PI MU EPSILON JOURNAL

VOLUME 9 FALL 1990 NUMBER 3

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Τὴν παίδευσιν καὶ τὰ μαθηματικὰ ἐπισπεύδειν

Editorial correspondence, Including books tor review, chapter reports, news items and manuscripts (two copies) should be mailed to PI MU EPSILON JOURNAL, Richard L. Puss, EDITOR, St. Norbert College, De Pere, WI 54115. Students submitting manuscripts are requested to identify their college or university and their class or expected graduation date. Others are requested to provide their affiliation, academic, or otherwise.

Problems for solution *and* solutions to problems should be mailed directly to the PROBLEM EDITOR. Puzzle proposals and puzzle solutions should be mailed to the EDITOR.

The PI MU EPSILON JOURNAL is published at St. Norbert College twice a year–Fall and Spring. One volume c[illegible]s of five years (10 issues) beginning with the Fall 19x4 or Fall 19x9 issue, starting in 1949. For rates, see inside back cover.

Editor's Note

The Pi Mu *Epsilon* Journal was founded In 1949 and Is dedicated to undergraduate and beginning graduate students Interested In mathematics. Submitted articles, announcements, and contributions to the Puzzle Section and Problem Department of the Journal should be directed toward **this** group.

Undergraduates and beginning graduate students are urged to submit papers to the Journal for consideration and possible publication. Student papers are given top priority. Expository articles by professionals In all areas of mathematics are especially welcome. Some guidelines are:

1. papers must be correct and honest
2. most readers of the Pi Mu Epsilon Journal are undergraduates; papers should be directed to them
3. with rare **exceptions,** papers should be of general interest
4. assumed definitions, concepts, theorems, and notations should be part of the average undergraduate curriculum
5. papers should not exceed 10 pages In length
6. figures provided by the author should be camera-ready
7. papers should be submitted In duplicate to the Editor

In each year that at least **five** student papers have been received by the Editor, prizes of $200, $100, and $50, known as Richard V. Andree Awards, are given to student authors. All students who have not yet received a Master's Degree, or higher, are eligible for these prizes.

There are four student papers in this issue of the Journal. The first is "Regular Polygon Targets," by Joel Atkins. Joel prepared the paper with the help of Professor Elton Graves, while he was a student at Rose-Hulman Institute of Technology.

The second paper is "A Family of Fields," by Eric Berg. Eric prepared this paper while he was **still** a student in high school.

The third student paper is "Convergent Ratios of Parallel Recursive Functions," by David Richter. David prepared the paper while he was a freshman at St. Cloud State University.

The final student paper is "More Applications of Full Coverings," by Karen **Klaimon.** Karen prepared this paper, under the supervision of Dr. John Marafino, while a student at James Madison University.

This issue is the first prepared by the newly elected Editor. On behalf of the officers, councilors, and all members of Pi Mu Epsilon, the Editor extends thanks to the Retiring Editor, Joseph D.E. Konhauser, for his outstanding work in editing the Journal during the period Fall, 1984 to Spring, 1990.

REGULAR POLYGON TARGETS

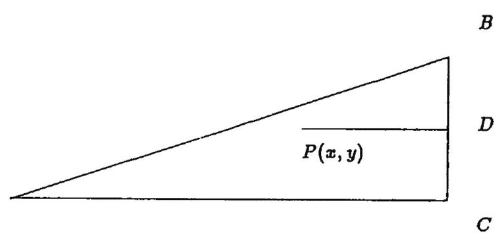
Joel Atkins
Rose-Hulman Institute of Technology

One of the problems on the 1989 William Lowell **Putnam** Mathematical **Competition** was:

> A dart, thrown at random, hits a square target Assuming that any two parts of the target of equal area are equally likely to be hit, find the probability that the point is nearer to the center than to any edge.

(The solution to this problem can be found on pp. 138-9 of the April, 1990, Issue of Mathematics Magazine.)

In this paper we will generalize the **problem** to a regular ngon target. This pmblem can be solved with geometry, trigonometry, calculus, and probability, making **It** an interesting problem for undergraduates. For "any two parts of... equal area [to be] equally likely to be hit" the probability distribution must be uniform, and so we need only find the proportion of the ngon that Is doser to the center than to the edge. By drawing lines from the center of the target to each vertex, we can divide the ngon into n conguent triangies. We can now construct the altitude of each of these triangles, from the center of the n-gon. This altitude connects the center of the ngon and the line segment which Is a side of both the n-gon and the triangle. Thus, we will now have $2n$ triangles. Each of these conguent triangles wiil be of the form shown In the diagram below.



Where $\angle A$ is $2\pi/2n$ and $\angle C$ is $\pi/2$. In each triangle, **A** wiil coincide with the center of the n-gon and $\overline{BC}$ will coincide with part of the perimeter of the n-gon.

Now that we have divided the n-gon into $2n$ triangles of this form. we will notice that any point, $P(x,y)$, in the polygon **will** lie in exactly one of these triangles. (If the point is on the edge of two triangles, we will assume It is in the more clockwise triangle.) From the symmetry of a-regular polygon, we can find the point, D, which is the closest point of the n-gon to $P(x,y)$. The point, D, wiil be on the common edge of the ngon and the triangle which encloses $P(x,y)$. Therefore, the problem is simplified to finding the proportion of the points In ΔABC which are closer **to** A than to $\overline{BC}$.

At this point, we can assume that $|\overline{AC}| = 1$ and $|\overline{BC}| = \tan(\pi/n)$. We can also choose our coordinate system so that **A** is at $(0,0)$ and **C** is at $(1,0)$ (the orientation in the diagram). The distance from a point, $P(x,y)$, to **A** will then be $(x^2 + y^2)^{1/2}$ while the distance from $P(x,y)$ **to** $\overline{BC}$ **will** be 1-x (when $P(x,y)$ is in ΔABC). Thus, the **area** which we want to measure, where $P(x,y)$ is doser to **A** than **to** $\overline{BC}$, wiil be the points which satisfy the inequality:

$$(x^2 + y^2)^{1/2} \le 1 - x \quad \text{or}$$

(1) $$x \le (1 - y^2)/2$$

For $P(x,y)$ to be in **the** ΔABC, it must **be** above $\overline{AC}$ and **below** $\overline{AB}$. This can be expressed by the inequalities:

(2) $$0 \le y \qquad \text{and}$$

(3) $$y\cot(\pi/n) \le x$$

Combining (1) and (3) we obtain the inequality:

(4) $$y\cot(\pi/n) \le (1 - y^2)/2$$

By solving for y, in **(4)**, we find that equality holds when $y = -\cot(\pi/n) \pm \csc(\pi/n)$. Since the **Inequality** Is true when $y = 0$, we know that $y \le \csc(\pi/n) - \cot(\pi/n)$. The inequality In (2) tells us that $0 \le y$. Now using (1) and (3) the area of the triangle which Is closer to A than to $\overline{BC}$ **is:**

$$\int_0^{\csc(\pi/n)-\cot(\pi/n)} \int_{y\cot(\pi/n)}^{(1-y^2)/2} dx\,dy$$

$$= \int_0^{\csc(\pi/n)-\cot(\pi/n)} (1/2-y^2/2-y\cot(\pi/n))\,dy$$

$$= \frac{1-\cos(\pi/n)}{2\sin(\pi/n)} - \frac{(1-\cos(\pi/n))^3}{6\sin^3(\pi/n)} - \frac{(1-\cos(\pi/n))^2\cos(\pi/n)}{2\sin^3(\pi/n)}$$

Dividing this by the area of ΔABC, which is equal to $\tan(\pi/n)/2$, gives a probability of:

$$\frac{(1-\cos(\pi/n))\cos(\pi/n)}{\sin^2(\pi/n)} - \frac{(1-\cos(\pi/n))^3\cos(\pi/n)}{3\sin^4(\pi/n)} - \frac{(1-\cos(\pi/n))^2\cos^2(\pi/n)}{\sin^4(\pi/n)}$$

$$= \frac{\cos(\pi/n)(2+\cos(\pi/n))}{3(1+\cos(\pi/n))^2}$$

For $n = 4$, the original problem, this probability is

$$\frac{1+2\sqrt{2}}{3(1+\sqrt{2})^2} = \frac{4\sqrt{2}-5}{3}$$

In the limiting case of a circle, where n approaches ∞ and π/n approaches 0, our probability approaches

$$\frac{(1)(2+1)}{3(1+1)^2} = \frac{1}{4}$$

This is intuitively correct, since the area we should want is a circle with the same center and half the radius. This limit is reached rapidly, as the probability becomes .22, .243, and .248 when n is 4, 8, and 16 respectively.

There were 2,137 initiates to Pi Mu Epsilon during the **1989-90** academic year. There are now 261 chapters of Pi Mu Epsilon

CONVERGENT RATIOS OF PARALLEL RECURSIVE FUNCTIONS

David Richter
St. Cloud State University

In the study of continued fractions, there is a notable sequence of numbers closely associated with the square root of two. It starts by letting $A_0(0) = 1$ and $A_1(0) = 0$, and continues by defining:

(1) $A_0(n+1) = A_0(n) + 2A_1(n)$, and

(2) $A_1(n+1) = A_1(n) + A_0(n)$,

yielding the following sequences:

n	$A_0(n)$	$A_1(n)$
0	1	0
1	1	1
2	3	2
3	7	5
4	17	12
5	41	29
6	99	70
7	239	169
...	...	...

For any n, take the number in the A_0 column and the number in the A_1 column to be the numerator and denominator respectively. It has been proven that as n Increases, these fractions, called **convergents**, become better approximations for $\sqrt{2}$. That is, $\lim_{n\to\infty} A_0(n)/A_1(n) = \sqrt{2}$.

I first encountered this method in a book, A Long Way from Euclid, by Constance Reid. According to the book, the algorithm was originally developed by the Pythagoreans. It makes one wonder about estimating square roots of other numbers, and one might also ask about estimating cube roots or fourth roots of numbers by modifying and generalizing (1) and (2). Answering these questions is the intent of this paper. The methods used are simply Idealized cases of a more general method found in The *Application* of Continued *Fractions* and *their Generalizations* to Problems In Approximation Theory, by Alexey Khovanskii. There, the author multiplies matrices by a single vector to obtain rational approxmations. That is precisely what I am doing with (1) and (2), but without explicitly using any matrix algebra.

Let us now ask what would happen F, in (1), the coefficient on $A_1(n)$ were any positive real number, say a, so that $A_0(n+1) = A_0(n) + aA_1(n)$, and $A_1(n+1) = A_1(n) + A_0(n)$. Writing down the first few terms, we find:

n	$A_0(n)$	$A_1(n)$
0	1	0
1	1	1
2	a+1	2
3	3a+1	a+3
4	a^2+6a+1	4a+4
5	$5a^2$+10a+1	a^2+10a+5
...	...	...

Note that the binomial coefficients appear as coefficients for various powers of a. In fact, it looks like

$$A_0(n) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n}{2k} a^k \quad A_1(n) = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{2k+1} a^k$$

where [x] denotes the greatest integer not exceeding x, and $\binom{m}{n} = m!/n!(m-n)!$ These conjectures are In fact strengthened if one carefully looks at $(1+\sqrt{a})^n$ for $n \geq 0$. That is,

n	$(1+\sqrt{a})^n$		
0	1		
1	1	+	$\sqrt{a}$
2	a+1	+	$2\sqrt{a}$
3	3a+1	+	$(a+3)\sqrt{a}$
4	a^2+6a+1	+	$(4a+4)\sqrt{a}$
5	$5a^2$+10a+1	+	$(a^2+10a+5)\sqrt{a}$
...	...		

Here, the binomial theorem was applied to expand $(1+\sqrt{a})^n$; then the terms were rearranged and factored to illustrate the fact that the coefficients of $\sqrt{a}$ are identical to the terms In the A_1 column of the sequences generated by (1) and (2). Similarly, the first columns are identical.

At **this point**, we cannot prove that $\lim_{n\to\infty} A_0(n)/A_1(n) = \sqrt{a}$, which is one of the main goals of this paper. However, it will be proven after (1) and (2) have been generalized and a non-recursive expression for $A_j(n)$ has been established.

To **generalize** (1) and **(2)**, we use more than two rules. For example, to find $a^{1/3}$, we let $A_0(0) = 1$, $A_1(0) = 0$, and $A_2(0) = 0$, and define: $A_0(n+1) = A_0(n) + aA_2(n)$,

$A_1(n+1) = A_1(n) + A_0(n)$, and

$A_2(n+1) = A_2(n) + A_1(n)$.

Letting a = 2, for instance, we have

n	$A_0(n)$	$A_1(n)$	$A_2(n)$
0	1	0	0
1	1	1	0
2	1	2	1
3	3	3	3
4	9	6	6
5	21	15	12
6	45	36	27
7	99	81	63
8	225	180	144
...	...	...	...

If this process Is carried out far enough, and one examines $A_0(n)/A_1(n)$ and $A_0(n)/A_2(n)$, then one sees we obtain fair approximations for $2^{1/3}$ and $4^{1/3}$, respectively.

The general case of a for three columns yields:

n	$A_0(n)$	$A_1(n)$	
0	1	0	0
1	1	1	0
2	1	2	1
3	a+1	3	3
4	4a+1	a+4	6
5	10a+1	5a+5	a+10
6	a^2+20a+1	15a+6	6a+15
...	...	...	...

