PI MU EPSILON JOURNAL

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PI MU EPSILON JOURNAL
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A GENERALIZATION OF ODD AND EVEN VERTICES IN GRAPHS

by Amy Dykstra and Michelle Schultz
Western Michigan University

1. Introduction. Perhaps the most familiar theorem in graph theory is that the sum of the degrees of the vertices of a graph is equal to twice the number of edges. From this, it follows that every graph has an even number of vertices having odd degree. Thus, if G is a graph having m vertices of odd degree and n vertices of even degree, then $m + n > 0$ and m is even. On the other hand, if we are given two nonnegative integers m and n such that $m + n > 0$ and m is even, then we can find a graph which has m vertices of odd degree and n vertices of even degree. The graph in Figure 1 shows how this can be done.

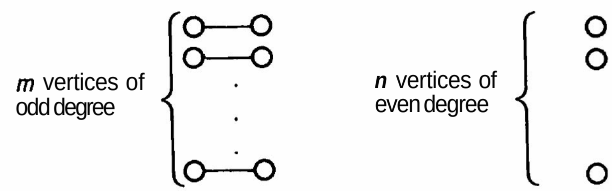


Figure 1

For a positive integer n and integers a and b, we write $a \equiv b \pmod{n}$ if $n \mid a - b$. So if $a \equiv 0 \pmod{2}$, then a is even; while if $a \equiv 1 \pmod{2}$, then a is odd. Therefore, the observation we stated above can be restated as follows:

Theorem A. *For every two nonnegative integers m and n such that $m + n > 0$ and m is even, there exists a graph G with m vertices whose degrees are congruent to 1 modulo 2 and n vertices whose degrees are congruent to 0 modulo 2.*

The goal of this paper is to present an extension of Theorem A to graphs whose vertices are congruent to the integers $0, 1, \ldots, n - 1$ modulo n, where $n \geq 3$.

2. Preliminary Definitions. In this paper, we are primarily concerned with graphs. A graph with four vertices and four edges is shown in Figure 2.

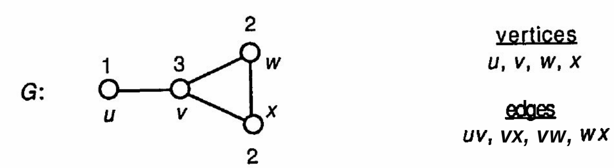


Figure 2

The degree deg v of a vertex v is the number of vertices that are joined to v. So G contains two odd vertices (vertices of odd degree) and two even vertices (vertices of even degree).

A complete graph K_p is a graph with p vertices, where every two vertices are adjacent (that is, joined by an edge). Thus, K_p contains $\binom{p}{2}$ edges and the deg $v = p - 1$ for every vertex v in K_p.

A graph that is contained in and has the same vertex set as graph G is called a *1-factor* of G if all of its vertices have degree 1. Thus, in order for a graph to contain a 1-factor, it is necessary that it have an even number of vertices. If the edges of a graph can be partitioned into 1-factors, then we say that the graph is 1-factorable. For example, K_6 is 1-factorable; that is, every edge of K_6 appears in exactly one of the 1-factors F_1, F_2, F_3, F_4, F_5. (See Figure 3.)

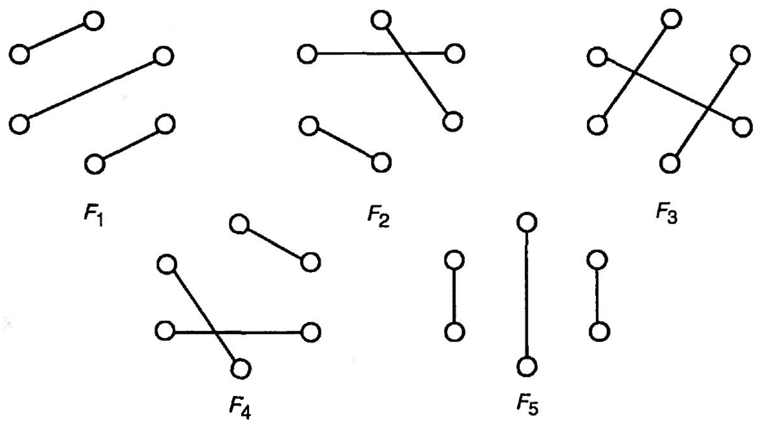


Figure 3

In fact, we have the following general result:

Theorem B. K_{2n} is 1-factorable for every positive integer n.

This says that the graph K_{2n} can be partitioned into $2n - 1$ 1-factors.

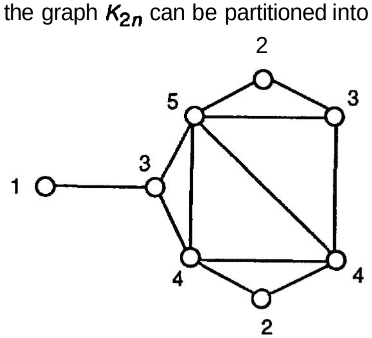


Figure 4

Recall that Theorem A stated that for every two integers x and y, where $x + y > 0$, such that x is even, there exists a graph having x vertices whose degrees are congruent to 1 modulo 2 and y vertices whose degrees are congruent to 0 modulo 2. We now consider the case where the degrees of the vertices are congruent to 0, 1, or 2 modulo 3.

The graph in Figure 4 has two vertices of degree 3 (that is, two vertices whose degrees are congruent to 0 modulo 3), one vertex of degree 1 and two vertices of degree 4 (that is, three vertices whose degrees are congruent to 1 modulo 3), and two-vertices of degree 2 and one vertex of degree 5 (that Is, three vertices whose degrees are congruent to 2 modulo 3). In this case there is no requirement that we have an even number of any of these types of vertices as there was when we were dealing with degrees of vertices modulo 2. This suggests that following question:

Question. Given nonnegative integers x, y, z such that $x + y + z > 0$, is there a graph having x vertices whose degrees are congruent to 0 modulo 3, y vertices whose degrees are congruent to 1 modulo 3, and z vertices whose degrees are congruent to 2 modulo **3?**

The answer is "yes" in almost all cases.

Theorem 1. For all triples (x, y, z) of nonnegative integers where $x + y + z > 0$, there exists a graph containing exactly x vertices whose degrees are congruent to 0 modulo 3, y vertices whose degrees are congruent to 1 modulo 3, and z vertices whose degrees are congruent to 2 modulo 3, with seventeen exceptions.

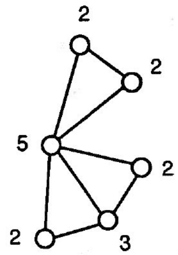
In the case where the degrees of the vertices are congruent to 0, 1, 2, or 3 modulo 4, every graph must contain an even number of vertices whose degrees are congruent to 1 or 3 modulo 4 because these are precisely the odd vertices of the graph. Stated below is the result for modulo 4.

Theorem 2. Given a sequence $S = (s_0, s_1, s_2, s_3)$ of nonnegative integers such that $\sum_{i=0}^{3} s_i > 0$ and $s_1 + s_3$ is even, there exists a graph containing s_i vertices whose degrees are congruent to i modulo 4 with twenty-four exceptions.

As a consequence of Theorems A, 1, and 2, it follows that there always exists a sequence $S = (s_0, s_1, \ldots, s_{N-1})$, where $N = 2, 3, 4$, such that if $A = (a_0, a_1, \ldots, a_{N-1})$ is any sequence for which $a_i \geq s_i$ $(0 \leq i \leq N-1)$, then there exists a graph containing exactly a_i **vertices** whose degrees are congruent to i modulo N (with the usual condition that the graph must contain an even number of odd vertices). It is the goal of this paper to show that such a sequence S exists for every $N \geq 2$.

3. The Main Result. For the purpose of presenting this result, it is convenient to introduce some additional terminology. For a graph G and positive integer N, the frequency sequence modulo N of G is that sequence $(a_0, a_1, \ldots, a_{N-1})$ such that G contains a_i vertices v such that deg $v \equiv i \pmod N$. For example, the frequency sequence modulo 4 for the graph in Figure 5 is (0, 1, 4, 1) because the graph contains no vertices whose degrees are congruent to 0 modulo 4, one vertex whose degree is congruent to 1 modulo 4, four vertices whose degrees are congruent to 2 modulo 4, and one vertex whose degree is congruent to 3 modulo 4.

We call a sequence $(a_0, a_1, \ldots, a_{N-1})$ realizable if it is the frequency sequence modulo N of some graph. So according to Theorem 1, all but seventeen sequences (a_0, a_1, a_2) of nonnegative integers with $a_0 + a_1 + a_2 > 0$ are realizable.



(0, 1, 4, 1)

Figure 5

We say that a sequence $A = (a_0, a_1, \ldots, a_{N-1})$ *majorizes* $B = (b_0, b_1, \ldots, b_{N-1})$ if $a_i \geq b_i$ for all i. It is known that $S_1 = (2, 0, 0, 0)$ and $S_2 = (0, 1, 2, 1)$ are realizable sequences. The sequence $(2, 0, 2, 0)$ majorizes S_1 but is not realizable. On the other hand, every sequence that majorizes S_2 is realizable.

We show that such a sequence S_2 exists for every positive integer N beginning with the case where N is even.

Theorem 3. *For every positive integer $N = 2n$, there exists a sequence $S = (s_0, s_1, \ldots, s_{2n-1})$ of length $2n$ such that if $A = (a_0, a_1, \ldots, a_{2n-1})$ is any sequence that majorizes S and $\sum_{i=1}^{n} a_{2i-1}$ is even, then A is realizable.*

Proof. Choose m to be an integer such that $2m \geq n - 1$. Then we show that the sequence $S = (0, 0, 0, 2m, \ldots, 2m)$ of length $2n$ is realizable. To do this we show that there exists a graph G containing exactly $2m$ vertices whose degrees are congruent to i modulo $2n$ for each i $(3 \leq i \leq 2n - 1)$ and no other vertices. We begin by considering a graph G^* that consists of two cycles C: $v_1, v_2, \ldots, v_{2n-3}, v_1$ and C': $v_1', v_2', \ldots, v_{2n-3}', v_1'$ together with all edges of the type $v_i v_j'$ where $j \geq i$. So the graph G^* has two vertices of each of the degrees $3, 4, \ldots, 2n - 1$. Then the graph G consisting of m copies of G^* contains exactly $2m$ vertices of each of the required types. Therefore, S is realizable.

We wish to show that any sequence that majorizes S and such that the sum of its odd terms if even is realizable. To do this we need only show that we can begin with a realizable sequence $B = (b_0, b_1, \ldots, b_{2n-1})$ that majorizes S and produce another realizable sequence in one of the three following ways:

(1) by adding 2 to some term b_i, where i is odd;
(2) by adding 1 to some term b_i, where i is even;
(3) by adding 1 to each of b_i and b_j, $i = j$, where i and j are both odd.

Since proofs of (1) and (2) are similar and the proof of (3) follows from (2), we present the proof of (1) only.

G^*:

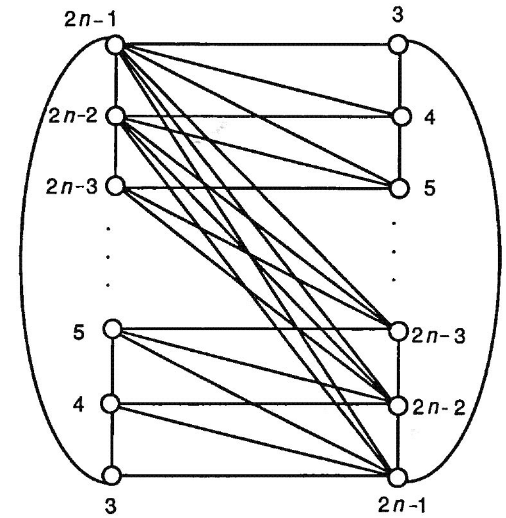


Figure 6

Since B is realizable, there exists a graph H with frequency sequence B, that is H has b_i vertices whose degrees are congruent to i modulo $2n$, $0 \leq i \leq 2n - 1$, and such that $b_i \geq 0$ for $i = 0, 1, 2$, and $b_i \geq 2m$ for $3 \leq i \leq 2n - 1$.

If we add K_2 to the graph, then the frequency sequence is $(b_0, b_1 + 2, b_2, \ldots, b_{2n-1})$. It remains to show that a graph with frequency sequence

$$B' = (b_0, \ldots, b_{2k}, b_{2k+1} + 2, b_{2k+2}, \ldots, b_{2n-1}),$$

where $3 \leq 2k + 1 \leq 2n - 1$ is realizable.

Let T be a set of $2m$ vertices of degree $2k + 1$ in H, and let $\langle T \rangle$ be the subgraph with vertex set T and whose edges join the vertices of T. Suppose that $\langle T \rangle$ contains t edges, where then

$$0 \leq t \leq \binom{2m}{2}. \qquad (1)$$

The graph H has the appearance shown in Figure 7 where Y denotes the set of edges joining H − T and T. The sum of the degrees in H of the vertices of T is $2m(2k + 1)$. Therefore the number of edges In Y is $2m(2k + 1) - 2t$.

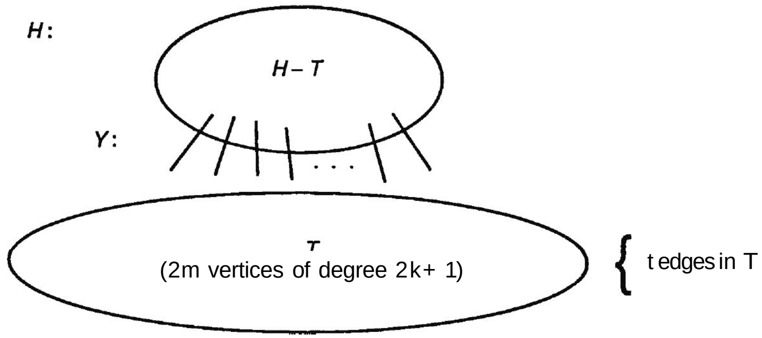


Figure 7

Our goal is to construct a graph G with frequency sequence B'. To do this, we first remove the t edges from $\langle T\rangle$. Then we add two new vertices to T and call the resulting set T'. (See Figure 8).

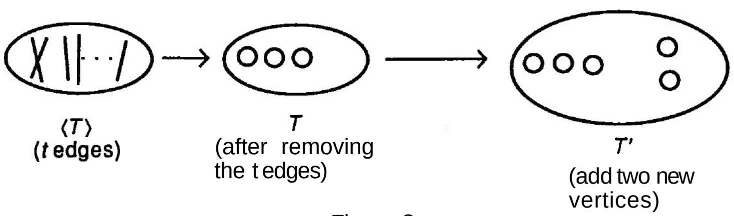


Figure 8

The next step in the construction of G is to show that it is possible to add edges among the vertices of T', say x of them. and possibly redistribute the edges of Y so that each vertex of T' has degree 2k+ 1 and the degree of every other vertex is not changed.

If it is possible to construct G in this manner, then the sum of the vertices of T' equals

$$(2m+2)(2k+1) = [2m(2k+1) - 2t] + 2x.$$

Solving for x, we find that $x = 2k+1+t \le 2k+1+\binom{2m}{2}$ where the inequality follows from (1). Since $2k+1 \le 2n-1$, it follows that $k \le n-1 \le 2m$ (or $2m \ge k$). So

$$\begin{aligned} x &\le 2k+1+\binom{2m}{2} \\ &= k+(k+1)+\binom{2m}{2} \\ &\le 2m+(2m+1)+\binom{2m}{2} = \binom{2m+2}{2}. \end{aligned}$$

This inequality shows that there is sufficient space to add the x edges among the vertices of T'. It remains to show that the X edges can be suitably placed among the vertices of T' to produce the desired result.

By Theorem B, the complete graph K_{2m+2} is 1-factorable. So K_{2m+2} can be factored into the 1-factors $F_1, F_2, \dots, F_{2m+1}$. Select the edges of $F_1, F_2, \dots, F_r$ for a suitably chosen r and possibly some of the edges of F_{r+1} so that a total of x edges have been chosen. After these edges are placed among the vertices of T', every vertex of T' will have degree r or r + 1.

Even though the sum of the degrees of the vertices of T' is correct, it is unlikely that each vertex has the correct degree. We next show that it is possible to redistribute the edges of Y so that each edge of Y leaves the same vertex of H − T but possibly goes to a different vertex in T'. Our goal is to show that we can do this in such a way that $\deg_G v = 2k+1$ for every $v \in T'$. Suppose that the edges of Y come from the vertices $u_1, u_2, \dots, u_s$ in H − T and the vertices of T' are given by $v_1, v_2, \dots, v_{2m+2}$, where those having degree r are listed first. Suppose u_i joins n_i vertices of T where $1 \le i \le s$. We redistribute the edges of Y leaving u_1 so that they now join $v_1, v_2, \dots, v_{n_1}$. Next we redistribute the edges of Y leaving leaving u_2 so that they join the next n_2 vertices of T'. After an edge is joined to v_{2m+2}, the next one will be joined to v_1, and we continue with this procedure until all the edges of Y have been redistributed. There is only one possible concern that we must consider. That is, do we have so many edges to redistribute from some vertex u_i that two of these edges now join the same vertex of T'? Since $n_i \le 2m$ for each i, every vertex u_i will be joined to distinct vertices of T'. We have thus constructed a graph G with frequency sequence B'.

A similar result holds for the odd case N = 2n + 1.

Theorem 4. For every positive integer $N = 2n+1$, there exists a sequence S of length $2n+1$ such that if A is any sequence that *majorizes S*, then A is realizable.

The proof of Theorem 4 is similar to that of Theorem 3, so we omit it except to say that the sequence $S = (4m, 4m, \dots, 4m)$ of length 2n + 1, where $4m \ge 4n+1$ satisfies the condition of the theorem.

There is still a lingering question that remains unanswered.

Conjecture. For each integer $N \ge 2$, every sequence $S = (s_0, s_1, \dots, s_{N-1})$ of nonnegative integers (with $\sum s_{2i-1}$ even if N is even) is realizable with a finite number of exceptions.

Of course, this conjecture is true for N = 2, 3, 4, and the truth of the conjecture in general implies our Theorems 3 and 4.

AN APPROXIMATION FOR THE NUMBER OF PRIMES BETWEEN K AND K^2, WHEN K IS A PRIME

by Randall J. Osteen
University of Central Florida

I thought originally to obtain a series where each term represented a percentage of the set of positive integers. I also wanted the sum of the series to equal one. I obtained that series and the following approximation for the number of primes between K and K^2, when K is a prime:

$$A(K) = \left[K^2 \prod_{p=2}^{p=K}{}'\left(1 - \frac{1}{p}\right)\right] - 1$$

where p takes on only prime values.

For example, when $K = 5$, then:

$$A(5) = \left[5^2 \prod_{p=2}^{p=5}{}'\left(1 - \frac{1}{p}\right)\right] - 1$$

$$= \left[25\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{5}\right)\right] - 1$$

$$= \frac{17}{3}.$$

There are actually six primes between ***5*** and ***25;*** namely ***7, 11, 13, 17, 19,*** and ***23.***

The result was derived as follows: Let K be a prime. The intent is to use an extraction process on the positive integers up to K^2. The multiples of ***2*** represent about half the set. When the multiples of ***2*** are removed, approximately half of our original set remains. Of the remaining one-half, about ***113*** are multiples of three. This is **1/3** of ***112,*** or ***116*** of the original set. If we continue this extraction process up to K, we will have accounted for all the non-primes with the exception of the integer ***1*** and the primes $\le K$. The series we obtain is:

$$\frac{1}{2} + \frac{2-1}{2\cdot 3} + \frac{(2-1)(3-1)}{2\cdot 3\cdot 5} + \frac{(2-1)(3-1)(5-1)}{2\cdot 3\cdot 5\cdot 7} + \dots$$

$$+ \frac{(2-1)(3-1)(5-1)\ \dots\ (K'-1)}{2\cdot 3\cdot 5\ \dots\ K'\cdot K} \qquad (1)$$

where K' is the prime which immediately precedes K. (The Kth partial sum of this series can be written as $S(K) = 1 -$ ***(112 · 2/3 · 4/5 · 617 ... (K–1)/K.)*** This represents the proportion of integers extracted (the non-primes, with the exception of ***1*** and the primes $\le K$). It follows that $1 - S(K)$ approximates the proportion of the numbers from the original set not extracted in the process, which Includes the integer ***1*** and the primes between K and K^2.

$$1 - S(K) = 1 - \left[1 - \left(\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\ \cdots\ \frac{K-1}{K}\right)\right]$$

$$= \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdots\frac{K-1}{K} = \prod_{p=2}^{p=k}{}'\left(1 - \frac{1}{p}\right),$$

where p takes on only prime values.

(1) is the aforementioned series that sums to one.

We multiply this percentage by K^2 and subtract one (since ***1*** was not accounted for in the extraction process) to obtain the desired results.

<u>Numerical **Data**</u>

K	$K*K$	$P(K)$	$A(K)$	% difference
5	25	6	5.67	5.50
7	49	11	10.20	7.27
11	121	25	24.14	3.44
13	169	33	31.42	4.79
17	289	54	51.17	5.24
19	361	64	60.74	5.09
31	961	151	145.89	3.38
41	1681	250	242.90	2.84
53	2809	393	381.27	2.98
59	3481	470	464.69	1.13
71	5041	655	643.23	1.80
79	6241	789	775.70	1.69
101	10201	1227	1214.20	1.04
199	39601	4119	4113.30	0.14
307	94249	9027	9154.40	-1.41
401	160801	14674	14928.70	-1.74
499	249001	21870	22311.90	-2.02
601	361201	30726	31503.20	-2.53
701	491401	40749	41816.10	-2.62
797	635209	51592	53124.70	-2.97
907	822649	65462	67518.50	-3.14
997	994009	77893	80479.00	-3.32

K – prime number
$P(K)$ – actual number of primes between K and K^2
$A(K)$ = calculated number of primes between K and K^2
% difference = $((P(K) - A(K))/P(K)) * 100$

Note: for $K \neq$ prime, the approximation equals

$$K^2 \sum_{p=2}^{(\langle k^2\rangle')^{\frac{1}{2}}} \left(1 - \frac{1}{p}\right)$$

where $\langle k^2\rangle'$ equals the prime squared nearest to K^2.

