulging in irony, painting the portrait of a thoroughly disagreeable man; he was doing the best that he could. Most mathematical obituaries concentrate on the deceased's mathematical work and go into personalities hardly at all, perhaps with good reason.

There are two, and only two, anecdotes about Euclid in to be found in [9] or in any other book:

> On being asked if there was any easier way to learn geometry than to go through all of the work in Euclid's ***Elements,*** Euclid answered, "There is no royal road to geometry."

Cold. Austere. Not cuddly.

> On being asked by a student what learning geometry was good for, Euclid said to his slave, "Give him a penny, since he must make a profit out of learning."

The mathematician—ever the master of the put-down! Did you notice that Euclid didn't even ***speak*** to the poor worm? That put him in his place!

Look in E. T. Bell's ***Men of Mathematics [3]*** and read the anecdotes: Gauss is unable to get along with his sons, Cauchy "loses" Galois' papers, . . Not one of the great mathematicians could be characterized as being genial. Music has Papa Haydn. Mathematics has ... Papa Euclid? Papa anybody?

Even the unrecorded anecdotes show nastiness. There is a contemporary mathematician who remarried after the death of his first wife. He always referred to his second wife by her given name, but when he said "my wife" he meant his first wife. He later married his third wife.

Scholars' vices—pride, ungenerosity, spleen, bile, self-righteousness—are no strangers to mathematicians. Back in 1969, W. G. Spohn wrote an article [11] deploring various tendencies in mathematical research and teaching (and the passage of time has shown, I think, that he was largely right). In [7] three replies were printed. One was

> I hope the teacher of the future will apply the proper German spelling of Gottfried Wilhelm Freiherr von Leibniz (1646-1716).

Professor Spohn had chosen to give the Americanized spelling "Leibnitz"; the writer of the reply pretended to assume that he knew no better and thus all of his ideas would be discredited by this ponderous put-down. The second reply was heavily ironic:

> One quiet evening in my study I was brought breathlessly to my feet with the article of William G. Spohn, Jr. "Can Mathematics be Saved?" I realized suddenly that his title had become an intensely burning question lurking somewhere unexpressed, or not fully expressed, deep in the recesses of my mental activity. My heart leaped as I read well-put phrases ...

And so on. The third included

> If this is true, there is little for the readers of the ***Notices*** to do but die (leaving instructions for our unspoiled successors in editorships to accept Dr. Spohn's papers). ... It is clear that Spohn has been troubled before now by "the artificial requirement that research be new." Had his present article been refereed, he would have been troubled again.

What venom! What nastiness! None of the replies mentioned the ideas of the original article: they were outpourings of emotions, nasty ones. And these letters were not only written, they were published. What could have been in the unpublished ones?

If you are convinced, partially at least, that mathematicians tend towards nastiness, it is time to consider the reasons for it. This can be only speculation, since there has been no research on the subject that I know of.

One reason for behavior like the replies to the Spohn is that mathematicians are unappreciated. It is not that they merely think they are unappreciated: they genuinely ***are.*** As Alfred Adler put it in [1], writing for a non-mathematical audience,

> Of course, money and power are only superficial rewards, and it is possible to live without them; less tangible rewards, those of recognition and understanding, are what make effort and accomplishment rich and fulfilling when things are going well, and effort and failure bearable when things are going badly. But mathematicians cannot expect these either. For example, it would be astonishing if the reader could identify more than two of the following names: Gauss, Cauchy, Euler, Hilbert, Riemann. It would be equally astonishing if he should be unfamiliar with the names of Mann, Stravinsky, de Kooning, Pasteur, John Dewey. The point is not that the first five arc the mathematical equivalents of the second five. They are not. They are the mathematical equivalents of Tolstoy, Beethoven, Rembrandt, Darwin, Freud.

There is no Nobel Prize in mathematics. The world is ignorant and uncaring. Is it not galling to know that no matter what you do, even if you settle Goldbach's Conjecture and the Riemann Hypothesis in the same year, there will never be for you the praise and recognition that others get for accomplishments that are orders of magnitude less worthy? Docs it not grate to know that all there will be is the grudging respect of colleagues—

grudging because after they read the proof of the theorem they can see how easy it was to prove—and perhaps a line or two in future histories of mathematics? As Adler said,

> In the company of friends, writers can discuss their books, economists the state of the economy, lawyers their latest cases, and businessmen their latest acquisitions, but mathematicians cannot discuss their mathematics at all. And the more profound their work, the less understandable it is.

A bleak prospect, which can do nothing but build frustration, and frustration bottled up can change to sourness, envy, jealousy, and nastiness. There seems to be no remedy.

But of course that cannot be the whole explanation, since botanists are also unappreciated (can you name one prominent American botanist?) but Howard Eves assures us that they are pleasant people. Can part of the explanation be that mathematicians peak early? Gauss said that his accomplishments were only the working out ideas he had before he was 20. In *A* Mathematician's *Apology* [5], *G. H.* Hardy eloquently expressed the melancholy that conies with the decline of powers. One of the bigger names in American mathematics said to me, when he was around 60, "I can only prove the trivial stuff now." It is sad, as Adler noted:

> It is easy to believe that life is long and one's gifts are vast—easy at the beginning, that is. But the limits of life grow more evident; it becomes clear that great work can be done rarely, if at all. Moreover, there are family responsibilities and professional sinecures. Hard work can certainly continue. But creativity requires more than steady, hard, regular, capable work. It requires total commitment over years, with the likelihood of failure at the end, and so the likelihood of a total waste of those years. It requires work of truly immense concentration. Such consuming commitment can rarely be continued into middle and old age, and mathematicians after a time do minor work.

It is not pleasant to know that you have nothing but decline to look forward to, and it does not contribute to a sunny disposition.

But of course even those two together cannot be the whole explanation. Athletes also have early primes, but aged athletes have no reputation for nastiness. There must be something else, and part of it, I think, is the nature of the discipline. As Adler rightly notes, there is no guarantee of success in mathematical research. In botany and other disciplines, life is easier. Work

will bring reward: enough time spent in the library, enough time spent in the laboratory or the greenhouse, enough effort gathering data, enough work analyzing and summarizing will bring something that can be published and might be useful. It is not that way in mathematics. It is possible to pick a problem that is too hard and that you will never be able to solve because you lack the necessary talent, but you do not know that when you start. You do know it when you finish, and to give up the attempt with nothing to show for it is not pleasant. It is a nasty experience.

There is another difference, one which shows when a mathematician succeeds in solving a problem. Unlike a botanist, for example, a mathematician is not investigating nature. A botanist's reaction to making a discovery or solving a problem might be, "How wonderful the world is!" whereas when a mathematician proves a theorem it is easy to for him to think, "How smart I am! Look what I have done!" There is a large difference between investigating the nature of plants and finding new theorems. The botanist is an observer, the mathematician a creator—how like a god! There is nothing in mathematics to induce humility; all the force is in the other direction, towards pride, arrogance, and nastiness. Also, mathematicians earn money by teaching, and there is none of the humanists's learning from their students in mathematics: teachers of mathematics know the subject, students do not; they have the right answers, all of them, and students will never win a mathematical argument with them (not that any student would try). It is not hard to make the transition from being always right in a classroom to being always right everywhere. The three replies to Spohn illustrate that: the writers *knew* they were right, so there was no need at all to argue ideas. Scorn and sneers sufficed.

Perhaps, in addition to the discipline, it is the kind of person who is attracted to it. Stanislaw Ulam [12, p. 120], said

> In many cases, mathematics is an escape from reality. The mathematician finds his own monastic niche and happiness in pursuits that are disconnected from external affairs. Sonic practice it as if using a drug. Chess sometimes plays a similar role. In their unhappiness over the events of this world, some immerse themselves in a kind of self-sufficiency in mathematics. (Some have engaged in it for this reason alone.)

Why does a mathematician need to escape from reality? As we have seen, mathematicians are smart. Among schoolchildren, smartness is not generally popular, so it is easy lor the future mathematician lo learn that gratification does not come from people. But numbers! Numbers are different. They are under our control, they do what we want them to, when we want them to. The world of numbers, so unlike the world of people, is full of gratifications that can be had, predictably, at any time, and with no stress or strain. Equations solved, square roots found, formulas applied: you can spend hours, happy rewarding hours, with pencil, paper, and numbers.

Speculation is fairly fruitless, especially since there are many, many examples of thoroughly un-nasty mathematicians at all levels up to and including the giants of the profession. Nevertheless, I think that a tendency to nastiness exists, and although I cannot claim to have solved the problem of why some mathematicians are nasty, I think that I have given some indications of the solution. In fact, I know I have. Indeed, I am right. You may disagree, but if you do you are wrong, quite wrong. You may argue, but it will do no good. Besides, if you do not agree that I am perfectly correct, I may turn nasty. Good heavens, can it be that I am nasty?

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N. L. Mackenzie has divided his career between teaching mathematics and doing actuarial work for insurance companies. He says that it is not true that if you cross an actuary with a gorilla what you will get is a very dull gorilla.

Chapter Report

The FLORIDA KAPPA Chapter (University of West Florida), cooperating with the student chapter of the Mathematical Association of America, assisted in organizing and hosting the conference of the International Linear Algebra Society, and in sponsoring the annual Math Counts contest, reports Professor James R. Weaver, faculty correspondent. Also jointly with the MAA student chapter, some members of the chapter received some support to attend the winter and summer mathematics meetings, and the Rocky Mountain Mathematics Consortium's meeting on matrix algebra.

E. M. Beesley

Edward Maurice Beesley died on October 20, 1993 at the age of 78. He was a professor of mathematics at the University of Nevada, Reno, for 41 years and chairman of the department for 37. Long active in Pi Mu Epsilon, he sewed as its President from 1981 to 1984.

Exhortation

Students! Have you written anything suitable for a general mathematical audience lately? If you have, send it to the Pi ***Mu Epsilon Journal.*** While publication in the ***Journal*** will probably not bring you fame, and certainly not fortune, it is nevertheless a feat accomplished each year by only about .000008% of the population of the United States and Canada, and thus is a distinction.

A NOTE ON ISOMORPHIC FACTOR GROUPS

Kandasamy Muthuvel
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Not many examples are known to show that the statement "if H and K are subgroups of a group G and $G/H \approx G/K$, then $H \approx K$" is false. The aim of this paper is to give an interesting uncommon example.

In many undergraduate abstract algebra textbooks, to show that $Z \oplus Z \not\approx Z$ is a standard problem and $R \oplus R \not\approx R$ is stated without proof. Since the internal direct sum of subgroups Z and aZ, where a is an irrational number, is isomorphic to the external direct sum of Z and aZ, and $aZ \approx Z$, it follows that $Z + aZ \approx Z \oplus Z \not\approx Z$. Factor groups R/Z and $R/(Z + aZ)$, where a is an irrational number, arc not isomorphic because R/Z has exactly one element, namely $112 + Z$, of order 2, while $R/(Z + aZ)$ has more than one element, namely $1/2 + Z + aZ$ and all $+ Z + aZ$, of order 2. However, we will show that $Q + aQ \not\approx Q$ and $R/Q \approx R/(Q + aQ)$, where a is an irrational number.