Again, the binomial coefflceints appear to be the coefficients of various powers of a. As before, we see that this is not surprising when we compare the above to $(1+a^{1/3})^n$. For n from 0 to 5, that Is,

n	$(1+a^{1/3})^n$		
0	1		
1	1	$+ a^{1/3}$	
2	1	$+ 2a^{1/3}$	$+ a^{2/3}$
3	a+1	$+ 3a^{1/3}$	$+ 3a^{2/3}$
4	4a+1	$+ (a+4)a^{1/3}$	$+ 6a^{2/3}$
5	10a+1	$+ (5a+5)a^{1/3}$	$+ (a+10)a^{2/3}$

To generalize further, we let the number of columns equal any natural number, say **m.** Then we define $A_0(0) = 1$, $A_j(0) = 0$,

(3) $A_0(n+1) = A_0(n) + aA_{m-1}(n)$, and

(4) $A_j(n+1) = A_{j-1}(n)$,

where $j \in N$, $0 < j < m$. For instance, letting $m = 4$ yields the following definitions for any $j \in \{1,2,3\}$:

$A_0(n+1) = A_0(n) + aA_3(n)$, and $A_j(n+1) = A_j(n) + A_{j-1}(n)$, with the described initial conditions. These yield

n	$A_0(n)$	$A_1(n)$	$A_2(n)$	$A_3(n)$
1	1	0	0	0
1	1	1	0	0
2	1	2	1	0
3	1	3	3	1
4	a+l	4	6	4
5	5a+1	a+5	10	10
6	15+1	6a+6	a+15	20
7	35a+1	21a+7	7a+21	a+35
8	a^2+70a+1	56a+8	28a+28	8a+56
...	...	...	...	...

Once again, the binomial coefficients appear, and they will appear with any number of columns, **m,** so that we can make the following proposition.

Let $j \in N$, $0 \le j < m$ and $n \in N$. Then

$$(5)\quad A_j(n) = \sum_{j=0}^{\left[\frac{n-j}{m}\right]} \binom{n}{mk+j} a^k$$

This simply asserts that with **m** sequences defined, we can find the nth term of any sequence using the nth row of Pascal's Triangle. Since (5) is a simplified form of another form of $A_j(n)$, **it** will be proven when the other form is discussed.

The comparison of $(1+\sqrt{a})^n$ and $A_j(n)$ with 2 columns and the analogous **similarities** between $(1+a^{1/3})^n$ **and** $A_j(n)$ with 3 columns as well as $(1+a^{1/4})^n$ and $A_j(n)$ for 4 columns are merely examples of a general conjecture. Specifically.

$$(6)\quad (1+a^{1/m})^n = \sum_{j=0}^{m-1} a^{j/m} A_j(n)$$

However, there is yet another, more general, identity that encompasses this statement. While ignoring **initial** conditions, the definitions (3) and (4) can be manipulated into independently defined recursive functions, so that for any $m \in N$, and any $j \in N$, $0 \le j < m$,

$$(7)\quad aA_j(n-m) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} A_j(n-k)$$

This was obtained in the following manner. We know, from **(4)**, that $A_j(n+1) = A_j(n) + A_{j-1}(n)$. By substituting n-m for n and $j+1$ for **j**, and subtracting $A_{j+1}(n-m)$ from each side, we have $A_j(n-m) = A_{j+1}(n-m+1) - A_{j+1}(n-m)$. Using (4) again yields

$A_j(n-m) = A_{j+2}(n-m+2) - 2A_{j+2}(n-m+1) + A_{j+2}(n-m)$.

Once more, if (4) is applied for another substitution, we obtain

$A_j(n-m) = A_{j+3}(n-m+3) - 3A_{j+3}(n-m+2) + 3A_{j+3}(n-m+1) - A_{j+3}(n-m)$.

Clearly the values of **m** and j dictate how far this can be carried out. However, once the subscripts on the right side reach **m-1,** (3) can be applied to produce the coefficient of a **in (7),** at which point (4) can be applied again as many times as necessary to obtain (7)

Now, suppose there is a non-recursive function, $A_j(n)$, that satisfies **(3), (4),** and hence (7). To satisfy **(7),** this function can obviously take on the form $A_j(n) = cx^n$, where c is some scalar, n is any natural number, and x is a root of the characteristic equation.

$$(8)\quad a = \sum_{k=0}^{m} (-1)^k \binom{m}{k} x^{m-k} = (x-1)^m$$

Upon examination, it will be seen that this equation is simply a polynomlal **equation** with the same coefficients as of (7). On the right side of this equation **I** have the polynomial factored so that after substractlng a from both **sides,** it should be clear that the **m** roots of this equation can be summarized by $x_k = 1+\omega^k a^{1/m}$, where $k \in N$, $0 \le k < m$ and $\omega = e^{2\pi i/m}$, a primitive mth root of unity.

It turns out that for any x_k,

$$(9)\quad x_k^n = \sum_{j=0}^{m-1} \omega^{jk} a^{j/m} A_j(n),$$

where $n \in N$. The case where $k = 0$ has already been expressed, **(6),** and Illustrated form $= 2$ and $m = 3$. To prove **(9), I'll** use induction on n:

i. For $n = 0$,

$$(10)\quad x_k^0 = \sum_{j=0}^{m-1} \omega^{jk} a^{j/m} A_j(0).$$

Since $A_j(0) = 0$ for all $j \in \mathbf{N}$, $0 < j < m$ and $A_0(0) = 1$, we can write $x_k^{\,0} = \omega^{0k}a^{0/m}$, but $\omega \neq 0$ and $a \neq 0$, so (10) holds for $n = 0$.

II. Assume (9) is true for some value **n;** then we need to show that the following is true as well:

$$(11)\quad x_k^{n+1} = \sum_{j=0}^{m-1} \omega^{jk} a^{j/m} A_j(n+1)\,.$$

This can be accomplished in the following manner

$$x_k^{n+1} = x_k x_k^n = (1+\omega^{k} a^{1/m}) \sum_{j=0}^{m-1} \omega^{jk} a^{j/m} A_j(n)$$

$$= \sum_{j=0}^{m-1} \omega^{jk} a^{j/k} A_j(n) + \sum_{j=1}^{m} \omega^{jk} a^{j/m} A_{j-1}(n)$$

Changing the indices of the second summand, we have

$$x_k^{n+1} = \sum_{j=0}^{m-1} \omega^{jk} a^{j/m} A_j(n) + \sum_{j=1}^{m} \omega^{jk} a^{j/m} A_{j-1}(n)$$

Now, if we apply (4), we get

$$x_k^{n+1} = A_0(n) + aA_{m-1}(n) + \sum_{j=1}^{m-1} \omega^{jk} a^{j/m} A_j(n+1)$$

From (3), we know $A_0(n) + aA_{m-1}(n) = A_0(n+1)$, so

$$x_k^{n+1} = \sum_{j=0}^{m-1} \omega^{jk} a^{j/k} A_j(n+1)$$

which is identical to (11). Since (9) is true implies (11) Is true. by the induction principle. (9) must be true for any n.

From (9), a closed form of $A_j(n)$ may be derived. That Is, we can show that for any $n \in \mathbf{N}$, **and for any** $j \in \mathbf{N}$, $0 \le j < m$,

$$(12)\quad A_j(n) = \sum_{k=0}^{m-1} \frac{\omega^{-jk} x_k^n}{m a^{j/m}}$$

We illustrate how this is obtained by using the following table when m = 5.

$$x_0^{\,n} = \omega^0 A_0(n) + \omega^0 A_1(n) a^{1/5} + \omega^0 A_2(n) a^{2/5} + \omega^0 A_3(n) a^{3/5} + \omega^0 A_4(n) a^{4/5}$$

$$x_1^{\,n} = \omega^0 A_0(n) + \omega^1 A_1(n) a^{1/5} + \omega^2 A_2(n) a^{2/5} + \omega^3 A_3(n) a^{3/5} + \omega^4 A_4(n) a^{4/5}.$$

$$x_2^{\,n} = \omega^0 A_0(n) + \omega^2 A_1(n) a^{1/5} + \omega^4 A_2(n) a^{2/5} + \omega^6 A_3(n) a^{3/5} + \omega^8 A_4(n) a^{4/5}.$$

$$x_3^{\,n} = \omega^0 A_0(n) + \omega^3 A_1(n) a^{1/5} + \omega^6 A_2(n) a^{2/5} + \omega^9 A_3(n) a^{3/5} + \omega^{12} A_4(n) a^{4/5}.$$

$$x_4^{\,n} = \omega^0 A_0(n) + \omega^4 A_1(n) a^{1/5} + \omega^8 A_2(n) a^{2/5} + \omega^{12} A_3(n) a^{3/5} + \omega^{16} A_4(n) a^{4/5}.$$

For any **j**, $A_j(n)$ can be isolated by adding each $x_k^{\,n}$ divided by a certain ω^{jk} necessary to produce real coefficients on $A_j(n)$. For example, with j = 3, we notice that $A_3(n)$ appears in each of these sums; however, **its** coefficient varies from sum to sum depending on what $x_k^{\,n}$ at which we are looking. If we divide each sum by ω^{3k}, then total, we will obtain $5A_3(n)a^{3/5}$ as one of the terms, and the other $A_j(n)$**'s** will drop out since their coefficients are primitive roots of unity whose sums total zero. They are zero because in the complex plane, these terms represent unit vectors that when laid tip-to-tail constitute the edges of a regular m-gon. Since this is a dosed path, the sum Is zero. (12) expresses this method for any **m** and **j**. However, at this point (12) Is only conjectural, so **I** will prove it, again using induction on n.

I. To prove this for n = 0, I want to consider two possibilities, one when j = 0, and the other when j > 0. With j = 0, we have to make sure $A_0(0) = 1$. If we use (12) with n = 0 and **j** = 0, we have

$$\sum_{k=0}^{m-1} \frac{\omega^{-0k} x_k^0}{m a^{0/m}} = \sum_{k=0}^{m-1} \frac{1}{m} = 1,$$

so (12) works for **j** = 0. With j > 0, we should get $A_j(0) = 0$, by definition, and indeed,

$$\sum_{k=0}^{m-1} \frac{\omega^{-jk} x_k^0}{m a^{j/m}} = \sum_{k=0}^{m-1} \frac{\omega^{-jk}}{m a^{j/m}} = 0$$

I have already slated why this sum is zero.

II. Suppose (12) is true for n. Then it can be applied to yield $A_j(n+1)$: $A_j(n+1) = A_j(n) + A_{j-1}(n)$

$$= \sum_{k=0}^{m-1} \frac{\omega^{-jk}x_k^n}{ma^{j/k}} + \sum_{k=0}^{m-1} \frac{\omega^{-(j-1)k}x_k^n}{ma^{(j-1)/m}}$$

Collecting terms, we get

$$A_j(n+1) = \sum_{k=0}^{m-1} \frac{\omega^{-jk}x_k^n+\omega^{-jk+k}a1/mx_k^n}{ma^{j/m}}$$

$$= \sum_{k=0}^{m-1} \frac{\omega^{-jk}(1+\omega^{k}a^{1/m})\,x_k^n}{ma^{j/m}} = \sum_{k=0}^{m-1} \frac{\omega^{-jk}x_k^{n+1}}{ma^{j/m}}$$

A similar argument will show that (12) **holds** for j = **0** as well. Thus, we obtain a closed form for $A_j(n)$. With a little algebra. (12) can be made to look like **(5),** so that (5) now holds true.

Looking back on the **unproven** limit **in** the **introduction,** we can now prove that for any real a greater than zero, and for any j, $0 < j < m$,

$$(13)\quad \lim_{n\to\infty} \frac{A_0(n)}{A_j(n)} = a^{j/m}$$

for **m** defined sequences, and that for any j, $0 \leq j < m$

$$(14)\quad \lim_{n\to\infty} \frac{a^{j/m}A_j(n)}{(1+a^{1/m})^n} = \frac{1}{m}$$

Actually, (13) can be found to be dependent on (14) by substituting the closed form of $A_j(n)$ so that

$$\frac{A_0(n)}{A_j(n)} = \frac{\sum_{k=0}^{m-1} x_k^n}{\sum_{k=0}^{m-1} \omega^{-jk}x_k^n}\,a^{j/m} = \frac{1 + \sum_{k=1}^{m-1} \frac{x_k^n}{x_0^n}}{1 + \sum_{k=1}^{m-1} \frac{\omega^{-jk}x_k^n}{x_0^n}}\,a^{j/m}$$

Therefore, **I will simply** prove (14). (14) can be rewritten as

$$\lim_{n\to\infty} \frac{ma^{j/m}A_j(n)}{x_0^n} = 1$$

Substituting the closed form for $A_j(n)$, **I** assert

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{m-1} \omega^{-jk}x_k^n}{x_0^n} = 1$$

But $\omega^{-0k} = 1$, so **subtracting** x_0^n/x_0^n from both sides, we need

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{m-1} \omega^{-jk}x_k^n}{x_0^n} = 0$$

Now, looking at Individual terms of this sum, we see

$$\frac{\omega^{-jk}x_k^n}{x_0^n} = \omega^{-jk}\left(\frac{x_k}{x_0}\right)^n = \omega^{-jk}\left(\frac{1+\omega^{k}a^{1/m}}{1+a^{1/m}}\right)^n$$

The absolute value of this expression is

$$\left|\omega^{-jk}\left(\frac{1+\omega^{k}a^{1/m}}{1+a^{1/m}}\right)^n\right| = \sqrt{\frac{1+2a^{1/m}\cos\left(\frac{2\pi k}{m}\right)+a^{2/m}}{1+2a^{1/m}+a^{2/m}}}^{\,n}$$

which is clearly less than 1 for any $k \in \mathbf{N}$, $0 < k < m$. Thus

$$\lim_{n\to\infty} \frac{\omega^{-jk}x_k^n}{x_0^n} = 0$$

and (13) and (14) hold. In other words, the expansion of $(1+a^{1/m})^n$ yields rational approximations for $a^{j/m}$. Unfortunately, the convergence is relatively slow, compared to methods **Khovanskii** discusses.

In any case, what was presented here represents an introduction to further generalizations of all of the definitions and theorems in this paper. For instance, all of the initial conditions **I** used were used because they yield convenient results; one might wonder what would happen if other, possibly complex, initial conditions are utilized. All of the recursive definitions can be generalized in the same way that the binomial theorem can be generalized to produce coefficients of powers of polynomials, so that is yet another line to pursue.

A Family of Fields

Eric Berg

In this paper we utilize properties of logarithms and exponents to recursively define a family of fields beginning with the real numbers.

A **field** is a set F together with **two** binary operations on F, usually called addition and multiplication and denoted by $+$ and $\times$, having the following properties:

1) Associativity

For all a,b,c in F, $(a + b) + c = a + (b + c)$

and $(a \times b) \times c = a \times (b \times c)$.