Editor's Note

The Pi Mu Epsilon Journal was founded in 1949 and is dedicated to undergraduate and beginning graduate students interested in mathematics. Submitted articles, announcements and contributions to the Puzzle Section and Problem Department of the Journal should be directed toward this group.

Undergraduate and beginning graduate students are urged to submit papers to the Journal for consideration and possible publication. Student papers are given top priority. Expository articles by professionals in all areas of mathematics are especially welcome. Some guidelines are:

1. papers must be correct and honest
2. most readers of the Pi Mu Epsilon Journal are undergraduates; papers should be directed to them
3. with rare exceptions, papers should be of general interest
4. assumed definitions, concepts, theorems and notations should be part of the average undergraduate curriculum
5. papers should not exceed 10 pages in length
6. figures provided by the author should be camera-ready
7. papers should be submitted in duplicate to the Editor.

In each year that at least **five** student papers have been received by the Editor, prizes of $200, $100 and $50, known as Richard V. Andree Awards, are given to student authors. All students who have not yet received a Master's Degree, or higher, are eligible for these prizes.

Two student papers appear in this issue of the Journal. The first is "A Generalization of Odd and Even Vertices in Graphs" by Amy Dykstra and Michelle **Schultz.** Amy and Michelle are students at Western Michigan University. The paper was prepared under the supervision of Professor Gary Chartrand and was presented by Amy and Michelle in August 1989 at the National Pi Mu Epsilon Meeting in Boulder.

The second paper is "An Approximation for the Number of Primes Between K and K^2, When K is a Prime" by Randall J. Osteen. Randall prepared the paper while an undergraduate at the University of Central Florida and presented his paper at the National Pi Mu Epsilon Meeting in Providence in August 1988.

THE BAYESIAN BUFFON NEEDLE PROBLEM ON CONCENTRIC CIRCLES

By H.J. Khamis
Wright State University

Introduction. **Buffon's** Needle Problem (1777) is one of the oldest problems in geometrical probability. Consider a board of large size that is ruled with equidistant parallel lines d units apart. When a needle of length $\ell < d$ is dropped at random on the board then the probability that the needle crosses a line is $2\ell/\pi d$.

A large number of variations of the **Buffon** Needle Problem have been studied. In Uspensky's (1937) classic text, a generalization called Laplace's Problem involves randomly dropping a needle of length ℓ onto a board covered with a set of congruent rectangles of dimensions a x b. If $\ell < a \le b$, then the probability that the needle intersects the boundary of one of the rectangles is

$$\frac{2\ell(a+b) - \ell^2}{\pi ab}.$$

This reduces to the classical **Buffon** Needle Problem if b tends to infinity.

Gnedenko (1962) generalized **Buffon's** Needle Problem first to n-sided convex polygons with diameter less than d, and then to convex closed curves with diameter less than d by considering such curves as limits of inscribed polygons, giving the probability of a "cross" as $P/\pi d$, where P is the perimeter of the convex curve.

Ramaley (1969) eliminated the assumptions on Gnedenko's curve (closed, convex, and with restricted diameter), called it a wet noodle, and considered the expected number of line-crossings. When such a wet noodle of length P is dropped randomly onto an infinite grid of parallel lines with common distance d between them, then the expected number of line-crossings is $2P/\pi d$.

Duncan (1967) studied the case in which a needle is randomly dropped onto a board containing a set of n radial lines with uniform angular spacing $2\pi/n$. The probability of a cross is expressed as a definite integral that must be estimated. The classical **Buffon** Needle Problem is obtained as a limiting case of this **problem.**

Khamis (1987) considered the problem of randomly dropping a needle of length ℓ onto a board containing N concentric circles, where the difference in the radii between any two consecutive such circles is a constant d, with $\ell < d$. The probability that the needle crosses one of the N circumferences is

$$\rho_N = \frac{\ell(N-1)}{\pi N d} + \frac{1}{\pi N^2 d^2} \sum_{k=1}^{N} \left(\ell\sqrt{k^2 d^2 - \ell^2/4} + 2k^2 d^2 \sin^{-1} \frac{\ell}{2kd} \right). \qquad (1)$$

The classical **Buffon** Needle Problem problem is obtained as a limiting case of this formula:

$$\rho = \lim_{N\to\infty} \rho_N = 2\ell/\pi d.$$

Other aspects of **Buffon's** Needle Problem have been discussed by Mantel **(1953),** Schuster (**1974**), **Perlman** and Wichura (**1975**), Diaconis (**1976**), **DeTemple** and Robertson **(1980),** and Robertson and Siege! (1986).

Problem. We now consider the problem of dropping the needle onto a set of concentric circles with the assumption that the distance between the center of the concentric circles and the midpoint of the needle is a Gaussian random variable. This problem is much more realistic than the case in which the needle is randomly dropped since there will generally be a tendency for the needle to be dropped close to the center of the concentric circles. This assumption requires the imposition of an a *priori* bivariate normal distribution for the coordinates of the midpoint of the needle, and subsequent derivation of the a ***posteriori*** distribution of the distance between the midpoint of the needle and the nearest circumference.

Solution. Let (U,V) be the rectangular coordinates of the midpoint, ***M***, of the needle in the two-dimensional Euclidean plane. Assume that U and V both have normal distributions with zero mean and standard deviation σ, that they are independent of each other so that their joint probability density function, pdf, is the bivariate normal distribution,

$$g(u,v) = (1/2\pi\sigma^2)\,exp(-(u^2 + v^2)/2\sigma^2);\ -\infty < u,v < \infty,\quad \sigma > 0.$$

Then the probability that M falls within a circle of radius z is

$$\phi(z) = 4\int_0^{\pi/2}\int_0^{z} (1/2\pi\sigma^2)\,exp(-r^2/2\sigma^2)\ rdrde = 1 - exp(-z^2/2\sigma^2).$$

Now, let X represent the distance between M and the nearest circumference as measured along the radius extending from the center (O) of the concentric circles through M. See Figure *1*.

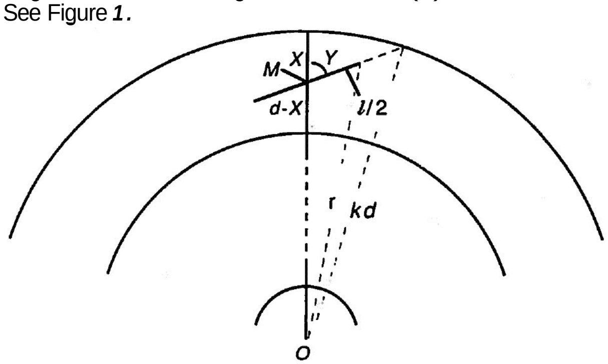


Figure *1*

Let Y represent the acute angle created by OM and the needle. These two random variables, X and Y, are independent. Assume that M falls within the kth annulus, $k = 1,2,3,...$; that is $(k - l)d < OM < kd$ (k=1 corresponds to the circle having radius d). Then there are two cases to consider:

CASE *1.* $(k - 1/2)d < \overline{OM} < kd,\ k = 1,2,3,...,$ and

CASE *2.* $(k - l)\,d < \overline{OM} < (k-1/2)d,\ k = 2,3,...$.

The first case corresponds to the event that M falls inside the "outer **half**" of **the kth** annulus and the second case to the event that M falls inside the "inner **half**" of the kth annulus. Let these two events be represented by O_k and I_k, respectively. We consider each case separately.

Case 1. If $X < kd - \sqrt{k^2d^2 - \ell^2/4}$ then the needle crosses the kth circumference for any value of Y. To see this, think of the needle as a chord of the circle (see Figure *2).* By the Pythagorean Theorem, the value of x_0 shown in Figure *2* is

$$x_0 = kd - \sqrt{k^2d^2 - \ell^2/4}.$$

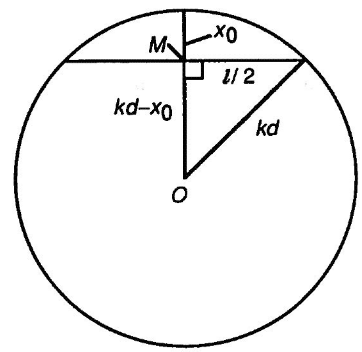


Figure *2*

So, when $X < x_0$, the needle crosses the kth circumference with probability one.

When $x_0 < X < 112$, the probability that the needle crosses the **kth** circumference depends on Y. By the law of cosines (see Figure 1),

$$r^2 = (kd - X)^2 + \ell^2/4 + (kd - X)\ell cos\ Y.$$

The needle crosses the kth circumference if and only if $r > kd$, which is equivalent to

$$Y < cos^{-1}\left[\frac{(k^2d^2 - \ell^2/4) - (kd - X)^2}{(kd - X)\ell}\right],\quad x_0 < X < \ell/2.$$

Now consider the joint **pdf** of X and Y. The acute angle Y is treated as a uniform random variable on $[0,\pi/2]$. Let $F_O k(x)$ be the posterior cumulative distribution function, **cdf,** of X, conditioned on the event O_k, $k = 1,2,...$ Note that X takes values in the interval $(0,d/2)$. Then the probability that M falls at a distance between zero and x from the kth circumference, $F_O k(x)$, is the ratio of the volume beneath $g(u,v)$ corresponding to the annulus having inner radius $kd - x$ and outer radius kd to the volume corresponding to the annulus having inner radius $(k - 1/2)d$ and outer radius kd; viz.,

$$F_{O_k}(x) = (\phi(kd) - \phi(kd - x))/(\phi(kd) - \phi(kd - d/2))$$

$$= \frac{exp(-(kd - x)^2/2\sigma^2) - exp(-k^2d^2/2\sigma^2)}{exp(-(kd - d/2)^2/2\sigma^2) - exp(-k^2d^2/2\sigma^2)}, \quad 0 < x < d/2.$$

Then the posterior **pdf** conditional on O_k is

$$f_{O_k}(x) = \frac{(1/\sigma^2)(kd - x)\ exp(-(kd - x)^2/2\sigma^2)}{exp(-(kd - d/2)^2/2\sigma^2) - exp(-k^2d^2/2\sigma^2)}, \quad 0 < x < d/2.$$

Define $a_k = (1/\sigma^2)/[exp(-(kd - d/2)^2/2\sigma^2) - exp(-k^2d^2/2\sigma^2)]$. Then

$$f_{O_k}(x) = a_k(kd - x)\ exp(-(kd - x)^2/2\sigma^2), \quad 0 < x < d/2.$$

Finally, the posterior conditional joint **pdf** of X and Y is

$$f_{O_k}(x,y) = (2a_k/\pi)(kd - x)\ exp(-(kd - x)^2/2\sigma^2), \quad 0 < x < d/2,\ 0 < y < \pi/2,$$
$$k = 1, 2, 3, \ldots .$$

The conditional probability that the needle crosses the kth circumference is

$$P(C_k|O_k)$$

$$= P_{O_k}(X < x_0) + P_{O_k}\left[(x_0 < X < \ell/2),\ Y < cos^{-1}\left(\frac{(k \quad - \ell^2/4) - (kd - X)^2}{(kd - X)\ell}\right)\right]$$

$$= \int_0^{x_0} a_k(kd-x)exp(-(kd - x)^2/2\sigma^2)dx \ +$$

$$(2a_k/\pi)\int_{x_0}^{1/2} (kd-x)exp(-(kd - x)^2/2\sigma^2)cos^{-1}\left(\frac{(k^2d^2 - \ell^2/4) - (kd - x)^2}{(kd - x)\ell}\right)dx$$

$$= a_k\sigma^2\left(exp(-(kd - x_0)^2/2\sigma^2) - exp(-k^2d^2/2\sigma^2)\right) + a_kA_1(k),$$

where $A_1(k) \equiv \pi^{-1} \displaystyle\int_{(kd-\ell/2)^2}^{(kd-x_0)^2} exp(-x/2\sigma^2)cos^{-1}\left(\frac{(k^2d^2 - \ell^2/4) - x}{\ell\sqrt{x}}\right)dx.$

Case 2. This is the case in which M falls inside the inner half of the kth annulus (I_k). See Figures 3 and 4. By the law of cosines,

$$r^2 = (h + X)^2 + \ell^2/4 - \ell(h + X)cos Y,$$

where $h \equiv (k-1)d$, $k = 2,3,\ldots$.

Now, let x_1 represent the value of X when the left endpoint of the needle coincides with the point at which the needle is tangent to the circle having radius h (see Figure 5). According to the Pythagorean Theorem,

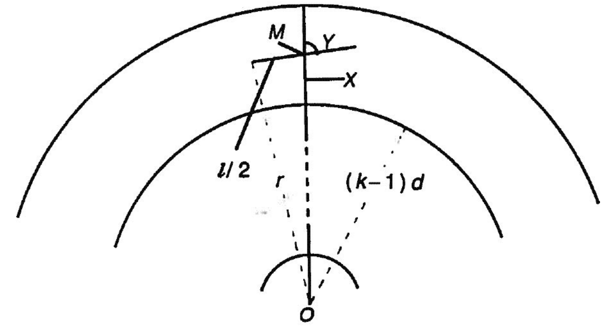


Figure 3

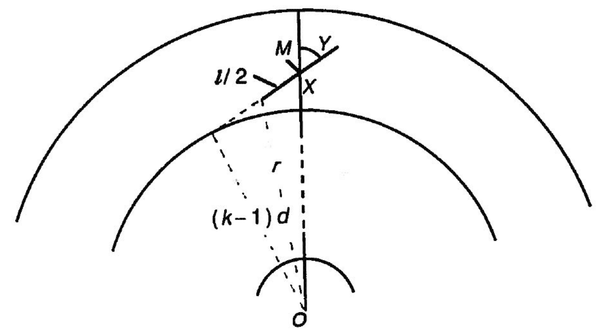
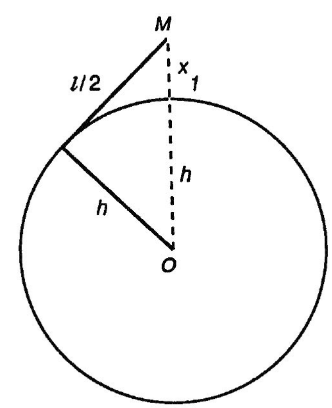


Figure 4



$h = (k-1)d$

Figure 5

$$x_1 = \frac{1}{2}\sqrt{4h^2 + \ell^2} - h.$$

Let $F_{I_k}(x)$ be the posterior cdf of X conditioned on the event I_k, k = 2,3,.... The probability that M falls at a distance between zero and x from the (k − l)th circumference, $F_{I_k}(x)$, is the ratio of the volume between beneath $g(u,v)$ corresponding to the annulus having inner radius h and outer radius h + x to the volume corresponding to the annulus having inner radius h and outer radius (k − 1/2)d; viz.,

$$F_{I_k}(x) = (\phi(h + x) - \phi(h)/(\phi(k - 1/2)d) - \phi(h)$$

$$= \frac{\exp(-h^2/2\sigma^2) - \exp(-(h + x)^2/2\sigma^2)}{\exp(-h^2/2\sigma^2) - \exp(-(kd - d/2)^2/2\sigma^2)}, \quad 0 < x < d/2.$$

Then the posterior pdf conditional on I_k is

$$f_{I_k}(x) = \frac{(1/\sigma^2)(h + x)\exp(-(h + x)^2/2\sigma^2)}{\exp(-h^2/2\sigma^2) - \exp(-(kd - d/2)^2/2\sigma^2)}, \quad 0 < x < d/2.$$

Define $b_k = (1/\sigma^2)/[\exp(-h^2/2\sigma^2) - \exp(-(kd - d/2)^2/2\sigma^2)]$. Then

$$f_{I_k}(x) = b_k(h + x)\exp(-(h + x)^2/2\sigma^2), \quad 0 < x < d/2.$$

So, the posterior conditional joint pdf of X and Y is

$$f_{I_k}(x,y) = (2b_k/\pi)(h + x)\exp(-(h + x)^2/2\sigma^2), \quad 0 < x < d/2,$$
$$0 < y < \pi/2, \quad k = 2,3,\ldots .$$

Case 2 will be handled in two parts:

Case 2a. $0 < X < x_1$ (see Figure 3), and

Case 2b. $x_1 < X < \ell/2$ (see Figure 4).

In case **2a**, the needle crosses the (k − 1)th circumference if and only if

$$r^2 < (k-1)^2d^2 + \left\{\ell/2 - \sqrt{((k-1)d + X)^2 - (k-1)^2d^2}\right\}^2$$

The right-hand side of this inequality is the value of r^2 when the needle in Figure 3 is tangent to the circle of radius h, or, equivalently,

$$Y < \cos^{-1}\left[\sqrt{1 - h^2/(h+X)^2}\right].$$

Then the conditional joint probability of C_k for the case $0 < X < x_1$ is

$$P(C_k, 0 < X < x_1 \mid I_k)$$

$$= \int_0^{x_1}\int_0^{\cos^{-1}\left[\sqrt{1 - h^2/(h+X)^2}\right]} (2b_k/\pi)(h + x)\exp(-(h + x)^2/2\sigma^2)\,dy\,dx$$

$$= (2b_k/\pi)\int_0^{x_1} (h + x)\exp(-(h + x)^2/2\sigma^2)\cos^{-1}\left[\sqrt{1 - h^2/(h+x)^2}\right]dx$$

$$= b_kA_2(k),$$

where $A_2(k) \equiv \pi^{-1}\int_{h^2}^{(h+x_1)^2} \exp(-x/2\sigma^2)\cos^{-1}\left[\sqrt{1 - h^2/x^2}\right]dx.$

In case **2b**, $x_1 < X < \ell/2$ and the needle crosses the (k − 1)th circumference if and only if r < h (see Figure 4), or, equivalently,

$$Y < \cos^{-1}\left[\frac{(X + h)^2 + (\ell^2/4 - h^2)}{(X + h)\ell}\right].$$

Then the conditional joint probability of C_k for the case $x_1 < X < 1/2$ is

$$P(C_k, x_1 < X < \ell/2 \mid I_k)$$

$$= \int_{x_1}^{\ell/2}\int_0^{\cos^{-1}\left[\frac{(x + h)^2 + (\ell^2/4 - h^2)}{(x + h)\ell}\right]} (2b_k/\pi)(h+x)\exp(-(h+x)2/2\sigma^2)\,dy\,dx$$

$$= (2b_k/\pi)\int_{x_1}^{\ell/2} (h+x)\exp(-(h+x)^2/2\sigma^2)\cos^{-1}\left[\frac{(x + h)^2 + (\ell^2/4 - h^2)}{(x + h)\ell}\right]dx$$

$$= b_kA_3(k)$$

where $A_3(k) \equiv \pi^{-1}\int_{(h+x_1)^2}^{(h+\ell/2)^2} \exp(-x/2\sigma^2)\cos^{-1}\left[\frac{x + (\ell^2/4 - h^2)}{\ell\sqrt{x}}\right]dx.$

Combining the results of case 2a and case 2b we have,

$$P(C_k|I_k) = P(C_k, 0 < X < x_1 \mid I_k) + P(C_k, x_1 < X < \ell/2 \mid I_k)$$

$$= b_k(A_2(k) + A_3(k)).$$

The probabilities of the events O_k and I_k are easily obtained by computing volumes beneath the bivariate normal distribution surfaces corresponding to annular areas:

$$P(O_k) = \phi(kd) - \phi((k-1/2)d) = 1/a_k\sigma^2,\ k = 1,2,\ldots, \text{ and}$$
$$P(I_k) = \phi((k-1/2)d) - \phi((k-1)d) = 1/b_k\sigma^2,\ k = 2,3,\ldots.$$

Now, define $s = \sigma/d$ and $\lambda = \ell/d$ so that s and λ denote, respectively, the standard deviation and the length of the needle relative to d. Then σ can be replaced by sd and ℓ can be replaced by λd in the expressions for $a_k, b_k, x_0, x_1, A_1(k), A_2(k)$, and $A_3(k)$, and the probabilities $P(C_k|O_k), P(C_k|I_k), P(O_k), P(I_k)$ are each independent of d.

The Bayesian probability that the needle crosses a circumference when it is dropped onto a set of concentric circles is, by the Law of Total Probabilities,

$$\rho^* = \sum_{k=1}^{\infty} P(O_k)P(C_k|O_k) + \sum_{k=2}^{\infty} P(I_k)P(C_k|I_k). \tag{2}$$

In computing ρ^* we set $d = 1$ so that

$$a_k s^2 = \left[exp(-(k-1/2)^2/2s^2) - exp(-k^2/2s^2)\right]^{-1},$$

$$b_k s^2 = \left[exp(-(k-1)^2/2s^2) - exp(-(k-1/2)^2/2s^2)\right]^{-1},$$

$$x_0 = k - \sqrt{k^2 - \lambda^2/4},$$

$$x_1 = \frac{1}{2} - \sqrt{4(k-1)^2 + \lambda^2} - (k-1),$$

$$A_1(k) = \pi^{-1} \int_{(k-\lambda/2)^2}^{(k^2-\lambda^2/4)} exp(-x/2s^2)\cos^{-1}\left(\frac{(k^2-\lambda^2/4) - x}{\lambda\sqrt{x}}\right) dx,$$

$$A_2(k) = \pi^{-1} \int_{(k-1)^2}^{(k-1)^2+\lambda^2/4} exp(-x/2s^2)\cos^{-1}\left(\sqrt{1-(k-1)^2/x}\right) dx,$$

and

$$A_3(k) = \pi^{-1} \int_{(k-1)^2+\lambda^2/4}^{(k-1+\lambda/2)^2} exp(-x/2s^2)\cos^{-1}\left[\frac{x + \lambda^2/4 - (k-1)^2}{\lambda\sqrt{x}}\right] dx.$$

Then, substituting into *(2)* we have

$$\rho^* = \sum_{k=1}^{\infty}\left[exp(-(k^2-\lambda^2/4)/2s^2) - exp(-k^2/2s^2)\right] +$$
$$s^{-2}\left[A_1(1) + \sum_{k=2}^{\infty}\left(A_1(k) + A_2(k) + A_3(k)\right)\right] \tag{3}$$

Limiting Case. Suppose the needle is dropped within a circle of radius N $(d=1)$, $N = 2,3,\ldots$. Then the truncated normal prior cdf is

$$(1 - e^{-a^2/2s^2})/\phi(N),\ 0 \le a \le N,$$

and the Bayesian probability of a cross becomes

$$\rho^*_N = (\phi(N))^{-1}\sum_{k=1}^{N}\left[exp(-(k^2-\lambda^2/4)/2s^2) - exp(-k^2/2s^2)\right]$$
$$+(\phi(N)s^2)^{-1}\left[A_1(1) + \sum_{k=2}^{N}\left(A_1(k) + A_2(k) + A_3(k)\right)\right]$$

Theorem 1. $\lim_{s\to\infty} \rho^*_N = \rho_N$, where ρ_N is given in (1).