The usual way of showing that $Z \oplus Z \not\approx Z$ is by noting that Z is cyclic but $Z \oplus Z$ is not. This argument cannot be applied to $Q \oplus Q$ and Q, as both are noncyclic groups. Suppose $f : Q \oplus Q \to Q$ is an isomorphism. Since $f(0, 0) = 0$, it follows that $f(1, 0) = a/b$, and $f(0, 1) = c/d$ for some nonzero integers a, b, c, d, Thus

$$f(bc, 0) = bc f(1, 0) = bc\frac{a}{6} = ac$$

and

$$f(0, ad) = ad f(0, 1) = ad\frac{c}{d} = ac.$$

Consequently $f(bc, 0) = f(0, ad) = ac$. Since f is one-to-one, $bc = 0$. This contradicts the requirement that b and c are nonzero integers. Thus $Q \oplus Q \not\approx Q$ and $Q + aQ \approx Q \oplus Q \not\approx Q$ for every irrational number a.

To show that $RIQ \approx R/(Q + aQ)$, let B be a basis of R containing $\{1, a\}$. (That is, every element in R can be expressed as a finite linear combination of elements from B with rational coefficients and B is linearly independent over the set of rationals.) It follows from the work in [1, p. 35], with suitable minor modifications, that such a basis B exists. Well-order $B\backslash\{1\}$ and $B\backslash\{1, a\}$ as (l_i) and (m_i) respectively. Let L be the set of all finite linear combinations of elements of (l_i) with rational coefficients and

let M be the set of all finite linear combinations of elements of (m,) with rational coefficients. Then it can be easily seen that R is an internal direct sum of Q and L and R is also an internal direct sum of $Q + aQ$ and M. Hence $RIQ \approx L$ and $R/(Q + aQ) \approx M$ [2, p. 188, ex. 12]. Since the napping $f : L \to M$ defined by $f\left(\sum_{i\in I} q_i l_i\right) = \sum_{i\in I} q_i m_i$ (where I is a finite set) is an isomorphism, $L \approx M$. Thus $R/Q \approx R/(Q + aQ)$ but $Q \not\approx Q + aQ$.

It is interesting to note in the above example that $L \approx M$ but $RIL \not\approx RIM$.

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Kandasamy *Muthuvel's* Ph. D. degree *in* mathematics (1985) *is from* the University of *Wisconsin—Milwaukee*, and lie has been at *UW—Oshkosh* since *1988*. He is the author *of* several papers and his research interests *include* set theory, general topology, real analysis, and group *theory*.

Why Study Mathematics?

Our future lawyers, clergy, and statesmen are expected at the University to learn a good deal about curves, and angles, and numbers and proportions; not because these subjects have the smallest relation to the needs of their lives, but because in the very act of learning them they arc likely to acquire that habit of steadfast and accurate thinking, which is indispensable to success in all the pursuits of life.—J. C. Fitch (1906), quoted in ***Memorabilia Mathematica*** by *R.* E. Moritz, 1914 edition reprinted by the Mathematical Association of America, 1993.

Solving Functional Equations

Piotr Krysta
Wrocław University

Functional equations arise often, in problems and elsewhere. In this note, we will solve the functional equation

(1) $$f(4x) - f(3x) = 2x, \quad x \in \Re$$

assuming that f is monotonic. We will not assume that f is continuous or differentiable. We will show two methods of solution.

First, we note that f is increasing, since it is monotonic and $f(4) - f(3) = 2$. If we let $h(x) = f(x) - 2x$, then

$$h(4x) = f(4x) + 8x = (f(3x) - 2x) - 8x = f(3x) - 6x = h(3x).$$

Thus $h(x) = h((3/4)x))$ and, by induction, $h(x) = h((3/4)^n x)$ for any positive integer n. If h were continuous, it would follow that $h(x) = \lim_{n \to \infty} h((3/4)^n x) = h(0)$ and so h would be constant. We do not assume the continuity of h, but we can show it is constant everywhere except at 0. Suppose that for some x_1, x_2 with $0 < x_1 < x_2$ that $h(x_1) < h(x_2)$. Let $\alpha_n = x_1(3/4)^n$. Since $\alpha_n \to 0$ as $n \to \infty$, there is a positive integer k such that $h(x_1) + 2\alpha_k < h(x_2)$. Thus

$$f(\alpha_k) = h(\alpha_k) + 2\alpha_k = h(x_1) + 2\alpha_k < h(x_2).$$

Let $\beta_n = x_2(3/4)^n$. There is a positive integer m such that $\beta_m < \alpha_k$. Thus

$$f(\beta_m) = h(\beta_m) + 2\beta_m = h(x_2) + 2\beta_m > f(\alpha_k) + 2\beta_m > f(\alpha_k).$$

Since f is increasing, this is impossible. If we suppose that $h(x_1) > h(x_2)$ for some x_1, x_2 with $0 < x_1 < x_2$ we get a similar contradiction. Thus, we conclude that $h(x) = a$ for all $x > 0$.

An analogous proof shows that $h(x) = b$ for $x < 0$. Since the original equation with $x = 0$ says that $f(0) - f(0) = 0$, the value $f(0) = c$ may be assigned arbitrarily, but since f is increasing c must be between a and b. This gives us

$$f(x) = \begin{cases} 2x + a, & x > 0 \\ c, & x = 0 \\ 2x + b, & x < 0, \end{cases}$$

with $b \le c \le a$.

On the other hand, it is easy to check that this function satisfies the functional equation. If we require f to be continuous, the solutions are $f(x) = 2x + a$ for some constant a.

The second method of solution comes from Stefan Banach. We will use the following result from analysis: if f is monotone with domain $\Re$ then f has a finite left limit and a finite right limit at 0.

Substituting $x/4$ for x in (1) gives

(2) $$f\left(4 \cdot \frac{x}{4}\right) - f\left(3 \cdot \frac{x}{4}\right) = 2 \cdot \frac{x}{4} \quad \text{or} \quad f(x) = f((3/4)x) + \frac{x}{2}.$$

Now we put $(3/4)x$ instead of x in (2) to get

(3) $$f((3/4)x) = f((3/4)^2 x)) + \frac{3}{4} \cdot \frac{x}{2}.$$

Substituting the right-hand side into (2) gives

(4) $$f(x) = f((3/4)^2 x) + \frac{3}{4} \cdot \frac{x}{2} + \frac{x}{2}.$$

Replacing x with $(3/4)x$ in (3), we obtain

(5) $$f((3/4)^2) = f((3/4)^3 x) + \left(\frac{3}{4}\right)^2 \cdot \frac{x}{2},$$

and we can substitute the right-hand side of (5) for $f((3/4)^2 x)$ in (4). Now we have

$$f(x) = f((3/4)^3 x) + \left(\frac{3}{4}\right)^2 \cdot \frac{x}{2} + \left(\frac{3}{4}\right)^1 \cdot \frac{x}{2} + \frac{x}{2}.$$

Continuing this iterative procedure, after $n - 1$ steps we will obtain

$$f(x) = f((3/4)^n x) + \left(\frac{3}{4}\right)^{n-1} \cdot \frac{x}{2} + \left(\frac{3}{4}\right)^{n-2} \cdot \frac{x}{2} + \dots + \left(\frac{3}{4}\right)^1 \cdot \frac{x}{2} + \left(\frac{3}{4}\right)^0 \cdot \frac{x}{2},$$

which can be shown to hold for $n = 1, 2, \dots$ by mathematical induction. Summing the geometric series, we have

$$f(x) = f((3/4)^n x) + \frac{x}{2} \sum_{i=0}^{n-1} (3/4)^n = f((3/4)^n x) + \frac{x}{2} \cdot 4(1 - (3/4)^n)$$

Now we take the limit as $n \to \infty$. Since f has a right and left limit at 0 we can write

$$\lim_{y \to 0^+} f(y) = a \quad \text{and} \quad \lim_{y \to 0^-} f(y) = b,$$

and so

$$f(x) = \begin{cases} 2x + a, & x > 0 \\ 2x + b, & x < 0. \end{cases}$$

As before, the value $f(0) = c$ may be assigned arbitrarily, but since f is monotone c must be between a and b.

Both methods can be applied to other equations of the same type.

Piotr *Krysta* is a student of computer science (called informatics in Poland) at Wroclaw University. He wrote the above note in July, 7997.

Filler Wanted

The laws of probability guarantee that at the end of every paper in this Journal there will be, on the average, one-half of a page that the paper does not occupy.

This space should not be left blank. Do you have anything that could fill it? If so, send it in. Surely you can do better than

> You can always tell mathematicians by looking at their scalps. Their hair has square roots.

Or than

THEOREM: 1 is the largest positive integer.

Proof (by contradiction): Suppose not. Then some other positive integer, n, $n > 1$ is the largest positive integer. But $n^2 > n$, so n is not the largest positive integer. This contradiction proves the theorem.

Feeble, are they not? Send in something that will better occupy these spaces. Full credit will be given, or anonymity if you choose.

COMPUTATION OF π USING DIFFERENCES

Andrew *Cusumano*
Great Neck, New York

An ancient method for finding an approximate value of TI, used by Archimedes, is to successively double the number of sides of a regular polygon inscribed in a circle. In this note, we will use this method and then, using differences, double the number of significant digits obtained.

In Figure 1, s_n is the length of the side of an n-sided regular polygon inscribed in a circle of radius 1 and s_{2n} is the corresponding length of the $2n$-sided polygon. From the law of cosines,

$$s_n = 1^2 + 1^2 - 2\cos\theta$$

and

$$s_{2n}^2 = 1^2 + 1^2 - 2\cos(\theta/2).$$

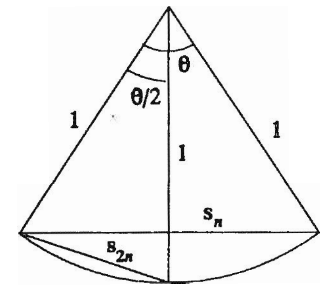


Figure 1

Using $\cos(\theta/2) = \sqrt{(1 + \cos\theta)/2}$, we find that $s_{2n} = \sqrt{2 - \sqrt{4 - s_n^2}}$.

The perimeter of the n-gon is u s , so T I = $ns_n/2$ provides an approximation to TI. If $n = 4$, the square inscribed in the circle has length $s_4 = \sqrt{2}$, and $\pi_4 = 2.828427\ldots$. Approximations can easily be generated by a computer.

If we make a table of differences of successive approximations, $d_n = \pi_{2n} -$ TI, and then of quotients of differences, $q_n = d_{2n}/d_n$, it is apparent that $\lim_{n\to\infty} q_n = 4$:

n	π_n	d_n	q_n
4	2.82842712	.23304033	3.88545009
8	3.06146746	.05997769	3.97115474
16	3.12144515	.01510333	3.99277564
32	3.13654849	.00378267	3.99819309
64	3.14033116	.00094609	
128	3.14127725		

Continuing the table by adding 4.00000000 to the last column and calculating backward from q_n to d_n and then to π_n, we are able to double the number of significant digits in the approximation:

3.14033116	.00094609	4.00000000
3.14127725	.00023652	4.00000000
3.14151377	.00005913	4.00000000
3.14157291	.00001478	
3.14158769		

The numbers in the first column approach 3.1415926... , correct to seven decimal places.