2) Commutativity

For all a,b in F, $a + b = b + a$ and $a \times b = b \times a$.

3) Existence of identities

There are elements z and i in F such that for all a in F,

$a + z = a$ and $a \times i = a$.

4) Existence of inverses

For all a,b in F, $b \neq z$, there are elements a' and b' in F such that $a + a' = z$ and

$b \times b' = i$.

5) Distributivity

For a,b,c in F, $a \times (b + c) = (a \times b) + (a \times c)$.

Three familiar examples of fields are the rational numbers, the real numbers, and the complex numbers with the usual addition and multiplication. Less familiar examples are sets of the form $\{0,1,2...p-1\}$ where p is a prime integer and addition and multiplication are done modulo p.

Our construction of a recursive family of fields is motivated by the observation that for any pair of positive real numbers, a and b, multiplication and addition are related by the condition $\ln(a \times b) = \ln a + \ln b$. We begin by defining F_0 as the field of real numbers with the operations ϑ_0 and ϑ_1 as addition and multiplication, respectively. Thus, our condition above becomes $\ln(a\, \vartheta_1\, b) = (\ln a)\, \vartheta_0\, (\ln b)$ for $a,b > 0$. For our next field F_1 we want ϑ_1 to play the role of addition and a new operation ϑ_2 to satisfy the condition $\ln(a\, \vartheta_2\, b) = (\ln a)\, \vartheta_1\, (\ln b)$. Expressed in terms of ordinary multiplication of real numbers, $\ln(a\, \vartheta_2\, b) = (\ln a)\, \vartheta_1\, (\ln b)$ becomes $a\, \vartheta_2\, b = \exp(\ln a \times \ln b)$. This is our definition of ϑ_2.

Is $F_1 = \{x \mid x > 0, x \text{ real}\}$ a field under ϑ_1 and ϑ_2? Yes. The definition of ϑ_2 shows that it is a binary operation on F_1. Since ϑ_1 is associative and commutative, so is ϑ_2. Direct calculations reveal that the identity for ϑ_2 is e and the inverse of any element a under ϑ_2 is $\exp(1/\ln a)$ with the exception of $a = 1$, the identity element under ϑ_1. To verify that ϑ_2 is distributive over ϑ_1 observe that

$$\begin{aligned} a\, \vartheta_2\, (b\, \vartheta_1\, c) &= \exp(\ln a \times \ln(b\, \vartheta_1\, c)) = \exp(\ln a \times \ln(b \times c)) \\ &= \exp(\ln a \times (\ln b + \ln c)) \\ &= \exp((\ln a \times \ln b) + (\ln a \times \ln c)) \\ &= \exp((\ln a \times \ln b) \times \exp(\ln a \times \ln c) \\ &= (a\, \vartheta_2\, b)\, \vartheta_1\, (a\, \vartheta_2\, c) \end{aligned}$$

This completes the proof that F_1 is a field under ϑ_1 and 0

To describe the general situation, let i_k denote the identity element for the operation ϑ_k. Then for $k \geq 1$ we define $F_k = \{x \mid x > i_{k-1}, x \text{ real}\}$ and for a,b in F_k we define $a\, \vartheta_{k+1}\, b = \exp(a\, \vartheta_k\, b)$. The table below provides details for F_0, F_1, and F_2.

i	F_i	Operations	Identities	Inverses
0	reals	$a + b$; $a \times b$	0; 1	$-a$, $1/a$
1	$\{x > 0\}$	$a \times b$; $\exp(\ln a \times \ln b)$	1; e	$1/a$; $\exp(1/\ln a)$
2	$\{x > 1\}$	$\exp(\ln a \times \ln b)$; $\exp(\exp(\ln(\ln a) \times \ln(\ln b)))$	e; $\exp(e)$	$\exp(1/\ln a)$; $\exp(\exp(1/\ln(\ln a)))$

Finally, we close with a question. Addition of positive real numbers has the geometric interpretation that it is the sum of lengths. Multiplication of positive real numbers can be interpreted as the area of a rectangle. Does ϑ_k have a geometric interpretation for $k > 1$?

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MORE APPLICATIONS OF FULL COVERING

Karen Klaimon
James Madison University

Using the concept of a full cover, one can unify in style and structure many proofs in analysis. The method is similar to proofs using compactness; however, a full cover argument results in finite covering by intervals which do not overlap except at the end points. Furthermore, the technique is more accessible to undergraduate students. In [1] the following theorems are proven by using a full covering argument: if $f(x)$ Is continuous on $[a,b]$ then f is bounded on $[a,b]$, the intermediate **Value** Theorem, if $f(x)$ is continuous on $[a,b]$ then f is uniformly continuous on $[a,b]$, the **Heine-Borel** Theorm and the **Bolzano-Weierstrass** Theorem. The purpose of this paper is to extend the application of full covering. We shall use the full cover definition and Thomson's lemma to prove: the **Max/Min** Theorem, Rolle's Theorem, **Dini's** Theorem, if the derivative of a function is zero on an interval then the function is constant, and if the derivative of a function is positive on an interval then the function is increasing. You will notice that the proofs of these theorems are not necessarily simplified, just similar in form. We now state the definition of a full cover as given in [1]:

Definition: Let $[a,b]$ be a given closed, bounded interval. A collection C of dosed subintervals of $[a,b]$ is a full cover of $[a,b]$ if, for each x ϵ $[a,b]$, there corresponds a number $\delta > 0$ such that every closed subinterval of $[a,b]$ that contains x and has length less that $\delta(x)$ belongs to C.

The following lemma is central to full covering arguments. It has been proved in both [1] and [2]; therefore, we will not display the proof In this paper.

Thompson's Lemma: If C is a full cover of $[a,b]$, then C contains a partition of $[a,b]$; i.e., there is a partition of $[a,b]$ all of whose subintervals belong to C.

The full covering technique involves defining a class C of subintervals having a local property and using Thompson's Lemma to extend the property to $[a,b]$.

Theorem: If $f(x)$ is a continuous function on $[a,b]$, then there exist points $m,n \in [a,b]$ such that $f(m) \geq f(x)$ for all x ϵ $[a,b]$, and $f(n) \leq f(x)$ for all x ϵ $[a,b]$.

Proof: We prove that a maximum value exists. The proof of the existence of a minimum value is similar and we omit It. From [1] we know that a continuous function on a closed interval is bounded. Let B be the least upper bound of $\{f(x) \mid x \in [a,b]\}$. By definition, $B \geq f(x)$ for all x $\in$ $[a,b]$. <u>Claim</u>: There exists an m ϵ $[a,b]$ such that $f(m) = B$. Suppose the claim is false, then $B > f(x)$ for all x ϵ $[a,b]$. Let

C = $\{I \mid I$ is a closed subinterval in $[a,b]$, and there exists $\epsilon_1 > 0$ such that $f(x) < B - \epsilon_1$ for all $x \in I\}$

Let $B - f(x) = N_x$ at a point x. Let $\epsilon = N_x/2$. By continuity, there exists $d = \delta(\epsilon,x)$ such that $|x - y| < \delta(\epsilon,x)$, it follows that $f(y) < f(x) + \epsilon = (B - N_x) + \epsilon = B - N_x$. Thus $f(y) < B - \epsilon$. Now let J be any dosed subinterval of $[a,b]$ containing x with $|J| < d$. Then for all $y \in J$, $f(y) < B - \epsilon_j$ where $\epsilon_j = N_x/2$. Thus J is in C and C is a full cover of $[a,b]$. Using Thompson's lemma, C contains a partition of $[a,b]$; that is, there exist $a < p_0 < p_1 < p_2 < \ldots < p_n = b$ such that $[p_{k-1},p_k]$ is an element of C for $k = 1,\ldots n$ and $[a,b] = \cup I_k = [p_{k-1},p_k]$. Thus on each I_k there exists ϵ_k such that $f(x) < B - \epsilon_k$. Let $\underline{\epsilon} = \min\{\epsilon_k \mid k = 1,\ldots n\}$. Let x be an element of $[a,b]$. Thus x Is in I_k for some $k = 1,\ldots n$, and so $f(x) < B - \epsilon_k < B - \underline{\epsilon} < B$. Since B is the least upper bound of $\{f(x) \mid x \in [a,b]\}$ and we have displayed a $Q = B - \underline{\epsilon}$ such that $Q < B$ and $Q > f(x)$ for all $x \in [a,b]$, we have a contradiction [3, p. 17]. Thus, $B = f(m)$ for some $m \in [a,b]$.

We will now use this technique to prove Rolle's Theorem. In proving **Rolle's** Theorem, we use the following result:

Theorem: If $f(x)$ is continuous on a dosed Interval $[a,b]$ and differentiable on (a,b) and $f(x)$ assumes either **its** maximum or minimum value at the interior point m of the interval then $f'(m) = 0$.

The proof of this theorem is not difficult and the theorem is typically needed in the proof of **Rolle's** Theorem [3, p. 75].

Roiie's Theorem: If $f(x)$ is continuous on the closed interval $[a,b]$, if $f(a) = f(b) = 0$, and if $f(x)$ is differentiable on the open interval (a,b), then there Is some point t of the open interval (a,b) such that $f'(t) = 0$.

Proof: Suppose no such point t exists on (a,b) such that $f'(t) = 0$. Let

C = $\{I \mid I$ is a dosed subinterval of $[a,b]$, and there exists a t which is an element of I such that for all $x,y \in I$, where $x < t < y$, either $f(x) < f(t) < f(y)$ or $f(x) > f(t) > f(y)\}$

Let s be an element of $[a,b]$. If $f'(s) > 0$, then there exists a $\delta_1(s) > 0$ such that if $0 < |h| < \delta_1(s)$, then $[f(s+h) - f(s)]/h > 0$. If $f'(s) < 0$ there exists a $\delta_2(s) > 0$ such that If $0 < |h| < \delta_2(s)$ then $[f(s+h) - f(s)]/h < 0$. Let $\underline{\delta}(s) = \min(\delta_1(s),\delta_2(s))$. We have the following for $0 < |h| < \delta(s)$:

If $f'(s) > 0$ and $h > 0$, then $f(s+h) > f(s)$.
If $f'(s) > 0$ and $h < 0$, then $f(s+h) < f(s)$.
If $f'(s) < 0$ and $h > 0$, then $f(s+h) < f(s)$.
If $f'(s) < 0$ and $h < 0$, then $f(s+h) > f(s)$.

Let J be a subinterval of $[a,b]$ where $|J| < \delta$ and $s \in J$. Let $x, y \in J$ and $x < s < y$. With $t = s$ we have $x = s - |h_1|$ and $y = s + |h_2|$ with $|h_1| < \underline{\delta}(s)$ and $|h_2| < \underline{\delta}(s)$. From above we have either $f(x) < f(s) < f(y)$ or $f(x) > f(s) > f(y)$. Thus J is an element of C and C is a full cover of $[a,b]$.

Using Thomson's Lemma we know C contains a partition of $[a,b]$; that is, there exist $a = p_0 < p_1 < \dots < p_n = b$ such that $[p_{k-1},p_k] \bullet C$ for $k = 1,2,\dots,n$. Thus for each k there exists a $t_k \bullet [p_{k-1},p_k]$ such that for all $x_k, y_k \in [p_{k-1},p_k]$, where $x_k < t_k < y_k$, either $f(x_k) > f(t_k) > f(y_k)$ or $f(x_k) < f(t_k) < f(y_k)$.

First suppose that $f(x_1) < f(t_1) < f(y_1)$. We will show that the direction of the inequality Is preserved on the remaining subintervals. Suppose to the contrary that $f(x_2) > f(t_2) > f(y_2)$. Then $f(t_1) < f(y_1)$ and $f(t_2) < f(x_2)$ and $t_1 < y_1 < x_2 < t_2$. Now $f($. is continuous on the interval $[t_1, t_2]$ and differentiable on (t_1,t_2). Thus $f(x)$ assumes a maximum value $f(m)$ on $[t_1,t_2]$ and from above $m \neq t_1$ and $m \neq t_2$. As a consequence $m \in (t_1,t_2)$ and $f'(m) = 0$. This contradicts our initial supposition, and thus $f(x_2) < f(t_2) < f(y_2)$. The same argument hdds for $[p_{k-1},p_k]$, $k = 3,\dots,n$. it follows that $f(a) < f(p_1) < \dots < f(p_n) = f(b)$. However, this contradicts the assumption that $f(a) = f(b)$, and so Case 1 cannot hold.

The same argument hdds if $f(x_1) > f(t_1) > f(y_1)$. Thus, the following statement: there exists a $t_k \in [p_{k-1},p_k]$ such that for all $x_k, y_k \in [p_{k-1},p_k]$ where $x_k < t_k < y_k$, either $f(x_k) < f(t_k) < f(y_k)$ or $f(x_k) > f(t_k) > f(y_k)$, is FALSE! Therefore, the initial supposition is false, and so there is a point $t \in (a,b)$ such that $f'(t) = 0$.

in [1, p.452], Botsko suggests using the full covering technique to prove the theorem stating that if the derivative of a function equals zero for ail points on a ciosed interval, then the function is constant. As a ciass, we completed a proof and some time later, a similar argument appeared in Botsko's second paper [2, p. 331]. This was inspirational to the ciass. We now present a proof of the result.

Theorem: If $f(x)$ is differentiable on $[a,b]$ and $f'(x) = 0$ for all $x \in [a,b]$, then $f(x) = K$ for all $x \in [a,b]$.

Proof: Let $e > 0$ be given. Let

$C = \{I \mid I$ is a closed subinterval of $[a,b]$ and there exists $t \in I$ such that for all $y \in I$, $\mid [f(y) - f(t)]/(y - t) \mid < e/3(b - a) \}$

Let x be in $[a,b]$. Since $f'(x) = 0$, there exists a $\delta(x) > 0$ such that when $0 < \mid h \mid < \delta(x)$

$$\left|\frac{f(x+h)-f(x)}{h}\right| < \frac{\epsilon}{3(b-a)}$$

Consider any subinterval J of $[a,b]$ with $x \in J$ and $\mid J \mid < \delta(x)$. Thus, for all $y \in J$

$$\left|\frac{f(x)-f(y)}{x-y}\right| < \frac{\epsilon}{3(b-a)}$$

With $t = x$ we have that $J \in C$ and thus C is a full covering of $[a,b]$. From Thomson's Lemma, there exist subintervals $[p_{k-1},p_k] = I_k$ for $k = 1,\dots,n$ such that I_k and which partition $[a,b]$. Thus for each

k there exists a $t_k \in I_k$ such that for all $y \in I_k$

$$\left|\frac{f(y)-f(t_k)}{y-t_k}\right| < \frac{\epsilon}{3(b-a)}$$

Now let x and y be in $[a,b]$ with $x < y$. Then $x \in [p_{j-1},p_j]$ and $y \in [p_{m-1},p_m]$ for some $j,m = 1,\dots,n$ with] $\leq m$.