Proof. By using L'Hôpital's Rule we have

$$\lim_{s\to\infty}\left\{(\phi(N))^{-1}\left[exp(-(k^2-\lambda^2/4)/2s^2) - exp(-k^2/2s^2)\right]\right\} = \lambda^2/4N^2, \text{ and}$$

$$\lim_{s\to\infty} (\phi(N)s^2)^{-1} = 2/N^2.$$

Exchanging the limit and the definite integral and using integration by parts we have

$$\lim_{s\to\infty} A_1(k) = (2\pi)^{-1}\left[\pi(k^2-\lambda^2/4) - 2k^2\cos^{-1}(\lambda/2k) + \lambda\sqrt{k^2-\lambda^2/4}\right],$$

$$\lim_{s\to\infty}(A_2(k) + A_3(k)) = (k-1)\lambda/\pi.$$

Then

$$\lim_{s\to\infty} \rho^*_N = \lambda^2/4N + (\pi N^2)^{-1}\left[\pi(1-\lambda^2/4) - 2\cos^{-1}(\lambda/2) + \lambda\sqrt{1-\lambda^2/4}\right]$$
$$+ 2N^{-2}\sum_{k=2}^{N}(k-1)\lambda/\pi$$
$$+ (\pi N^2)^{-1}\sum_{k=2}^{N}\left[\pi(k^2-\lambda^2/4) - 2k^2\cos^{-1}(\lambda/2k) + \lambda\sqrt{k^2-\lambda^2/4}\right]$$
$$= \lambda^2/4N + 2N^{-2}\sum_{k=2}^{N}(k-1)\lambda/\pi$$
$$+ (\pi N^2)^{-1}\sum_{k=1}^{N}\left[\pi(k^2-\lambda^2/4) - 2k^2\cos^{-1}(\lambda/2k) + \lambda\sqrt{k^2-\lambda^2/4}\right].$$

Upon summing and using the trigonometric identity $\cos^{-1}x = \pi/2 - \sin^{-1}x$, we have

$$\lim_{s\to\infty} \rho^*_N = \frac{\lambda(N-1)}{\pi N} - \frac{1}{\pi N^2}\sum_{k=1}^{N}\left[\lambda\sqrt{k^2-\lambda^2/4} + 2k^2\sin^{-1}(\lambda/2k)\right].$$

Conclusion. If the expression for n given in (1) is looked upon as the probability of a cross in the case of a bivariate uniform prior, then Theorem 1 is intuitively reasonable. It is seen, then, that ρ^* tends to $2l/\pi d$, the classical Buffon Needle Problem probability, when s tends to infinity and the normal prior converges in distribution to the uniform prior. This can be seen in Table 1, which lists ρ^* to four decimal place accuracy for s = 0.5(0.5)5.0,10.0 and ∞, and λ = 0.1(0.1)1.0. In obtaining the values in Table 1 the Cautious Adaptive Romberg Extrapolation method in the IMSL library is used to estimate the definite integrals in (3).

The Bayesian Buffon Needle Problem probability, ρ^*, given in (3) involves three definite integrals which must be estimated. This is perhaps to be expected since the problem is based on the normal prior, a pdf that does not admit an antiderivative in closed form. Other prior distributions can be considered, such as the bivariate double exponential, however the advantages to using a normal prior are that it is a natural choice for characterizing the propensity of the needle to fall close to the center, and it provides a convenient, closed form expression for $\phi(z)$, and hence for the posterior distributions $f_{O_k}(x,y)$ and $f_{I_k}(x,y)$.

Table 1 Values of ρ^*

s	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.5	.0347	.0700	.1063	.1442	.1842	.2265	.2716	.3197	.3709	.4252
1.0	.0581	.1161	.1742	.2324	.2905	.3488	.4070	.4653	.5237	.5820
1.5	.0612	.1225	.1837	.2449	.3060	.3671	.4281	.4889	.5497	.6103
2.0	.0623	.1246	.1869	.2491	.3113	.3734	.4355	.4974	.5592	.6209
2.5	.0628	.1256	.1884	.2511	.3138	.3764	.4390	.5015	.5638	.6261
3.0	.0630	.1261	.1891	.2522	.3151	.3781	.4409	.5037	.5664	.6291
3.5	.0632	.1264	.1896	.2528	.3159	.3790	.4421	.5051	.5680	.6309
4.0	.0633	.1266	.1899	.2532	.3165	.3797	.4429	.5060	.5691	.6321
4.5	.0633	.1267	.1901	.2535	.3168	.3801	.4434	.5067	.5698	.6330
5.0	.0634	.1268	.1903	.2537	.3171	.3805	.4438	.5071	.5704	.6336
10.0	.0634	.1270	.1906	.2542	.3178	.3814	.4450	.5085	.5721	.6356
∞	.0637	.1273	.1910	.2546	.3183	.3820	.4456	.5093	.5730	.6366

REFERENCES

1. G. Buffon, *L'Histoire Naturelle,* 4 Supplement, 1777

2. Duane W. DeTemple and Jack M. Robettson, Constructing Buffon curves from their distributions, *American Mathematical Monthly,* 87(1980), 779-784.

3. Persi Diaconis, Buffon's problem with a long needle, *Journal of Applied Probability* 13 (1976), 614-618.

4. R. L. Duncan, A Variation of the Buffon needle problem, *Mathematics Magazine* 40 (1967), 36-38.

5. B. V. Gnedenko, *Theory of Probability,* Chelsea, New York, 1962.

6. H. J. Khamis, On Buffon's needle problem using concentric circles, *Pi Mu Epsilon Journal,* 8 (1987), 368-376.

7. Nathan Mantel, An extension of the Buffon needle problem, *Ann. Math. Stat.* 24(1953), 674-677.

8. M. Perlman and M. Wichura, On sharpening Buffon's needle, *American Statistician,* 28 (1975), 157-163.

9. J. F. Ramaley, Buffon's noodle problem, *American Mathematical Monthly,* 76(1969), 916-918.

10. Jack M. Robettson and Andrew F. Siegel, Designing Buffon's needle for a given crossing distribution, *American Mathematical Monthly,* 93 (1986), 116-119.

11. E.F. Schuster, Buffon's needle experiment, *American Mathematical Monthly,* 81(1974), 26-29.

12. J. V. Uspensky, *Intmduction to Mathematical Probability,* McGraw-Hill, New York, 1937.

DESARGIAN* SEGMENTS

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We begin with the idea that motivated this paper, mainly to illustrate the evolution of that idea. If one erects equilateral triangles on certain segments, then the third vertices of these triangles are collinear. It seemed reasonable to investigate this idea. It quickly generalized. We state the original definition and the first immediate theorem, whose proof led to the generalization that follows.

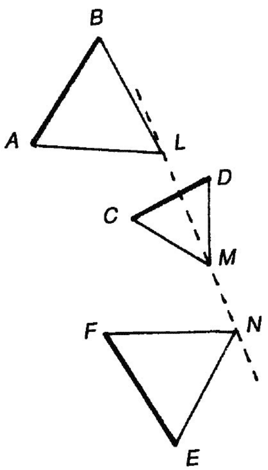

Figure 1

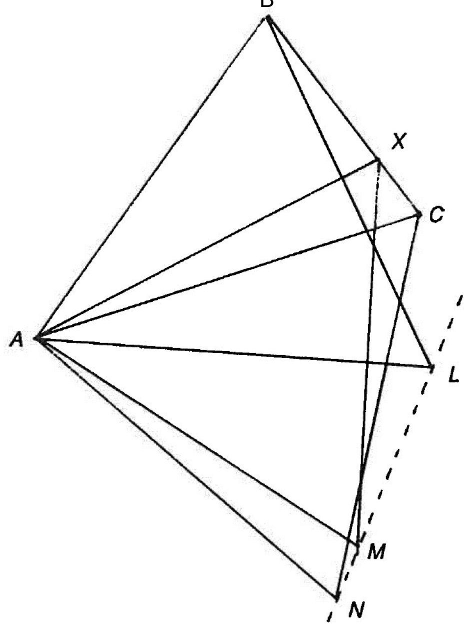

Figure 2

Definition 0. Three line segments, ***AB, CD,*** and EF are ***equi-Desargian*** if the third vertices of equilateral triangles erected in the same sense on the three segments are **collinear.** See Figure 1.

Theorem 0. **If** X is a **point** lying on side BC of triangle ABC, then ***AB,*** AX, and AC are equi-Desargian.

---

* Purists may prefer the spelling "Desarguesian". -- C. W. D.

Proof. Let directly similar equilateral triangles ABL, AXM, and ACN be erected on ***AB,*** AX, and ***AC,*** as shown in Figure 2. It is readily seen that a **60°** rotation about the point A carries triangle ABC to triangle ALN. Hence it carries any point X on side BC to the corresponding point M on the image LN of BC.

The proof above shows that we have simply performed a rotation. We have rotated **line** BC about point A through **60°.** Then the image line LMN will intersect **line** ***BXC*** in that same **60°** angle, too. There is nothing special here about a **60°** angle; any angle will do. Thus the idea of **equi-Desargian** might be generalized to apply to more **general** rotations and also to homotheties or stretches. That is, let us examine the sltuations when we replace the equilateral **triangle** by any general triangle. Thus we shall use the following definition:

Definition 1. Let a given triangle ***PQR*** be called the base triangle. Then three given line segments ***AB, CD,*** and EF are called Desargian if the third vertices ***L, M,*** and N of triangles ABL, CDM, and EFN, erected on the segments ***AB, CD,*** and ***EF*** **with** each triangle directly similar to triangle ***PQR,*** are collinear. See Figure 3.

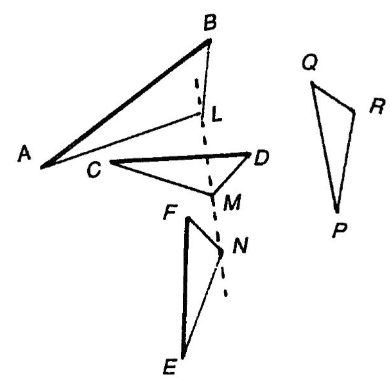

Figure 3

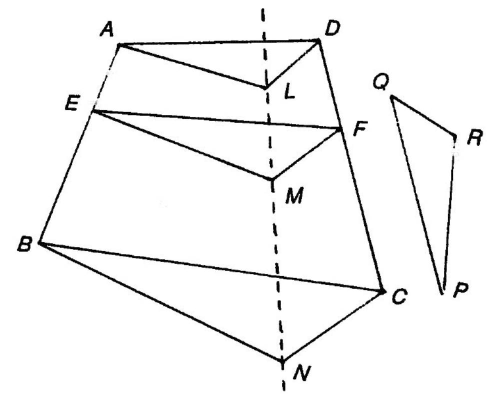

Figure 4

The proof of Theorem **0** and the comments that follow it prove the following restatement of that theorem.

Theorem 1. Take any base triangle ***PQR.*** **If** X is a point lying on side BC of any triangle ABC, then ***AB,*** AX, and AC are Desargian.

Corollary. **If** L, **M,** and N are the third vertices of the triangles constructed on the segments ***AB,*** AX, and AC of Theorem **1,** then the angle between lines BXC and LMN is equal to the angle QPR of the base triangle, the angle of the rotation-homothety that carries ***PQ*** to ***PR, AB*** to ***AL, AX*** **to** AM, and AC to AN.

Definition 2. Line LMN of the Corollary to Theorem **1** is called the Desargian line of BC relative to A.

Next suppse that pint A of Theorem 1 is allowed to move. Will Theorem **1 still** hold? One possibility follows.

Theorem 2. Select any base triangle PQR. Let $ABCD$ be a quadrilateral and let E and F be points on AB and CD respectively such that $AE/EB = DF/FC$, where all segment measurements are taken directed. That is, let E divide AB in the same ratio that F divides DC. Then AD, EF, and BC **are Desargian.**

Proof. Place the figure in the complex plane. **Let** u and v be complex numbers such that $r = pu + qv$. (Then u and v describe the rotation-homothety performed on a segment by appending triangle PQR to that segment.) **Since** erected triangles ADL, EFM, and BCN are each directly similar to triangle PQR, we have that

$$l = au + dv, \quad m = eu + fv, \quad \text{and } n = bu + cv.$$

Since E and F divide AB and DC in the same ratio, there are real numbers r and s such that $r + s = 1$ and

$$e = ra + sb \quad \text{and} \quad f = rd + sc.$$

Then we have

$$rl + sn = r(au + dv) + s(bu + cv) = u(ra + sb) + v(rd + sc) = ue + vf = m,$$

so M **divides** LN in the same ratio that E **divides** AB.

In the above proof it was assumed that r and s are real numbers such that $r + s = 1$ in order to guarantee that A, M, and B are collinear. The proof, however, never uses that restriction, so we have proved the theorem true in the far more general situation that if AEB and DFC are any two directly similar triangles, then LMN is directly similar to these two triangles. Also observe that the quadrilateral $ABCD$ need not be convex; it could even be a cross quadrilateral. See Figure 5.

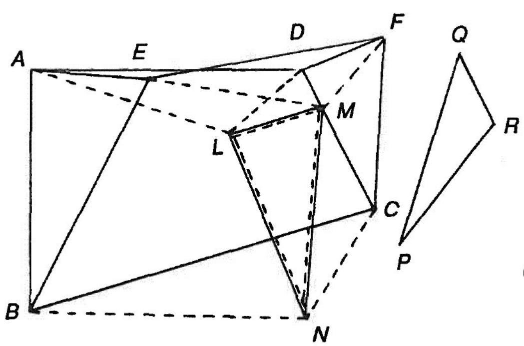

Figure 5

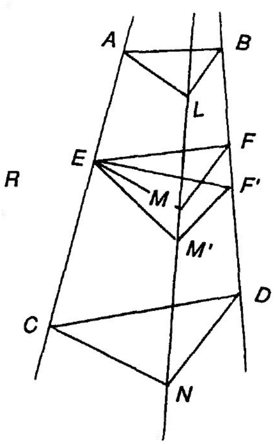

Figure 6

Theorem **1** is the special case of Theorem 2 in which segment or triangle AEB has shrunk to zero. Furthermore, the concept of Desarglan is a special case of the more general idea that if AEB and DFC are any directly similar triangles, then LMN is similar

to them both. We shall not introduce a new term for that generalization.

Notice that Theorem 1 is self-converse. That is, if triangles ABL, AXM, and ACN are all directly similar to some base triangle, then B, X, and C are collinear if and only if L, M, and N are collinear. This is clear when one considers that the **rotation-**homothety that carries PR to PQ of the base triangle PQR is just the inverse of the rotation-homothety that carries PQ to PR. Let us consider an inverse for Theorem 2. The unequal angle condition in Theorem 3 below was motivated by a proof using complex numbers. The condition is necessary, as the reader may which to show by drawing a figure to illustrate, but fortunately the messy, awkward complex number proof has been replaced by a simpler synthetic one.

Theorem 3. Let PQR be a base triangle; let A, E, and C be collinear; let B, F, and D be collinear; and suppose segments AB, EF, and CD are Desargian. That is, suppose also that L, M, and N are collinear, where triangles ABL, EFM, and CDN are all directly similar to triangle PQR. **If** angle BAL is not equal to either angle between lines BFD and LMN, or if angle ALB is not equal to either angle between lines AEC and BFD, or if angle LBA is not equal to either angle between lines LMN and AEC, then E, F, and M divide segments AC, BD, and LN respectively in the same ratio.

Proof. Suppose, for example, that angle BAL is not equal to either angle between lines BFD and LMN. See Figure 6. Then $ABDC$ is a quadrilateral. Take point F' on BD so that $AE/EC = AF'/F'D$. Then, by Theorem 2, point M' such that triangles $EF'M'$ and PQR are directly similar is collinear with L and N and $AE/EC = AF'/F'D = LM'/M'N$. **If** $F = F'$, then also $M = M'$ since triangles EFM and $EF'M'$ are directly similar, and we are done. **If** $F \neq F'$, then, by Theorem **1**, line FF' **maps** to line MM' by a rotation-homothety centered at E and with angle equal to angle QPR, which is equal to angle BAL. Hence the lines $BFF'D$ and $LMM'N$ meet at angle BAL, violating the hypothesis of the theorem. Hence we cannot have $F \neq F'$ and the theorem follows.

Parallelism can be defined in theorems of the Desargian property, as shown in the next theorem.

Theorem 4. For given lines x and y, choose a point A not on either line and draw a pair of lines ACE to cut x at C and y at E, and ADF **to** cut x at D and y at F, as shown in Figure 7. Choose a base triangle PQR and take points M and N such that triangles CDM and EFN are directly similar to triangle PQR. Then lines x and y are parallel if and only if A, M, and N are collinear.

Proof. Apply Theorems **2** and 3, where triangle ABL has shrunk to just point A, since $AE/EC = AF/FD$ if and only if lines x and y are parallel.

Our last theorem was again first proved by **complex** numbers, another messy proof taking two or three pages. Again, fortunately, a synthetic proof that is far less complicated has been found, so we shall use the latter and spare you the horrors of the former. We first present a lemma.

Lemma. Let ABC be a triangle and θ an angle less than **180°. In** the same direction through angle θ rotate ray BC about B and rotate ray AC about A and let the image lines meet at G as shown in Figure 8. Then

$$CG = \frac{BC \sin GAC}{\sin BAC}.$$

Proof. Note that $\angle GAC = \angle BAC \pm \angle\theta$ **and** $\angle ACG = \angle ACB \mp \angle\theta$, according to the direction of the rotation. Also $\angle ABC = \angle AGC$. Now apply the law of sines to triangles ACG

and ACB. We get that

$$\frac{CG}{\sin GAC} = \frac{AC}{\sin G} = \frac{AC}{\sin B} = \frac{BC}{\sin BAC},$$

from which the Lemma readily follows.

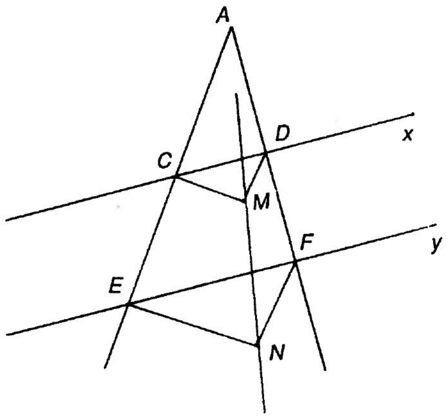

Figure 7

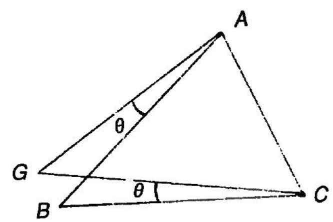

Figure 8

Theorem 5. Let PQR be a base triangle in which side PQ is mapped to side PR by a rotation though angle θ and homothety of ratio k. Let ABC be a given triangle. Let x be the Desargian line for BC relative to A, y that for CA relative to B, and z the line for AB relative to C. Let x and y meet at C', y and z at A', and z and x at B'. Then triangle $A'B'C'$ is directly similar to triangle ABC and has perimeter that is $|3k - 2\cos\theta|$ times the perimeter of ABC.

Proof. By the corollary to Theorem 1, the angle between BC and $B'C'$ is equal to 0, as is also the angle between $A'B'$ and AB, and also the angle between $C'A'$ and CA. Hence $\angle A' = \angle A$, etc., so that triangle $A'B'C'$ is similar to triangle ABC. It remains now to find their ratio of similarity. As in Figure 9, let the rotation-homothety about A map triangle ABC to AHJ. Then $HJ = k \cdot BC$. To obtain the length of $B'C'$ we must find also the lengths of $B'H$ and JC'. Let the rotation-homothety about C map triangle ABC to MNC, and let AH and CN meet at G. Then $GHB'N$ is a parallelogram, so $B'H = NG$. Also

$$B'H = NG = NC - CG = k \cdot BC - CG$$

$$= k \cdot BC - \frac{BC \sin GAC}{\sin A}.$$

by the Lemma. Similarly, let triangle ABC map to triangle LBK in the rotation-homothety about B, and let BK and AJ meet at F. Then we have

$$JC' = FK = BK - BF = k \cdot BC - BF$$

$$= k \cdot BC - \frac{BC \sin FAB}{\sin A}$$

Observe that one of angles GAC and FAB is $A + \theta$ and the other is $A - \theta$, so that

$$\begin{aligned} & \sin GAC + \sin FAB \\ = \; & \sin (A + 0) + \sin (A - 0) \\ = \; & \sin A \cos\theta + \cos A \sin\theta + \sin A \cos\theta - \cos A \sin\theta \\ = \; & 2 \sin A \cos\theta. \end{aligned}$$

Now we have that

$$\begin{aligned} B'C' &= B'H + HJ + JC' \\ &= k \cdot BC - \frac{BC \sin GAC}{\sin A} + k \cdot BC + k \cdot BC - \frac{BC \sin FAB}{\sin A} \\ &= 3k \cdot BC - \frac{BC(\sin GAC + \sin FAB)}{\sin A} \\ &= BC\left(3k - \frac{2 \sin A \cos\theta}{\sin A}\right) \\ &= BC(3k - \cos\theta) \end{aligned}$$

Finally, we take absolute values because we are interested in the nondirected length of segment $B'C'$.