It is interesting to note that if the process of taking differences and quotients is repeated by calculating $d_n' = q_{2n} - q_n$ and $q_n' = d_{2n}'/d_n'$, then $\lim_{n\to\infty} q_n'$ is also 4. If the process is carried out again, $\{q_n''\}$ behaves similarly. This could be used to further accelerate the convergence of the approximations.

***Andrew Cusumano** has been a frequent contributor to the Journal, with both articles and problems.*

Forty Years Ago in the *Journal*

In Problem 58 (this ***Journal,*** 1 (1949-54) #10, 413-414), C. W. Trigg gave twenty-nine anagrams which, when unscrambled, were the names of "twenty-nine mathematicians." How many can you identify? I could not get all twenty-nine:

Poincaré, Iamblichus, Mersenne, Uspensky, Einstein, Plato, Salmon, Ibn Ezra, L'Hopital, Oughtred, Niconiachus, Jourdain, Osgood, Underwood, Recorde, Napier, Archibald, Lindemann, Mascheroni, Apollonius, Tchebychef, Hippocrates, Eudoxus, Monge, Abel, Tschirnhausen, Infeld, Cantor, Stirling.

AN INDEFINITE INTEGRAL FOR [x]

Steven W. Sy
Michigan Technological University

To compute the value of the integral of a function, we can use the definition of the integral or, knowing the function's antiderivative, apply the Fundamental Theorem of Calculus. Though we have no methods for finding antiderivatives of functions involving greatest-integer functions, nevertheless we can give a simple formula for $\int [x]\,dx$.

If a and b are integers, to calculate $\int_a^b [x]\,dx$ we add the areas of rectangles with unit width. For example,

$$\int_1^4 [x]\,dx = \int_1^2 1\,dx + \int_2^3 2\,dx + \int_3^4 3\,dx = 1 + 2 + 3.$$

In general,

$$\int_a^b [x]\,dx = \sum_{i=a}^{b-1} \int_i^{i+1} i\,dx = \sum_{i=a}^{b-1} i.$$

Let $j = i - a + 1$ to get

$$\sum_{i=a}^{b-1} i = \sum_{j=1}^{b-a} (j + a - 1) = \sum_{j=1}^{b-a} j + (a-1)\sum_{j=1}^{b-a} 1.$$

Summing, we get

$$\int_a^b [x]\,dx = \frac{(b-a)(b-a+1)}{2} + (a-1)(b-a) = \frac{(b-a)(b+a-1)}{2}$$

To find $\int_c^d [x]\,dx$ when c and d are real, we will take the wider interval $[[c],[d]+1]$ (because it has integer limits) and then subtract the excess. We have

$$\int_c^d [x]\,dx = \int_{[c]}^{[d]+1} [x]\,dx - \int_{[c]}^{c} [x]\,dx - \int_d^{[d]+1} [x]\,dx.$$

In the second integral, since $[x]$ is constant and equal to $[c]$, we have

$$\int_{[c]}^{c} [x]\,dx = (c - [c])[c].$$

Dealing similarly with the third integral, we get

$$\int_c^d [x]\,dx = \int_{[c]}^{[d]+1} [x]\,dx - (c-[c])[c] - ([d]+1-d)[d].$$

The integral on the right side has integer limits, so

$$\int_c^d [x]\,dx = \frac{([d]+1-[c])([d]+1+[c]-1)}{2}$$
$$+ [c]([c]-c) + [d](d-[d]-1).$$

This can be written

$$\int_c^d [x]\,dx = \frac{(2d[d]-[d]-[d]^2)}{2} - \frac{(2c[c]-c-[c]^2)}{2} = \left.\frac{(2x[x]-[x]-[x]^2)}{2}\right|_c^d$$

which may be expressed as

$$\int [x]\,dx = \frac{2x[x]-[x]-[x]^2}{2} + C.$$

It is interesting to note that if we let $<x> = x - [x]$, the fractional part of x, then the above formula gives

$$\int <x>\,dx = \frac{<x>^2}{2} + \frac{[x]}{2} + c,$$

similar to $\int x\,dx = x^2/2 + C$.

alumna
alumni

Steven Sy is currently a sophomore at Michigan Tech, majoring in mathematics and interested in the theory of integration and differential equations. He developed this representation as a freshman while taking an integral calculus class.

Solution to Mathacrostic *37,* by Patti Vahedi (Fall, *1993).*

Words:

A. right-hand rule	N. trident of
B. Otto III	O. hidebound
C. bodhisattva	P. Eratosthenes
D. eigenvalue	Q. $\sqrt{2}$ over 5
E. Ross	R. Aryabhata
F. the Twenty	S. power series
G. Pavlov's dog	T. of the wits
H Riemann sum	U. complex plane
I. Eis	V. R. H. Brown
J. Schrödinger's cat	W. yueh-ch'in
K. two-body	X. prime number
L. obiter dictum	Y. Hubble
M. Newton's apple	Z. Augustin Cauchy

Author and title: Robert Preston, The $\sqrt{2}$ *Apocrypha.*

Quotation: $\sqrt{2}$, an irrational number, cannot be expressed as the ratio of two whole numbers. Tippasus of Metapontum supposedly discovered this in the 5th century BC while traveling by boat with some Pythagoreans who, believing everything in nature could be reduced to such a ratio, threw him overboard in disgust.

Solvers: THOMAS BANCHOFF, Brown University, JEANETTE BICKLEY, St. Louis Community College—Meramec, PAUL S. BRUCKMAN, Everett, Washington, CHARLES DIMMINIE, St. Bonaventure University, ROBERT FORSBERG, Lexington, Massachusetts, META HARRSEN, Durham, North Carolina, THEODOR KAUFMAN, Brooklyn, New York, HENRY S. LIEBERMAN, Waban, Massachusetts, CHARLOTTE MAINS, Rochester, New York, DON PFAFF, University of Nevada—Reno, NAOMI SHAPIRO, Georgian Court College, DAVID S-SHOBE, New Haven, Connecticut, and STEPHANIE SLOYAN, Georgian Court College.

In word U, "39" should have been "29". As pointed out by Stephanie Sloyan, Arybhata's dates are AD and not BC. Some solvers had word Q not quite correct.

Mathacrostic 38, proposed by THEODOR KAUFMAN, follows. To be listed as a solver, *send* your solution to the editor.

A. In fencing, a parry which is **connected with a riposte (hyph)** 332 263 185 34 303 163 138 211

B. In marine insurance, when the cost of recovering and repairing damaged property exceeds its value **(2)** 301 289 161 26 236 139 261 72 65

C. The monkshood (hyph) 245 214 331 141 25 291 300 177 107

D. **Tophetic** 128 99 102 55 227 254 207 44

E. It is used to find the greatest common divisor of two integers **(2)** 250 137 297 260 286 203 319 224 192 222 325 125 237 166 206 181

F. A line over a group of elements indicating it is to be considered as a whole, as $\overline{a+b+2}$ 123 184 21 148 213 330 111 239

G. To insert intermediate terms in a mathematical series according to the law of the series 98 307 19 124 104 42 22 82 172 90 295

H. Someone who shafts when he or she should shalt not 126 43 100 80 337 142

I One-seventh, for one **(2)** 73 173 29 96 217 122 89 249 308 112 281 200 61 169 193

J. PPS, for an example 70 132 274 216 154 310 294 9 199 81 118 1

K. Type of equation of the first degree between two variables 299 329 32 95 157 76

L. Administers 174 317 194 75 257 97 18 233 202 49

M. In it, the unknown function occurs under $\int$ 92 248 33 103 262 47 17 170 3 198 231 278 235 144 56 221

N. Severance; cleavage 230 318 67 165 145 265 320 251

O. **In geometry, any surface of** constant negative curvature, especially that generated by the revolution of a tractrix about its axis 27 54 275 252 110 52 6 152 267 290 324 119

1 J	2 #	3 M		5 S	6 O	7 Z		9 J	10 X		12 +	13 Q	14 Z	15 R	16 T	17 M	18 L
19 G		21 F	22 G	23 @		25 C	26 B	27 O	28 P	29 I	30 Q	31 Y	32 K	33 M	34 A	35 T	36 #
37 S	38 V	39 W	40 X		42 G	43 H	44 D	45 +	46 R	47 M	48 #	49 L		51 @	52 O		54 O
55 D	56 N	57 @		59 X	60 §	61 I	62 +	63 @	64 V	65 B		67 H	68 V		70 J		72 B
73 I	74 +	75 L	76 K	77 U	78 X	79 @	80 H	81 J	82 G	83 Q		85 R	86 U	87 T	88 +	89 I	90 G
	92 M	93 U	94 X	95 K	96 I	97 L	98 G	99 D	100 H		102 D	103 M	104 G	105 P	106 T	107 C	108 W
	110 O	111 F	112 I		114 @	115 S		117 #	118 J	119 O		121 S	122 I	123 F	124 G	125 E	126 H
127 @	128 D	129 R		131 F	132 J		134 Y	135 S	136 Z	137 E	138 A	139 B		141 C	142 H	143 +	144 M
145 N	146 P	147 W	148 F		150 §	151 Z	152 O	153 +	154 J	155 X	156 Y	157 K	158 Q		160 W	161 B	162 V
163 A	164 U	165 N	166 E	167 S	168 P	169 I	170 M		172 G	173 I	174 L		176 P	177 C	178 Y	179 S	180 V
181 E	182 X	183 Z	184 F	185 A	186 Q	187 +		189 +	190 §	191 Y	192 E	193 I	194 L		196 U	197 S	198 M
199 J	200 I	201 Y	202 L	203 E		205 @	206 E	207 D	208 Y		210 Y	211 A	212 U	213 F	214 C	215 @	216 J
217 I		219 T	220 P	221 M	222 E		224 E	225 T	226 U	227 D	228 #		230 N	231 M	232 +	233 L	
235 M	236 B	237 E	238 Y	239 F	240 V	241 Z	242 +		244 X	245 C		247 X	248 N	249 I	250 E	251 N	252 O
253 P	254 D	255 *		257 L	258 @		260 E	261 B	262 M	263 A	264 Y	265 N	266 W	267 O	268 Q	269 X	
271 X	272 Y	273 Z	274 J	275 O	276 W	277 V	278 M	279 §		281 I	282 +	283 X	284 W	285 Y	286 E	287 V	288 Z
289 B	290 O	291 C		293 #	294 J	295 G		297 E	298 X	299 K	300 C	301 B	302 P	303 A	304 V		306 T
307 G	308 I		310 J	311 #	312 V		314 +	315 Q	316 X	317 L	318 N	319 E		320 N	321 W		323 S
324 O	325 E	326 V	327 U	328 W	329 K	330 F	331 C	332 Z	333 P		335 A	336 P	337 H	338 V	339 S	340 U	

P. One of the earliest methods of color printing	302 253 146 336 131 168 105 28 220 176 333
Q. Profound	315 13 83 30 268 186 158
R. Light, circular movable tent consisting of skins or felt stretched over a lattice framework	129 46 85 15
S. Cubic curve whose Cartesian equation is $x(x^2+y^2)=y(ax-cy)$	37 323 179 135 115 5 339 167 121 197
T. A set. such that two standard definitions of its dimension give different answers	225 16 106 87 35 306 219
U Discharge of official duty involving frequent change of residency	164 226 77 93 86 196 327 212 340
V. Set of eleven rods, usually made of wood, devised for the purpose of making numerical calculations	287 162 240 277 180 64 304 326 338 38 312 68
W. Some composers never get to this (2)	321 147 284 276 266 328 39 160 108
X	283 94 298 40 269 244 10 271 182 316 78 155 59 247
Y Linear polar coordinate of a variable point (2)	264 272 285 191 238 208 134 31 156 178 210 201
Z Sensitive method for determining whether arsenic is present in a solution (2)	151 288 273 136 241 183 7 14 332
*. Well	12 314 187 255
# More remote	117 2 36 293 311 48 228
. Ravel, not Maurice	127 23 51 63 215 114 57 79 258 205
+. Science of midwifery	282 62 242 143 88 45 74 153 232 189
§. Overwhelm (slang)	279 60 190 150

PROBLEM DEPARTMENT

Edited by Clayton W. Dodge
University of Maine

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

All communications should be addressed to C. W. Dodge, 5752 Neville/Math, University of Maine, Orono, ME 04469-5752. E-mail: dodge@gauss.umemat.maine.edu. Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly identified with name and address. Solutions to problems in this issue should be mailed by December 15, 1994.