We know that

$$f(x)-f(y) = (f(x)-f(t_j))+(f(t_m)-f(y))+(f(t_j)-f(t_m))$$

$$= (f(x)-f(t_j))+(f(t_m)-f(y))+\sum_{k=j+1}^{m}[f(t_{k-1})-f(t_k)]$$

$$|f(x)-f(y)| < |f(x)-f(t_j)|+|f(t_m)-f(y)|+$$

$$\sum_{k=j+1}^{m}|f(t_{k-1})-f(p_{k-1}| + \sum_{k=j+1}^{m}|f(p_{k-1})-f(t_k)|$$

$$< \frac{\epsilon(b-a)}{3(b-a)} \quad \frac{\epsilon(b-a)}{3(b-a)} \quad \frac{\epsilon(b-a)}{3(b-a)} = \epsilon + \underline{\qquad}$$

Since e was arbitrary, we have $f(x) = f(y)$ for all $x,y \in [a,b]$. Thus $f(x)$ is constant on $[a,b]$.

Dini's Theorem: If $f_n(x)$ is a sequence of continuous functions on $[a,b]$ and $f_n(x) < f_{n+1}(x)$ for all n and for ail $x \in [a,b]$, then $f_n(x)$ converges uniformly to $f(x)$.

Proof: Let $e > 0$ be given. Let

$C = \{I \mid I$ is a ciosed subinterval of $[a,b]$ and there exists an N_I such that for $n > N_I$, $\mid f_n(x) - f(x) \mid < e$ for ail $x \bullet I\}$.

We show that C is a fuil cover of $[a,b]$. Let x be an element in $[a,b]$. Since $f_n(x) \to f(x)$ pointwise, there exists an $N = N(x,e)$ such that when $n \geq N$, then $\mid f_n(x) - f(x)) \mid < e/3$. Also, since f is continuous at x, there exists a $\delta_1 > 0$ such that if $\mid x - y \mid < \delta_1$, then $\mid f(x) - f(y) \mid < e/3$. Furthermore, since f_N is continuous at x, there exists a $\delta_2 > 0$ such that if $\mid x - y \mid < \delta_2$, then $\mid f_N(y) - f(y) \mid < \mid f_N(y) - f_N(x) \mid + \mid f_N(x) - f(x) \mid + \mid f(x) - f(y) \mid < e$. Now for $n > N$ we know $f_N(y) < f_n(y) < f(y)$ for all y e $[a,b]$ and so it follows that if $\mid x - y \mid < \delta$ and $n > N$, $\mid f_n(y) - f(y) \mid < e$. Let J be any ciosed subinterval of $[a,b]$ such that $\mid J \mid < \delta$ and J contains x. Then for all $y \in J$, $\mid f_n(y) - f(y) \mid < e$ for all $n > N_1$ (=N). Thus J is in C and C is a full cover of $[a,b]$. Using Thomson's Lemma, C contains a partition of $[a,b]$; that is, there exists

$a = p_0 < p_1 < \dots < p_n = b$ such that $[p_{k-1},p_k] = I_k$ is in C for $k = 1,2,\dots,n$.
Let $\tilde{N} = \max\{N_{I,k} \mid 1 \le k \le n\}$. For any $x \in [a,b]$, x is in some I_j, $j = 1,\dots,n$ and so
$| f_n(x) - f(x) | < \epsilon$ for $n > \tilde{N}$. Thus $f_n(x)$ converges uniformly to $f(x)$ on $[a,b]$.

We close this paper with our final application of a full covering argument. in a student seminar class, we were challenged to come up with a point definition of increasing, prove some standard calculus results with this definition and then show that if this definition holds at each point of the interval (a,b) then the function is increasing in the usual sense on the interval.

Definition 1: A function f is increasing at a point p if and only if there exists a neighborhood $N_\epsilon(p) = (p - \epsilon, p + \epsilon)$ of p such that f is defined on $N_\epsilon(p)$ and for all $x,y \in (p - \epsilon, p + \epsilon)$ with $x < p < y$, $f(x) < f(p) < f(y)$.

This is a more general definition of the concept of increasing. We state the usual definition of increasing.

Definition 2: A function f is increasing on (a,b) if for all $x,y \in (a,b)$ with $x < y$, $f(x) < f(y)$.

Theorem: If Definition 1 holds at every point on (a,b), then Definition 2 holds.

Proof: Let x and y be elements of (a,b) with $x < y$. Let

$C = \{I \mid I$ is a closed subinterval of $[x,y]$ and there exists $t \in I$ with the property that if $f(m) < f(t) < f(n)\}$

Let s be In $[x,y]$. Then there exists $\epsilon > 0$ such that f is defined on $N_\epsilon(s) = (s - \epsilon, s + \epsilon)$ and for all m and n in $(s - \epsilon, s + \epsilon)$ with $m < s < n$, one has $f(m) < f(s) < f(n)$. Let J be a closed subinterval of $[x,y]$ with $| J | < \epsilon$ and let $s \in J$. Then for all m, n $\in$ J such that $m < s < n$, $f(m) < f(s) < f(n)$. With $t = s$ we see that $J \in C$ and so C Is a full cover of $[x,y]$.

By Thomson's lemma, there is a partition of $[x,y]$ contained in C; that is, there exists $x = p_0 < p_1 < \dots < p_n = y$ with $I_k = [p_{k-1},p_k]$ for $k = 1,\dots n$, such that $[x,y] = \bigcup I_k$, $I_k \in C$. From this we have that $f(x) = f(p_0) < f(p_1) < f(p_2) < \dots < f(p_n) = f(y)$. Thus the function f is increasing on (a,b).

It can also be shown that if $f'(x) > 0$ on (a,b) then f is increasing at each point of (a,b) according to Definition 1. By our last theorem it follows that $f(x)$ is increasing in the usual sense on (a,b). We thus have

Corollary 1: If $f'(x) > 0$ on (a,b), then $f(x)$ is increasing on (a,b).

Corollary 2: If $f'(x) < 0$ on (a,b), then $f(x)$ is decreasing on (a,b).

Note that the proofs of these corollaries avoid the use of the Mean Value Theorem. Further theorems which can be proved using this method can be found in any calculus text. The interested reader is challenged to find and prove some of those theorems.

REFERENCES

[1] M. W. Botsko, "A Unified Treatment of Various Theorems in Elementary Analysis," American Mathematical Monthly, 94 (1987), 450-452.

[2] M. W. Botsko, 'The Use of Full Covers in Real Analysis," American Mathematical Monthly, 96 (1989), 328-333.

[3] J. Olmsted, Advanced Calculus, Prentice-Hall, Inc. New Jersey, 1961.

LETTER TO THE EDITOR

Dear Editor:

In a recent note. The AM-GM Inequality: A Calculus Quickie" by Norman Schaumberger (Spring, 1990. p. 111), the author gives a non-elegant proof of the AM-GM inequality by first showing via the calculus that if $a_1, a_2, \dots a_n$ are nonnegative numbers then

(1) $$a_k \ge k(a_1a_2 \dots a_k)^{1/k} - (k-1)(a_1a_2 \dots a_{k-1})^{1/(k-1)}.$$

Firstly, the equality condition he gives is incorrect. Secondly, there are a number of elegant non-calculus proofs of the AM-GM inequality. Thirdly, (1) follows immediately from the AM-GM inequality. Let $y = a_1a_2 \dots a_{k-1}$ and $x = a_k$, then $\{x + (k-1)y^{1/(k-1)}\}/\{1+(k-1)\} \ge k(xy)^{1/k}$, with equality iff $x = y^{1/(k-1)}$ or $a_k = (a_1a_2 \dots a_{k-1})^{1/(k-1)}$.

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A CONFERENCE ON HISTORY, GEOMETRY, AND PEDAGOGY
At the University of Central Florida, Orlando, Florida
May 9-11, 1991
In honor of the 80th birthday of Howard Eves

Howard Eves's career interests in teaching, history, and geometry provide an ideal setting within which mathematics teachers and university professors can discuss their experiences and research. It is a fitting tribute in the year which marks the 80th birthday of Howard Eves that a conference be organized which brings together representatives of these diverse groups to discuss their common Interest so that each can learn from the perspectives of the others. Major speakers will include Professors Clayton Dodge, Peter Hilton, Murray Klamkin, Bruce Meserve, Fred Rickey, Marjorie Senechal and, of course, Howard Eves. There will also be parallel sessions for contributed papers and workshops.

For more information concerning the conference address all inquiries to the Conference Director. Professor Joby Anthony, Department of Mathematics, University of Central Florida, Oriando, FL 32816-6990. Phone (407) 823-2700 or FAX (407) 281-5156.

THE ZERO-ONE AFTERMATH OF CERTAIN INTEGRAL PATTERNS

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Consider a two-way classification of the positive integers such as square and non-square. Consider also an unending decimal x of the form

$$.d_1d_2d_3d_4...d_i...$$

so that if i conforms to the square classification above, $d_i = 1$. Otherwise, $d_i = 0$. Hence, in reference to the square and non-square classification of the positive integers, x = .1001000010000001000000001000000000010000... . In this case, ones appear in square place positions and zeros elsewhere. Such a number, consisting entirely of ones and zeros in which the digits denote conformity or non-conformity to a positive integer classification, will be called a ZERO-ONE number. As x is an unending, non-repeating decimal, it is thus irrational and cannot be represented as the quotient of two integers.

The zero-one number which stems from classifying the positive integers as cubes and non-cubes is

.100000010000000000000000001000... .

It is likewise irrational as are corresponding zero-one numbers for higher powers.

A famous zero-one number happens to be the first known transcendental number. Such numbers, which by definition cannot occur as roots of algebraic equations, form a challenging area of present day mathematical pursuits. **Transcendentals** include ln 2, sin 1, $2^{\sqrt{3}}$, and $5^{\sqrt{2}}$ as well as the remarkable π and e. In particular, consider a classification of the positive integers as factorials or otherwise. If 1 denotes conformity to the factorial property and 0 denotes non-conformity, then the corresponding zero-one number say, L, becomes

.110001000000000000000001000.. .

Such a number is not only irrational, it is also transcendental as was proved by Joseph D. **Liouville** in the mid-part of the last century.

An intriguing number is the **zero-one** number P based on a classification of the positive Integers as odd primes or otherwise. Consider thus the unending decimal

P = .001010100010100010100010000010100000100...

where odd prime places are filled by ones and the remaining places are filled by zeros. What properties does P have? First consider the matter of rationality or irrationality.

Suppose P is rational with a minimal repeating block of b digits. Both zeros and ones must appear in this repeating block. At least one single one must appear as the set of primes is Infinite; zeros appear in this block because of alternating even placed positions. Let a particular 1 in the repeating block have position q, where q is, of course, a prime. Then q + b, q + 2b, q + 3b, and in general, q + nb, are also prime as the repeating block consists of b digits. But no arithmetic progression yields only primes. That is, if q + nb is extended sufficiently far, n will eventually become a multiple of q and thus make the number q + nb composite. By contradiction, the number P is Irrational.

In the decimal expansion of **P**, 1's appear to be **relatively** scarce. It is easy to describe intervals of enormous length among the integers containing no primes whatever. Consider for example **100!** + 2, **100!** + 3, ..., 1001 + 100. This lengthy list of consecutive composites reveals a hundred or more consecutive zeros in the expansion of P. With slight modification, an interval containing a googol of zeros could be identified.

Various unanswered questions surround the number P. Included are the following:

1. Does the sequence 101 appear infinitely many times in P? If so, the set of prime twins is infinite. Note: the sequence 10101 appears but once.
2. Between any two zeros which denote exact squares in the expansion of P, can a 1 always be found? This concerns the unsolved problem as to the unfailing occurrence of primes between consecutive squares.
3. Is any zero in P which symbolizes a factorial immediately followed by a 1 but a finite number of times? Unanswered today is the question concerning the number of primes of the form $n! + 1$.
4. Consider any 0 which corresponds to an exact square. Is such a "square" zero followed immediately by a 1 infinitely many times? This question concerns the number of primes of the form $x^2 + 1$.

Some conjectures permit quick and easy dispositions. The following is typical:

> A prime which is not an element of a set of prime twins is called Isolated. Note that all primes of the form $(15n + 8)$ are isolated as both $15n + 6$ and $15n + 10$ are algebraically factorable. As a consequence of this, the sequence 00100 appears infinitely many times.

Unresolved also is the problem as to whether or not P is transcendental. If In fact P Is algebraic, it would prove most interesting to find an algebraic equation having P as a root.

Shifting the problem of the distribution of the primes from its basic setting to some seemingly unrelated mathematical form may eventually solve the mystery of the primes. Graphic representations based on the shading of prime-numbered squares in a cartesian framework have been pursued lately so as to give the problem a geometric flavor. Primes in such a setting seem to exhibit a diagonal consistency feature. Such is the attempt below. That is, the primes may be viewed in the context of the dig'tal pattern of an irrational, possibly transcendental, number.

EUCLIDEAN PRIMES, ZERO-ONE NUMBERS, AND TRANSCENDENCE

The testing of a number for transcendence is extremely difficulty. Even today such numbers as e^e, $\pi + e$, and Euler's constant remain **unclassified.** A **fairly** convenient number F permits a look at a transcendental testing procedure. Such a number we have chosen to call Euclidean.