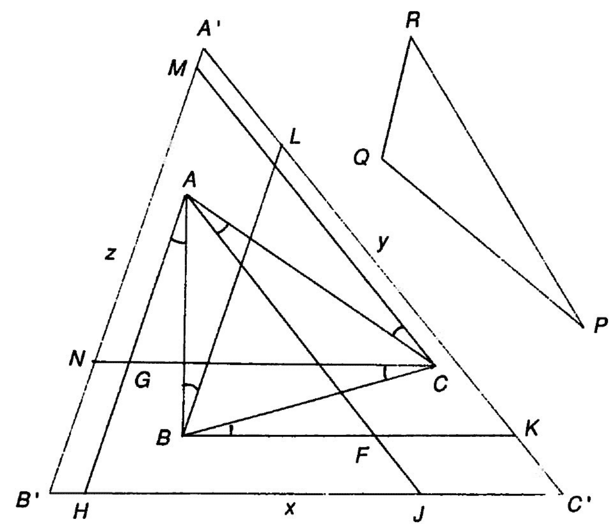

Figure 9

NUMBERS WHICH MERGE, MIX, AND MINGLE

By Richard L. Francis
Southeast Missouri State University

In August of 1989, it was announced that pi had been expressed to one billion decimal places (USA Today, 1989). Perceived patterns in the case for pi quickly suggest intriguing relationships to number patterns encountered elsewhere in mathematics. Among these are similarities in decimal expansions and the overlapping or merging of digital forms. A mixing of numbers by some out-of-the-ordinary selection process may likewise give rise to numbers of a mish-mash kind which strongly defy classification. Even basic patterns within the set of integers (well known to the secondary and the two-year college student) often suggest appealing decimal forms, these forms ranging over rational, irrational, and transcendental quantities (Weisner, 1938). An exploration of various number arrays in a merging, mixing, and classifying manner raises many questions, some answered, and some as of yet unanswered. Let us look first at numbers of a merging kind.

Merging Numbers. The remarkable number π (pi) has been carried out at present to 1,011,196,691 decimal places. Likewise, e has been represented decimally to an impressive number of places (in excess of a hundred thousand digits). Each, based on nineteenth century discoveries, is known to be transcendental and thus irrational. Various comparisons and contrasts emerge as these lengthy and available decimal expansions are examined. The far-reaching and enormously difficult questions of normality quickly arise. Another consideration is that of irrational numbers with unending, identical sequences of digits, apart from their initial or leading digits.

Note for example (Davis, 1961) that

$$\pi = 3.14159265358979323 \ldots 0198 \ldots$$

and 997th place

$$e = 2.71828182845904523 \ldots 0198 \ldots$$

623rd place

agree in a certain four-digit sequence. That is, the four-digit sequence beginning with the 997th place of π is identical to the four-digit sequence beginning with the 623rd place of e. Is it possible that π and e can have identical decimal sequences from a certain place on, as perhaps suggested above, or do the definitions of π and e preclude this? Such a question, here unresolved, nevertheless poses problems as to key number comparisons. Many of these prove within our reach.

Suppose x and y are irrational numbers and that the decimal sequence of x, beginning with the mth place, is identical to the digital sequence of y, beginning at nth place. Then x and y are merging numbers with index $|m - n|$. For example, let

$$x = 5.7401001000100001000001 \ldots \quad \text{and}$$
$$y = 213.696480100100010000100000 1 \ldots$$

In this case, the digital sequence of x beginning at the 4th place is the same as the digital sequence of y beginning at the 9th place. Accordingly, x and y are merging numbers with an index of 9 – 4 or 5.

Merging numbers thus differ in their digital sequences only by a finite number of initial digits. It follows that $10^k(x)$ and $10^j(y)$ must have an integral difference for some choice of k and j. In the example above, $10^5(y) - 10^2(x)$ is an integer. That is, 21369648 – 574 = 21369074.

it should be noted that certain changes in merging numbers do not alter the merging relationship. In particular, if x and y are merging and if x or y is changed by any finite number of digits so as to yield x' and y', these resulting numbers x' and y' are also of the merging type. The proof is fairly obvious. All that is required to establish that x' and y' are merging is to ignore the leading digits which reflect the change. The remaining, unending sequences of digits will of course be identical. Hence, if a million digits are changed in x as given above and a billion digits changed in y, the resulting numbers will still merge. Digits which are changed in this process need not be consecutive. The merging relationship can be pictured rather vividly as in Figure 1 which shows the irrational numbers z and w coalescing in an endless stream of digits.

$$z = 342.14202416912598988988898888988888988888898888888 \ldots$$

$$w = 35.29164370532498988988898888988888988888898888888 \ldots$$

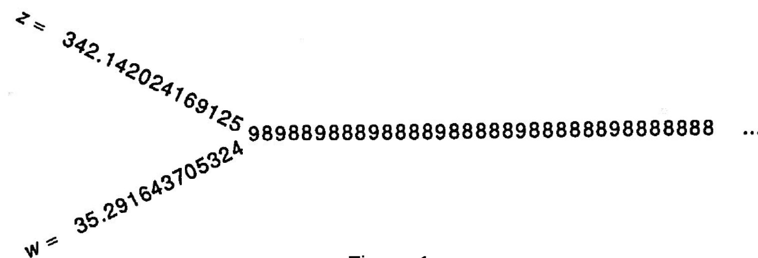

Figure 1

Multiples, Roots, and Density. Relatively short expansions of certain frequently encountered numbers, such as $\sqrt{2}$ and $\sqrt{3}$, may easily raise the merging question. Note that $\sqrt{2} = 1.4142 \ldots 6887 \ldots$ and that $\sqrt{3} = 1.7320 \ldots 6887 \ldots$. Suppose then that $\sqrt{2}$ and $\sqrt{3}$ are merging numbers. Then $10^k(\sqrt{3}) - 10^j(\sqrt{2})$ is an integer r for some integral choice of k and j. That is,

$$10^k\left(\sqrt{3}\right) - 10^j\left(\sqrt{2}\right) = r.$$

Squaring,

$$3(10^{2k}) - 2(10)^{k+j}\left(\sqrt{6}\right) + 2(10^{2j}) = r^2.$$

Rearranging,

$$3(10^{2k}) + 2(10^{2j}) - r^2 = 2(10)^{k+j}\left(\sqrt{6}\right).$$

This forces $\sqrt{6}$ to be the quotient of two integers and thus rational, which is of course a contradiction. The numbers $\sqrt{2}$ and $\sqrt{3}$ are therefore not of a merging kind. Though $\sqrt{2}$ and $\sqrt{3}$ do not merge, is it the case that like sequences of digits of impressive length actually exist? For example, do $\sqrt{2}$ and $\sqrt{3}$ have identical, digital sequences of a trillion digits somewhere in their expansions? The question is today unanswered.

One may wish to apply the argument above to other pairs of square roots which give perhaps some suggestion of merging. Note that $\sqrt{2} = 1.4142 \ldots$ and $\sqrt{51} = 7.14142 \ldots$, or that $\sqrt{80} = 8.944 \ldots 9999 \ldots$ and $\sqrt{204} = 14.282 \ldots 9999 \ldots$. By now a generalized question has likely surfaced. Should 2 and 3 in the argument above be replaced in a cautious manner, a more general result is obtained:

> Consider a and b which are positive inexact squares such that ab is not a square. Then the irrational numbers &– and $\sqrt{b}$ cannot merge.

An interesting problem concerns radical quantities in which the indices are not alike. Consider the question as to whether $\sqrt{2}$ and $\sqrt[3]{3}$ are merging numbers. If they are, then

$$10^k\left(\sqrt{2}\right) - 10^j\left(\sqrt[3]{3}\right) = r$$

where k, j, and r are integers. Multiplying both members by $\sqrt{2}$ yields

$$2(10)^k - \sqrt[6]{72}\ (10)^j = r\left(\sqrt{2}\right)$$

$$\sqrt[6]{72}\ (10)^j = 2(10)^k - r\left(\sqrt{2}\right).$$

Raising both members to the sixth power gives

$$72\left(10^{6j}\right) = P\left(\sqrt{2}\right) + I$$

where P and I are integers. Hence, $P\left(\sqrt{2}\right)$ is an integer thus forcing $\sqrt{2}$ to be rational. Such a contradiction reveals that $\sqrt{2}$ and $\sqrt[3]{3}$ are not merging numbers. Note that P is not zero in the above expansion.

A generalization in this case, not here pursued, must be approached carefully. For example, $\sqrt{2}$ and $\sqrt[4]{4}$ not only merge, but are also equal.

Still another question concerns whether or not an irrational number can merge with an integral multiple of itself other than by a power of ten. Consider the square root of 3 and the square root of 12 which is the double of the square root of 3.

$$\sqrt{3} = 1.73205080 \ldots 7446 \ldots$$

$$\sqrt{12} = 3.46410161 \ldots 7446 \ldots$$

Is there perhaps some suggestion of a merging relationship? Suppose that an irrational number x merges with $2x$. Then

$$10^k(x) - 10^j(2x) = r$$

where k, j, and r are integers. Solving for x, we obtain

$$x = \frac{r}{10^k - 2(10)^j}.$$

This last result is clearly rational, thus contradicting the fact that x is irrational. Accordingly, no irrational number can merge with its double.

It is easy to demonstrate, using a slight variation on the argument above, a highly generalized result. That is, if x is irrational, then x and px cannot merge for any rational value of p where $p \neq 10^k$. If, however, the multiplier p can itself be irrational, the situation changes drastically. Should, for example, $x = 0.1010010001 \ldots$ and $y = 6.871010010001 \ldots$ (which merge beginning with the first 1 in the decimal representations), a value of the multiplier p could be found by dividing x by y.

As the set of irrational numbers has the density property (Niven, 1961). that is, between any two irrational numbers, there is another irrational number,, merging numbers also make a subtle appearance. In particular, between any two (irrational) numbers which merge, say x and y, there is a number z which merges with either. Consider

$$x = 2.741010010001000010000001 \ldots$$

and

$$y = 2.751010010001000010000001 \ldots.$$

Suppose further that the one-millionth 1 in x is changed to a 3, thus forming a new number z. As a single digit change does not alter the merging relationship and as z, an irrational number, is greater than x but less that y, it follows that z is a number between x and y which merges with x and y. The process can be continued without difficulty. Changing the one-billionth 1 to a 3 yields z', changing the one-trillionth 1 to a 3 yields z'', and so on.

More generally. if x and y merge and if $x < y$, a merging number z can be formed by increasing a single digit in x but keeping the number thus formed less than y. Accordingly, infinitely many numbers exist between merging numbers x and y with the property that each merges with both x and y.

The Non-Merging of Algebraic and Transcendental Numbers. A number is called transcendental if it cannot occur as a root of algebraic equation (Lovitt, 1939), that is, a polynomial equation of the form

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n = 0$$

having integral coefficients. Otherwise, the number will be called algebraic.

Accordingly, $\sqrt{243}$ is algebraic as it is a root of $x^2 - 243 = 0$. The number π was proved transcendental (non-algebraic) in 1882 by C. F. Lindemann, a result which remarkably resolved the long-standing circle-squaring problem in the negative. Is there perhaps some suggestion of the merging of a transcendental number and an algebraic number by the expansion which follows?

$$\sqrt{243} = 15.58845726 \ldots 1989 \ldots$$

$$\pi = 3.14159265358979323 \ldots 1989 \ldots$$

More precisely, can a transcendental number and an algebraic number ever merge? In order to establish that the answer is "no." techniques from the theory of equations are needed.

One involves writing an algebraic equation whose roots differ respectively from those of an original equation by some integer. Another involves writing an equation whose roots are multiples of those of some given equation. Thus it is possible (see Lovitt, pages 80 - 85):

1. to transform an algebraic equation into another whose roots are those of the given equation each diminished or increased by an integer.

2. to transform an algebraic equation into another whose roots are those of the given equation each multiplied or divided by a non-zero integer.

If c is transcendental, then $(10^k)c$ cannot occur as a root of an algebraic equation. Nor can $(10^k)c$ diminished or increased by an integer occur as a root. Yet, if d is algebraic, any integral multiple of it, say $(10^j)d$, can occur as a root of an algebraic equation. Accordingly, $(10^k)c$ cannot differ from $(10^j)d$ by an integer and the given numbers c and d will not merge. Very simply, no transcendental number can merge with an algebraic number.

The numbers given earlier, namely π and $\sqrt{243}$, are now known not to merge. Nor do e and $\sqrt{2}$. Should a number be transcendental, any number merging with it is also transcendental. If an irrational number is algebraic, any number merging with it is also algebraic.

Though the above argument proves insightful in the case of the non-merging of transcendental and algebraic numbers, a simpler version of the procedure should be noted. If a transcendental number T merges with an algebraic number A, then $T - A = R$ where R is rational. Then $T = R + A$. As $R + A$ is necessarily algebraic, a contradiction follows. Hence no transcendental number can merge with an algebraic number.

Not all transcendental numbers merge, one with another. The first known transcendental number $L = 0.11000100000\ldots$ (in which 1's appear in the **factorially** placed positions and **0's** elsewhere) cannot merge with $2(L)$. Note that $2(L)$ is not only transcendental, but contains no 1's in its expansion (Francis, 1986).

Today, with the powerful assist of the computer, vast expanses of decimal representations come to view. Such a view surpasses the dreams of early twentieth century mathematicians. Imagine pi visibly represented to more than a billion decimal **places!** In such an explicit setting, remarkable questions naturally arise. Some concern digital frequency. For example, is there a place in the decimal expansion of pi where a trillion consecutive sevens appear? Today, no one knows. Other concerns relate to number comparison. Do certain notable numbers have digital representations of vast finite length (or even infinite length) which overlap? Questions such as these suggest the intriguing pursuits of numbers which merge, mix, and mingle.

REFERENCES

Davis, Philip J. Lore of large numbers. Random House: New York, 1961.

Francis, Richard L. From none to infinity. The College Mathematics Journal, 17 (1986), 226-230.

Lovitt, William V. Elementary theory of equations. Prentice-Hall: New York, 1939.

Niven, Ivan M. Numbers, rational and irrational. Random House: New York, 1961.

USA Today, A really thin slice of pi. Section A, page 1, August 30, 1989.

Weisner, L. Introduction to the theory of equations. Macmillan: New York, 1938.

LETTER TO THE EDITOR

Dear Editor,

Prem N. Bajaj, *Pi Mu Epsilon Journal* 9 (1989) 32–33, suggested, that a fallacy in probability exists when the computation of $P(K|Q^c)$ (here K represents the event that a second card is a king and Q^c the event that the first card is not a queen) is made as (4/52)(3/51) + (44/52)(4/51). The first probability in each term represents the conditioning that the first card is or is not a king, <u>noting that the first card is not a queen</u>. Computing this way, a false result is obtained.

The solution proposed is to avoid this fallacy by computing $P(K|Q^c) = P(Q^cK)/P(Q^c)$. However, the true fallacy is that the conditioning is done incorrectly. That is, $P(K_1|Q^c) = 4/48$ and not 4/52. Similarly, $P(K_1^c|Q^c) = 44/48$ and not 44/52. (Here K_1 represents the event that the first card is a king.)

Sincerely,

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THE SYNCOPATED MATHEMATICAL WORKS OF JOSEPH SOLOMON DELMEDIGO

by Sandra M. Pulver
Pace University

Joseph Solomon Delmedigo, Rabbi, **talmudist,** philosopher mathematician, astronomer, and physician, was born in Candia, Crete, on June 16, 1591 to a scholarly family which was well-to-do, intellectually gifted, and which often assumed the leadership of Candia. Besides his religious training, he was given a thorough classical education through tutors who instructed him in Greek, Italian, Latin, Spanish, French, German and Polish **[Barzilay, 4].**

At the age of fifteen, he was accepted into the University of Padua, where he studied mathematics and astronomy under **Galileo (1564-1642),** and also philosophy and medicine.

Like Cardano, he received his medical degree at Padua [Ore, **7]** and practiced medicine in many of the lands of the Near East and Eastern and Central Europe during the course of the following three decades, all the while writing his mathematical and scientific texts — five works under the comprehensive title ***Elim.***

Joseph Solomon Delmedigo (1591-1655) was the author of numerous scientific works, most of them known only from the author's references to them in his own writings, since they are no longer extant. All of his known works were written in the Hebrew language. His main work on mathematics, ***Elim,*** composed of five books which constitute the answers to twelve general and seventy specific queries posed to him by a pupil of his, Zerah ben Natan, has, however, been handed down to us.

He is of historical, mathematical, and educational interest since he was one of the first in the Jewish world to attempt to integrate the new secular scientific knowledge into the religious aspects of Jewish life and culture and to advocate the spread of these new ideas and values among his people.

History of Mathematical Symbolism. Prior to the sixteenth century, the only man who had consciously introduced symbolism to make algebraic writings more effective was Diophantus [Kline, **259].** Even Leonardo of Pisa (Fibonacci) possessed no algebraic symbolism [Cajori, **125].**

In the fifteenth and sixteenth centuries, the common style of scientific writings was still rhetorical. Rhetorical writings are those in which no symbols are used, and everything is written out in words [Kline, **259].**

Mathematicians of that era wrote in literary language, or else used symbols that were arbitrary or insufficient. Indeed, one of the great advances of the mathematical renaissance was the creation of syncopated, and later, symbolic, algebra.

In syncopated algebra, everything is written out in words, except that abbreviations are used for certain frequently occurring operations. In symbolic algebras, all terms and operations are represented by a fully developed algebraic symbolism [Sarton, 38 and Cajori, **111].**

Syncopated algebra formed a natural transition. Instead of describing an equation by means of a long sentence, some kind of shorthand symbolism was employed.

Perhaps the first abbreviations were found in 1494 in the first printed edition of the ***Summa de arithmetica, geometrica, proportion! et proportionalita,*** written by the monk Luca Pacioli (1445-ca. 1517). The algebra was syncopated by the use of ***"p"*** (from ***piu,*** or more) for plus; ***"m"*** (for ***meno,*** or less) for minus; ***"co"*** (from ***cosa,*** or thing) for the unknown, x; ***"ce"*** (from ***censo)*** for x^2; and ***"cece"*** (from ***censo censo)*** for x^4 [Cajori, **128].**

The symbols "+" and "–" were introduced in an arithmetic text, ***Rechenung auff alien Kauffmanschafft*** [Boyer, **308],** published in Leipzig in 1489, by Johann **Widman (b.** ca. 1460). These signs were not generally adopted for some time, however, and the letters "*p*" and "*m*" continued in Italy until the beginning of the seventeenth century.

In Robert Recorde's (1510-58) algebra text, ***The Whetstone of Witte*** (England, **1557),** the modern symbol of equality was used for the first time [Cajori, **140].** The radical sign was introduced in 1525 by Christoff **Rudolff** (b. **1475),** in his algebra text, ***Coss*** [Eves, **217].** The "+"for division was used first by Johann **Heinrich** Rahn **(1622-76),** a Swiss, in his ***Teutsche Algebra,*** thirty years after Delmedigo published ***Elim*** [Cajori, **140].**

François Vieta (**1540-1603),** in his ***Canon mathematicus*** (1579) and ***In artem analyticam isaqoque*** **(1591),** was the first to use letters purposefully and systematically, not just to represent an unknown or powers of an unknown, but as general coefficients, to express an entire class of functions. Vieta used vowels to represent unknowns and consonants to represent positive unknown quantities [Cajori, **139].** (The present custom of using the latter letters of the alphabet for unknowns and early letters for knowns, was introduced by Descartes in 1637.) **Vièta** used the same letter for various powers of a quantity, as ***A, A quandrantum, A cubum,*** (or x, x^2, x^3). Prior to Vihta, different letters were used for the various powers of a quantity [Eves, **223].** Exponents and our equality symbol were not yet in use, but **Vièta** employed "+" as the shorthand symbol for addition and the "–" for subtraction.

Delmedigo's Use of Mathematical Symbolism. Delmedigo's mathematical writings are partly symbolic, mostly rhetorical, not having yet advanced to the use of all symbols employed at his time, but using the same type of symbolism as that of Luca Pacioli over a century before. His expositions are therefore wearying for the reader of today who is accustomed to modern mathematical symbols, and they do not therefore impress the reader with their mathematical strength.

Delmedigo uses the **sign** ᔑ for *toseef* or "add on" in Hebrew, for a positive quantity, **and** ə for *pahot* or "make less" in Hebrew, for a negative quantity, writing his "equations" and algebraic expressions to the right, contrary to Hebrew usage, but according to the convention of the mathematical community.

For example, Delmedigo writes the algebraic expression:

$4x^4 - 12x^3 + 9x^2$ as 4 ᑫ ə 12 ट ᔑ 9 ʔ

His operations of addition, subtraction, and multiplication are syncopated. For example,

$$(5-2)(7-3) \text{ is written as } \begin{matrix} 5 & 2 \\ a & a \\ 7 & 3 \end{matrix}$$

while its solution is given as:

35 ə **29**ᔑ **6,** or + 35 – 29 + 6, that is, 12.

In this exposition, x is written as 1 /c, $1x^2$ as 1ʔ, $1x^3$ as 'ו, and so on, where the ट, ʔ, /c represent the exponent, not the unknown. Delmedigo uses no symbol for the unknown.

The known quantity is marked by means of a ○, (the letter ○ or ***samach*** in

Hebrew) possibly from the Hebrew word ***mispar*** or "number." Between both sides of his equations, Delmedigo writes the word שוה , ***shaveh*** or "equals." He uses no symbol for equality.