Problems for Solution

823. *Proposed by Alan Wayne, Holiday, Florida.*

Find all solutions to the multiplication alphametic

$$(I)(DINE) = ENID.$$

That is, find the form(s) taken by all solutions in all bases.

824. *Proposed by Joel L. Brenner, Palo Alto, California.*

Prove that there are no real integral solutions to the set of equations

$$(x^3 + 6x^2 - 159)y = 160,$$
$$(y^3 + 6y^2 - 159)z = 160,$$
$$(z^3 + 6z^2 - 159)x = 160.$$

You may not assume that a putative solution would possess any symmetry.

825. *Proposed by Leon Bankoff, Los Angeles, California.*

Let O be a point inside the equilateral triangle ABC whose side is of length s. Let OA, OB, OC have lengths a, b, c respectively. Given the lengths a, b, c, find length s.

826. *Proposed by M. A. Kahn, Lucknow, India.*

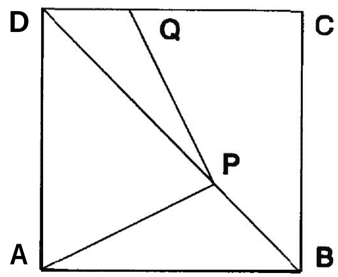
Prove or disprove that the product

$$P = \prod_{k=1}^{n} (1 + k^2)$$

is a perfect square only for $n = 3$ and for no other positive integer.

827. *Proposed by Stanley Rabinowitz, MathPro Press, Westford, Massachusetts.*

Let P be a point on diagonal BD of square $ABCD$ and let Q be a point on side CD such that APQ is a right angle. Prove that $AP = PQ$.



828. *Proposed by Rex H. Wu, Brooklyn, New York*

Evaluate the integral

$$\int \ln(e^x + 1)\, dx.$$

829. *Proposed by Seung-Jin Bang, Seoul, Republic of Korea.*

Let f be a function such that $f, f', \ldots, f^{(n)}$ are all continuous, $f(0) = f'(0) = \cdots = f^{(n-1)}(0) = 0$ and $f^{(n)}(0) \neq 0$. Let

$$f_1(x) = \int_0^x f(t)\, dt \quad \text{and} \quad f_k(x) = \int_0^x f_{k-1}(t)\, dt \quad k = 2, 3, \ldots, n.$$

Evaluate the limit

$$\lim_{x \to 0} \frac{f_m(x)}{x^m f(x)}$$

830. *Proposed by David Iny, Baltimore, Maryland.*

Let

$$w_k = \sum_{n=0}^{\infty} \frac{x^{4n+k-1}}{(4n+k-1)!} \qquad k = 1, 2, 3, 4.$$

a) Prove that $[(w_1 + w_3)^2 - (w_2 + w_4)^2][(w_1 - w_3)^2 + (w_2 - w_4)^2] = 1.$

b) Can you find similar identities with $p \geq 2$ for

$$w_k = \sum_{n=0}^{\infty} \frac{x^{pn+k-1}}{(pn+k-1)!} \qquad k = 1, 2, \ldots, p?$$

This problem is a generalization of a 1939 Putnam Exam problem, which considered the case of $p = 3$.

831. *Proposed by Paul S. Bruckman, Everett, Washington.*

Solve exactly and completely

$$x^6 - 8x^5 + 18x^4 - 6x^3 - 12x^2 + 2x + 1 = 0.$$

832. *Proposed by David Iny, Westinghouse Electric Corporation, Baltimore, Maryland.*

The *taxicab distance* between points (a, b) and (c, d) is $|a - c| + |b - d|$. Determine the circumference in taxicab space of the circle whose equation is $x^2 + y^2 = 1$.

833. *Proposed by Seung-Jin Bang, Seoul, Republic of Korea.*

Define a function f by $f(0) = 1$ and

$$f(m) = \binom{n+m-2}{m} + \binom{n+m-3}{m-1}.$$

Find the value of the sum $\sum_{m=0}^{k} f(m)$.

834. *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.*

Let T and T' denote two triangles with respective sides (a, b, c) and (a', b', c') where $a'^2 = 2a(s - a)$, $b'^2 = 2b(s - b)$, and $c'^2 = 2c(s - c)$. Prove that

(i) $s \geq s'$, (ii) $R \geq R'$, (iii) $r' \geq r$, and (iv) $F'/s'^2 \geq F/s^2$

where $s = (a + b + c)/2$ is the semiperimeter, R the circumradius, r the inradius, and F the area of triangle ABC, and similarly for triangle $A'B'C'$.

835. *Proposed by the Alma College Problem Solving Group, Alma College, Alma, Michigan.*

Let $P(x)$ be a polynomial of degree $n \geq 2$ with real coefficients whose leading three terms are $ax^n + bx^{n-1} + cx^{n-2}$. All remaining terms are of degree $n - 3$ or less. If $b^2 \leq 2ac$, then prove that $P(x)$ cannot have n distinct real roots. (This problem is a generalization of Problem 4 from the

February 1990 issue of *Problem Solving Newsletter* by Dr. Hugh Montgomery of the University of Michigan.)

Solutions

797. [Spring 1993] *Proposed by Alan Wayne, Holiday, Florida.*

Restore the enciphered digits of the addends in the following base four addition:

$$A + RAP + AT + A + RAT = 1230.$$

By what means was the RAP caused?

I. *Solution by Matt Collins, St. Joseph's University, Philadelphia, Pennsylvania.*

From the ones' column we get $24 + 2T + P \equiv 0 \pmod 4$. Since this sum is at most 11, it follows that it is either 4 or 8. If it is 4, then the lours' column now reads $1 + 3A \equiv 3 \pmod 4$, which implies $A = 2$. But then $2T + P = 0$, which is impossible. Thus $24 + 2T + P = 8$ and the fours' column becomes $2 + 3A \equiv 3 \pmod 4$, whence $A = 3$ and 2 is carried to the sixteens' column. Thus $2 + 2R = 6$, so $R = 2$. Returning to the ones' column, we see that P cannot bc odd, so $P = 0$ and $T = 1$. Since $P = 0$, $T = 1$, $R = 2$, and $A = 3$, the *RAP* was caused by a *TRAP* (= 1230).

II. *Comment by Jason Beck and Mike Hillis, Hendrix College, Conway, Arkansas.*

We determined that the RAP was caused by (ready for the overbull) a *RAT* eating cheese from a rat trap somewhere between 2:30 and 2:31 *in* the morning. When the rat ate the cheese, the trap killed the rat, waking the homeowner from a mathematical dream. From out of the blue the homeowner then scribbled this problem down on a piece of paper and decided to use it for the everlasting entertainment of Calculus II students everywhere. Obviously a problem that everyone should know and love.

Also solved by CHARLES ASHBACHER, *Cedar Rapids, IA,* SEUNG-JIN BANG, *Seoul, Korea,* FRANK P. BATTLES. *Massachusetts Maritime Academy, Buzzards Bay,* JASON BECK and MIKE HILLIS, *Hendrix College, Conway, AR,* SCOTT H. BROWN, *Auburn University, AL,* PAUL S. BRUCKMAN, *Edmonds, WA,* WILLIAM CHAU, *New York, NY,* BILL CORRELL, JR., *Denison University, Cincinnati, OH.* STACEY DACOSTA and PHILIP J. DARCY, *St. Bonaventure University. NY,* CLINTON B. EDWARDS and ALLISON LAMAR, *Hendrix College, Conway, AR,* MARK EVANS, *Louisville, KY,* VICTOR G. FESER, *University of Mary, Bismarck, ND,* SEAN FORBES, *Drake University, Des Moines, IA,* STEPHEN I. GENDLER, *Clarion University of Pennsylvania,* RICHARD I. HESS, *Rancho Palos Verdes, CA,* CARL LIBIS, *Idaho State University, Pocatello,* KAREN MCNIECE, *Hendrix College, Conway, AR,* YOSHINOBU MURAYOSHI, *Okinawa, Japan,* MICHAEL R. PINTER, *Belmont University, Nashville, TN,* BOB PRIELIPP, *University of Wisconsin-Oshkosh,* THOMAS F. SWEENEY, *Russell Sage College, Troy, NY,* ADRIENNE N. TILEY, *Hendrix College, Conway, AR,* KENNETH M. WILKE, *Topeka, KS,* REX H. WU, *Brooklyn, NY, and the* PROPOSER.

***798.** [Spring 1993] *Proposed by Dmitry P. Mavlo, Moscow, Russia.*

Since *1993* is a prime year, it seems reasonable to ask which is larger,

$$\frac{10^{1992}-1}{10^{1993}-1} \text{ or } \frac{10^{1993}-1}{10^{1994}-1}?$$

Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.

More generally, we show that the function

$$F(x) = \frac{e^{ax} - e^{bx}}{e^{a(x+c)} - e^{b(x+c)}},$$

where $a > b \geq 0$ and $c > 0$, is increasing in x. It suffices to show that $F'(x) > 0$ for all x. To that end we find that $F'(x) =$

$$\frac{(e^{a(x+c)} - e^{b(x+c)})(ae^{ax} - be^{bx}) - (e^{ax} - e^{bx})(ae^{a(x+c)} - be^{b(x+c)})}{(e^{a(x+c)} - e^{b(x+c)})^2}$$

$$= \frac{(a-b)(e^{ac} - e^{bc})e^{x(a+b)}}{(e^{a(x+c)} - e^{b(x+c)})^2},$$

which is always greater than zero.

The given problem corresponds to the case $b = 0$, $c = 1$, and $e^a = 10$, and we see that the right hand expression is the greater one.