A Euclidean number is one of the form $(p_1)(p_2)\ldots(p_m) + 1$ where p_i denotes the i^{th} prime. The first few primes of this form are 2, 7, **31,** 211, and 2311 as suggested below:

$$2 + 1 = 3$$
$$(2)(3) + 1 = 7$$
$$(2)(3)(5) + 1 = 31$$
$$(2)(3)(5)(7) + 1 = 211$$
$$(2)(3)(5)(7)(11) + 1 = 2311.$$

The next Euclidean number, namely $(2)(3)(5)(7)(11)(13) + 1$ or 30031, is not prime as 30031 equals **(59)(509).** Nor is the next as **510511** is equal to **(19)(97)(277).** Unresolved at present is the question of the cardinality of the set of Eudidean primes. Consider the **zero-one** number based on Euclidean primes which is

$$F = .00100010000000\ldots1000\ldots010\ldots01000\ldots.$$

3 7 31 211 2311

Should the Eudldean primes form but a finite set, then F is dearly rational as it becomes a terminating decimal. If the Euclidean primes form an infinite set, more challenging questions arise. Note that F can be expressed concisely **in** this latter case by the symbol

$$F = \sum_{j=1}^{\infty} 10^{-p_j}$$

where p_j denotes the j^{th} Euclidean Prime.

To establish the irrationality of F under the assumption of the infinitude of the set of Euclidean **primes,** suppose that F **is** rational, having a repeating block of b digits. Consider any 1 in this repeating block, representing say p_m, a Euclidean prime. Then $p_m + b$, $p_m + 2b$, $p_m + 3b, \ldots$ are all prime as the coefficients of b range over the positive integers. This, as established earlier, is impossible. Accordingly, F would be irrational.

The more difficult question of transcendence next arises. In establishing that F Is transcendental, the approach **will** again be Indirect. Recall that F = **.0010001000... .** Let G = F^* where F^* Is formed by taking a finite number of initial digits in F and **terminating** the representation with a **1.** More specifically,

$$G = \sum_{j=1}^{k} 10^{-p_j}$$

or, by adding fractions

$$G = \frac{W}{10^{p_k}}$$

G, which can be expressed as

$$G = 10^{-p_1} + 10^{-p_2} + \ldots + 10^{-p_k}$$

is rational and can be made very close to F in value. That is,

$$F - G = 10^{-p_{k+1}} + 10^{-p_{k+2}} + 10^{-p_{k+3}} + \ldots$$

and satisfies the condition that

$$F - G < \frac{3}{10^{p_{k+1}}}$$

The numerator could have been chosen in various ways, but 3 proves a fairly convenient choice.

Assume that F is alaebraic and let

$$h(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_{,,} = 0$$

be the algebraic equation of lowest degree satisfied by F. Then **h(F)** = 0. Next note that **h(G)** is not zero. Should G be a root of **h(x)** = 0, then **(x - G)** **would** be a factor of **h(x).** That is, **h(x)** = (x - G) · **q(x)** where **q(x)** has coefficients which are rational and a degree 1 less than **h(x).** So **h(F)** = (F - G) · **q(F)** = 0. But (F - G) is not zero. Hence **q(F)** is zero. This is, of course, impossible as F cannot satisfy an algebraic equation of degree less than n. Therefore, **h(G)** is not equal to zero.

It can be established that a number M exists, relying only on the degree of **h(x)** = 0 and its coefficients, such that

$$|h(F) - h(G)| < M|F-G|$$

Recall that F - G can be made as small as desired.

It can also be established that

$$|h(F)-h(G)|10^{np_k}$$

is a positive integer regardless of the value assigned to k. The number $|h(F)-h(G)|$ is the same as $|h(G)|$. To establish the above, one must substitute

$$G = \frac{W}{10^{p_k}}$$

Into $h(x)$. Multiplication of this expression by 10^{np_k} yields a sum of integers. As $h(G)$ is not zero, it follows that

$$|h(F)-h(G)|10^{np_k}$$

Is a positive integer. But. based on assuming that F Is algebraic, we can also show that

$$|h(F)-h(G)|10^{np_k}$$

is a number between 0 and 1. Such a contradiction is critical in showing that F is transcendental. Note that

$$|h(F)-h(G)| < M|F-G|$$

and that

$$|h(F)-h(G)|10^{np_k} < M|F-G|10^{np_k}$$

But

$$F - G < \frac{3}{10^{p_{k+1}}}.$$

So

$$|h(F)-h(G)|10^{np_k} < M|F-G|10^{np_k} < \frac{3M\ 10^{np_k}}{10^{p_{k+1}}}$$

Writing this last fraction as

$$\frac{3M}{10^{p_{k+1}-np_k}}$$

we note that the denominator can be made extremely large by choosing k very iarge. Recall that **n,** the degree of **h(x) = 0,** is not a variable. Hence, in the **factor part** of p_k, a largest prime q appears. That is, $p_k = (2)(3)(5) \ldots (q) + 1$. in the factor part of p_{k+1}, not **only** does q appear, but also a larger prime r. So,

$$p_{k+1} - n\,p_k = [(2\cdot3\cdot5\cdot\ldots\cdot q\cdot r) + 1] - n\,[(2\cdot3\cdot\ldots\cdot q) + 1]$$
$$= (2\cdot3\cdot5\cdot\ldots\cdot q)\,(r-n) + (1-n).$$

By making q, and thus, r sufficiently iarge, the denominator in

$$\frac{3M}{10^{p_{k+1}-np_k}}$$

becomes iarge without bound.

The fraction will then tend to zero through positive values.

Our original assumption that F is algebraic has **led** to the contradiction that

$$|h(F)-h(G)|10^{np_k}$$

is a positive integer as well as a number between 0 and 1. Accordingly, F is not algebraic but rather transcendental. It should again be stressed that the premise of the argument is the infinitude of the set of Euclidean primes.

The reader may wish to pursue zero-one numbers which are generated by other classifications of the positive integers. These include various subsets of the primes such as **Fermat** or Mersenne, perfect numbers, abundant or deficient numbers, and pseudoprimes. Some of these sets are known to be infinite (pseudo-primes, for example) and others are unresolved. A challenging problem is that of classifying the super-number S which relies on a designation of the positive integers as superpowers or otherwise. Superpowers clearly form an infinite set.

Consider:

x	x^x	SUPERPOWERS
1	1^1 or 1	
2	2^2 or 4	
3	3^3 or 27	
4	4^4 or 256	
5	5^5 or 3125	
6	6^6 or 46656	
7	7^7 or 823543	
8	8^8 or 16777216	
9	9^9 or 387420489	
10	10^{10} or 10000000000	and so on.

Based on the above,

$$
\begin{array}{l}
S = .100100 \ldots 0100 \ldots 0100 \ldots 010 \ldots 0100 \ldots 0100 \ldots \\
1 \quad 4 \qquad 27 \qquad 256 \qquad 3125 \qquad 46656 \qquad 823543
\end{array}
$$

In prophetic anticipation of transcendental numbers, Euler once remarked "they transcend the power of algebraic methods." Of course, Euler died in the late eighteenth century, well prior to the discovery of the first known number of a transcendental kind. Today, we can do more than point to specific examples of transcendental numbers though mathematicians of the glorious eighteenth century could identify none. Based on the colossal efforts of Cantor (efforts praised by **Hilbert),** it is known that the set of transcendental numbers proves uncountable. Building too on the works of **Hermite, Lindemann, Gelfond,** Thue, **Siegel,** Roth, and others, the surface has at least been scratched. The methods required to probe deeply suggest all the more forcefully the limitless expanse of the wonderland of numbers.

References

[1] D. **M.** Burton, The History of Mathematics, **Allyn** and Bacon, Inc., Boston, 1985.

[2] D. **Costellanos,** The Ubiquitous Pi, Part 1," Mathematics Magazine 61, (April, **1988),** 67-100.

[3] D. Costellanos, The Ubiquitous Pi, Part **2,"** Mathematics Magazine 61, (June, **1988), 148-163.**

[4] **R.** L. Francis, "From None to Infinity," The College Mathematics Journal 17, (May **1986),** 226-230.

[5] **R.** L. Francis, "Mathematical Haystacks: Another Look at **Repunit** Numbers," The College Mathematics Journal 19, (May, **1988), 240-246.**

[6] W. V. **Lovitt,** Elementary Theory of Equations, Prentice-Hall, Inc., New York, 1939.

[7] **I. Niven,** Numbers. Rational and Irrational, Random House, Inc., New York, 1961.

[8] K. H. **Rosen,** Elementary Number Theory and Its Applications. Addison-Wesley, Reading, **1988.**

[9] **I. Niven** and H.S. Zuckerman, An Introduction to the Theory of Numbers, (4th Edition), John **Wiley** and Sons, New York, 1980.

[10] L. **Weisner,** Introduction to the Theory of Equations, **Macmillan,** New York, 1938.

THE **3-4-5** TRIANGLE

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There was an Interesting article on **this** topic by Michael Eisenstein in the Spring, 1990, issue of this Journal **(p.112),** and it tempted me to put the algebraic proof in geometric form.

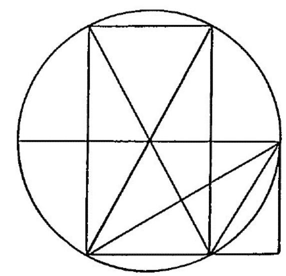
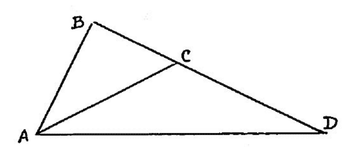


Figure 1 shows the construction (from Euclid, **IV, 10) for** a golden rectangle. Put two squares together to form a **rectangle, and** draw the circumscribing circle, then put in the other lines as shown. A golden rectangle makes it possible to draw a regular pentagon, because the sides and diagonals are in the golden ratio. (For more details, see **Coxeter's** Introductory to Geometry, page 161.)

From Figure 1 we see that if β is the angle between the diagonal and the longer side of a **golden** rectangle then $\tan(2\beta) = 2$. The relation with the **3-4-5** triangle is then dear. In fact, more generally, if ϕ is any angle with its tangent rational, then 2ϕ is an angle of a Pythagorean triangle (right-angled with integer sides). See Figure 2 where
$AB = 2pq$, $AC = CD = p^2 + q^2$ and $BC = p^2 - q^2$.



These ideas lead one to ask questions about the additive group of angles whose tangents are rational. For example, suppose we write a_n for the inverse tangent of **n.** It is known that the subgroup generated by a_1 and a_2 includes a_3 but not a_4. Does the subgroup generated by a_1, a_2 and a_3 include a_5, a_6, and a_7?

COUNTING WITH CONVEX COORDINATES

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Introduction. We have found that knowing a bit about convex coordinates **allows** us to attack a surprising variety of problems at an elementary level. Our solutions. in those cases in which our attempts have been successful, have not always been the most elegant available. But for problems in Euclidean geometry involving concurrence of **lines, collinearity of** points, or areas and volumes, computations with convex coordinates often **lead** directly to workmanlike solutions and interesting results. **[1,2,3]**

Suppose that P is a point of A $V_1V_2V_3$, the dosed triangular region having vertices V_1V_2 and V_3. Then P has convex coordinates $(\alpha_1, \alpha_2, \alpha_3)$ with respect to $V_1V_2V_3$ in that order as defined by

$$\alpha_1 = \text{Area A } PV_2V_3/\text{Area A } V_1V_2V_3,$$
$$\alpha_2 = \text{Area A } PV_3V_1/\text{Area A } V_1V_2V_3,$$
$$\alpha_3 = \text{Area A } PV_1V_2/\text{Area A } V_1V_2V_3$$

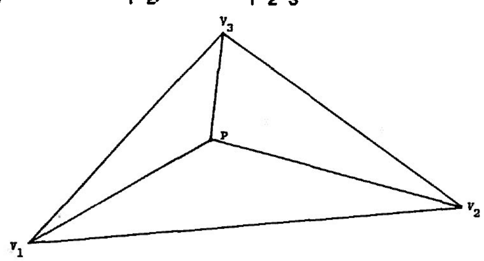

Figure 1. Point P ● A $V_1V_2V_3$.

it is clear that each convex coordinate is nonnegative and that $\alpha_1 + \alpha_2 + \alpha_3 = 1$. For example, we note that the convex coordinates of the centroid of A $V_1V_2V_3$ are **(1/3, 1/3, 1/3)** and that the convex coordinates of the vertex V_i are $\alpha_i = 1$, $\alpha_j = 0$ for $j \neq i$. If P has Cartesian coordinates **(x,y),** we can also write $x = \alpha_1x_1 + \alpha_2x_2 + \alpha_3x_3$ and $y = \alpha_1y_1 + \alpha_1y_2 + \alpha_1y_3$ where (x_i, y_i) are the Cartesian coordinates of vertex V_i.

it can be easily shown that if A $P_1P_2P_3 \subset$ A $V_1V_2V_3$ then the area of A $P_1P_2P_3$ is given by

$$\det \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix} \text{Area (A } V_1V_2V_3). \ [4]$$

Convex coordinates also admit of interpretations as probabilities and percentages of constituents in additive mixtures. Such Interpretations in combination with computations suggested by the properties of convex coordinates as **listed** above have **led** to nontrivial observations concerning the additive mixing of colors, hypothesis testing, random walks, and electrical circuits. **[5,6,7]**

in our work which follows, we give an approximate solution to a counting problem by rephrasing that probiem in terms of convex coordinates and then using the properties of those coordinates.

A Countina Problem. Let **F(n)** be the number of distinct triangles modulo congruence having sides of integral lengths **s, t,** u with $s + t + u = n \geq 3$. Let us use convex coordinates to write a function **L(n)** which gives the asymptotic behavior of **F(n).** That is, **let** us find a function **L(n)** such that $\lim_{n\to\infty} F(n)/L(n) = 1$.

We begin by considering the isosceies right triangle shown in Figure 2. The **legs** have length **n,** the integral perimeter of the triangie having sides **s, t,** u. The isosceles triangle is the convex **hull** of an array of $(n+1)(n+2)/2$ points from a square grid having density of points $\rho = 1$ **point/(unit** area).