Elsewhere, Delmedigo sets up proportions using syncopated notation such as:

$$\overset{2}{\text{Ask}} \text{ is to } \overset{3}{?} \text{ so is } \overset{4}{ꝣ} \text{ to the fourth term, which is } \overset{6}{3}.$$

Most problems and their calculations and solutions, however, are described rhetorically in Delmedigo's work. In his writings Delmedigo explains proportions in similar triangles and the use of the Pythagorean theorem using a completely rhetorical exposition. In addition, Delmedigo illustrates the classical method of finding the approximate square root of a number, known to the Babylonians, namely;

$$\sqrt{a^2 + b} \approx a + \frac{b}{2a},$$

only through words. No symbols are used for the discourse on the square of a binomial, nor for the computation of the sum of successive integers from 1 to n, where he actually employs the formula

$$\text{sum} = \left[\frac{n-1}{2} \cdot n\right] + n.$$

In his exposition on negative numbers, he writes a quadratic equation completely in words, that is "The object was, to discover a number, not equal to two, such that when it is subtracted from its square, the difference would be two." That is, he means

$$(x^2 - x = 2) \cap (x = 2).$$

His discussions on square and cube roots are completely rhetorical. For example, he states "Three times the square root of eight, which is irrational, is the square root of seventy two, and the sum of the cube root of one hundred and sixty two and the cube root of two thousand and fifty eight will be the cube root of six thousand," that is

$$3 \times \sqrt{8} = \sqrt{72}, \text{ and } \sqrt[3]{162} + \sqrt[3]{2058} = \sqrt[3]{6000}.$$

Delmedigo's notation had not advanced to the forefront of the mathematical notation used at his time, thus putting constraints on the clarity and effectiveness of his exposition and making it long-winded and verbose. While an improved symbolism does not directly advance mathematics, it does free scholars from undue preoccupations with mere techniques, and makes possible further advances in theory by simply providing greater clarity, and ease and power of expression.

The greater part of Delmedigo's work is still buried in his one extant published text and has only been investigated by historians of mathematics and astronomy to a slight extent. Much needs to be done to explore the mathematical works of Joseph Solomon Delmedigo which heretofore have not been adequately analyzed to provide an additional body of knowledge with regard to the original contributions of Jewish scholars in the history of mathematics and science.

REFERENCES

Barzilay, Isaac. ***Yoseph Shlomo Delmedigo, Yashar of Candia,*** Leiden: E. S. Brill, 1974.

Boyer, Carl. A ***History of Mathematics,*** New York: John Wiley & Sons, 1968.

Cajori, Florian. ***History of Mathematics.*** New York: The MacMillan Company, 1953.

Eves, Howard. ***An Introduction to the History of Mathematics,*** New York: Holt, Rinehart and Winston, 1963.

Kline, Morris. ***Mathematical Thought from Ancient to Modem Times,*** New York: Oxford University Press, 1972.

Ore, Oystein. ***Cardano, The Gambling Scholar,*** Princeton University Press, 1953.

Sarton, George. ***The Study of the History of Mathematics;*** The Study of the History of Science, 2 vols. bound as one. New York: Dover Publications Inc., 1957.

Award Certificates

Your chapter can make use of the Pi Mu Epsilon Award Certificates available to help you recognize mathematical achievements of your students. Contact Professor Robert Woodside, Secretary-Treasurer.

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PHILIP HALL — A FAMOUS MATHEMATICIAN

by J. L. Brenner
Palo Alto, California

P. Hall (1904-1982) "was a mathematician of great influence." He was an algebraist.

In a Lie Algebra, the "bracket product" [x, y] Is defined as xy − yx, and multiplication is associative but not commutative. In such a structure, the "Jacobi identity"

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$$

holds. Here there are two operations, multiplication and subtraction. In a group, multiplication is the natural operation. Hall defined exponentiation by $c^d := d^{-1}cd$, and used [a, b] to abbreviate the commutator $a^{-1}b^{-1}ab$. He then discovered an analog in groups to the Jacobi identity in Lie algebras, namely

$$[[x, y^{-1}], z]^y \, [[y, z^{-1}], x]^z \, [[z, x^{-1}], y]^x = 1$$

The commutator [a, b] measures the quotient $(ba)^{-1}(ab)$ by which ba "differs" from ab, that is, a and b fail to be commutative. Hall invented an elaborate commutator calculus that he used to investigate the structure of groups.

Hall generalized the Eulerian totient (defined on the group Z/mZ) to investigate group generators. (The group elements $x_1, \ldots, x_s$ "generate" the group **G** if no proper subgroup contains all of them.)

One of Hairs first papers concerns combinatorics, and he may be considered to be the founder of a certain important branch of combinatorics, a branch that is connected with matrix theory through "systems of representatives" and permanents.

All of Hall's papers and lectures were seminal. He wrote about 30 papers, so that he founded about 30 branches of mathematics. A biography and critique of his mathematical work was published in the Bulletin of the London Mathematical Society 16 (1984) 603-626.

Hall's intellectual prowess was impressive. Assigned to a cipher unit (with A. Turing) during the war, he learned Japanese, but not Russian. I have studied and forgotten both, so can certify that Japanese is the more difficult of the two.

Hall was "reticent rather than shy." He was a fine patron to those who sought him out, but he tended not to be outgoing. B. H. Neumann wrote me that Hall and a colleague (not an acquaintance) from Cambridge both attended a conference in Manchester. The colleague gave a talk in which he conjectured the existence of a group with certain properties, but found himself unable to construct an example. Hall immediately rose and gave the relevant example. Although the two men had worked at the same university for years, they had to travel 60 miles to arrange for their mathematics to touch.

Olga Taussky-Todd relates that when she visited Cambridge after the war, she had to introduce Hall to more than one colleague in his own department.

Hall had a fabulous sense of humor, all of it deadpan. When I received an article in Russian to review, I asked Hall if he would take my place, since the subject was in his domain rather than mine. He replied, "I can't do it; Russian is Greek to me."

Another time I asked him for an example of a group G in which the commutator

subgroup H (the smallest subgroup containing all commutators [x, y] in G) contains an element that is not a commutator. He replied that he was retired for a number of years, and thought about groups "only with reluctance." The next day, before he had mailed that denial, he appended a two-page postscript containing the desired example. Incidentally, his letters were meticulously written out in longhand.

REFERENCE

Obituary – Philip Hall, by J. E. Roseblade. Bull London Math Soc 16 (1984) 603-626.

THE AM-GM INEQUALITY: A CALCULUS QUICKIE

by Norman Schaumberger
Bronx Community College

If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers, then

$$a_k \ge k\left(a_1 a_2 \ldots a_k\right)^{\frac{1}{k}} - (k-1)\left(a_1 a_2 \ldots a_{k-1}\right)^{\frac{1}{k-1}} \qquad (*)$$

for k = 2, 3, ..., n and equality holds iff $\left(a_1 a_2 \ldots a_k\right)^{\frac{1}{k}} = \left(a_1 a_2 \ldots a_{k-1}\right)^{\frac{1}{k-1}}$.

If b > 0, the function $f(x) = x^k - kb^{k-1}x$ has an absolute minimum for x > 0 at $x = b$, because $f'(x) = kx^{k-1} - kb^{k-1}$ vanishes for x > 0 iff $x = b$, and $f''(b) = k(k-1)b^{k-2}$ is positive. Hence $x^k - kb^{k-1}x \ge (1-k)b^k$ or $\frac{x^k}{b^{k-1}} \ge kx - (k-1)b$, with equality iff $x = b$. Putting $x = \left(a_1 a_2 \ldots a_k\right)^{\frac{1}{k}}$ and $b = \left(a_1 a_2 \ldots a_{k-1}\right)^{\frac{1}{k-1}}$ gives (*).

To prove that $\frac{1}{n}\left(a_1 + a_2 + \ldots + a_n\right) \ge \left(a_1 a_2 \ldots a_n\right)^{\frac{1}{n}}$, we use (*) to get

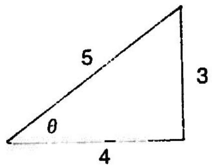
$$\sum_{k=1}^{n} a_k \ge a_1 + \sum_{k=2}^{n}\left[k\left(a_1 a_2 \ldots a_k\right)^{\frac{1}{k}} - (k-1)\left(a_1 a_2 \ldots a_{k-1}\right)^{\frac{1}{k-1}}\right] =$$

$n\left(a_1 a_2 \ldots a_n\right)^{\frac{1}{n}}$. There is equality iff for each k = 2, 3, ..., n, $\left(a_1 a_2 \ldots a_k\right)^{\frac{1}{k}} = \left(a_1 a_2 \ldots a_{k-1}\right)^{\frac{1}{k-1}}$; that is, iff $a_1 = a_2 = \ldots = a_n$.

A RELATIONSHIP BETWEEN THE *3 4 5* RIGHT TRIANGLE *AND* THE GOLDEN RATIO

by Michael Eisenstein
San Antonio, Texas

Theorem. Given the ***3-4-5*** right triangle:



$\theta = \operatorname{Arccos}\frac{4}{5}$ and the **golden** ratio $\varphi = \frac{1+\sqrt{5}}{2}$, then $\tan\left(\frac{\theta + \frac{\pi}{2}}{4}\right) = \frac{1}{\varphi}$

Proof.

$$\tan\left(\frac{\theta + \frac{\pi}{2}}{4}\right) = \frac{\sin\left(\frac{\theta}{2} + \frac{\pi}{4}\right)}{1 + \cos\left(\frac{\theta}{2} + \frac{\pi}{4}\right)} = \frac{\sin\frac{\theta}{2}\frac{\sqrt{2}}{2} + \cos\frac{\theta}{2}\frac{\sqrt{2}}{2}}{1 + \cos\frac{\theta}{2}\frac{\sqrt{2}}{2} - \sin\frac{\theta}{2}\frac{\sqrt{2}}{2}}.$$

Now

$$\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{1-\frac{4}{5}}{2}} = \frac{\sqrt{10}}{10}, \qquad \text{and}$$

$$\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}} = \sqrt{\frac{1+\frac{4}{5}}{2}} = \frac{3\sqrt{10}}{10}.$$

Hence

$$\tan\left(\frac{\theta + \frac{\pi}{2}}{4}\right) = \frac{\frac{\sqrt{10}}{10}\frac{\sqrt{2}}{2} + \frac{3\sqrt{10}}{10}\frac{\sqrt{2}}{2}}{1 + \frac{3\sqrt{10}}{10}\frac{\sqrt{2}}{2} - \frac{\sqrt{10}}{10}\frac{\sqrt{2}}{2}} = \frac{4\sqrt{20}}{20 + 2\sqrt{20}}$$

$$= \frac{2\sqrt{5}}{5+\sqrt{5}} = \frac{2}{\sqrt{5}+1} = \frac{1}{\varphi}.$$

MORE APPLICATIONS OF THE MEAN VALUE THEOREM

by Norman Schaumberger
Bronx Community College

1. A basic integration formula. If $b > 0$ and α is any positive integer, it follows from the definition of the definite integral that

$$\int_0^b x^{\alpha}\,dx = \lim_{n\to\infty} b^{\alpha+1}\left(\frac{1^{\alpha} + 2^{\alpha} + \ldots + n^{\alpha}}{n^{\alpha+1}}\right). \tag{1}$$

A familiar set of textbook exercises use rational formulas for $\sum_{k=1}^{n} k$, $\sum_{k=1}^{n} k^2$ and $\sum_{k=1}^{n} k^3$ to evaluate the right side of ***(1)*** when $\alpha = 1$, 2 and 3, respectively. However, in the general case, the computation of $\int_0^b x^{\alpha}\,dx$ cannot easily be accomplished in this manner. (See **R.** Courant, ***Differential and Integral Calculus.*** Vol. ***1,*** Interscience Publishers, Inc. NY, ***1961,*** pp. ***85 - 86.)***

We use the mean value theorem **(MVT)** for derivatives to obtain the formula

$$\int_0^b x^{\alpha}\,dx = \frac{b^{\alpha+1}}{\alpha+1}$$

where α is any positive integer. Since the MVT is usually presented before the definite integral this approach can be used in a standard calculus course.

If $f(x) = \frac{x^{\alpha+1}}{\alpha+1}$, the MVT gives

$$\frac{k^{\alpha+1}}{\alpha+1} - \frac{(k-1)^{\alpha+1}}{\alpha+1} = c^{\alpha}$$

where $c \in (k-1, k)$. It follows that

$$(k-1)^{\alpha} < \frac{k^{\alpha+1}}{\alpha+1} - \frac{(k-1)^{\alpha+1}}{\alpha+1} < k^{\alpha}.$$

Thus

$$\sum_{k=1}^{n} (k-1)^{\alpha} < \frac{n^{\alpha+1}}{\alpha+1} < \sum_{k=1}^{n} k^{\alpha}$$

or

$$\frac{1}{\alpha+1} < \left(\frac{1^\alpha + 2^\alpha + \dots + n^\alpha}{n^{\alpha+1}}\right) < \frac{1}{\alpha+1} + \frac{1}{n}.$$

Letting $n \to \infty$ gives the desired result.

2. An estimate for a familiar product. Euler's summation formula can be used to show that

$$1^1 \cdot 2^2 \cdot 3^3 \dots n^n \cong A \cdot n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}} e^{-\frac{n^2}{4}} \qquad (2)$$

where A has the value:

$$A = 2^{\frac{1}{36}} \pi^{\frac{1}{6}} e^{\left[\frac{1}{3}\left(-\frac{1}{4}C + \frac{1}{3}S_2 - \frac{1}{4}S_3 + \frac{1}{5}S_4 - \frac{1}{6}S_5 + \dots - \dots\right)\right]}$$

(See K. Knopp, *Theory and Application of Infinite Series,* Blackie & Son Ltd., *1963,* pp. *521, 553, 555.)* Here C is Euler's constant and S_k denotes the sum $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^k}$.

This asymptotic formula is credited to the English mathematician James W. L. Glaisher *(1848 - 1928).*

We show that

$$n^{\frac{n^2}{2}} e^{\frac{-n^2+1}{4}} < 1^1 \cdot 2^2 \cdot 3^3 \dots n^n < n^{\frac{n^2}{2}+n} e^{\frac{-n^2+1}{4}} \qquad (3)$$

Although *(3)* is not as good an estimate as (2), it is easy to derive and it can carry the student a long way.

Example: **Evaluate** $\lim_{n\to\infty}\left(1^1 \cdot 2^2 \cdot 3^3 \dots n^n\right)^{\frac{1}{n^3}}$. *(3)* can be written as

$$n^{\frac{1}{2n}} e^{-\frac{1}{4n}+\frac{1}{4n^3}} < \left(1^1 \cdot 2^2 \cdot 3^3 \dots n^n\right)^{\frac{1}{n^3}} < n^{\frac{1}{2n}} \cdot n^{\frac{1}{n^2}} \cdot e^{-\frac{1}{4n}+\frac{1}{4n^3}}.$$

Since $n^{\frac{1}{2n}}$, $n^{\frac{1}{n^2}}$, and $e^{-\frac{1}{4n}+\frac{1}{4n^3}}$ each approach *1* as $n \to \infty$, it follows that

$$\lim_{n\to\infty}\left(1^1 \cdot 2^2 \cdot 3^3 \dots n^n\right)^{\frac{1}{n^3}} = 1.$$

Using the MVT with $f(x) = \frac{x^2 \log x}{2} - \frac{x^2}{4}$, we have

$$\frac{k^2 \log k}{2} - \frac{k^2}{4} - \left(\frac{(k-1)^2 \log(k-1)}{2} - \frac{(k-1)^2}{4}\right) = c \log c$$

where $k \geq 2$ and $c \in (k-1, k)$. Hence

$$(k-1)\log(k-1) < \left(\frac{k^2 \log k}{2} - \frac{k^2}{4}\right) - \left(\frac{(k-1)^2 \log(k-1)}{2} - \frac{(k-1)^2}{4}\right)$$

$$< k \log k$$

or

$$\sum_{k=2}^{n} \log(k-1)^{k-1} < \frac{n^2 \log n}{2} - \frac{n^2}{4} + \frac{1}{4} < \sum_{k=2}^{n} \log k^k.$$

Consequently

$$\log n^{\frac{n^2}{2}} e^{\frac{-n^2+1}{4}} < \log 2^2 \cdot 3^3 \dots n^n < \log n^{\frac{n^2}{2}+n} e^{\frac{-n^2+1}{4}}$$

which is *(3).*

Editor's Note

For more applications of the mean value theorems by the same author, see "Another Application of the Mean Value Theorems," this *Journal,* Vol. 6, No. *9,* Fall *1978,* page *513;* "Three Familiar Results via the Mean Value Theorem," this ***Journal,*** Vol. 8, No. 4, Spring *1986,* pages *250-1* and "Another Approach to $e^\pi > \pi^e$," this *Journal,* Vol. 8, No. 4, page *251.*

Errata

The Editor extends sincere apologies to **Prof. Subhash** C. Saxena for having transposed pages *20* and *21* of his paper "Some Shortcuts for Finding Absolute Extrema," this *Journal,* Volume *9,* No. *1.* The Editor regrets that the error caused embarrassment to the author.

PUZZLE SECTION

Edited by *Joseph D. E. Konhauser*
Macalester College

The PUZZLE SECTION is for the enjoyment of those readers who are addicted to working doublecrostics or who find an occasional mathematical puzzle or word puzzle attractive. We consider mathematical puzzles to be problems whose solutions consist of answers immediately recognizable as correct by simple observation and requiring little formal proof. Material submitted and not used here will be sent to the Problem Editor if deemed suitable for the PROBLEM DEPARTMENT.

Address all proposed puzzles and puzzle solutions to Professor Joseph D. E. Konhauser, Mathematics and Computer Science Department, Macalester College, St. Paul, MN 55105. Deadlines for puzzles appearing in the Fall Issue will be the next March 15, and for the puzzles in the Spring issue will be the next September 15.

PUZZLES FOR SOLUTION

1. ***Proposed by Alan Wayne, Holiday, Florida.***

(Alphametic "by hand".) Without calculator or computer, solve the base seven alphametic:

ONE x ONE + TWO =THREE.

2. ***Suggested by S. S. A. Case.***

In quadrilateral ABCD, angle A = angle C and AB = CD. Must ABCD be a square?

3. ***Proposed by the Editor.***

Sixteen of the 64 squares of an **8** x **8** grid of squares are chosen so that in each row and in each column there are exactly two squares. Is it possible to color eight of the squares red and the other eight green so that in each row and in each column there will be exactly one of each color?

4. ***Posed in a lecture by Sherman K. Stein, University of California at Davis.***

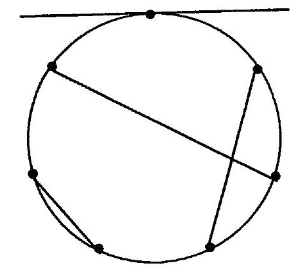
(Monotone matrices.) In the 5 x 5 grid of squares on the next page, the eleven numbers 1, 1, 1, **2, 2, 3, 3, 4, 4, 5, 5** are arranged so that

a. for two filled-in squares in a row, the square further to the right contains the larger entry,

b. for two filled-in squares in a column, the higher square contains the larger entry, and

c. for two filled-in squares which contain the same entry, the square further to the right is higher.

	3		4	5
			1	2
	1	5		
3		4		
1		2		



Given a 6x6 grid of squares, what is the largest number of squares that can be filled with the numbers 1, 2, **3, 4, 5,** 6 subject to the same conditions a, b. and c.?

5. ***Proposed by the Editor.***

In the sketch (above right) a tangent line to the circle is drawn at one of seven equally-spaced points on the circle. The remaining six points are joined by three unequal chords such that no two are parallel and such that none is parallel to the tangent line. Are you able to do likewise with nine equally-spaced points on a circle?

6. ***Proposed by the Edimr.***

Three ***a*** x ***b*** x ***c*** bricks (with ***a/b, b/c*** and ***c/a*** irrational) can be arranged to form a rectangular solid in just three different ways. In how many different ways can **28** ***a*** x ***b*** x ***c*** bricks be arranged to form rectangular solids? How about ***n*** bricks?

7. ***Proposed by the Editor.***

Except for the Puzzle Section, the last bit of copy for this issue of the ***Journal*** was completed on March **30, 1990.** In writing the date as **3130190, I** noticed that **3** x **30** = **90.** The next date for which the product of the number of the month and the date equals **90** will be May **18, 1990.** In which year of the 20th century does the largest number of dates with the "product" property occur?

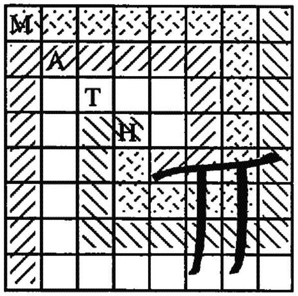
COMMENTS ON PUZZLES 1-7, FALL 1989

TOM KREMER and RICHARD I. HESS submitted the identical solution (the only solution?) shown on the upper left of the next page for Puzzle **#1.** For Puzzle **#2,** CHARLES ASHBACHER, RON GRAMS and RICHARD I. HESS submitted the response **1/sin** a, or its equivalent, for the number of times a cone with generating angle a will revolve about its axis if the cone is rolled on a plane through a complete circle about its apex. For Puzzle #3, CHARLES ASHBACHER submitted

$$9 + 9\cdot9 + 9\cdot9\cdot8 + 9\cdot9\cdot8\cdot7 + \dots + 9\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2 + 9\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1$$

$$= 9\cdot9! \sum_{i=0}^{9} 1/i! = 9[9!e] = 8877690$$

as the number of positive integers with base ten representations consisting of distinct digits **(0** through **9).** Several incorrect responses were received. Only RICHARD **I. HESS** submitted a response to the "timely variation on a familiar theme" in Puzzle **#4.** HESS observed that the numbers in the given array were entries in the "addition" table below

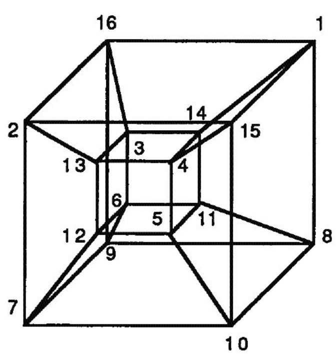


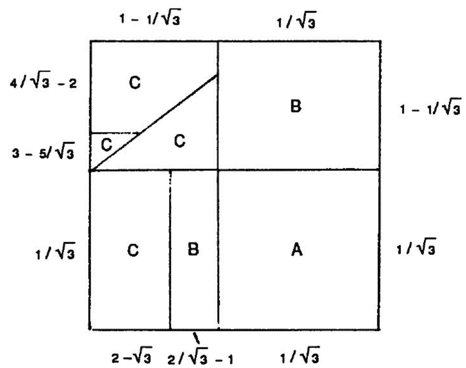
+	0	80	142	–36	284
164	164	224	306	128	448
268	268	348	410	232	552
387	387	467	529	351	671
425	425	505	567	389	709
276	276	356	418	240	560

The word "timely" in the title of the puzzle was meant to be a hint and refers to the unobserved facts that

1. the sum of the numbers in the row and column headings is 1990, and
2. no matter how five numbers are selected from the table, no more than one from any row or column, the sum of the numbers will be 1990.