Also solved by AVRAHAM ADLER, *Monsey, NY,* CHARLES ASHBACHER, *Cedar Rapids, IA,* MOHAMMAD K. AZARIAN, *University of Evansville, IN,* SEUNG-JIN BANG, *Seoul, Korea,* FRANK P. BATTLER *Massachusetts Maritime Academy, Buzzards Bay,* JAMES D. BRASHER,

Teledyne Brown Engineering, Huntsville, AL, PAUL S. BRUCKMAN, ***Edmonds, WA,*** BARRY BRUNSON, ***Western Kentucky University, Bowling Green,*** PHILIP A. D. CASTORO, ***Queens College, Flushing, NY,*** WILLIAM CHAU, ***New York, NY,*** BILL CORRELL, JR., ***Denison University, Cincinnati, OH,*** PHILIP J. DARCY, ***St. Bonaventure University, NY,*** DAVID DELSESTO, ***Smithfield, RI,*** CHARLES R. DIMINNIE, ***St. Bonaventure University, NY,*** GEORGE P. EVANOVICH, ***Saint Peter's College, Jersey City, NJ,*** MARK EVANS, ***Louisville, KY,*** SEAN FORBES, ***Drake University, Des Moines, IA,*** JAYANTHI GANAPATHY, ***University of Wisconsin-Oshkosh,*** STEPHEN I. GENDLER, ***Clarion University of Pennsylvania,*** RICHARD I. HESS, ***Rancho Palos Verdes, CA,*** CARL LIBIS, ***Idaho State University, Pocatello,*** CHIA SIEN LIM, ***Oklahoma State University, Stillwater,*** PETER A. LINDSTROM, ***North Lake College, Irving, TX,*** DAVID E. MANES, ***SUNY College at Oneonta,*** YOSHINOBU MURAYOSHI, ***Okinawa, Japan,*** KANDASAMY MUTHUVEL, ***University of Wisconsin-Oshkosh,*** WILLIAM MYERS, ***Belmont Abbey College, NC,*** MICHAEL R. PINTER, ***Belmont University, Nashville, TN,*** BOB PRIELIPP, ***University of Wisconsin-Oshkosh,*** JOHN F. PUTZ, ***Alma College, MI,*** LAURA SILVA, ***Albany, CA,*** ANA STANGL, ***Drake University, Des Moines, IA,*** THOMAS F. SWEENEY, ***Russell Sage College, Troy, NY,*** KENNETH M. WILKE, ***Topeka, KS, and*** REX H. WU, ***Brooklyn, NY.***

The comment that 1993 was a prime year apparently became a red herring for some solvers. It was irrelevant to the problem and simply an interesting fact.

799. [Spring 1993] ***Proposed by Stan Wagon, Macalester College, St. Paul, Minnesota.***

a) Find all years that are palindromes in both the standard and the Hebrew calendars. (To get the Hebrew year, add 3761 if it is after the Jewish New Year in September, add 3760 otherwise. A ***palindrome*** is a number, such as 1221, that reads the same backwards and forwards.)

b) Find all positive integers x such that there are infinitely many positive integers n for which n and $n + x$ are palindromes.

Solution by William H. Peirce, Delray Beach, Florida.

a) All palindromes with an even number of digits are divisible by 11. Let H and S denote the Hebrew and standard years, so that $H = S + 3760$ or $H = S + 3761$. Since neither 3760 nor 3761 is divisible by 11, then H and S cannot both have an even number of digits.

If S has one digit, then $3760 \le H \le 3770$ and H cannot be a palindrome.

Also S cannot have 2 digits since then H would have 4 digits, an even number.

Let $S = aba$ with $a \ne 0$, a three-digit palindrome. Then $3861 \le H \le 4760$ and H is divisible by 11. If the addend is $3760 \equiv 9 \pmod{11}$, then $a \equiv 3$ or 4 and $S \equiv 2 \pmod{11}$. Hence $S = 343$ or 464. Only the latter produces a solution, specifically $4224 = 464 + 3760$. If the addend is $3761 \equiv 10 \pmod{11}$, then $a = 2$ or 3 and $S \equiv 1 \pmod{11}$. Hence $S = 232$ or 353, yielding the two solutions $3993 = 232 + 3761$ and $4114 = 353 + 3761$.

If S has an even number n of digits, $n \ge 4$, then H must have $n + 1$ digits and will be of the form $1xx...x1$, which requires that the addend be 3760 and S be of the form $1y...y1$. Since the sum $3760 + 1y...y1$ can have no more than n digits, no solution occurs.

If S has an odd number n of digits, $n \ge 5$, the argument of the preceding paragraph shows that H cannot have $n + 1$ digits. If $S = abcba$, a 5-digit number, since the addend is 3760 or 3761, then the tens' digit of H is $b + 6$ or $b + 7 \pmod{10}$ whereas the thousands digit is $b + 3$ or $b + 4 \pmod{10}$. Hence the tens' and thousands' digits cannot be equal and H is not a palindrome. If $S = abc...cba$ has more than 5 digits, then the tens' digit of H is still $b + 6$ or $b + 7 \pmod{10}$ while its second digit from the left will be b or $b + 1 \pmod{10}$. Again H is not a palindrome.

The only solutions to part (a) are therefore $4224 = 464 + 3760$, $3993 = 232 + 3761$, and $4114 = 353 + 3761$.

b) Since there are infinitely many n such that n and $n + x$ are palindromes, we may assume, without loss of generality, that n has at least twice as many digits as x. If $n + x$ has more digits than n, then it has just one more digit, its first two digits are 10, and the first digit of n is 9. To get the necessary carry for $9 + 1 = 10$, the first half (at least) of n must be all 9s. Since it is a palindrome, n is a string of 9s. Similarly, $n + x$ is 100...01 and $x = 2$.

It is easily seen that if x is a solution whose ones' digit is 0, then $x/10$ is also a solution. Just delete the first and last digits of any n that works with x and the result will work with $x/10$. Thus we assume that the ones' digit of x is not 0. Again considering n much larger than x, we see that the ones' of x is 1 since the leftmost digits of n and $n + x$ can differ by at most 1. Therefore, the ones' digit of n is a digit from 1 to 8 and there is no carry into the tens' column in the sum $n + x$. To have the leftmost digits of n and $n + x$ differ by 1, there must be a carry from the preceding columns, which requires that the middle digits of n be all 9s and those of $n + x$ all 0s. This occurs only for $x = 11$.

The only other solutions to part (b) are extensions of this last solution. Some illustrations include 9 + 2 = 11, 99 + 2 = 101, 999 + 2 = 1001, 898 + 11 = 909, 2992 + 11 = 3003, 799997 + 11 = 800008, 28982 + 110 = 29092, and 6662992666 + 11000 = 6663003666.

Also solved by PAUL S. BRUCKMAN, Edmonds, WA, RICHARD I. HESS, Rancho Palos Verdes, CA, and the PROPOSER.

800. [Spring **1993]** Proposed by Michael D. Williams, Wake Forest University, Winston-Salem, North Carolina.

Prove that for positive integral n,

$$(2^n)! = \prod_{i=1}^{n} \binom{2^i}{2^{i-1}}^{2^{n-i}}.$$

Solution by ***Philip*** A. D. Castoro, student at Queens College, City University of New York, Flushing, New ***York.***

Since $2^k - 2^{k-1} = 2^{k-1}$, we have that for positive integral k,

$$\binom{2^k}{2^{k-1}} = (2^k)!/[(2^{k-1})!]^2.$$

Now we have

$$\prod_{i=1}^{n} \binom{2^i}{2^{i-1}}^{2^{n-i}} = \left(\frac{(2^1)!}{[(2^0)!]^2}\right)^{2^{n-1}} \cdot \left(\frac{(2^2)!}{[(2^1)!]^2}\right)^{2^{n-2}} \cdot \ldots \cdot \left(\frac{(2^n)!}{[(2^{n-1})!]^2}\right)^{2^0}$$

$$= \frac{[(2^1)!]^{2^{n-1}}}{[(2^0)!]^{2^n}} \cdot \frac{[(2^2)!]^{2^{n-2}}}{[(2^1)!]^{2^{n-1}}} \cdot \frac{[(2^3)!]^{2^{n-3}}}{[(2^2)!]^{2^{n-2}}} \cdot \ldots \cdot \frac{[(2^n)!]^{2^0}}{[(2^{n-1})!]^{2^1}},$$

a telescoping product that collapses to $(2^n)!$ since $(2^0)! = 1$.

Also solved by SEUNG-JIN BANG, Seoul, Korea, PAUL **S.** BRUCKMAN, Edmonds, WA, WILLIAM CHAU, New York, *NY,* BILL CORRELL, JR., ***Denison*** University, Cincinnati, OH, MARK EVANS, Louisville, *KY,* SEAN FORBES, Drake University, Des Moines, *IA,* RICHARD I. HESS, Rancho Palos Verdes, CA, CARL **LIBIS,** Zdaho State University, Pocatello, DAVID E. MANES, ***SUNY*** College at Oneonta, **ROLLINS COLLEGE/UNIVERSITY** OF CENTRAL FLORIDA GRAPH GROUP, Orlando, MANUEL SILVA, Albany, CA, KENNETH M. WILKE (two solutions), Topeka, *KS,* REX H. WU, Brooklyn, *NY,* and the

PROPOSER.

A few solvers used mathematical induction to establish the formula, although the collapsing product technique was most common.

801. [Spring **1993]** Proposed by Norman ***Schaumberger,*** Bronx Community College, Bronx, New York.

If a, b, and c are real numbers, then prove that

$$e^a(a-b)+e^b(b-c)+e^c(c-a) \ge 0 \ge e^a(c-a)+e^b(a-b)+e^c(b-c).$$

I. Solution by Richard **I.** ***Hess,*** Rancho Palos Verdes, California.

Let

$$f(a, b, c) = e^a(a-b) + e^b(b-c) + e^c(c-a)$$

Then $f(a, b, c) = f(b, c, a) = f(c, a, b)$. Without loss of generality we assume that $a \ge b, c$. Then we have

$$\frac{\partial f}{\partial a} = e^a - e^c + e^a(a-b) \ge 0,$$

so, as a increases, then $f(a, b, c)$ increases. Because

$$f(a, b, a) = (e^a - e^b)(a-b) \ge 0 \text{ and } f(a, a, c) = (e^a - e^c)(a-c) \ge 0,$$

with equality if and only if a = b and a = c, then $f(a, b, c) \ge 0$ with equality if and only if a = b = c.

To prove the right-hand inequality, we note that

$$0 \ge -f(c,b,a) = e^a(c-a) + e^b(a-b) + e^c(b-c).$$

II. Solution by David E. Manes, State University of New York College at Oneonta, Oneonta, New ***York.***

We use the inequality

$$\textbf{(1)} \qquad x \ln x \ge x - 1$$

for $x > 0$ (see **e.g.,** N. Schaumberger, Extending a familiar inequality, this Journal, 9 (**1989-94**) #6, 384-385). Successively substituting e^{a-b}, e^{b-c}, e^{c-a} for x in (1) yields

$$e^{a-b}(a-b) \ge e^{a-b} - 1$$

and two similar expressions. Multiplying each of these inequalities by e^b, e^c, and e^a respectively gives

$$e^a(a - b) \geq e^a - e^b$$

and two similar expressions. Add these three inequalities to get

$$e^a(a - b) + e^b(b - c) + e^c(c - a) \geq e^a - e^b + e^b - e^c + e^c - e^a = 0.$$

Similarly, substituting e^{a-c}, e^{b-a}, e^{c-b} for x in *(1)* yields the right-hand inequality.