If we associate the vertices of the isosceies right triangle with the lengths **s, t, u,** we can solve our problem by counting grid points having convex coordinates **(s/n, t/n, u/n),** $s + t + u = n$, subject to the constraints to be given. Vertices **V(s), V(t), V(u)** represent the sides **s, t, u** of the triangle of integral perimeter and the convex coordinates represent the percentage contributions of the sides to the total perimeter.

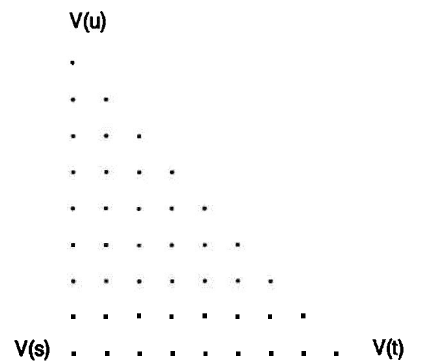

Figure 2. The Isosceles Right Triangle **With** n = 8.

By drawing the medians as shown in the next figure, we restrict our count to grid points In the shaded region or on the boundary of the shaded region to avoid repetitions of the sort **(6/15, 5/15, 4/15)** and **(5/15, 6/15, 4/15)**, assuming that n = 15 rather than 8 for the purpose of this illustration.

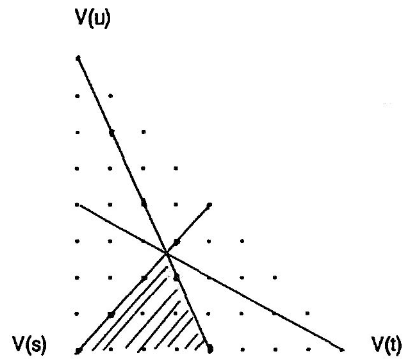

Figure 3. Drawing the Medians to Restrict Our Count.

The lengths **s, t, u,** must also satisfy the triangle inequality. However, we replace $s < t + u$ with $s < (s + t + u)/2 = n/2$ to restrict the shaded region even further. We accomplish the restriction by drawing the segment between the midpoints of the sides **V(s)V(t)** and **V(s)V(u).** The grid points on this segment should not be counted since that would replace the triangle inequality with an equality on the segment.

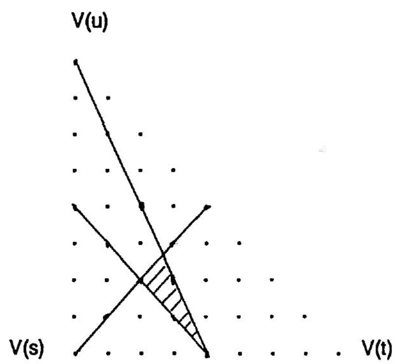

Figure 4. A Further Restriction From the Triangle Inequality

We might now find **F(n)** by counting grid points In the shaded region of Figure 3. However, we elect to estimate the number Instead. We begin by noting that the vertices of the shaded region with their convex coordinates are **A:(1/2, 1/2, 0), B:(1/3, 1/3, 1/3), C:(1/2, 1/4, 1/4).**

The area of A ABC Is $\det\begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/3 & 1/3 & 1/3 \\ 1/2 & 1/4 & 1/4 \end{bmatrix} n^2/2 = n^2/48$. Since the density of points is $\rho = 1$ **point/(unit** area), we estimate that there are **INT(n^2/48)** points of the grid in the shaded region, A ABC, for large n. Thus **L(n) = INT(n^2/48).**

A computer generated comparison of **L(n)** is given below.

n	F(n)	L(n)	F(n)/L(n)
50	52	52	1
100	208	208	1
150	469	468	1.0021
200	833	833	1
250	1302	1302	1
300	1875	1875	1
350	2552	2552	1
400	3333	3333	1.0002
450	4219	4218	1
500	5208	5208	1

Table 1. A Comparison of **L(n)** and **F(n)**

References

[1] J.N. Boyd and **P.N.** Raychowdhury, ***Mathematics*** and ***Computer Education*** 17 **(1983)**, pp. **113-8.**

[2] J.N. Boyd and P.N. Raychowdhury, The Mathematical Gazette 71 **(1987)**, pp. 209-14.

[3] J.N. Boyd, **R.W. Farley,** and P.N. Raychowdhury, Pure and Applied Mathematical Sciences XXV (No. 1-2, **1987), pp.** 11-16.

[4] J.N. Boyd and P.N. **Raychowdhury,** Mathematics and ***Computer Education*** 18 **(1984)**, pp. 121-4.

[5] J.N. Boyd, Pi Mu ***Epsilon*** Journal 8 **(1988)**, pp. **508-15.**

[6] J.N. Boyd and P.N. Raychowdhury, College Mathematics Journal 18 **(1987)**, pp. 186-94.

[7] J.N. Boyd and P.N. Raychowdhury, College Mathematics Journal 20 **(1989)**, pp. **385-92.**

Multiple Applications of Integration by **Parts**

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One of the most common mistakes an undergraduate student makes in evaluating an integral using more than one application of integration by parts is reversing the proper choice of factors for differentiation and integration. Many textbooks (see, for example, Thomas & Finney [1]) have shown methods which eliminate such errors. Essentially, they show a good organization of the solution in a tabular form from which the ultimate answer can be written down. Unfortunately. none of them are complete in the sense that they fall to show an organization for a certain type of problem. For such problems, they recommend going back to several applications of Integration by parts, resulting in a possible mistake of switching factors for integration and differentiation. The purpose of the present paper Is to address this problem. We propose to show a proper organization of integration by parts to avoid all possible errors. The reader may question the use of this because of the symbolic integrators available to students. However, the method proposed in this paper will give the students an illustration on how to organize mathematical calculations into a readable form.

After this work was accepted for publication, the author learned of the paper by **Horowitz** [2] where a tabular integration by parts is briefly discussed with applications to several problems, but termination and alteration of tabular integration by parts was not explained. The present work focuses on **these** aspects.

Recall that the integration by parts uses a simple formula

$$\int uv\,dx = uA(v) - \int D(u)A(v)\,dx$$

where **A(v)** is an antiderivative of v and **D(u)** is the derivative of u. This formula can be adjusted into a table **consisting** of two rows and two columns with entries **u,** v in the first row (whose product constitutes the integrand of the original problem), and **D(u), A(v)** in the second row (thus, the second row obtained by differentiating the left entry above and integrating the previous right entry) so that the integral $\int uv\,dx$ is the product of the diagonal entries minus the integral of the product of the entries in the last row. Addition of the product of diagonal entries Is indicated by an arrow from u to **A(v)** labelled with a + sign and the subtraction of the integral of the last row Is indicated

by a back arrow (from **A(v)** to **D(u))** with a minus sign.

$$\begin{tikzcd} u \arrow[dr, "+"] & v \\ D(u) & A(v) \arrow[l, "-"'] \end{tikzcd}$$

In a second application of integration by parts to the problem of finding $\int uvdx$, the second row becomes the original row for the integral $\int D(u)A(v)dx$ and the above process shows that we obtain the following table where $D^2(u)$ is the second derivative of u and $A^2(v)$ is an **antiderivative** of **A(v):**

$$\begin{tikzcd} D(u) \arrow[dr, "+"] & A(v) \\ D^2(u) & A^2(v) \arrow[l, "-"'] \end{tikzcd}$$

$$\int D(u)A(v)\,dx = D(u)A^2(v) - \int D^2(u)A^2(v)\,dx$$

Combining these two tables we arrive at a two-column table

$$\begin{tikzcd} u \arrow[dr, "+"] & v \\ D(u) \arrow[dr, "-"] & A(v) \\ D^2(u) & A^2(v), \arrow[l, "+"'] \end{tikzcd}$$

with three rows whose left column is obtained by successive differentiation of u and right column contains successive integrals of v and notice that the signs on the arrows alternate starting with + sign. And the table reads

$$\int uvdx = uA(v) - D(u)A^2(v) + \int D^2(u)A^2(v)\,dx$$

Of course, we use the convention that a signed forwarding arrow means the product of the functions on the ends of the arrow with the associated sign and a signed back arrow translates into the integral of the product of the end functions with the corresponding signs.

Now, n applications of Integration by parts to the integral $\int uvdx$ can be put into a table with two columns and **(n+1)-rows** whose initial row has entries u and v and other rows are obtained by successive differentiation of u and successive integration of v. Thus each entry in the left column is obtained by differentiating the previous entry in the left column and each entry in the right column is an integral of the entry on the right above. The arrows are placed from left to right one step down except for the last row where a back arrow is placed instead. Moreover, the **arrows** are labelled signs + or - alternately, starting with + sign for the arrow initiating from **u.** Now the integral ∫ uvdx Is read from the table:

$$\begin{array}{ccc} u & & v \\ & \searrow^{+} & \\ D(u) & & A(v) \\ & \cdots\cdots & \\ D^{n-1}(u) & \searrow^{(-1)^{n-1}} & A^{n-1}(v) \\ & \searrow^{(-1)^{n}} & \\ D^{n}(u) & \longleftarrow & A^{n}(v) \end{array}$$

where the general block in the table (rand r + 1 rows) looks like:

$$\begin{array}{ccc} D^{r-1}(u) & & A^{r-1}(v) \\ & \searrow^{(-1)^{r-1}} & \\ D^{r}(u) & & A^{r}(v) \end{array}$$

And so we obtain the result:

$$\int uv\ dx = \sum_{r=1}^{n} (-1)^{r-1} D^{r-1}(u) A^{r}(v) \ + \ (-1)^{n} \int D^{n}(u) A^{n}(v)\, dx$$

Obviously we need to stop the process at some **level.** The above formula certainly tells us a good criterion: The process can be terminated if the integral ∫ $D^n(u)A^n(v)dx$ is easy to find using methods other than integration by parts or if it is a multiple of the integral we started with so that we obtain a linear equation for the integral which can be **solved.** In other words, the process **terminates** at the row where the row product is either easily integrated by other integration techniques or is a constant multiple of the first row. In particular, if zero occurs in the first column then we can terminate the process, moreover, a back arrow is not necessary in the last row because it merely translates to an addition of an integration constant in the formula for the original integral.

We also observe that in order to apply the process, we have to select a factor from the Integrand that can be easily differentiated (it will be placed on the left) and the remaining factor's successive integrals should be easily found (they will be placed on the right). For instance, for any integrals of the type (see **[1]**):

$$\int x^{n} f(x)\, dx$$

the x^n can be taken as a left function **provided** successive Integrals of **f(x)** are **easily** found; otherwise, hoping that derivatives of **f(x)** are simple, we have to take x^n as a right function. Let us consider the following examples:

(1) $\int x^{n} e^{x} dx$

$$\begin{array}{ccc} x^{n} & & e^{x} \\ & \searrow^{+} & \\ nx^{n-1} & & e^{x} \\ & \searrow^{-} & \\ n(n-1)x^{n-2} & & e^{x} \\ & \cdots\cdots & \\ & \searrow^{(-1)^{n-2}} & \\ n(n-1)\cdots 2x & & e^{x} \\ & \searrow^{(-1)^{n-1}} & \\ n(n-1)\cdots 2\cdot 1 & & e^{x} \\ & \searrow^{(-1)^{n}} & \\ 0 & \xleftarrow{(-1)^{n+1}} & e^{x} \end{array}$$

which results in:

$$\int x^{n} e^{x} dx = e^{x}\left(\sum_{r=0}^{n} (-1)^{r} ({}^{n}P_{r})\, x^{n-r}\right) + C,$$

where ${}^{n}P_{0} = 1$, and ${}^{n}P_{r} = n(n-1)(n-2)\ldots(n-r+1)$ for $1 \le r \le n$.

(2) $\int x^{n} \ln x\ dx$

Since ∫ ln x dx can be found by integration by parts and in general is not known before with other techniques, we have to choose ln x as the left function. Thus we obtain a table,

$$\begin{array}{ccc} \ln x & & x^{n} \\ & \searrow^{+} & \\ \dfrac{1}{x} & \xleftarrow{-} & \dfrac{x^{n+1}}{(n+1)} \end{array}$$

which terminates in the second row because the integral, of the row product can be found using the power rule. Thus,

$$\int (\ln x)x^n dx = \frac{x^{n+1}}{n+1}\ln x - \int \frac{1}{x}\,\frac{x^{n+1}}{n+1}\,dx$$

$$\int (\ln x)x^n dx = \frac{x^{n+1}}{n+1}\ln x - \frac{x^{n+1}}{(n+1)^2} + C$$

Of course this argument is valid only for $n \neq -1$, and for the case $n = -1$, the integral Is found by the substitution of a new variable $u = \ln x$. Thus.

$$\int x^n \ln x\, dx = \begin{cases} \dfrac{x^{n+1}}{n+1}\ln x - \dfrac{x^{n+1}}{(n+1)^2} + C, & \text{if } n \neq -1 \\[2ex] \dfrac{(\ln x)^2}{2} + C, & \text{if } n = -1 \end{cases}$$

in particular, $n = 0$ gives $\int \ln x\, dx = x \ln x - x + C$.