Other headings can be obtained for the addition table by adding the same number to each column heading and subtracting that number from each row heading. For Puzzle **#5,** ROB FLEMING submitted the solution, below at the left, for the labelling of the vertices of the hypercube so that all **24** faces have vertex sums which are equal. ROB used the numbers 1 through 16 and got the familiar magic constant of **34.** RICHARD **I.** HESS remarked that "this is not the answer you want but you can label each vertex with the same number so that each of the **24** faces have the same sum." The only solution received for Puzzle **#6** was that of RICHARD **I.** HESS, who produced the seven-piece dissection shown below.





In responding to **Puzzle #7,** CHARLES ASHBACHER, ROB FLEMING and RICHARD **I.** HESS agreed that it is possible to publish readers' solutions (two correct from each of the seven contributors) so that exactly one from each of the contributors will appear. FLEMING cited Hall's Marriage Theorem and remarked that solutions to the puzzle exist for any number of solvers.

Editor's Note

The Editor apologizes to Dr. **Theodor** Kaufman, Brooklyn, New York, for having overlooked his solution to Puzzle **# 4** of the Spring 1989 issue. Dr. Kaufrnan submitted

ONE - ORE - ORT - OAT - TAT - TAO - TWO

which is shorter than that published in the Fall 1989 issue.

Solution to Mathacrostic No. 29 (Fall 1989)

WORDS:

A. **Intarsia**
B. Affine Plane
C. Newton's Cradle
D. Smale's Horseshoe
E. Taine
F. Ebb Tide
G. Whiffletree
H. **Awlwort**
I. Rubbersheet
J. Two Cents Worth
K. Dada
L. Out-And-Out
M. Eidos
N. Space Music Man
O. Gaia
P. Outer Content
Q. Dawn
R. **Peano** Continua
S. Liberated Spirits
T. Around The World
U. Yardstick
V. Deep Thought
W. Icarian
X. **Cash Cow**
Y. Euhemerist

AUTHOR AND TITLE: IAN STEWART DOES GOD PLAY DICE?

QUOTATION: **...** we seek not to destroy chaos, but to tame it. In the distant past of our race, nature was considered a capricious creature, and the absence of pattern in the natural world was ascribed to the whims of the powerful and incomprehensible deities who ruled it. Chaos reigned and law was unimaginable.

SOLVERS: THOMAS A. BANCHOFF, Brown University; JEANETTE BICKLEY, St. Louis Community College at Meramec; CHARLES R. DIMINNIE, St. Bonaventure University; ROBERT FORSBERG, Lexington, MA; META HARRSEN, New Hope, PA; MICHELE HEIBERG, Herman, MN; JOAN AND DICK JORDAN, Indianapolis, IN; DR. **THEODOR** KAUFMAN, Brooklyn, **NY;** HENRY S. LIEBERMAN, **Waban,** MA; CHARLOTTE MAINES, Rochester, NY; PHILIP PARKER, Wichita State University; STEPHANIE SLOYAN, Georgian Court College; MICHAEL J. TAYLOR, Indianapolis Power and Light, Co., IN; and ALAN WAYNE, Holiday, FL.

Mathacrostic No. 30

Proposed by Joseph D. E. Konhauser

The 261 letters to be entered in the numbered spaces in the grid will be identical to those in the 26 keyed words at the matching numbers. The key numbers have been entered in the diagram to assist in constructing the solution. When completed, the initial letters of the Words will give the name and an author and the title of a book; the completed grid will be a quotation from that book.

Definitions	Words
A. magnetomotive force equals magnetic flux times reluctance **(2 wds.)**	154 177 3 43 164 214 58 202 87 256 101
B. a unit lor measuring the apparent loudness of a sound	114 96 254 9
C. with a wagon spring, ratchets and imagination, he gave the world a lift **(2 wds.)**	195 42 81 19 152 14 227 157 213 190
D. nickname for rubber substitute with unusual properties, developed in the **1940's** (2 wds.)	91 178 144 204 27 230 161 12 59 175
E. language of the **Grisons** in Switzerland	147 229 184 68 217 31 139
F. out of the way (comp.)	115 176 245 171 74 234 22 8
G. a 36-card game also known as "Catch the Ten" (2 wds.)	109 162 70 242 38 122 6 173 225 143 181
H. harmonious proportion	155 117 125 237 45 200 62 246
I. incidental joke or witticism (theatrical)	128 231 46 49 11 73 54 220 100
J. in science, she was "Among the women of **antiquity what Sappho was in poetry and** what **Aspasia** was in philosophy and eloquence - the chiefest glory of her sex." Mozans **(1913)**	110 221 145 102 168 201 41
K. bleak; unpleasant (comp.)	140 86 72 199 132 21 248 212 218
L. in the Luneberg geometry, their **non-parallel**-ism shows that binocular visual space is **non**-Euclidean **(2 wds.)**	186 88 40 247 191 32 28 163 63 71 106 146 235 260 97 120 129
M. three-dimensional analogue of **Sierpinski's** carpet **(2 wds.)**	94 83 196 153 123 259 185 134 25 51 165 208 7
N. A. K. Dewdney's hypothetical two-dimensional space	78 236 23 211 126 188 174 137 118 75
O. a term used to describe crystals having a well-formed characteristic shape	167 89 182 111 4 80 241 226
P. winding	205 142 47 34 261 216 119 55 238 15
Q. eccentric **(3 wds.)**	10 135 50 131 60 252 219 156 187 24
R. the critical distance at which a body with no tensile strength would be torn **apart by** tidal forces **(2 wds.)**	99 258 233 169 61 33 57 29 82 150 121
S. a topological equivalent of a pinched sphere (2 wds.)	203 151 192 37 104 249 17 67 180 85 2 56 93 222
T. a cinematic movement characterizedby improvisation, abstraction and subjective symbolism (2 wds.)	65 76 257 149 170 194 48
U. technique, devised **by Eudoxus,** exploited with consummate skill by Archimedes	189 77 13 210 52 127 138 243 198 116
V. claptrap (comp.)	253 64 103 26 107 133 244 224
W. the name of an octagon **in Dionys** Burger's Sphereland **(2 wds., 1** abbr.)	141 53 223 113 207 166 92 5
X. initial	193 84 16 1 105 90 39 232 250 159
Y. on a line, the smallest set of points that **contains, for each three of its members, the** harmonic conjugate of each with respect to the other two **(3** wds.)	179 79 148 98 255 66 18 172 251 35 239 206 160 108 197 44
Z. a model of the universe in which both time and space are curved **(2** wds.)	112 183 20 130 95 36 209 158 240 30 136 69 215 124 228

1 X	2 S	3 A		4 O	5 W		6 G	7 M		8 F	9 B	10 Q	11 I	
12 D	13 U	14 C	15 P		16 X	17 S	18 Y	19 C	20 Z	21 K	22 F	23 N	24 Q	
25 M	26 V	27 D	28 L	29 R	30 Z	31 E		32 L	33 R		34 P	35 Y	36 Z	
37 S	38 G	39 X	40 L	41 J	42 C	43 A	44 Y		45 H	46 I	47 P	48 T		49 I
50 Q		51 M	52 U	53 W		54 I	55 P	56 S	57 R	58 A		59 D	60 Q	61 R
	62 H	63 L	64 V	65 T		66 Y	67 S	68 E	69 Z	70 G	71 L	72 K		73 I
74 F	75 N		76 T	77 U	78 N	79 Y	80 O	81 C	82 R	83 M	84 X	85 S	86 K	87 A
	88 L	89 O	90 X	91 D	92 W	93 S	94 M		95 Z	96 B	97 L	98 Y	99 R	100 I
	101 A	102 J	103 V		104 S	105 X	106 L		107 V	108 Y	109 G	110 J	111 O	112 Z
	113 W	114 B	115 F	116 U		117 H	118 N		119 P	120 L		121 R	122 G	123 M
124 Z	125 H	126 N	127 U	128 I	129 L		130 Z	131 Q		132 K	133 V	134 M		135 Q
136 Z	137 N		138 U	139 E	140 K		141 W	142 P	143 G	144 D		145 J	146 L	147 E
148 Y		149 T	150 R	151 S	152 C		153 M	154 A	155 H	156 Q	157 C		158 Z	159 X
160 Y	161 D	162 G	163 L	164 A	165 M	166 W	167 O		168 J	169 R	170 T	171 F		172 Y
173 G	174 N	175 D		176 F	177 A	178 D	179 Y	180 S		181 G	182 O	183 Z	184 E	185 M
186 L	187 Q	188 N	189 U	190 C		191 L	192 S	193 X	194 T	195 C	196 M		197 Y	198 U
	199 K	200 H	201 J	202 A		203 S	204 D	205 P	206 Y	207 W	208 M	209 Z		210 U
211 N	212 K		213 C	214 A		215 Z	216 P	217 E	218 B		219 Q	220 I	221 J	222 S
	223 W	224 V	225 G	226 O	227 C	228 Z	229 E	230 D	231 I	232 X	233 R	234 F	235 L	236 N
237 H		238 P	239 Y	240 Z	241 O	242 G	243 U	244 V	245 F	246 H	247 L	248 K	249 S	
250 X	251 Y	252 Q	253 V		254 B	255 Y		256 A		257 T	258 R	259 M	260 L	261 P

PROBLEM DEPARTMENT

Edited By Clayton W. Dodge
University of Maine

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied By solutions if available and By any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

All communications should be addressed to C. W. Dodge, Math. Dept., University of Maine, Orono, ME 04469. Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly identified with name and address. Solutions to problems in this issue should be mailed by December 15, 1990.

PROBLEMS FOR SOLUTION

719. *Proposed by John M. Howell, Littlerock, California.*

Professor E. P. B. Umbugio translated Problem 626 [Fall 1986, Fall **1987**] into Spanish, as shown below. Since he didn't like zeros because they reminded him of his score on an IQ test, he used **only** the nine nonzero digits. He found solutions in which 2 divides ***DOS,*** 3 divides ***TRES,*** and 6 divides ***SEIS.*** Find that solution in which also 7 divides **both** ***DOS*** and ***SEIS.***

$$\begin{array}{r} UNO \\ DOS \\ \underline{+TRES} \\ SEIS \end{array}$$

720. *Proposed By the Late Charles W. Trigg, San Diego, California.*

In base 4, find two repdyads, one the reverse of the other, whose squares are concatenations of two repdyads. A **repdyad** has the form ***abab...ab.*** For example, a base ten solution is

$$8989^2 = 80802121 \quad\text{and}\quad 9898^2 = 97970404.$$

721. *Proposed By Robert C. Gebhardt, Hopatcong, New Jersey.*

Evaluate the integral

$$\int \frac{b - \cot ax}{1 + b \cot ax}\, dx.$$

722. *Proposed By Robert C. Gebhardt, Hopatcong, New Jersey.*

On Interstate 84 in Connecticut a **roadsign** indicating a route number change reads:

NOTICE
66
IS NOW
322

.

This, of course, is startling news to mathematicians. But consider: in what base would the number 66 equal 322 in what other base?

723. *Proposed by John L. Leonard, University of Arizona, Tucson, Arizona.*

Show that, for any positive integers n and k, the product

$$\left(1 + n\right)\left(1 + \frac{n}{2}\right)\left(1 + \frac{n}{3}\right) \cdots \left(1 + \frac{n}{k}\right)$$

is always an integer.

724. *Proposed By Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.*

Which of the following triangle inequalities, if any, are valid?

$$(1) \qquad \max\{h_a, h_b, h_c\} \ge \min\{m_a, m_b, m_c\},$$

$$(2) \qquad \max\{w_a, w_b, w_c\} \ge \min\{m_a, m_b, m_c\},$$

$$(3) \qquad \min\{w_a, w_b, w_c\} \ge \min\{m_a, m_b, m_c\}.$$

As usual, h_a, m_a, w_a, etc., denote the altitude, median, and angle bisector, respectively, to side a.

725. *Proposed By Seung-Jin Bang, Seoul, Korea.*

Let A, B, C be vectors. Let $||A||$ denote the usual norm of A, and let p and q be real numbers such that $p + q = 1$. Show that

$$||(p^2 + q^2)A + 2pqB + C||^2 - (p^2 + q^2)||A + C||^2 - 2pq\,||B + C||^2$$

is independent of C.

726. *Proposed by Jack Garfunkel, Flushing, New York.*

Given that $x, y, z > 0$ and $x + y + z = 1$, prove that

$$\sqrt[3]{1 + x} + \sqrt[3]{1 + y} + \sqrt[3]{1 + z} \le \sqrt[3]{36}.$$

727. *Proposed by Jack Garfunkel, Flushing, New York.*

If A, B, C are the angles of a triangle ABC, prove that

$$2 + \Pi \cos\frac{B - C}{2} \ge 2\,\Sigma \cos A.$$

728. *Proposed by Dmitry P. Mavlo, Moscow, U.S.S.R.*

The distance between towns ***A*** and ***B*** is 5 km. A straight road passes through town ***A*** and forms the angle $a = \arccos(4/5)$ with the line ***AB***. Two hikers leave town ***A*** at the same time and arrive at town ***B*** simultaneously. The first hiker goes by the direct route at 4 km/hr. The second hiker first travels along the road at 6 km/hr and then turns off the road and goes directly to ***B*** at 4 km/hr. Find the distance traveled by the second hiker.

729. *Proposed by Jack Garfunkel, Flushing, New York.*

Given a non-obtuse triangle ***ABC*** with altitude ***CD*** = ***h***, drawn to side ***AB***, denote the inradii of triangles ***ACD***, ***BCD***, and ***ABC*** by r_1, r_2, and r_3, respectively. Prove that if $r_1 + r_2 + r_3 = h$, then triangle ***ABC*** is a right triangle with right angle at ***C***.

730. *Proposed by R. S. Luthar, University of Wisconsin Center, Janesville, Wisconsin.*

Solve in integers the equation

$$2xy + 13x - 5y - 11 = 4x^3.$$

731. *Proposed by Roger Pinkham, Stevens Institute of Technology, Hoboken, New Jersey.*

a) Show that on the lattice points in the plane having integer coordinates one cannot have the vertices of an equilateral triangle.

*b) What about a tetrahedron in 3-space?

SOLUTIONS

691. [Spring 1989] *Proposed by Charles W. Trigg, San Diego, California.*

Find the smallest possible ***FACE*** on the largest possible ***CUBE*** of this addition alphametric:

$$\begin{array}{r} SIX \\ +\underline{FACE} \\ CUBE \end{array}$$

Solution by Katharine Vance, Hope College, Holland, Michigan.

Clearly, $X = 0$. Now, for the largest ***CUBE***, ***C*** should be 9. Then F must be 8. Now ***U*** cannot be 7 or 6 or 5 since then S or ***A*** would have to be 8 or 9, so $U = 4$. To make ***A*** as small as possible, take $S + A + 1$ (carried from the tens) = 14, and take $A = 6$ and $S = 7$. Since $C = 9$, then ***B*** and ***I*** are consecutive integers. The largest permissible value for ***B*** is thus 2, and $I = 3$. Finally we take $E = 5$ to obtain the solution

$$\begin{array}{r} 730 \\ +\underline{8695} \\ 9425 \end{array}.$$

Also solved by CHARLES ASHBACHER, *Mount Mercy College, Cedar Rapids, IA,* FRANK P. BATTLES, *Massachusetts Maritime Academy, Buzzards Bay,* JAMES E. CAMPBELL, *Indiana University at Bloomington,* WILLIAM CHAU, *Amherst, NY,* VICTOR G. FESER, *University of Mary, Bismarck, ND,* RICHARD I. HESS, *Rancho Palos Verdes, CA,* CARL LIBIS, *Granada Hills, CA,* THOMAS MITCHELL, *Southern Illinois University-Carbondale,* L. J. UPTON, *Mississauga, Ontario, Canada,* KENNETH M. WILKE, *Topeka, KS,* MARK YOUNG and DAVID LADEHOFF, *Drake University, Des Moines, IA, and the* PROPOSER.

Young and Ladehoff found the largest ***CUBE*** for the smallest ***FACE***, obtaining 960 + 1425 = 2385. Hess also found this same solution to be the smallest ***FACE*** on any ***CUBE***. In addition to finding the above solutions, Mitchell found the corresponding base 9 solutions.

692. [Spring 1989] *Proposed by Mohammad K. Azarian, University of Evansville, Evansville, Indiana.*

Solve the equation

$$3(30^x) - 6(15^x) - 3(6^x) + 6(3^x) + 2(5^x) - 10^x + 2^x - 2 = 0.$$

I. *Amalgam of essentially identical solutions by Robert C. Gebhardt, Hopatcong, New Jersey, and Kenneth M. Wilke, Topeka, Kansas.*

The given equation can be rewritten as

$$0 = 2^x 3^{x+1} 5^x - 2(3^{x+1})(5^x) - 2^x 3^{x+1} + 2(3^{x+1}) + 2(5^x) - 2^x 5^x + 2^x - 2$$

$$= 2(2^{x-1} - 1)(3^{x+1} - 1)(5^x - 1).$$

In order that this product be zero, one of its factors must be zero, which requires that one of its exponents be zero. Hence the solutions are $x = -1, 0, 1$.

II. *Comment By Damon L. Hitt, The Johns Hopkins University, Baltimore, Maryland.*

The real solutions to the equation are $x = -1, 0, 1$; but note that there are infinitely many complex solutions

$$x = -1 + 2k\pi i/\ln 3,\ 2k\pi i/\ln 5,\ 1 + 2k\pi i/\ln 2,$$

where k is any integer.

Also solved by JOHN T. ANNULIS, *University of Arkansas-Monticello,* STEVE ASCHER, *McNeil Pharmaceutical, Spring House, PA,* CHARLES ASHBACHER, *Hiawatha, IA,* SEUNG-JIN BANG, *Seoul, Korea,* FRANK P. BATTLES, *Massachusetts Maritime Academy, Buzzards Bay,* BARRY BRUNSON, *Western Kentucky University, Bowling Green,* JAMES E. CAMPBELL, *Indiana University at Bloomington,* WILLIAM CHAU, *Amherst, NY,* DAVID DELSESTO, *North Scituate, RI,* GEORGE P. EVANOVICH, *Saint Peter's College, Jersey City, NJ,* VICTOR G. FESER, *University of Mary, Bismarck ND,* IEM HENG, *Providence College, RI,* RICHARD I. HESS, *Rancho Palos Verdes, CA,* DARREN L. HITT, *The Johns Hopkins University, Baltimore, MD,* FRANK HUBENY, *University of Maine, Orono,* RALPH E. KING, *St. Bonaventure University,* NY, CARL LIBIS, *Granada Hills, CA,* HENRY S. LIEBERMAN, *Waban, MA,* PETER A. LINDSTROM, *North Lake College, Irving, TX,* BRO. HOWARD LOHREY, SM, *Moeller High School, Cincinnati,* Off, CHRISTINE J. MENTELE, *Rockford College, IL,* THOMAS MITCHELL, *Southern Illinois University-Carbondale,* YOSHINOBU MURAYOSHI, *Portland, OR,* LEV S. NAKHAMCHIK, *Willowdale, Ont., Canada,* OXFORD RUNNING CLUB, *University of Mississippi, University,* BOB PRIELIPP, *University of Wisconsin-Oshkosh,* JOHN PUTZ, *Alma College, MI,* HARRY SEDINGER, *St. Bonaventure University, NY,* WADE H. SHERARD, *Furman University; Greenville, SC,* TIMOTHY SIPKA, *Alma College, MI,* BETTY SWIFT, *Cerritos College, Norwalk, CA,* UNDERGRADUATE PROBLEM SOLVING LAB, *University of Arizona, Tucson,* JEB WILSON, *Lafayette, LA,* TARA YANEY, *College Park, MD,* MARK YOUNG, *Drake University, Des Moines, IA, and the* PROPOSER.

693. [Spring **1989**] *Proposed by Barry Brunson,* Western *Kentucky* University, *Bowling Green, Kentucky.*

Solve the equation

$$3 \cdot 2^{\log_x(3x-2)} + 2 \cdot 3^{\log_x(3x-2)} = 5 \cdot 6^{\log_y(3x-2)},$$

where $y = x^2$ (from Problem Book in High School Mathematics. A. **Prilepko,** ed., Mir Publishers, Moscow, 1982, p. 43, partial solution on p. 184).

I. *Solution by Yoshinobu Murayoshi, Portland,* Oregon.

Note that

$$\log_y(3x-2) = \frac{\ln(3x-2)}{2\ln x} = \frac{1}{2}\log_x(3x-2).$$

Let $a = \log_x(3x-2)$ so that $x^a = 3x-2$. Then the given equation can be rewritten as

$$3 \cdot 2^a + 2 \cdot 3^a = 5 \cdot 6^{a/2},$$

which factors into

$$(2^{a/2} - 3^{a/2})(3 \cdot 2^{a/2} - 2 \cdot 3^{a/2}) = 0.$$

Hence $a = 0$ or $a = 2$. Now $1 = x^0 = 3x - 2$ or $x^2 = 3x - 2$, so $x = 1$ or $x = 2$. Since 1 cannot be a base for logarithms, only $x = 2$ is a solution.

II. Comment *by Harry Sedinger,* St. *Bonaventure* University, St. *Bonaventure, New York.*

Usually the function $\log_1 x$ is undefined since its domain would consist of only the number 1. Since $3x - 2 = 1$ when $x = 1$, here $x = 1$ is a plausible solution.

I . Comment 5 the proposer.

The problem is not new. One interesting and rather humbling thing is its source. It is one of a host of similar exercises in this Russian high school problem book.