Also solved by SEUNG-JIN BANG, *Seoul, Korea,* PAUL S. BRUCKMAN, *Edmonds, WA,* WILLIAM CHAU, *New York, NY,* BILL CORRELL, JR., *Denison University, Cincinnati, OH,* MARK EVANS, *Louisville, KY,* JAYANTHI GANAPATHY, *University of Wisconsin-Oshkosh,* DAVID INY, *Westinghouse Electric Corporation, Baltimore, MD,* MURRAY S. KLAMKIN *(two solutions) University of Alberta, Canada,* CHIA SIEN LIM, *Oklahoma State University, Stillwater,* KANDASAMY MUTHUVEL, *University of Wisconsin-Oshkosh,* REX H. WU, *Brooklyn, NY, and the* PROPOSER.

802. (Spring 1993] *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.*

Let a and b be positive real numbers. Determine the maximum value of

$$f(x) = (a - x)(x + \sqrt{x^2 - b^2})$$

over all real x with $x^2 \geq b^2$. A non-calculus solution is requested,

Solution by William H. Peirce, Defray Beach, Florida.

By straightforward algebra solve $y = f(x)$ for x to get

$$(1) \qquad x = g(y) = \frac{a(b^2 + y) \pm \sqrt{(a^2 - b^2)y^2 - 2y^3}}{b^2 + 2y}$$

All (x, y) satisfying $y = f(x)$ will also satisfy $x = g(y)$, but the converse is not true. For example, the expression for $g(y)$ is not valid when $y = -b^2/2$.

We show that the portion of $y = f(x)$ that lies to the left of the y-axis is in the third quadrant and has a rightmost point $(-b, -b(a + b))$, which is a minimum in that quadrant. To that end, rationalize the numerator in $y = f(x)$ and note that $x < 0$ so that $\sqrt{(x^2)} = -x$, whence

$$\sqrt{x^2 - b^2} = -x\sqrt{1 - b^2/x^2}.$$

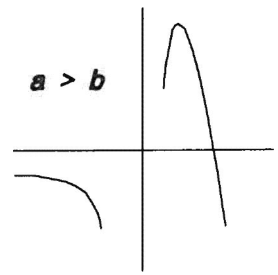
We get that

$$y = f(x) = \frac{-b^2 + ab^2/x}{1 + \sqrt{1 - b^2/x^2}} = -\frac{b^2}{1 + \sqrt{1 - b^2/x^2}} + \frac{ab^2}{x(1 + \sqrt{1 - b^2/x^2})},$$

which shows that fix) approaches $-b^2/2$ as x approaches $-\infty$. When $x \leq -b$, then $ab^2 \leq -abx$ and

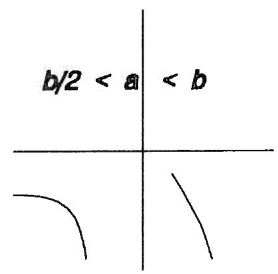
$$ab^2 \leq -abx + x(-ab - b^2)\sqrt{1 - b^2/x^2}$$

since the last term on the right is always positive. Now divide through by x, reversing the sense of the inequality of course, add $-b^2$ to each side, and then divide through by the radical to get that $y \geq -b(a + b)$ whenever $x \leq -b$. We have shown that $-b(a + b) \leq y < -b^2/2$ when x is negative.



We now consider three cases.

Case I: $a \geq b$. Since the expression under the radical in $g(y)$ cannot be negative, then $y = (a^2 - b^2)/2$ is a possible maximum for fix). Then *(1)* gives $x = (a^2 + b^2)/2a$, which is greater than or equal to b and does satisfy $y = f(x)$. Hence the point $((a^2 + b^2)/2a, (a^2 - b^2)/2)$ is the maximum point on the curve. When $a = b$, the point becomes $(a, 0)$.



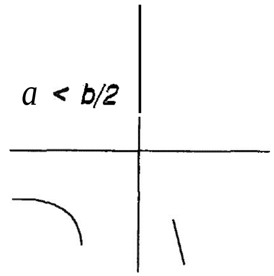
Case II: $b/2 \leq a < b$. Here the maximum of $f(x)$ on the right branch of the curve is not less than the asymptotic maximum on the left, that is, $(a - b)b \geq -b^2/2$, so the maximum is $(a - b)b$, achieved at $x = b$.

Case III: $a < b/2$. Here the maximum value of $f(x)$ in the fourth quadrant is less than the asymptotic value in the third quadrant. That is, $(a - b)b < -b^2/2$, so there is no maximum in this case. The accompanying figures show the graphs of $y = f(x)$ for these three cases.

Also solved by PAUL S. BRUCKMAN, *Edmonds, WA,* JAYANTHI

GANAPATHY, *University of Wisconsin-Oshkosh,* DAVID INY, *Westinghouse Electric Corporation, Baltimore, MD, and the* PROPOSER.

Plaudits to Peirce and to all three listed also-solvers for spotting the three cases in the solution to this problem.



803. [Spring **1993]** *Proposed by R. S. Luthar, University of Wisconsin Center, Janesville, Wisconsin.*

In any triangle *ABC* prove that

$$\sum \sqrt{\tan (A/2)} < \sqrt{3} \sum \sqrt{\csc A} .$$

(In a triangle *ABC*, $\Sigma f(A)$ means $f(A) + f(B) + f(C)$.)

*Solution by **Seung-Jin** Bang, Seoul, Republic of Korea.*

We prove the sharper inequality

$$(1) \qquad \sum \sqrt{\tan (A/2)} \le \frac{1}{\sqrt{2}} \sum \sqrt{\csc A}$$

with equality if and only if triangle ***ABC*** is equilateral. It is easy algebra to check, for nonnegative *a, b, c,* that

$$\sqrt{a} + \sqrt{b} + \sqrt{c} \le \sqrt{3(a + b + c)}.$$

We apply this result to the known equation

$$\sum \tan (A/2) = \frac{4R + r}{s},$$

where *R, r,* and *s* and the circumradius, **inradius,** and semiperimeter of the triangle, to obtain

$$\sum \sqrt{\tan (A/2)} \le \sqrt{3(4R + r)/s} .$$

Again, easy algebra shows, for nonnegative *a, b, c,* that

$$\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} + \frac{1}{\sqrt{c}} \ge \frac{3\sqrt{3}}{\sqrt{a + b + c}} .$$

Since $\sin A = a/2R$, we have that

$$\sum \sqrt{\csc A} = \sum \sqrt{2R/a} \ge \sqrt{2R} \left(\frac{3\sqrt{3}}{\sqrt{2s}} \right) = 3 \sqrt{3R/s} .$$

From $R \ge 2r$ we have

$$\sqrt{3(4R + r)/s} \le \frac{3}{\sqrt{2}} \sqrt{3R/s}$$

and finally

$$\sum \sqrt{\tan (A/2)} \le \frac{1}{\sqrt{2}} \sum \sqrt{\csc A} .$$

It is easy to check that equality holds if ***ABC*** is an equilateral triangle. Since $R > 2r$ when ***ABC*** is not equilateral, equality in **(1)** holds if and only if ***ABC*** is equilateral.

Also solved by PAUL S. BRUCKMAN *(who conjectured the stricter inequality), Edmonds, WA,* BILL CORRELL, JR., ***Denison** University, Cincinnati, OH,* DAVID INY, *Westinghouse Electric Corporation, Baltimore, MD,* YOSHINOBU MURAYOSHI, *Okinawa, Japan,* REX *H.* WU, *Brooklyn, **NY,** and the* PROPOSER.

*Correll and **Iny** each proved that the* **√3** *could be replaced by* **√2,** *yielding a slightly stronger inequality.*

804. [Spring **1993]** *Proposed by Robert C. Gebhardt, Hopatcong, New Jersey.*

Show that

$$4 \arctan \frac{1 - x}{1 + x} = \pi - 4 \arctan x.$$

Student solutions are especially invited.

Solution by Sammy Yu, age 13, and Jimmy Yu, age 11, special students at University of South Dakota, Vermillion, South Dakota.

This statement of the problem is correct only for $x > -1$. It is meaningless for $x = -1$ and we have

$$4 \arctan \frac{1 - x}{1 + x} = -3\pi - 4 \arctan x \quad \text{if } x < -1.$$

Let $\alpha = \arctan x$ and $\beta = \arctan [(1 - x)/(1 + x)]$, so $-\pi/2 < \alpha, \beta < \pi/2$. Then

$$\tan(\alpha + \beta) = \frac{x + \dfrac{1-x}{1+x}}{1 - x\cdot\dfrac{1-x}{1+x}} = 1.$$

Since the tangent function is not one-to-one, we cannot simply take the inverse function, arctangent, to get $\alpha + \beta$. We must first figure out the range of $\alpha + \beta$, which depends upon the value of x.

If $x > -1$, then $(1 - x)/(1 + x) > 0$, so $-\pi/4 < \alpha < \pi/2$ and $0 < \beta < \pi/2$. Then

$$-\frac{\pi}{4} < \alpha + \beta < \pi, \quad \text{whence} \quad \alpha + \beta = \frac{\pi}{4}.$$

If $x < -1$, then $(1 - x)/(1 + x) < 0$, so $-\pi/2 < \alpha < -\pi/4$ and $-\pi/2 < \beta < 0$. Now

$$-\pi < \alpha + \beta < -\frac{\pi}{4}, \quad \text{whence} \quad \alpha + \beta = -\frac{3\pi}{4}.$$

Hence $4\alpha + 4\beta = \pi$ or -3π, according as $x > -1$ or $x < -1$, and the corrected theorem follows.

Also solved by CHARLES ASHBACHER, *Cedar Rapids, IA,* MOHAMMAD K. AZARIAN, *University of Evansville, IN,* SEUNG-JIN BANG, *Seoul, Korea,* FRANK P. BATTLES, *Massachusetts Maritime Academy, Buzzards Bay,* SCOTT H. BROWN, *Auburn University, AL,* PAUL S. BRUCKMAN, *Edmonds, WA,* WILLIAM CHAU, *New York, NY,* BILL CORRELL, JR., *Denison University, Cincinnati, OH,* STACEY DACOSTA and PHILIP J. DARCY, *St. Bonaventure University, NY,* RUSSELL EULER *(two solutions), Northwest Missouri State University, Maryville,* GEORGE P. EVANOVICH, *Saint Peter's College, Jersey City, NJ,* MARK EVANS, *Louisville, KY,* VICTOR G. FESER, *University of Mary, Bismarck, ND,* JAYANTHI GANAPATHY, *University of Wisconsin-Oshkosh,* STEPHEN I. GENDLER, *Clarion University of Pennsylvania,* RICHARD I. HESS, *Rancho Palos Verdes, CA,* DAVID INY, *Westinghouse Electric Corporation, Baltimore, MD,* CHIA SIEN LIM, *Oklahoma State University, Stillwater,* PETER A. LINDSTROM, *North Lake College, Irving, TX,* DAVID E. MANES, *SUM College at Oneonta,* YOSHINOBU MURAYOSHI, *Okinawa, Japan,* KANDASAMY MUTHUVEL, *University of Wisconsin-Oshkosh,* MICHAEL R. PINTER, *Belmont University, Nashville, TN,* GEORGE W. RAINEY, *Los Angeles, CA,* HENRY J. RICARDO, *Medgar Evers College, Brooklyn, NY,* REX H. WU, *Brooklyn, NY, and the* PROPOSER.