(3) $\int e^x \sin x\, dx$

Here e^x or $\sin x$ are equally good for the left function.

$$\begin{array}{ccc} \sin x & & e^x \\ & \searrow + & \\ \cos x & & e^x \\ & \searrow - & \\ \sin x & \xleftarrow{+} & e^x \end{array}$$

We terminate at the third row because the product of this row Is a multiple of the integrand we started with. Thus,

$$\int e^x \sin x\, dx = \sin x\, e^x - \cos x\, e^x - \int \sin x\, e^x \&$$

which is a linear equation in the unknown $\int \sin x\, e^x dx$ whose solution is

$$\int e^x \sin x\, dx = \frac{1}{2}e^x(\sin x - \cos x),$$

and to include all possible antiderivatives, we add the Integration constant C.

$$\int e^x \sin x\, dx = \frac{1}{2}e^x(\sin x - \cos x) + C.$$

Notice that, in the tabular organization of the multiple integration by parts, continuing to the next row means applying Integration by parts to the product of the present row in which the selection of left and right functions Is already made. At this leave, if we desire, we can alter the selection of the left and right functions in such a way that the next row has a simpler expression. In other words, we can rearrange the factors in one level and start a new table for the arrow method. A method of organizing this alteration is illustrated In the following examples:

(4) $\int (\ln x)^2 x^n dx$

$$\begin{array}{ccc} (\ln x)^2 & & x^n \\ & \searrow + & \\ (2\ln x)\dfrac{1}{x} & & \dfrac{x^{n+1}}{n+1} \end{array}$$

$$\begin{array}{ccc} \dfrac{2\ln x}{n+1} & & x^n \\ & \searrow - & \\ \dfrac{2}{n+1}\,\dfrac{1}{x} & & \dfrac{x^{n+1}}{n+1} \end{array}$$

$$\int (\ln x)^2 x^n dx = \frac{x^{n+1}}{n+1}(\ln x)^2 - \frac{2}{(n+1)^2}x^{n+1}\ln x + \frac{2}{(n+1)^2}\int \frac{1}{x}\, x^{n+1} dx$$

$$= \frac{x^{n+1}}{n+1}(\ln x)^2 - \frac{2}{(n+1)^2}x^{n+1}\ln x + \frac{2}{(n+1)^3}\, x^{n+1} + C.$$

Notice that the third row is a rearrangement of the second row replacing it in the further application of integration by parts. An obvious advantage of this is avoiding the product rule when taking the derivative to arrive at the third row. This method can also be applied to avoid back arrows. For example, the above solution can also be arranged as follows:

$$\begin{array}{ccc} (\ln x)^2 & & x^n \\ & \searrow^{+} & \\ (2\ln x)\frac{1}{x} & & \frac{x^{n+1}}{n+1} \end{array}$$

$$\begin{array}{ccc} \frac{2\ln x}{n+1} & & x^n \\ & \searrow^{-} & \\ \frac{2}{n+1}\,\frac{1}{x} & & \frac{x^{n+1}}{n+1} \end{array}$$

$$\begin{array}{ccc} \frac{2}{(n+1)^2} & & x^n \\ & & \\ 0 & & \frac{x^{n+1}}{n+1} \end{array}$$

As another example of this technique, consider:

(5) $\int \text{Arcsin } x \, dx$

$$\begin{array}{ccc} \text{Arcsin } x & & 1 \\ & \searrow^{+} & \\ \frac{1}{\sqrt{1-x^2}} & & x \end{array}$$

$$\begin{array}{ccc} -\frac{1}{2} & & \frac{-2x}{\sqrt{1-x^2}} \\ & \searrow^{-} & \\ 0 & & 2\sqrt{1-x^2} \end{array}$$

$$\int \text{Arcsin } x \, dx = x \text{ Arcsin } x + \sqrt{1-x^2} + C.$$

By now, it should be dear how the arrow method is applied. The arrow method consists of the following steps to Integrate by parts.

Step **1**: Select a factor from the integration which Is easily differentiated (call this the left function) so that the remaining factor can be easily integrated (call **thls** the right function). Place the left function at the top of the first column and the right function at the top of the second column.

Step 2: Obtain the next row by differentiating the function on the left and integrating the function on the right (do not add any integration constant).

Step 3: Repeat step **2** until the row **product** is easily integrated by other integration techniques (for example, the product is a multiple of a power function) or a multiple of the original integrand.

Step **4**: Place arrows diagonally from left to right, one step down, for all rows except for the last. Place a back arrow for the last row.

Step 5: Label all arrows with signs + or - starting with + sign fo the top arrow and switch signs as you do down to the last (back) arrow. Thus, the signs on the arrows should alternate.

Step 6: Write the equation, stating that the original integral is the sum of the signed product of the ends of the arrows and the integral of the signed product of the ends of the back arrow.

Step 7: If the Integral of the product of last back arrow is a multiple of the original integral, then solve for this integral and add an integratlon constant. Otherwise, find the last integral using other integratlon techniques.

Remark: In the case when an integral for a row needs to be found using other integration techniques, it may be possible to rearrange the row into a new form for which the arrow method can be continued. The new form will start a new table which carries the sign of the back arrow for the first arrow. This of course avoids the back arrow and the final answer can be read from the table as illustrated in the examples **4** and 5.

References:

[1] G. B. Thomas, Jr., and R. L. Finney, Calculus and Analytic Geometry, 7th edition, Addison-Wesley Publishing Company, 1988.

[2] D. Horowltz, "Tabular Integration by Parts," The College Mathematics Journal, **21, 1990,** pp. **307-311.**

1990 National Pi Mu Epsilon Meeting

The Annual Meeting of the Pi Mu Epsiion National Honorary Mathematics Society was held at The Ohio State University in Columbus, August 8 through August 10. As in the past, the meeting was held in conjunction with the national meeting of the American Mathematical Society and the Mathematical Association of America.

The meeting began with a joint reception for MAA Student Chapters and Pi Mu Epsilon.

The MAA-PME Invited Address "Problems for All Seasons" was presented by Ivan Niven, University of Oregon.

The J. Sutherland Frame Lecturer was Ronald Graham, AT & T Bell Laboratories. He spoke on "Combinatorics and Computers: Coping with Finiteness."

The Annual Banquet was highlighted by several special presentations: J. Sutherland Frame gave a brief history of Pi Mu Epsilon. William Jaco, Executive Director of the AMS, presented Pi Mu Epsilon with a videotape of the 1989 AMS-MAA-PME lecture by Joseph Gailian. (To obtain this tape, see page 185). Marvin Wunderlich, Director of the Mathematical Sciences Program for the National Security Agency, spoke a few words of support for Pi Mu Epsilon. The NSA has again provided a generous grant to help the students who spoke at the meeting.

There were 30 student papers presented at the meeting:

PROGRAM - STUDENT PAPER SESSIONS

The Fibonacci Numbers Modulo 10	Tammy Anderson North Carolina Delta East Carolina University
Multiplication Models for Failure	Joel Atkins Indiana Gamma Rose-Hulman Institute of Technology
Conditions for a Perfect Join	Marjorie August Ohio Zeta University of Dayton
Applying Extrapolation for Archimedes's Approximation for Pi	James Baglama Ohio Xi Youngstown State University
TEX for Senior Projects	James M. Banoczi Ohio Xi Youngstown State University
On the Number of Independent Sets in a Graph	Wing Chan New York Pi SUNY at Fredonia
Duals of Two-Normed Spaces	Catherine Crosby Pennsylvania Rho Dickinson College
The Sum of the First n Integers	Beth-Allyn Eggens Ohio Xi Youngstown State University
The Hat Problem Revisited	Steve Elkins Arkansas Beta Hendrix College
Going in Cycles	Anna S. Fiehler Ohio Delta Miami University
The Game-Theoretic Analysis of Superior Beings	Francis Y.C. Fung Kansas Beta Kansas State University
Properties that Survive the Line Graph Operation	Colleen Galligher Ohio Zeta University of Dayton
Tensor Products and Finite Abelian Groups	David Gebhard Ohio Zeta University of Dayton
Greatest Common Divisors and Least Common Multiples of Graphs	Lisa Hansen Michigan Epsiion Western Michigan University
Approximation of the Trajectory of a Golf Ball	Richard Kinkela Ohio Xi Youngstown State University
Testing for Heteroscedasticity in Ordinary Least Squares	Robert E. Krulish South Carolina Gamma College of Charleston
Mathematical Models of Radiative Transfer Systems	Mark P. Kust Michigan Epsilon Western Michigan University
Expected Dimensions of a Vector Space	Mark Lancaster Arkansas Beta Hendrix College

Honey, I Shrunk the Bits!	Michael **Lang** Wisconsin Delta St. Norbert College
A Simplified **Smale's** Horseshoe Map	Derek **Ledbetter** Florida Delta University of Florida
Highly Regular Maps	Chikako Mese Ohio Zeta University of Dayton
Modem Cryptographic Methods	Lisa **Muccillo** Ohio Xi Youngstown State University
Linear Programming Versus Integer Programming	Scott **Odierno** Massachusetts Alpha Worcester Polytechnic Institute
Numerical Methods in Calculus of Variations	Lisa A. Pederson North Dakota Alpha North Dakota State University
Group Theory Through Card Shuffling	Donna E. Peers Illinois **Iota** Elmhurst College
Inversions and Adjacent Transpositions	Amy **Pinegar** Tennessee Gamma Middle Tennessee State University
Fibonacci Numbers	Brenda **L. Shultz** Ohio Xi Youngstown State University
On the Convergence of Ardength	John **Terilla** Ohio Delta Miami University
Fuzzy Controllers	John **Tverbakk** Texas Nu University of **Houston-**Downtown
Using a Spreadsheet to Generate Caley Tables for Groups	Wayne Young Texas Delta Stephen F. Austin State University

For the second consecutive year, the American Mathematical Society has given Pi Mu Epsilon a grant to be used as prize money for excellent student presentations. This year, five prizes of $100 each were awarded. The winning speakers were:

Anna **Fiehler,** Miami University,
"Going in Cycles"

Francis **Fung,** Kansas State University,
"The Game-Theoretic Analysis of Superior Beings"

Lisa **Hansen,** Western Michigan University,
'Greatest Common Divisors and Least Common Multiples of Graphs"

Richard **Kinkela,** Youngstown State University,
'Approximating the Trajectory of a Golf Ball"

Chikako **Mese,** University of Dayton,
"Highly Regular Graphs"

The AMS also presented Pi Mu Epsilon with a videotape of the 1989 AMS-MAA-PME address by Joseph **Gallian, entitled "The** Mathematics of Identification Numbers."

Message from the Secretary-Treasurer

Copies of the new, revised Constitution and Bylaws are now available. The prices are: $1.50 for each of the first four copies and $1 for each copy thereafter. **I.e., $(1.50** n) for $n < 4$ and $(n + 2) for $n \geq 4$.

The videotape of Professor Joseph A. **Gallian's** AMS-MAA-PME Invited Address. **"The** Mathematics of Identification Numbers," given as part of **PME's** 75th Anniversary Celebration at Boulder, CO, in August, 1989, is also now available. The tape may be borrowed free of charge by PME chapters, and by others upon an advance payment of $10. Please contact my office If you desire to borrow the tape, telling me the date on which you would like to use it. I prefer to mail the tape directly to faculty advisors, and expect them to take responsibility for returning **it** to my office. Please submit your request in writing and include a phone number and a time that I might reach you If there are problems. Robert M. Woodside, Secretary-Treasurer, Department of Mathematics, East Carolina University, Greenville, NC 27858.

PUZZLE SECTION

Edited by Joseph D. E. Konhauser
Macalester College

The PUZZLE SECTION Is for the enjoyment of those readers who ***are*** *addicted to working* ***doublecrostics*** *or who find an occasional mathematical puzzle or word puzzle* ***attractive.*** *We consider mathematical puzzles to be problems whose solutions consist of answers immediately recognizable as correct by simple observation and requiring little formal proof. Material submitted and not used here will be sent to the Problem Editor if deemed* ***suitable*** *for the PROBLEM DEPARTMENT.*

Address all proposed puzzles and puzzle solutions to Professor Joseph ***D. E.*** *Konhauser, Mathematics* ***and*** *Computer Science Department,* ***Macalester*** *College, St. Paul, MN 55105. Deadlines for puzzles appearing in the Fall Issue will be the next March 15, and for the puzzles In the Spring issue will be the next September 15.*

PUZZLES FOR SOLUTION

1. *Proposed by Clark Kinnaird, Flemlngton, NJ..*

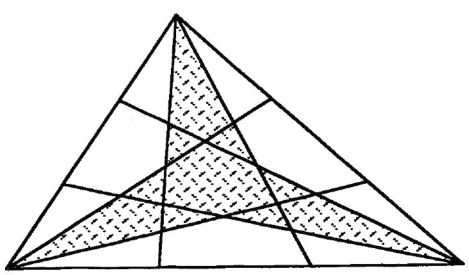
Find a fraction with value different from 1 which retains its **value** when turned upside down.

2. *Proposed by Clark Kinnaird, Flemington, NJ.*

In a three-horse race the odds on horses A. B and C are even, 2 to 1 and 10 to 1, respectively. How should one place bets on all three horses so that the bettor will come out exactly $5 ahead no matter which horse wins?

3. *Suggested by a problem in a work of John Pottage.*

As in the sketch (below left), lines drawn from the vertices of a triangle to the points of trisection of the opposite sides form a three-pointed "star." What fraction of the area of the triangle is covered by the interior of the three-pointed "star?"



0	1	−2	3
1	1	−1	−5
2	−6	0	4
−4	4	8	−4
	....		

4. *Proposed by the Editor of the Puzzle Section.*

Guess a pattern of formation for the four-column array (above right) and determine the elements of the 100th row.

5. *Proposed by the Editor of the Puzzle Section.*

(An oldie.) The three-member set **{2, 3, 5}** has the property that the product of any two members leaves a remainder of 1 when divided by the third. Are there **any other** triples of distinct positive integers with the same property?

6. *Contributed.*

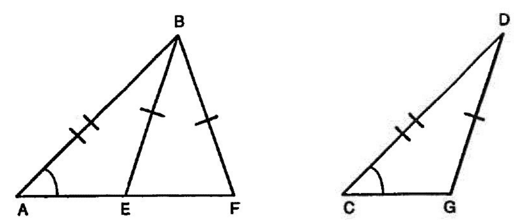
The rules for a **two-person** game played on a 3x3 board are as follows. Players take turns marking squares. In each turn a player marks one, two or three squares - as many as the player wishes, provided the squares are not already marked and provided the squares are in the same horizontal row or in the same vertical column. The squares need not be adjacent. The player marking the last square is the winner. Devise a strategy for the second player that will ensure a win for the second player.

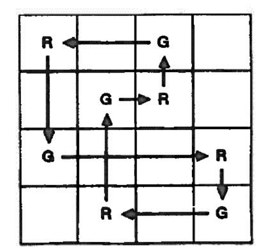
7. *Contributed.*

What is the smallest number of bishop's moves required to move a bishop from the upper left (white) corner of an 8x8 chessboard to the lower right (white) corner if each of the **32** white squares is to be occupied at least one time?