IV. Comment *by Elizabeth Andy, Limerick, Maine.*

A problem in old high school math
Had a stumbling stone in its path:
Its root extraneous
Proved most **inhumaneous,**
And filled many solvers with wrath.

Also solved by STEVE ASCHER, *McNeil Pharmaceutical,* Spring *House, PA,* SEUNG-JIN BANG, Seoul, *Korea,* FRANK P. BATTLES, Massachusetts *Maritime Academy,* Buzzards Bay, JAMES E. CAMPBELL, *Indiana University* at *Bloomington,* WILLIAM CHAU, *Amherst, NY,* DAVID DELSESTO, *North Scituate, RI,* GEORGE P. EVANOVICH, Saint *Peter's College, Jersey* City, *NJ,* VICTOR G. FESER, University *of Mary, Bismarck, ND,* ROBERT C. GEBHARDT, *Hopatcong, NJ,* RICHARD I. HESS, *Rancho Palos Verdes, CA,* DARREN L. HITT, *The Johns Hopkins* University, *Baltimore, MD,* RALPH E. KING, St. *Bonaventure University,* **Iff,** PETER A. LINDSTROM, *North Lake College, Irving, TX,* BRO. HOWARD LOHREY, SM, *Moeller High School,* Cincinnati, *OH,* CHRISTINE J. MENTELE, *Rockford College, IL,* THOMAS MITCHELL, *Southern Illinois University-Carbondale,* LEV S. NAKHAMCHIK, *Willowdale, Ont.,* Canada, OXFORD RUNNING CLUB, University *of Mississippi, University,* BOB PRIELIPP, *University* of *Wisconsin-Oshkosh,* JOHN PUTZ, *Alma College,* MI, GEORGE W. RAINEY, *Los Angeles, CA,* HARRY SEDINGER, St. *Bonaventure* University, **Iff,** WADE H. SHERARD, *Furman University, Greenville,* SC, TIMOTHY SIPKA, *Alma College,* MI, KENNETH M. **WILKE,** *Topeka, KS, and the* PROPOSER.

Several solvers overlooked the inadmissibility of $x = 1$, but most immediately saw the difficulty when it was pointed out to them that the equation did not have two solutions.

694. [Spring **1989**] *Proposed by Russell Euler, Northwest* Missouri State University, *Maryville, Missouri.*

An ellipse has its foci at the vertices of a hyperbola and its vertices at the hyperbola's foci. Under what conditions, if any, will the ellipse and the hyperbola be orthogonal at their points of intersection?

Solution by George **T.** *Evanovich,* Saint Titer's *College,* Jersey City, *New* Jersey.

It is not possible for the ellipse and the hyperbola to be orthogonal at their intersection points. Consider the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Then the ellipse satisfying the given condition has the equation

$$\frac{x^2}{c^2} + \frac{y^2}{b^2} = 1 \quad \text{where } b^2 = c^2 - a^2.$$

In the first quadrant the hyperbola and ellipse intersect at

$$\left[\frac{ac\sqrt{2}}{a^2 + c^2}, \frac{b^2}{a^2 + c^2}\right],$$

at which point their slopes are $(c/a)\sqrt{2}$ and $-(a/c)\sqrt{2}$, respectively. Since the product of these slopes is -2 and not -1, the curves are not orthogonal.

Also solved by CHARLES ASHBACHER, *Hiawatha, IA,* SEUNG-JIN BANG, Seoul, *Korea,* FRANK P. BATTLES *and* LAURA L. KELLEHER, *Massachusetts Maritime Academy, Buzzards* Bay, JAMES E. CAMPBELL, *Indiana* University at *Bloomington,* DRAKE UNIVERSITY PROBLEM SOLVING GROUP, *Des Moines, IA,* ROBERT C. GEBHARDT, *Hopatcong, NJ,* DARREN L. HITT, *The Johns Hopkins* University, *Baltimore, MD,* HENRY S. LIEBERMAN, *Waban, MA,* LEV S. **NAKHAMCHIK,** *Willowdale,* Ont., Canada, *and the* PROPOSER.

695. [Spring **1989**] *Proposed by Jack Garfunkel, Flushing, New York.*

If ABC is a triangle, prove that

$$\sqrt{\sin A} + \sqrt{\sin B} + \sqrt{\sin C} \geq 6\sqrt{\sqrt{3}\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}.$$

Solution by Murray S. Klamkin, University of *Alberta, Edmonton, Alberta,* Canada.

Since by the arithmetic-geometric mean inequality,

$$\sqrt{\sin A} + \sqrt{\sin B} + \sqrt{\sin C} \geq 3(\sin A \sin B \sin C)^{1/6},$$

it suffices to prove the stronger inequality

$$3(\sin A \sin B \sin C)^{1/6} \geq 6\left(\sqrt{3}\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)^{1/2}.$$

Raising both sides to the 6th power and replacing sin A by $2\sin\frac{A}{2}\cos\frac{A}{2}$, etc., we have to show that

$$\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2} \geq 8\cdot 3^{3/2}\ \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

The latter now follows from the two known inequalities [1]

$$\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2} \geq 3\sqrt{3} \text{ and } 1 \geq 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

There is equality if and only if the triangle is equilateral.

Reference

1. O. Bottema et al, *Geometric Inequalities*, Wolters–Noordhoff, Gronlgen, 1969, 2.12 and 2.42.

Also solved by BOB PRIELIPP, *University of Wisconsin-Oshkosh,* YOSHINOBU MURAYOSHI, *Portland, OR, and the* PROPOSER.

696. [Spring 1989] *Proposed by Robert C. Gebhardt, Hopatcong, New Jersey.*

Consider $\iint xy\sqrt{x^2+y^2}\,dA$, where the integral is taken over the region bounded by the y–axis and the semicircle $x = \sqrt{1-y^2}$ that lies in the first and fourth quadrants.

In rectangular coordinates we have

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}} xy\sqrt{x^2+y^2}\,dx\,dy = \int_{-1}^{1}\frac{1}{3}y(x^2+y^2)^{3/2}\Big|_0^{\sqrt{1-y^2}}dy$$
$$= \frac{1}{3}\int_{-1}^{1} y(1-y^3)\,dy = -\frac{2}{15}.$$

In polar coordinates, however, we have

$$\int_{-\pi/2}^{\pi/2}\int_{0}^{1} r^4\cos\theta\sin\theta\,dr\,d\theta = \frac{1}{5}\int_{-\pi/2}^{\pi/2}\cos\theta\sin\theta\,d\theta = 0.$$

What is the correct answer? Explain the discrepancy.

Solution by Oxford Running Club, University of Mississippi, University, Mississippi.

The correct answer is zero. The discrepancy in the first integral is that

$$\frac{y}{3}(x^2+y^2)^{3/2}\Big|_0^{\sqrt{1-y^2}} = \frac{y}{3}(1-|y|^3)$$

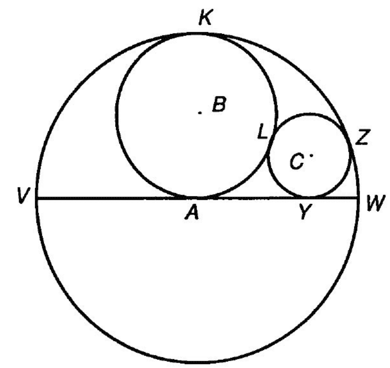
since $(y^2)^{3/2} = |y|^3$. This is a good example to show students why we need the definition

$$|y| = (y^2)^{1/2}.$$

Also solved by SEUNGJIN BANG, *Seoul, Korea,* MARTIN BAZANT, *University of Arizona Problem Solving Lab, Tucson,* CHARLOTTE L BROWN, *University of Arizona, Tucson,* BARRY BRUNSON, *Western Kentucky University, Bowling Green,* JAMES E. CAMPBELL, *Indiana University at Bloomington,* WILLIAM CHAU, *Amherst,* Iff. STEPHEN I. GENDLER, *Clarion University of Pennsylvania,* RICHARD I. HESS, *Rancho Palos Verdes, CA,* RALPH E. KING, *St. Bonaventure University, NY,* HENRY S. LIEBERMAN, *Waban, MA,* PETER A. LINDSTROM, *North Lake College, Irving, TX,* LEV S. NAKHAMCHIK, *Willowdale,* Ont., *Canada*, ROBERT PENDZIMAZ *and* SCOTT BESLIN, *Nicholls State University, Thibodaux, LA,* HARRY SEDINGER, *St. Bonaventure University, NY,* WADE H. SHERARD, *Furman University, Greenville, SC, and the* PROPOSER.

697. [Spring 1989] *Proposed by Keith Goggin, Joseph Puthoff and John Ruebusch, St. Xavier High School, Cincinnati, Ohio.*

Circle (B) is internally tangent to circle (A) at K and to diameter VA W at center A. Circle (C) is internally tangent to circle (A) at Z, externally tangent to circle (B) at L, and tangent to segment AW at Y, as shown in the figure below. Find the ratios of the areas of the three circles to one another. (This problem was adapted from an MAA test.)



Solution by Richard I. Hess, Rancho Palos Verdes, California.

Without loss of generality we take $AV = AW = 1$ for the radius of circle (A). Then $BA = BK = 1/2$ is the radius of circle (B). Let $r = CY = CZ$ be the radius of circle (C). Since $\angle BAY = 90°$,

$$\cos\angle BAC = \sin\angle CAY = CY/AC = r/(1-r).$$

We now apply the law of cosines to triangle ABC to get that

$$BC^2 = AB^2 + AC^2 - 2\cdot AB\cdot AC\cdot\cos\angle BAC,$$

$$(r + \frac{1}{2})^2 = \frac{1}{4} + (1 - r)^2 - (1 - r)r/(1 - r),$$

from which it follows that $r = 1/4$. Therefore the ratio of the areas of the three circles is $1 : 1/4 : 1/16$.

Also solved by FRANK P. BATTLES *and* LAURA L. KELLEHER, *Massachusetts Maritime Academy, Buzzards Bay,* JAMES E. CAMPBELL, *Indiana University at Bloomington,* WILLIAM CHAU, *Amherst, NY,* JACK GARFUNKEL, flushing, *NY,* RALPH E. KING, *St. Bonaventure University,* NY, HENRY S. LIEBERMAN, *Waban, MA,* WADE H. SHERARD, *Furman University, Greenville, SC,* KENNETH M. WILKE, *Topeka, KS, and the* PROPOSER. *Partial solution by* STEPHEN I. GENDLER, *Clarion University of Pennsylvania.*

698. [Spring 1989] *Proposed by John M. Howell, Littlerock, California.*
Find all solutions in integers
a) to the equation

$$2^n = n^x.$$

*b) to the simultaneous inequalities

$$2^n > n^x \quad \text{and} \quad 2^{(n-1)} < (n-1)^x.$$

I. *Solution to Part (a) by James E. Campbell, Indiana University at Bloomington, Bloomington, Indiana.*

If we accept that $0^0 = 1$, then $(n,x) = (0,0)$ is a solution. Assume that $n > 0$. Then

$$n = x \log_2 n, \quad \text{or } x = \frac{n}{\log_2 n}.$$

Since x and n are integers, then $\log_2 n$ must be rational. Now either $\log_2 n$ is integral or it is irrational for integral n. Hence we must have $n = 2^m$ for integral m, so $x = (2^m)/m$, which means that $m = 2^k$ for integral k. Thus we have

$$n = 2^{2^k}, \ x = 2^{2^k - k}$$

where k Is a positive integer in order to make both n and k integers.
Next consider $n < 0$ and let $n = -m$. Then we see that

$$x = -\frac{m}{\log_2 m}, \text{ whence } m = 2^{2^k} \text{ and } x = -2^{2^k - k}.$$

Thus the solutions are $(n,x) = (0,0)$ and $(\pm 2^{2^k}, \pm 2^{2^k - k})$.

II. *Solution to Part (b) by Charles Ashbacher, Hiawatha, Iowa.*

It is easy to check that there are no solutions when $n = 1, 2,$ or -1. Also, when $n = 0$, x can be any even positive integer.
Consider $n > 2$. Since the natural logarithm function is 1-1 and increasing, the two stated inequalities become

$$\frac{(n-1)\ln 2}{\ln(n-1)} < x < \frac{n \ln 2}{\ln n}.$$

Now, for each positive integer x, there is a smallest n that satisfies the right inequality. Except when either side is an integer, that is, except when $n = 2^{2^k}$ or when $n = 2^{2^k} - 1$, both inequalities are satisfied. Some solution pairs are $(x,n) = (3,10), (5,23)$, and $(6,30)$.

For $n < -1$, the right stated inequality demands that x be even. Then for each positive solution (x,n) given above, in which x is even, there is a corresponding solution $(-x, 1-n)$, since the following statements are equivalent:

$$2^{1-n} > (1-n)^{-x}, \quad 2^{n-1} < (1-n)^x$$

$$2^{n-1} < (n-1)^x \quad \text{(since } x \text{ is even)},$$

and these statements are equivalent:

$$2^{(1-n)-1} < ((1-n)-1)^{-x}, \quad 2^{-n} < (-n)^{-x}$$

$$2^n > (-n)^x = n^x \quad \text{(since } x \text{ is even)}.$$

The first few negative solutions are $(x,n) = (-6,-29), (-8,-43)$, and $(-10,-58)$.

Also solved by CHARLES ASHBACHER *(part a also), Hiawatha, IA,* SEUNG-JIN BANG *(part b, partial solution to part a), Seoul, Korea,* FRANK P. BATTLES *(part a), Massachusetts Maritime Academy, Buzzards Bay,* WILLIAM CHAU (both parts), *Amherst, NY,* RICHARD I. HESS *(both parts), Rancho Palos Verdes, CA,* BRO. HOWARD LOHREY, SM *(part a), Moeller High School, Cincinnati, OH,* PROBLEM SOLVING GROUP *(both parts), University of Arizona, Tucson, and the* PROPOSER *(part a and partial solution to part b).*

699. [Spring 1989] *Proposed by Peter A Lindstrom, North Lake College, Irving, Texas.*
Let $A(k)$ be the unpaid principal on a loan of $\$A$ after the kth payment out of n equal payments has been made, so that $A(0) = \$A$ and $A(n) = \$0$. Let i be the interest rate per payment period and I be the total interest paid over the life of the loan. Show that $\sum_{k=0}^{n-1} A(k) = I/i$.

Amalgam of essentially identical solutions by Charles Ashbacher, Hiawatha, Iowa, Frank P. Battles and Laura L. Kelleher, Massachusetts Maritime Academy, Buzzards Bay, Massachusetts, William Chau, Amherst, New York, and Henry S. Lieberman, Waban, Massachusetts.

The amount of interest credited to the kth payment is $i \cdot A(k)$, and so the total interest paid is given by

$$I = \sum_{k=0}^{n-1} i \cdot A(k).$$

Dividing through by i gives the desired result.

Also solved by FRANK P. BATTLES *and* LAURA L. KELLEHER *(second solution), Massachusetts Maritime Academy, Buzzards Bay,* STEPHEN I. GENDLER, *Clarion University of Pennsylvania,* RICHARD I. HESS, *Rancho Palos Verdes, CA,* HENRY S. LIEBERMAN *(second solution), Waban, MA,* ERIK RASMUSSEN, *North American Life & Casualty, Minneapolis, MN,* WADE H. SHERARD, *Furman University, Greenville, SC,* TIMOTHY SIPKA, *Alma College,* *MI, and the* PROPOSER.

700. [Spring **1989**] ***Proposed by R. S. Luthar, University of Wisconsin Center, Janesville, Wisconsin.***

Let a, b, c, p, q, r be any positive numbers satisfying the equation $1/p + 1/q + 1/r = 1$. Prove that

$$\frac{a^p}{p} + \frac{b^q}{q} + \frac{c^r}{r} \ge abc\,.$$

I. *Solution By Norman Schaumberger, Bronx Community College, Bronx, New York.*

Since $1/p + 1/q + 1/r = 1$, it follows from the weighted AM–GM inequality

$$(1) \qquad w_1x_1 + w_2x_2 + \ldots + w_nx_n \ge x_1^{w_1}\, x_2^{w_2} \ldots x_n^{w_n},$$

where $x_i > 0$, $w_i > 0$, and $\Sigma w_i = 1$, that

$$\frac{a^p}{p} + \frac{b^q}{q} + \frac{c^r}{r} \ge (a^p)^{1/p}(b^q)^{1/q}(c^r)^{1/r}.$$

II. *Solution By Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.*

Inequality (1) is a special case of Jensen's inequality for convex functions F, that is

$$\Sigma w_i\, F(x_i) \ge F(\Sigma w_i x_i)\,.$$

Just let $F(t) = -\ln t$

Also solved by SEUNG–JIN BANG, ***Seoul, Korea,*** MURRAY S. KLAMKIN (***second solution***), ***University of Alberta, Canada,*** BOB PRIELIPP, ***University of Wisconsin-Oshkosh, and the PROPOSER.***

701. [Spring 1989] ***Proposed by Dmitry P. Mavlo, Moscow, U.S.S.R.***

Let L and B be nonnegative numbers such that $L\sqrt{3} + 9B = 9\sqrt{3}$. Prove that in any triangle ABC

$$\frac{\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}}{\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}} \ge L\left(\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}\right)^2 + B\left(\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}\right)$$

with equality if and only if the triangle is equilateral.

Solution by Yoshinobu Murayoshi, Portland, Oregon.

Use the known identities

$$\Sigma \tan\frac{A}{2} = \frac{4R + r}{s},\ \Sigma \cot\frac{A}{2} = \frac{s}{r},\ \text{and}\ \Pi \tan\frac{A}{2} = \frac{r}{s}\,,$$

where R, r, and s are the triangle's **circumradius**, inradius, and semiperimeter, to rewrite the stated inequality in the form

$$\frac{4R + r}{s} \div \frac{s}{r} \ge L\left[\frac{r}{s}\right]^2 + B\left[\frac{r}{s}\right],$$

which reduces to

$$4R + r \ge Lr + Bs.$$

Since [1] states that $4R + r \ge \sqrt{3}s$, then we have to show that

$$\sqrt{3}s \ge Lr + Bs.$$

Since $L\sqrt{3} + 9B = 9\sqrt{3}$, the last inequality reduces to the known inequality [1],

$$s^2 \ge 27\, r^2.$$

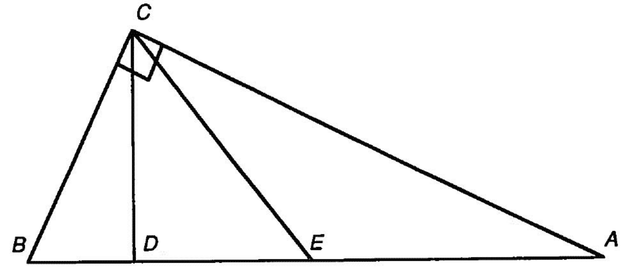
Reference

1. O. Bottema et al, Geometric Inequalities, Wolters–Noordhoff, Gronigen, 1969, *5.5* and 5.11.

Also solved by SEUNG–JIN BANG, ***Seoul, Korea,*** WILLIAM CHAU, ***Amherst, NY,*** JACK GARFUNKEL, ***Flushing, NY,*** MURRAY S. KLAMKIN, ***University of Alberta, Canada,*** BOB PRIELIPP, ***University of Wisconsin-Oshkosh, and the*** PROPOSER.

702. [Spring **1989**] ***Proposed By Dmitry P. Mavlo, Moscow, U.S.S.R.***

In right triangle ABC with right angle at C the altitude CD and the median CE are drawn. Determine the ratio of the sides containing the right angle if $AB = 3 \cdot DE$ (from the SYMP–86 Entrance Exam).



Solution by Wade H. Sherard, Furman University, Greenville, South Carolina.

Since E is the midpoint of hypotenuse AB, then

$$AE = BE = CE = \frac{1}{2}AB, \text{ so } BD = \frac{1}{6}AB \text{ and } AD = \frac{5}{6}AB.$$

Applying the Pythagorean theorem to triangle CDE, we get

$$CD^2 = CE^2 - DE^2 = \left(\frac{1}{2}AB\right)^2 - \left(\frac{1}{3}AB\right)^2 = \frac{5}{36}AB^2\,,$$

so that $CD = (\sqrt{5}/6)\ AB$. Since triangles ADC and CDB are similar, then we have

$$\frac{AC}{BC} = \frac{AD}{BD} = \frac{5}{6}AB \div \left(\frac{\sqrt{5}}{6}AB\right) = \sqrt{5}.$$

Also solved by STEVE ASCHER, *McNeil Pharmaceutical, Spring House, PA,* CHARLES ASHBACHER, *Hiawatha, IA,* JIM BOYD, *St. Christopher's School, Richmond, VA,* WILLIAM CHAU, *Amherst, NY,* JACK GARFUNKEL, *Flushing, NY,* STEPHEN I. GENDLER, *Clarion University of Pennsylvania,* RICHARD I. HESS, *Rancho Palos Verdes, CA,* HENRY S. LIEBERMAN, *Waban, MA,* LEV S. NAKHAMCHIK, *Willowdale, Ont., Canada,* BOB PRIELIPP, *University of Wisconsin-Oshkosh,* UNDERGRADUATE PROBLEM SOLVING LAB, *University of Arizona, Tucson,* KATHARINE VANCE, *Hope College, Holland, MI,* KENNETH M. WILKE, *Topeka, KS, and the* PROPOSER.

703. [Spring 1989] *Proposed by Christopher Stuart, New Mexico State University, Mesilla Park, New Mexico.*

If fix) $\neq 0$ and differentiable, find $\lim_{h\to 0}\left[\frac{f(x+h)}{f(x)}\right]^{1/h}$

Solution by Frank P. Battles, Massachusetts Maritime Academy, Buzzards Bay, Massachusetts.