"And a little child shall lead them." The only other solvers to find the discontinuity in the solution were Iny, Muthuvel, Pinter, and Wu. The editor also gets a rap on the knuckles for missing this one.

805. [Spring 1993] *Proposed by David Iny, Baltimore, Maryland.*

a) For all integers $k \geq -2$ evaluate the integral

$$I_k = \int_0^1 \left(\frac{y-1}{\ln y}\right)^k dy.$$

*b) Can you evaluate the integral for other values of k?

I. *Solution by J. S. Frame, East Lansing, Michigan.*

Clearly $I_0 = 1$. For positive integers k we set $y = e^{-x}$ to get that

$$I_k = \int_0^\infty [(1 - e^{-x})^k e^{-x}]\, x^{-k}\, dx.$$

We integrate by parts $k - 1$ times, taking $dv = x^{-k}\,dx$ the first time, $dv = x^{-(k-1)}\,dx$ the second time, ..., and noting that in each case the integrated part vanishes at both end points, obtaining

$$I_k = \frac{1}{(k-1)!}\int_0^\infty \left(\frac{d}{dx}\right)^{k-1}[(1 - e^{-x})^k e^{-x}]\, x^{-1}\, dx$$

$$= \frac{1}{(k-1)!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\int_0^\infty (-j-1)^{k-1} e^{-(j+1)x} x^{-1}\, dx$$

$$= \frac{1}{(k-1)!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(j+1)^{k-1}\ln(j+1)$$

because

$$\ln n = \int_1^n\int_0^\infty e^{-sx}\,dx\,ds = \int_0^\infty (e^{-x} - e^{-nx})\,x^{-1}\,dx.$$

Thus $I_0 = 1$, $I_1 = \ln 2 = 0.69315$,
$I_2 = 3\ln 3 - 4\ln 2 = 0.52325$,
$I_3 = 22\ln 2 - (27/2)\ln 3 = 0.41797$,
$I_4 = (1/3!)[\ln 1 - 32\ln 2 + 162\ln 3 - 256\ln 4 + 125\ln 5] = 0.34714$,
$I_5 = 0.29651$, etc.

Next for negative k, we set $-m = k$, so $m > 0$ and we get

$$I_k = I_{-m} = \int_0^1 (1-y)^{-m}(-\ln y)^m\,dy = \sum_{j=0}^{\infty}\binom{-m}{j}\int_0^1 (-y)^j(-\ln y)^m\,dy$$

$$= \sum_{j=0}^{\infty} \binom{j+m-1}{j} \int_0^1 y^j(-\ln y)^m\, dy = \sum_{i=1}^{\infty} \binom{i+m-2}{i-1} \int_0^1 y^{i-1}(-\ln y)^m\, dy.$$

We integrate by parts m times, taking $dv = y^{i-1}\, dy$ each time and noting that the integrated parts vanish at 0 and at **1.** We get that

$$\int_0^1 y^{i-1}(-\ln y)^m\, dy = \left[\frac{y^i}{i}(-\ln y)^m\right]_0^1 + \frac{m}{i}\int_0^1 y^{i-1}(-\ln y)^{m-1}\, dy$$
$$= \frac{m!}{i^m}\int_0^1 y^{i-1}\, dy = \frac{m!}{i^{m+1}},$$

so that

$$I_k = \sum_{i=1}^{\infty} \binom{i+m-2}{i-1} \frac{m!}{i^{m+1}} = m \sum_{i=1}^{\infty} \frac{(i+m-2)!}{i!\, i^m}.$$

We express these sums in terms of the zeta functions $\zeta(s) = \sum_{i=1}^{\infty} i^{-s}$, $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$. Thus

$I_{-1} = \pi^2/6,$
$I_{-2} = \pi^2/3,$
$I_{-3} = 3[\zeta(2) + \zeta(3)],$
$I_{-4} = 4[\zeta(2) + 3\zeta(3) + 2\zeta(4)],$
$I_{-5} = 5[\zeta(2) + 6\zeta(3) + 11\zeta(4) + 6\zeta(5)]$, and
$I_{-6} = 6[\zeta(2) + 10\zeta(3) + 35\zeta(4) + 50\zeta(5) + 24\zeta(6)].$

Also solved by PAUL **S.** *BRUCKMAN, Edmonds, WA, and the PROPOSER.*

Bruckman's *solution evaluates the integral for all real numbers k, but is not presented here because of space limitations. If you desire a copy of that solution please send a request to the Problems Editor at the address given at the beginning of the Problem Department.*

806. [Spring **1993]** *Proposed by Robert C. Gebhardt, Hopatcong, New Jersey.*

The integral

$$I = \int \frac{dx}{(x^{1/3} - x^{4/3})^{3/2}}$$

was evaluated by one student as follows:

$$I = \int \frac{dx}{x^{1/2} - x^2} = \int \frac{dx}{x^{1/2}} - \int \frac{dx}{x^2} = 2x^{1/2} + \frac{1}{x} + C.$$

Provide a correct evaluation. Student solutions arc especially invited.

I. *Solution by Bill Correll, Jr., student at* ***Denison*** *University, Granville, Ohio.*

Using the substitution $x = u^{-1}$, $dx = -u^2 du$, we get

$$I = \int \frac{dx}{x^{1/2}(1-x)^{3/2}} = \int \frac{-u^{-2}\, du}{u^{-1/2}(1-u^{-1})^{3/2}} = -\int \frac{du}{(u-1)^{3/2}}$$
$$= \frac{2}{\sqrt{u-1}} + C = \frac{2\sqrt{x}}{\sqrt{1-x}} + C,$$

valid for $0 < x < 1$.

II. *Solution by Murray S.* ***Klamkin,*** *University of Alberta, Edmonton, Alberta, Canada.*

More generally, we evaluate

$$I = \int \frac{dx}{(x^{1/n} - x^{(n+1)/n})^{n/2}}, \quad n \text{ integral.}$$

If $x = \sin^2 9$, the integral reduces to

$$I = 2\int \sec^{n-1}\theta\, d\theta.$$

This integral can be evaluated by repeated use of the reduction formula

$$(m-1)\int \sec^m\theta\, d\theta = \tan\theta \sec^{m-2}\theta + (m-2)\int \sec^{m-2}\theta\, d\theta$$

whenever $n > 3$. For the special case $n = 3$,

$$I = 2\tan\theta + C = 2\sqrt{x}/\sqrt{(1-x)} + C.$$

If $n < 1$, then $1 - n$ is a positive integer and

$$I = 2\int \cos^{1-n}\theta\, d\theta,$$

which is handled by similar reduction formulas.

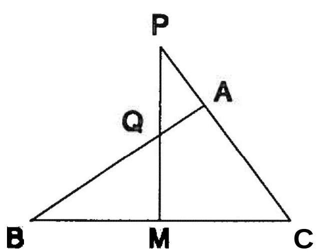
Also solved by AVRAHAM **ABLER,** ***Monsey,*** *NY, CHARLES* ASHBACHER, *Cedar Rapids,* ***IA,*** SEUNG-JIN BANG, *Seoul, Korea,* LEE J. *BARTON,* ***Warminster,*** *PA, FRANK* P. BATTLES, *Massachusetts*

Maritime Academy, Buzzards Bay, **JASON BECK and MIKE HILLIS,** *Hendrix College, Conway, AR,* **JAMES D. BRASHER,** *Teledyne Brown Engineering, Huntsville, AL,* **SCOTT H. BROWN,** *Auburn University, AL,* **PAUL S. BRUCKMAN,** *Edmonds, WA,* **MICHAEL CALLAHAN,** *Elmhurst College, EL,* **PHILIP A. D. CASTORO,** *Queens College, Flushing, NY,* **WILLIAM CHAU,** *New York, NY,* **STACEY DACOSTA and PHILIP J. DARCY,** *St. Bonaventure University, NY,* **MIKE ECKLES and LAURA WARD,** *Hendrix College, Conway, AR,* **RUSSELL EULER,** *Northwest Missouri State University, Maryville,* **MARK EVANS,** *Louisville, KY,* **VICTOR G. FESER,** *University of Mary, Bismarck, ND,* **JAYANTHI GANAPATHY,** *University of Wisconsin-Oshkosh,* **STEPHEN I. GENDLER,** *Clarion University of Pennsylvania,* **RICHARD I. HESS,** *Rancho Palos Verdes, CA,* **CHIA SIEN LIM,** *Oklahoma State University, Stillwater,* **DAVID E. MANES,** *SUNY College at Oneonta,* **KAREN MCNIECE,** *Hendrix College, Conway, AR,* **KANDASAMY MUTHUVEL,** *University of Wisconsin-Oshkosh,* **WILLIAM MYERS,** *Belmont Abbey College, NC,* **KARTIK PARIJA,** *Drake University, Des Moines, IA,* **GEORGE W. RAINEY,** *Los Angeles, CA,* **HENRY J. RICARDO,** *Medgar Evers College, Brooklyn, NY,* **ADRIENNE TILEY,** *Hendrix College, Conway, AR,* *REX H.* **WU,** *Brooklyn, NY,* **SAMMY YU and JIMMY YU,** *University of South Dakota, Vermillion, and the* **PROPOSER.** *One incorrect solution was received.*

Most solvers used the trigonometric substitution method.

807. [Spring *19931 Proposed by Florentin Smarandache, Phoenix, Arizona.*

In terms of the lengths a, b, **and** c **of the sides of a given triangle** ABC, **find the length of the segment** PQ **of the normal to side** BC **at its midpoint** M **cut off by the other two sides. See the accompanying figure.**



Solution by William Chau, New York, New York.

Let $x = PQ = |PM - QM|$. **Since** $PM = CM \tan C$ **and** $QM = BM \tan B$, **we have**

$$x = \frac{a}{2}|\tan C - \tan B|. \tag{1}$$

Let K **denote the area of triangle** ABC, **so that**

$$\sin C = \frac{2K}{ab} = \frac{4K}{lab}.$$

By the law of cosines we have

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab},$$

so that

$$\tan C = \frac{4K}{a^2 + b^2 - c^2} \quad \text{and similarly} \quad \tan B = \frac{4K}{c^2 + a^2 - b^2}$$

Substitute these expressions into (1) and simplify to get

$$x = 4aK\frac{|c^2 - b^2|}{(a^2 + b^2 - c^2)(c^2 + a^2 - b^2)}$$

Since

$$K = \sqrt{s(s-a)(s-b)(s-c)}, \quad \text{where} \quad s = (a+b+c)/2,$$

we may write that

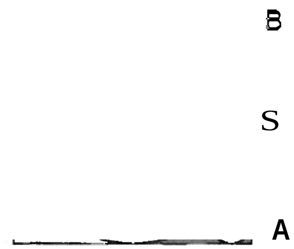
$$x = \frac{a|c^2 - b^2|\sqrt{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}}{(a^2 + b^2 - c^2)(c^2 + a^2 - b^2)}$$

Also solved by **SEUNG-JIN BANG,** *Seoul, Korea,* **SCOTT H. BROWN,** *Auburn University, AL,* **PAUL S. BRUCKMAN,** *Edmonds, WA,* **BILL CORRELL, JR.,** *Denison University, Cincinnati, OH,* **MARK EVANS,** *Louisville, KY,* **RICHARD I. HESS,** *Rancho Palos Verdes, CA,* **DAVID INY,** *Westinghouse Electric Corporation, Baltimore, MD,* **DAVID E. MANES,** *SUNY College at Oneonta,* **REX H. WU** *(two solutions), Brooklyn, NY,* **SAMMY YU and JIMMY YU,** *University of South Dakota, Vermillion, and the* **PROPOSER.**

There are many different ways to write the solution, perhaps the simplest of which is $(ha/2)\,|\,l/d(b^2 - h^2) - l/d(c^2 - h^2)\,|$, *where* h *is the altitude of the triangle to vertex A, although this expression is not strictly in terms of the sides only.*

808. [Spring **1993]** Proposed by Scott H. Brown, *Stuart* Middle *School,* Stuart) Florida.

Student solutions are especially solicited. A circle (R) is inscribed in the unit square ABCD and touches the sides of the square at S, T, U, and V, as shown in the accompanying figure. Another circle (r) is inscribed in the region *ASV* **outside** circle (R) and inside the square at vertex A.



a) Find the area of the shaded region inside **region** *ASV* **and** outside circle (r). Give the answer in radical, not just decimal, form.

*b) If the sequence of smaller circles is continued indefinitely, each successive circle inscribed between the preceding circle and the corner A of the square, find the limit of the shaded region. That is, **find** the area of region ASV less the sum of the areas of the circles in the resulting infinite chain.

Solution *by* Stacey *DaCosta and Philip* **J.** Darcy, students at St. *Bonaventure* University, St. *Bonaventure,* New York.

Let $r_0 = RS = 112$, the radius of circle (R) and r_1 the radius of circle (r). Since RA $= r_0\sqrt{2}$, $rA = r_1\sqrt{2}$, and $r_0 + r_1 + r_1\sqrt{2} = r_0\sqrt{2}$, then

$$r_1 = (3 - 2\sqrt{2})r_0.$$

Similarly, if r_2 is the radius of the next circle in the chain, then

$$r_2 = (3 - 2\sqrt{2})r_1 = (3 - 2\sqrt{2})^2 r_0.$$

In general, then,

$$r_n = (3 - 2\sqrt{2})^n r_0.$$

Hence the area K_n of the nth circle in the chain is

$$K_n = \pi r_n^2 = \frac{\pi}{4}(3 - 2\sqrt{2})^{2n}$$

a) Now the area of the shaded region is 1/4 the area of the square less 114 the area of circle (R) less the area of circle (r):

$$\frac{1}{4}(1) - \frac{1}{4}\pi\left(\frac{1}{2}\right)^2 - \frac{\pi}{4}(17 - 12\sqrt{2}) = \frac{4 + 3\pi(16\sqrt{2} - 23)}{16} \approx 0.03053.$$

b) The sum of the areas of **all** the circles $r_1, r_2, \ldots$ is

$$\sum_{n=1}^{\infty} \frac{\pi}{4}(17 - 12\sqrt{2})^n = \frac{\pi}{4}\cdot\frac{17 - 12\sqrt{2}}{1 - (17 - 12\sqrt{2})} = \frac{\pi}{32}(3\sqrt{2} - 4).$$

Therefore the desired area is

$$\frac{1}{4} - \frac{\pi}{16} - \frac{\pi}{32}(3\sqrt{2} - 4) = \frac{8 + (2 - 3\sqrt{2})\pi}{32} \approx 0.02983.$$

Also solved *by* **AVRAHAM ADLER,** Monsey, *NY,* PAUL S. BRUCKMAN, Edmonds, ***WA,*** BILL **CORRELL,** JR., ***Denison*** University, ***Cincinnati,*** OH, **WILLIAM** CHAU, New *York, NY,* MARK EVANS, Louisville, *KY,* SEAN FORBES, Drake University,; Des *Moines, IA,* **RICHARD** I. HESS, Rancho Palos Verdes, ***CA,*** GEORGE W. RAINEY, ***Los*** Angeles, CA, REX H. WU, Brooklyn, *NY,* SAMMY YU and JIMMY YU, University of South Dakota, Vermillion, and the PROPOSER- One incorrect solution was received.

809. [Spring **1993]** Proposed *by* David Iny, Baltimore, ***Maryland.***

In triangle ***ABC*** **let** ***AD*** and BE be any two **cevians** intersecting at a point F. (A cevian AD for triangle ABC is a line through the vertex A of the triangle and intersecting the opposite side BC, perhaps extended, at a point D, different from both B and C.) Find the ratios BDIDC **and** ***AF/FD*** in **terms** of the ratios ***AE/EC*** and ***BF/FE.***

Solution *by* William ***Chau,*** New York, New York

Recall that if D lies on segment BC, then BDIDC > 0; if D lies outside segment BC, then ***BD/DC*** < 0. That is, BDIDC is positive or negative **according as** BD and DC have the same or opposite directions. **Let**

$$x = \frac{BD}{DC},\quad y = \frac{AF}{FD},\quad a = \frac{AE}{EC},\quad \text{and}\quad b = \frac{BF}{FE}.$$

Applying **Menelaus'** theorem to triangle BCE with **transversal** AFD and also to triangle ACD with transversal BFD, we have

$$(1)\qquad \frac{BD}{DC}\cdot\frac{CA}{AE}\cdot\frac{EF}{FB} = -1 \quad\text{and}\quad \frac{AE}{EC}\cdot\frac{CB}{BD}\cdot\frac{DF}{FA} = -1.$$

Since

$$\frac{CA}{AE} = -\frac{CE + EA}{EA} = -\frac{CE}{EA} - 1 = -\frac{1}{a} - 1 \quad\text{and}\quad \frac{EF}{FB} = \frac{1}{b},$$

then the left equation in (1) can be solved for x to give

$$x = \frac{BD}{DC} = \frac{ab}{a+1}.$$

Since

$$\frac{CB}{BD} = -\frac{CD + DB}{DB} = -\frac{CD}{DB} - 1 = -\frac{1}{x} - 1 \quad \text{and} \quad \frac{DF}{FA} = \frac{1}{y},$$

then the right equation in (1) can be solved for y to give

$$y = \frac{AF}{FD} = \frac{1 + a + ab}{b}$$

Also solved by PAUL S. BRUCKMAN, *Edmonds,* WA, **YOSHINOBU MURAYOSHI**, *Okinawa,* Japan, REX H. WU, Brooklyn, *NY,* and the PROPOSER.

Late solution to Problem 786 by ***Yoshinobu Murayoshi, Okinawa,*** Japan.

Why Study Mathematics? (continued)

Mathematics, while giving no quick remuneration, like the art of stenography or the craft of bricklaying, does furnish the power for deliberate thought and accurate statement, and to speak the truth is one of the most social qualities a person can possess. Gossip, flattery, slander, deceit, all spring from a slovenly mind that has not been trained in the power of truthful statement, which is one of the highest utilities.—S. T. Dutton (1900).

Mathematics is the gate and key of the sciences. Neglect of mathematics works injury to all knowledge, since he who is ignorant of it cannot know the other sciences or the things of this world. And what is worse, men who are thus ignorant are unable to perceive their own ignorance and so do not seek a remedy—Roger Bacon (13th century).

Quoted in Memorabilia ***Mathematica*** by **R.** E. Moritz, 1914 edition reprinted by the Mathematical Association of America, 1993.

The Richard V. Andree Awards

The Richard V. Andree awards are given annually to the authors of the three papers, written by students, that have been judged by the officers and councilors of the Pi Mu Epsilon to be the best to have appeared in the Pi Mu Epsilon Journal in the past year.

Richard V. Andree was, until his death in 1987, Professor Emeritus of Mathematics at the University of Oklahoma. He had served Pi Mu Epsilon long and well in many capacities: as president, as secretary-treasurer, and as editor of the Journal.

The first-prize winner for 1993 is JENNIFER **DEBOER,** for her paper **"Non-existence** of certain unitary perfect numbers" (this Journal, 9 **(1989-94) #9, 601-606).**

The second prize is awarded to JAMES R. MURPHY and **MOHAMMAD P. SHAIKH,** for their paper **"Uniform** embeddings of graphs" (this Journul, 9 **(1989-94) #8, 504-512).**

There was a tie for the third Andree award, which will be shared by **CAROL CLIFTON,** for "Some operations on matrix-valued expressions" (this Journal, 9 **(1989-94) #8, 494-499,** and JEREMY M. DOVER, for **"Outerplanar** graphs and matroid isomorphism" (this Journul, 9 **(1989-94) #8,** 500-503.

At the times the **papers** were written, Ms. **DeBoer** was a student at Michigan Technological University, Mr. **Murphy** and Mr. Shaikh at Western Michigan University, Ms. Clifton at Middle Tennessee State University, and Mr. Dover at Worcester Polytechnic Institute.

The officers and councilors of the Society congratulate the winners on their achievements and wish them well in their futures, whether or nit they involve mathematics.

Award for Distinguished Service to J. Sutherland Frame

The Mathematical Association of American has named J. Sutherland Frame the recipient of its Yueh-Gin Gung and Dr. Charles Y. Hu **Award** for Distinguished Service. Professor Frame has been instrumental in the growth of Pi Mu Epsilon, personally installing more than fifty chapters. He was president of Pi Mu Epsilon for nine years **(1957-66)** and served as secretary during the organization's period of most rapid growth **(1951-54).** He was also instrumental in founding this Journal in 1949. He is still mathematically active as is shown by, among other things, his solution to problem 805, printed in this issue.

For more details, see the citation by David W. **Ballew** in the American Mathematical Monthly for February, 1994 **(101** (1994) **#2, 107-108),** whose cover contains a photograph of Professor Frame.

Letter to the Editor

In a recent note by J. D. Bomberger, "On the solution of $a^a = b^b$" **(this** Journal, 9 (1989-94) **#9, 571),** he notes that the equation $x^y = y^x$ had been solved by Euler and then proceeds to an ab initio derivation of the solution to $a^a = b^b$. It should be noted that by letting $a = 1/x$ and $b = 1/y$, $a^a = b^b$ becomes $x^y = y^x$ so there is no need for another derivation of the solution.

Murray S. **Klamkin**
University of Alberta

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(solution 7, 481); **#35,** 7, **481-483** (solution 8, 541).
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Referees

Thanks are due to the following, who served as referees for one or more **papers** that appeared in volume 10:

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