Let L denote the desired limit. Then

$$\ln L = \ln \lim_{h\to 0}\left[\frac{f(x+h)}{f(x)}\right]^{1/h}$$

$$= \lim_{h\to 0} \ln\left[\frac{f(x+h)}{f(x)}\right]^{1/h}$$

$$= \lim_{h\to 0} \frac{\ln f(x+h) - \ln(f(x))}{h}$$

$$= \frac{d}{dx}\ln f(x) = \frac{f'(x)}{f(x)},$$

so that

$$L = e^{\frac{f'(x)}{f(x)}}$$

Also solved by SEUNG-JIN BANG, *Seoul Korea,* BARRY BRUNSON, *Western Kentucky University, Bowling Green,* WILLIAM CHAU, *Amherst, NY,* DRAKE UNIVERSITY PROBLEM SOLVING GROUP, *Des Moines, IA,* RUSSELL EULER, *Northwest Missouri State University, Maryville,* GEORGE P. EVANOVICH, *St. Peters College, Jersey City, NJ,* STEPHEN I. GENDLER, *Clarion University of Pennsylvania,* IEM HENG, *Providence College, RI,* RICHARD I. HESS, *Rancho Palos Verdes, CA,* HENRY S. LIEBERMAN, *Waban, MA,* PETER A. LINDSTROM, *North Lake College, Irving, TX,* THOMAS MITCHELL, *Southern Illinois University-Carbondale,* OXFORD RUNNING CLUB, *University of Mississippi, University,* BOB PRIELIPP, *University of Wisconsin-Oshkosh,* GEORGE W. RAINEY, *Los Angeles, CA,* WADE H. SHERARD, *Furman University, Greenville, SC,* TIMOTHY SIPKA, *Alma College, MI, and the* PROPOSER. One incorrect solution was received.

GLEANINGS FROM THE CHAPTER REPORTS

CONNECTICUT GAMMA (Fairfield University). During the Fall semester the chapter sponsored the talk "**What Is the Probability that Two Group Elements Commute?**" by **Anthony Gaglione** of the United States Naval Academy. In the spring, the annual initiation ceremony was part of a week-long celebration of the 75th Anniversary of Pi Mu Epsilon. **Robert Decker** of the University of Hartford (CT Beta) began the celebration with his talk on "**Chaos.**" "**Symmetry Revisited**" by **Marjorie Senechal** of Smith College was the title of the Pi Mu Epsilon Lecture during the induction ceremony. The celebration concluded with **Joseph McKenna**'s presentation on "**Swinging Suspension Bridges.**" He represented the CT Alpha chapter at the University of Connecticut at Storrs. During the Annual Arts and Sciences Awards Ceremony, three members, **Lisa Notaroberto**, **Shruti Rajan** and **Lori Schmidt**, received recognition for their outstanding performance in mathematics. Each was given a Pi Mu Epsilon certificate of achievement, *The History of Mathematics* by Boyer and one-year memberships in the Mathematical Association of America.

GEORGIA BETA (Georgia Institute of Technology). At the 1989 Honors Program, books were awarded to **Christopher de Castro**, **Richard Dunlap** and **John Rafter**, outstanding graduates in mathematics. These students received the degree B. S. in Applied Mathematics and had earned a grade point average of at least 3.7 (A = *4.0)* in all mathematics courses taken. Each recipient was given a mathematics book of his choice.

GEORGIA EPSILON (Valdosta State College). At the Fall quarter meeting, **Dr. Barry Flannery**, Department of Mathematics and Computer Science, spoke on "**Common Sense Statistics.**" During the Winter quarter the chapter held its first annual mathematics contest. All students enrolled during the Winter quarter at Valdosta State College were eligible. The winner was **Anna Lewis**. At the Spring quarter meeting, **Charles Row**, a student member of Pi Mu Epsilon, talked on "**PERT/CPM and Applications.**" The talk was followed by the initiation ceremony for ten students and three faculty members.

ILLINOIS IOTA (Elmhurst College). Members of Pi Mu Epsilon participated in the Mathematics/Computer Science/Physics clubs. Four students presented papers at the Associated Colleges of the Chicago Area (ACCA) Student Spring Symposium. One student attended the national meetings of Pi Mu Epsilon in Providence. Chapter members served as ushers/guides at the dedication of the new Computer Science and Technology building on the Elmhurst campus, and, as part of the Elmhurst Mathematics Club, hosted students from Illinois Benedictine College at a Mathematics Department seminar which was followed by discussion and pizza.

KANSAS GAMMA (The Wichita State University). Activities included lectures by **D. foster** on "**Superconductors,**" by **Joel Haack** (Oklahoma State University) on "**mathematical Rspects of Escher's Prints,**" by **Robert Eslinger** (Hendrix College) on "**Rolling Cones: A Case Study in Undergraduate Research,**" by **William Perel** (WSU) on "**Mathematical Invective,**" by **K. Taylor** on "**Seen Any Good Films Lately? An Introduction to Some of the [lotions of Geometric measure Theory**" and by **G. Sarhangi** on "**Study of Linear Singularly Perturbed Systems.**" At the Pi Mu Epsilon National Meetings in Providence, **Dewi T. Saleh** contributed a talk titled "**The Effect of Intra-cluster Correlations on the**

Regression Estimation in the Finite Population Inference." Earlier, **Dewi** had given an expanded version of her talk to chapter members.

MINNESOTA GAMMA (Macalester College). **James Colliander** was awarded the 1989 Mathematics Achievement Prize in "recognition of an outstanding academic record coupled with a demonstrated dedication to and interest in mathematics." At graduation, **Jim** achieved highest honors for his paper "**The marriage of fllgebra and Geometry.**" He presented part of his paper at the National Pi Mu Epsilon Meeting in Boulder. Ezra Camp Prizes were awarded to **Peter Bergstrom** and to **David Scamehorn.** These prizes are "for meritorious performance in mathematics or computer science as a preceptor or intern, activity in Pi Mu Epsilon or the ACM, or some other noteworthy activity in the department." Invited speakers for the Fall semester were **Don. Crowe,** University of Wisconsin, who lectured on art and mathematics, and **Jon. Ward** who spoke on "**An Application of Petri Nets to Distributed Computing: The Occurrence Graph model.**" Guest speaker at the annual initiation banquet in the spring was **Hubert Walczak,** College of St. Thomas, who presented "**The Tan Greatest Theorems of all Time.**" During the Fall semester, the Nova video "The Music of the Spheres," from the series "The Ascent of Man" was shown. Fall and Spring picnics were part of the chapter's social activities, along with a "game" night and a "career" night at the homes of faculty members.

MINNESOTA ZETA (Saint Mary's College). **Paul Weiner** demonstrated what is exciting in the number business when he spoke about perfect numbers, Egyptian fractions, weird numbers and other friends in "**The Factors of a number: Adding It All Up.**" **Michelle Kust,** an 1982 SMC alumna who is currently a member of the technical staff at Bell Labs, spoke on how she uses mathematics and computer science in her job. She also addressed the status of women in industry with backgrounds in mathematics and computer science. **Michelle Schimek** gave the honors presentation at the initiation ceremonies. The title of her talk was the "**Ubiquity of Phi.**" Additional events included a talk by **fir. Jon. Lemke** of the University of Iowa, sponsorship of the presentation of the Brother Louis de La Salle Award to top freshman mathematics students and the showing of the film "Stand and Deliver."

MICHIGAN DELTA (Hope College). Programs and activities sponsored by Pi Mu Epsilon and/or the Mathematics Department included talks by **Professor K. Alan Loper** on "**Continued Fractions,**" "**Prime numbers**" and "**fllgebraic and Transcendental numbers;**" **Professor Tim Pennings** on "**Building a Better Slide or The Brachistochrone Problem**" and "**Fractals; Beauty and mystery Emerge from Chaos;**" **Professor Elliot Tanis** on "**An Escher Expedition: Its History and Participants;**" recent graduate and Pi Mu Epsilon member **David. Kraay,** now at the University of Pennsylvania Wharton School of Business, on "**Optimal Pacing of Trains in Freight Railroads: model Formulation and Solution;**" **Professor Martha Weaver** on "**152001 16221605091916331915241402190224090 9302 and Other methods of Coding;**" **Professor John Van Iwaarden** on "**mailmen, Königsberg's Bridges and Leonhard Euler - A Look at mathematical modeling;**" **Professor Richard Vandervelde** on "Error **Correcting Codes**" and "**Fields, Rings, and Other Things;**" **Professor Jay Treiman** (Western Michigan University) on "**More Than One Derivative;**" **Professor Sam Weaver** on "**Recurrence Relations;**" **Professor Gordon Stegink** on "**Iterated Function Systems and Fractals:**" **Professors frank Sherburne** and **Gordon Stegink** on "**Apollonius meets the Computer:**" Pi Mu Epsilon member **Bruce Brown** on "**The Simplicity of A_n for $n \geq 5$;**" **Scott Schell** on "**Sylow's Theorem**" and Pi Mu Epsilon member **Jeff Toppen** on "**Abstract fllgebra on the Computer.**" Members of Pi Mu Epsilon and the student chapter of the Mathematical Association of America (MAA) visited Haworth Corporation in Holland.

Members of their quality assurance staff outlined several ways in which they use statistics in industry. The officers of Pi Mu Epsilon helped to recruit 26 students and organize a student chapter of the MAA. At the National Pi Mu Epsilon Meeting in Boulder, student **Jeff Van. Eeuwen** presented "**Pseudo-Orbit Shadowing on the Unit Interval.**"

NEW MEXICO ALPHA (New Mexico State University). The chapter hosted its first high school math contest. Called the NMSU Math Challenge, it consisted of an individual competition and a team competition. First prize, a $200 scholarship to NMSU, went to **Abel Jaramillo.** Second prize, a HP 20S scientific calculator, went to **William Skinner.** Third prize, a copy of M. C. *Escher Kaleidocycles* by Schattschneider and Walker was awarded to **Marcy Shoberg.** All were students at Las Cruces High School. At the awards ceremony, **Professor Edward S. Gaughan** spoke on "**mathematics and mysticism.**"

NEW YORK OMEGA (St. Bonaventure University). Lectures given during the schpol year were "**What is Division?**" by **Professor Carl Kohls** (Syracuse University), "**Why 1 + 2 + 4 + ... = -1**" by **Dr. Francis Leary** (SBU), "**The Actuarial Profession from a Strategic Point of View**" by **Selig Erhlich,** FSA, Equitable Life, and "**Continued Fractions**" by **Dr. Marry Sedinger** (SBU). The Myra J. Reed Award was presented to **John P. Holcomb, Jr.** with honorable mention to **Cynthia Lawton** and **Lori Stinebrickner.** The winner was presented with a check for $50 and an award certificate at the induction ceremony, and he received a medal at the Honors Banquet.

NEW YORK PHI (Potsdam College). **Ronald Bousquet** received the 1989 Senior Award of $100 in mathematics books or journal subscriptions. **Ronald** was selected on the basis of mathematical excellence and service to Pi Mu Epsilon and the Mathematics Department and Potsdam College. Major events during the fall included a faculty/student mixer, a guest speaker, the induction ceremony and the holiday party. Speaker was **Dr. Larry Lardy** from Syracuse University who spoke on "**Ill-Posed Problems.**" Keynote speaker at the Fall Induction was **Dr. Patricia. Rogers** from York University in Toronto, Canada. Guest speaker at the Spring Induction ceremony was **Bill Martin.,** an alumnus of Potsdam College. He spoke on "**mathematics as Technology.**"

NORTH CAROLINA LAMBDA (Wake Forest University). At the initiation ceremony in the fall **Dr. Richard Carmichael** (WFU) spoke on "**What do math and Computer Science Teachers Do When They Are Not Teaching?**" The Gentry Lecture was given by **Dr. Lance** Small (University of California, San Diego) on "**Rings and Things.**" Other lectures given during the year included **Jim Kite,** Sara Lee Hosiery Co., on "**Contemporary Data Processing,**" student **Mike Williams** on "**Probabilities in the Game of Risk,**" **Or. timer Hayashi** (WFU) on "**How Fast Can the Nth Fibonacci number be Computed Recursively?**", student **Doug Chatham** on "**A Fibonacci-like Sequence That Converges,**" **Dr. Fred Howard** (WFU) on "**The Five Color Theorem,**" graduate student **Ramona Ranalli** on "**What It's Like to Be a Graduate Student at WFU,**" **Dr. David 'John**(WFU) on "**Synchronization with PVchunk,**" students **Phillip Hansberry** and **Robin Warlick** on "**Student Teaching**" and **Robert Camp,** Reynolds Tobacco Company, on "**Operations Research.**" Students **Michael D. Williams** and **David S. Brown** spoke on their honors papers "**Counting Twin Primes with a Sieve method**" and "**Symmetry: The Art of Science,**" respectively. An April picnic sponsored by Pi Mu Epsilon rounded out the social activities for the year.

OHIO NU (University of Akron). **Robert Carnahan, Brian Gilbey, Jennifer Mowrey** and *Richard Slocum* were awarded one-year memberships in the American Mathematical Society. *Kelly Battle, Marvin Hartzler, David Koler* and *Donald Wakefield* were given one-year memberships in the Association of Computing Machinery. *Scott Dudek, Jeffrey Evanko* and *Miklos Jalics* received one-year memberships in the Society of Industrial and Applied Mathematics. *Gary Griffith* and *Pamela Miller* were awarded one-half year memberships in the American Statistical Association. Samuel Selby Scholarships in the amount of $150 were awarded to *Beth Moore* and *Jennifer Mowrey.* Western Reserve Science Day Winner - Mathematical Sciences Category was *Erik Kangas. Erik* received a $50 savings bond.

OHIO OMICRON (Mount Union College). The chapter sponsored a trip to the 15th annual Pi Mu Epsilon Conference at Miami University. Chapter member *Karl Kauffman* gave a talk entitled **"From Bohr to Schrodinger."** Charles *Hofmeister* spoke on **"Discrete Chaotic Dynamics: An Undergraduate Research Project."** Invited speaker at the initiation ceremony was *Professor Louise Moses,* Computer Science Department, Mount Union.

OHIO ZETA (University of Dayton). Major events were the annual Pi Mi Epsilon Conference at Miami University in Oxford, Ohio, the Mathematical Competition in Modeling 1989 (MCM) and the Spring meeting of the Mathematical Association of America at Ohio State. Four students presented papers at Miami University. *Matt Davison* spoke on **"Some Properties of Pixley-Roy Spaces,"** *Save Diller* discussed **"Properties of Quartics,"** *Colleen Galligher* addressed **"Highly Composite numbers"** and Men-In. *Prenger* presented **"A Missionary's Dilemma."** The team of *Matt Davison, Cave Diller* and *David, Jessup* received highest honors in the MCM. Their paper will appear in the UMAP Journal. Seven students presented papers at the Spring MAA meeting in Columbus, Ohio. Relating to the MCM, *Matt Davison* addressed **"How to Please most of the People Most of the Time,"** *'JohrGeorge* discussed **"Just What Eastern Airlines Ordered"** and *Lisa Tsui* presented **"The Ups and Downs of midges."** *Dave Diller* spoke on **"Self-Complementary Graphs,"** *Nancy Egbers* presented **"The King Chicken Theorem,"** *Colleen Galligher* discussed **"Counting Homomorphisms"** and *Marla Prenger* addressed **"Probability and the Geometric Series."** *Colleen Galligher* received the 1989 Pi Mu Epsilon Award for Excellence in the Sophomore Class.

PENNSYLVANIA BETA (Bucknell University). The chapter sponsored the 17th Professor John Steiner Gold Mathematical Competition. Winning teams were State College Area Senior High School, Freedom High School-Bethlehem and Montoursville Area High School. These schools received plaques. Individual winners were *Susan Goldstine, Tyler Sundred, Tom Weston, Peter Stone* and *Chris Stone. Susan, Tom* and *Chris* are all from the State College Area Senior High School. *Tyler* is from Susquehanna Township High School and *Peter* is from Freedom High School-Bethlehem.

PENNSYLVANIA OMICRON (Moravian College). The Third Annual Moravian College Student Mathematics Conference organized by the Pennsylvania Omicron Chapter of Pi Mu Epsilon was held on Saturday, February 18, 1989 at Moravian College. Twenty colleges and universities in eastern Pennsylvania and New Jersey were represented by the 100 attendees. *Dr. Adi Sen-Israel* from the Rutgers Center for Operations Research (RUTCOR) was the invited keynote speaker. The title of his address was **"Generalized Inverses of matrices: Theory and Applications."** The remainder of the day was devoted to 16 student talks presented by students from Cedar Crest College, Dickinson College, Franklin and Marshall College, Gettysburg College, Kutztown University, Lafayette College, Messiah College, Moravian College, Rutgers University at Newark, the

College of St. Elizabeth and Ursinus College. Financial support for the conference was provided by the Lehigh Valley Association of Independent Colleges (LVAIC) and the Moravian College United Student Government.

SOUTH CAROLINA GAMMA (College of Charleston). Featured speakers included *Dr. Joel Brawley,* Clernson University, on coding theory; *Rob Fowler,* a local meteorologist, on **'math and Physics in Meteorology"** and *Burton Moore,* a local actuary, on actuarial science. The College of Charleston hosted the annual Math Meet in which contestants from high schools in several states competed.

TENNESSEE GAMMA (Middle Tennessee State University). The chapter announced the Pi Mu Epsilon Mathematics Project Award. The purpose of the new award is to promote mathematical and scholarly development of mathematics students at MTSU by encouraging independent study projects culminating in oral presentations. First place winner was *Melanie Butt.* Her project was entitled **"Automorphism Groups of Hasse Subgroup Diagrams for Groups of Low Order."** Second Place winner was *Franklin Mason. Franklin's* project was entitled **"Coxeter Groups and Dynkin Diagrams."** Both projects were presented at the Southeastern Section MAA meeting. Speakers during the year included *Dr. Vatsala Krishnamani* (MTSU) on **"mathematical Ideas in Biology,"** *Dr. Harold Spraker* (MTSU) on **"Paradoxes in mathematics,"** *Dr. Richard Arenstorf* (Vanderbilt) on **"The Three-Body Gravitation Problem"** and *Lora Brewer* (MTSU) on **"Careers in Mathematics."** One meeting was devoted to the movie "Stand and Deliver." Chapter members worked as proctors of the annual Junior High Mathematics Contest which tested 375 area students. The year ended with a combined picnic with members from The Physics Society and ACM.

TEXAS DELTA (Stephen F. Austin State University). The first annual Joe Neal Memorial Book Award was presented to *Patricia Peterson.* Patricia received J. R. Newman's four-volume *The* World of *Mathematics.*

WISCONSIN DELTA (St. Norbert College). In August 1988, *Summer Quimby* and *Katie, Coenen* presented papers at the Pi Mu Epsilon National Meeting in Providence, Rhode Island. Also in attendance was *'Jennifer Spence.* In September, *Paul Koleske, Annette Kunesh, Summer Quimby, Mark Schoenleber, Jennifer Spence* and *Becky Vande Key,* along with *Dr. Richard Poss,* attended the Miami University Regional Conference where *Jennifer* and *Summer* presented papers. In April, *Christine Ferriter, Paul Koleske, Colleen Weyers* and *Becky Vande Hey,* along with *Dr. Poss,* attended the St. John's University Regional Math Conference, where *Becky* and *Christine* presented papers. In April, *Summer Quimby* presented a paper at the meeting of the Wisconsin secion of the MAA. St. Norbert welcomed a variety of speakers during the 1988-89 academic year. Among these were *Dr. Terry Nyman* (UW-Center Fox Valley) speaking on **"The Surreal numbers: Creating Something Out of nothing,"** *Dr. Bob Hollister* (UW-Oshkosh) on **"Winning Take Away Games,"** *Neil Olsen* (Northeast Wisconsin Technical College) on **"Finite State machines"** and *Bob DeGroot* (West DePere High School) on **"A Common Sense Approach to General Math."** Highlights for the 1988-89 year were the 3rd annual High School Math Competition held in conjunction with Sigma Nu Delta (SNC Math Club). The math competition was the largest ever held by St. Norbert College with over 250 participants. The Conference attendance of 92 participants included eight students presenting papers (six of them St. Norbert students). The invited speaker was *Dr. Philip Straffin* (Beloit College) who presented the addresses **"Comparing Voting methods: The axiomatic Approach"** and **"The Geometry of Voting: Spatial models of Voting Power and Voting Outcomes."**

SEVENTEENTH ANNUAL PI MU EPSILON STUDENT CONFERENCE
MIAMI UNIVERSITY, OXFORD, OHIO

Call for student papers and guests
Friday and Saturday, September 28 and 29, 1990
Held in conjunction with
The Conference on Linear Algebra and its Applications
featuring Charles Curtis, Gilbert Strang, and Hans Zassenhaus

WE INVITE YOU TO JOIN US. THERE WILL BE SESSIONS OF THE STUDENT CONFERENCE ON FRIDAY EVENING AND SATURDAY AFTERNOON. FREE OVERNIGHT LODGING WILL BE ARRANGED WITH MIAMI STUDENTS. EACH STUDENT SHOULD BRING A SLEEPING BAG. ALL STUDENT GUESTS ARE INVITED TO A FREE FRIDAY EVENING PIZZA PARTY SUPPER, AND SPEAKERS WILL BE TREATED TO A SATURDAY AFTERNOON PICNIC LUNCH. TALKS MAY BE ON ANY TOPIC RELATED TO MATHEMATICS, STATISTICS, OR COMPUTING. WE WELCOME ITEMS RANGING FROM EXPOSITORY TO RESEARCH, INTERESTING APPLICATIONS, PROBLEMS, SUMMER EMPLOYMENT, ETC. PRESENTATION TIME SHOULD BE FIFTEEN OR THIRTY MINUTES.

We need your title, presentation time (15 or 30 min.), preferred date (Fri. or Sat.) and a 50 (approx.) word abstract by September 21. Please send to Professor Milton D. Cox, Department of Mathematics and Statistics, Miami University, Oxford. Ohio 45056

The Conference on "Linear Algebra and its applications" begins Friday afternoon, September 28. Contact us for more details.

TIME

St. Norbert College
Fifth Annual

Pi Mu Epsilon Regional Conference

November 2-3, 1990

Featured Speaker: Jeanne LaDuke
Depaul University

Sponsored by the St. Norbert College Chapter of $\Pi M E$ in conjunction with the St. Norbert Math club $\Sigma N A$

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