COMMENTS ON PUZZLES 1-7, SPRING I990

Ten readers responding to Puzzle # 1 gave the correct response 143x143 + 261 = 25033. CHARLES **ASHBACHER,** MARK EVANS and VICTOR G. FESER gave complete analyses and established uniqueness. For Puzzle **#** 2 nine readers responded. One said "No" and gave the rhombus as a counterexample. Two remarked **"... could** be a parallelogram **."** Two others said **"...** must be a **parallelogram."** A sixth "proved" that the quadrilateral with angle $A =$ angle C and $AB = CD$ must **be** a parallelogram. CARL **LIBIS,** VICTOR G. FESER and RICHARD **I.** HESS **gave examples** of auadrilaterals which are not parallelograms. Here is **Feser's: " ...** the sketch (below left) illustrates the two solutions of an ambiguous case; angle $A =$ angle C, $AB = CD$, $BF = DG$. The triangle on the right can now be **rotated** and moved so that D coincides with F and G with B. The resulting quadrilateral meets the conditions of the puzzle and need not be a parallelogram."





Seven readers responded to Puzzle **#** 3. Most showed that it is possible to select sixteen squares, two In each row and two in each column of an 8x8 board and color them **using** just **two** colors so that in each row and in each column there will be exactly one of each color. But this was not the point of the puzzle. The challenge was to show that the coloration is possible for every set of sixteen squares satisfying the conditions "two in each row and two in each column." RICHARD I. HESS responded this way: "Yes it will always be possible on any nxn grid. Pick one of the chosen squares and label it red, move down its column and label the remaining marked square green, move across its row and label the remaining marked square red, and so on. An example for 4x4 is shown (above right). If a path completes, start again with an uncolored square and continue until all are labeled. Since at the start there are only two squares to be colored in each row and column, the path is forced and will eventually return to the start along a row pointing from green to red." The result is a consequence of the theorem that a 2-regular bipartite graph can be decomposed into the product of two 1-factors. Only DOUG GROVE and EMIL SLOWINSKI responded to Puzzle # 4. Both succeeded in filling fourteen squares satisfying the specified conditions. Their solutions are reproduced below. Can anyone do better? Can anyone fill more than 19 squares for a 7x7 array?

				5	6
5			6		
			1	3	4
3	4	6			
		1			2
1	2				

Grove

			3	5	6
	3				4
3		6			
	1		2	4	
2		5			
1		4			

Slowinski

For Puzzle # 5, one faulty solution was submitted - two chords were parallel. Five other contributors claimed correctly that the construction was impossible but provided arguments too lengthy to reproduce here. The puzzle is well-known. The Editor first saw it in the May, 1957, Mathematical Gazette, Vol. XLI. It was presented by D. J. Behrens who linked it with the design of movements for duplicate bridge competitions. A solution exists if the number of points is not divisible by any Fermat prime. For discussions of the problem, see the October, 1958, Mathematical Gazette, Vol. XLII. For Puzzle # 6, BILL BOULGER, DOUG GROVE, RICHARD I. HESS and EMIL SLOWINSKI supplied the correct response of 14 for the number of different ways 28 axbxc bricks can be arranged to form rectangular solids. For n bricks, for the number of different ways, Hess gave the formula

$$\prod_{i=1}^{k} \frac{(a_i+1)(a_i+2)}{2}, \text{ where } n = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$$

Nine readers responded to Puzzle # 7. Five gave the correct result - in 1924 there are seven dates satisfying the "product" condition (1/24, 2/12, 3/8, 4/6, 6/4, 8/3 and 12/2). Two readers gave 1960, but in 1960 there are only six occurrences. Other years with six are 1912. 1930. 1936. 1948 and 1972. MARK EVANS remarked that 1960 would have seven dates if February had 30 days rather than 29." VICTOR FESER pointed out that the puzzle has a history, an important part of which is his paper "Product Dates" in the October, 1972, issue of the Journal of Recreational Mathematics. Feser also remarked that the puzzle appears in H. E. Dudeney's 536 Puzzles and Curious Problems, edited by Martin Gardner, Scribner's Sons, New York, 1967, pages 72, 279-80.

Solvers: Charles Ashbacher (1, 2, 3, 5, 7), Bill Boulger (1, 2, 3, 5, 6, 7), Mark Evans (1, 2, 3, 5, 7), Mark R. Fahey (1, 2, 3, 7), Victor G. Feser (1, 2, 7), Doug Grove (1, 2, 3, 4, 5, 6, 7), Richard I. Hess (1, 2, 3, 5, 6, 7), Michael W. Lanstrum (7), Carl Libis (1, 2), Tom Monikowski (1) and Emil Siowinski (1, 2, 3, 4, 5, 6, 7).

Solution to Mathacrostic No. 30 (Spring 1990)

WORDS:

A. Rowland's law
B. Phon
C. Elisha Otis
D. Nutty putty
E. Romansh
F. Off-track
G. Scotch whist
H. Eurythmy
I. Throwaway
J. Hypatia
K. East-windy
L. Equidistant alleys
M. Menger's sponge
N. Planiverse
O. Euhedral
P. Roundabout
Q. Off the wall
R. Roche's limit
S. Strangled torus
T. New wave
U. Exhaustion
V. Wish-wash
W. Mr. Puncto
X. inchoative
Y. Net of rationality
Z. DeSitter's cosmos

AUTHOR AND TITLE: R. PENROSE THE EMPEROR'S NEW MIND

QUOTATION: How do we know that classical physics is not actually true of our world? The main reasons are experimental. Quantum theory was not wished upon us by theorists. It was (for the most part) with great reluctance that they found themselves driven to this strange, and, in many ways, philosophically unsatisfying view of a world.

SOLVERS: THOMAS F. BANCHOFF, Brown University, Providence, RI; JEANETTE BICKLEY, St. Louis Community College at Meramec, MO; CHARLES R. DIMINNIE, St. Bonaventure University, NY; ROBERT FORSBERG, Lexington, MA; META HARRSEN, New Hope, PA; MICHELE HEIBERG. Herman, MN; DR. THEODOR KAUFMAN, Brooklyn, NY; CHARLOTTE MAINES, Rochester, NY; DON PFAFF, University of Nevada - Reno; ALLEN J. SCHWENK, Western Michigan University, Kalamazoo; STEPHANIE SLOYAN, Georgian Court College, Lakewood, NJ; and JOSEPH C. TESTEN, Mobile, AL.

Mathacrostic No. 31

Proposed by Joseph D. E. Konhauser

The 251 letters to be entered in the numbered spaces in the grid will be identical to those in the 28 keyed words at the matching numbers. The key numbers have been entered in the diagram to assist in constructing the solution. When completed, the initial letters of the Words will give the name and an author and the title of a book; the completed grid will be a quotation from that book.

Definitions	Words
A. Lorenz-given name to phenomenon of sensitive dependence on initial conditions (2 wds.)	111 239 183 87 97 221 159 31 72 119 168 13 204 19 83
B. belief that no set of morals can be established scientifically, and hence all are equally valid	43 113 235 135 69 195 146 123 205 179
C. a key notion in Klein's Erlanger Programm	121 98 28 236 57 160 248 145 137 210
D. a nickname for the Mandelbrot set (2 wds.)	228 37 176 187 243 18 51 77 200 193 4 103 162 125
E. configurations which can only appear as initial states of an automaton (3 wds.)	177 60 109 244 238 166 75 147 16 27 128 47 223
F. device used to measure the "speed" of a green (golf) (2 wds.)	169 219 42 126 199 96 78 131 3 211
G. a cipher found in the Old Testament (used in Jewish mystical and allegorical writing)	20 99 184 133 222 49 70
H. seasickness	218 110 64 241 230 35 174 40 130
I. kind of structure capable of maintaining its identity only by remaining continually open to the flux and flow of its environment	118 141 107 101 50 234 34 1 175 209 129
J. ensemble of points corresponding to the states of a dynamic system (2 wds.)	61 117 127 233 95 215 139 132 120 80
K. a name for the hexadecimal digit whose decimal equivalent is 14	150 206 41 212
L. In Pythagoreanism, a planet which "shielded" the earth from the direct rays of the central fire"	190 6 173 158 59 225 21 138 124 214
M. the bearing of the name of a natural object or animate being by a human group	196 185 203 71 155 100 11 89
N. used to excite luminous discharges in glass vacuum apparatus (2 wds.)	163 53 91 194 66 116 140 9 148
O. a point on a surface at which the curvature has the same value for all normal sections	17 38 220 88 182 62 161
P. the upward curve at the foot of a square sail (naut.)	48 76 154 189 2
Q. nickname for Turing machine which "writes" the maximum number of symbols (say 1's) for a given number of states (2 wds.)	249 191 178 5 14 90 202 79 58 44
R. loops without crossings	108 142 231 24 198 56 151
S. an attracting set to which orbits or trajectories converge and upon which the dynamics are periodic (2 wds.)	157 15 171 245 32 237 197 10 52 208
T. quasi-periodic warming of the upper ocean off Peru and Equador with sometimes disastrous effects on the climate (2 wds.)	105 82 67 22 227 93
U. Native American language utilized in World War II for battlefield communication	240 30 94 7 188 170
V. basaltic glass	143 81 224 33 251 112 54 216 39
W. double decomposition	104 45 68 217 247 84 55 192 164 25
X. constants	102 92 8 165 26 63 180 250 122
Y. reason	201 152 229 144 167 246 213 136 73 86 29
Z. iterative sound	242 115 36 85 156 181 106 153
a. in a watch, a smoothed jeweled bearing	23 65 226 149 172
b. rolled backward or downward	134 232 46 12 74 186 207 114

1 I	2 P	3 F		4 D	5 Q	6 L	7 U	8 X	9 N	10 S	11 M		12 b	13 A
	14 Q	15 S	16 E	17 O	18 D	19 A	20 G	21 L	22 T	23 a	24 R	25 W		26 X
27 E	28 C	29 Y	30 U	31 A		32 S	33 V	34 I	35 H		36 Z	37 D	38 O	39 V
	40 H	41 K		42 F	43 B	44 Q	45 W	46 b	47 E	48 P	49 G	50 I	51 D	52 S
53 N		54 V	55 W	56 R		57 C	58 Q	59 L	60 E	61 J	62 O	63 X	64 H	65 a
66 N	67 T	68 W		69 B	70 G	71 M	72 A		73 Y	74 b	75 E	76 P		77 D
78 F	79 Q	80 J	81 V	82 T		83 A	84 W	85 Z	86 Y		87 A	88 O	89 M	90 Q
91 N		92 X	93 T	94 U	95 J	96 F	97 A	98 C	99 G		100 M	101 I		102 X
103 D	104 W	105 T	106 Z	107 I	108 R	109 E	110 H	111 A	112 V	113 B		114 b	115 Z	116 N
117 J		118 I	119 A	120 J	121 C	122 X	123 B	124 L	125 D		126 F	127 J	128 E	129 I
	130 H	131 F		132 J		133 G	134 b	135 B	136 Y	137 C	138 L		139 J	140 N
141 I	142 R	143 V		144 Y	145 C	146 B	147 E	148 N	149 a	150 K	151 R		152 Y	153 Z
	154 P	155 M	156 Z	157 S	158 L	159 A	160 C	161 O	162 D	163 N	164 W	165 X	166 E	
167 Y	168 A		169 F	170 U	171 S	172 a	173 L	174 H	175 I	176 D	177 E		178 Q	179 B
180 X	181 Z	182 O		183 A	184 G	185 M	186 b	187 D	188 U		189 P	190 L	191 Q	192 W
193 D	194 N	195 B	196 M	197 S		198 R	199 F	200 D	201 Y	202 Q	203 M	204 A	205 B	
206 K	207 b		208 S	209 I	210 C	211 F	212 K		213 Y	214 L	215 J	216 V	217 W	218 H
219 F		220 O	221 A	222 G	223 E	224 V	225 L	226 a	227 T	228 D		229 Y	230 H	231 R
232 b	233 J		234 I	235 B	236 C	237 S	238 E		239 A	240 U	241 H	242 Z	243 D	244 E
245 S	246 Y	247 W	248 C	249 Q	250 X	251 V								

PROBLEM DEPARTMENT

Edited by Clayton W Dodge
University of Maine

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also Invited. Proposals should be accompanied by solutions if available and by any Information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

All communications should be addressed to C. W. Dodge, Math. Dept., University of Maine, Orono, ME 04469. Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly identified with name and address. Solutions to problems in this issue should be mailed by July 1, 1991.

Problems for Solution

719 [Spring 1990]. Corrected. Proposed by John M. Howell, Littlerock, California.

Professor E. P. B. Umbugio translated Problem 626 [Fall 1986, Fall 1987] into Spanish, as shown below. Since he didn't like zeros because they reminded him of his score on an IQ test, he used only the nine nonzero digits. He found solution- in which 2 divides DOS, 3 divides TRES, and 6 divides SEIS. Find that solution in which also 7 divides SEIS and 9 divides DOS.

UNO + DOS + TRES = SEIS.

732. Proposed by Man Wayne, Holiday, Florida.

The following is a partially enciphered multiplication:

(AY)(HARD) = 21340.

Restore the dig'ts. Of whom might it have been said that his mathematics was "AY HARD?"

733. Proposed by Roger Pinkham, Stevens Institute of Technology, Hoboken, New Jersey.

If $p(x)$ is a polynomial and $p(x) \geq 0$ for all x, then

$$p + p' + p'' + \dots \geq 0$$

for all x.

734. Proposed by Mohammad K. Azarian, University of Evansville, Evansville, Indiana.

Let f and g be two real-valued functions defined on the set of positive integers with the following properties:

a) $f(1) = g(1)$ and $f(2) = g(2)$;
b) $f(n) > g(n)$ for $n \geq 3$;
c) there are Infinitely many pairs (m,n) such that $f(m) = g(n)$ and $m > n > 2$; and
d) $\lim_{n\to\infty} f(n) = \lim_{n\to\infty} g(n) = L$, a finite real number.

Show that there are infinitely many functions f and g satisfying these conditions and find formulas for them.

*735. Proposed by Robert C. Gebhardt, Hopatcong, New Jersey.

If a and b are roots of the equation $x^2 + 7x - 3 = 0$, prove that

$$a^3 + b^3 + 7(a^2 + b^2) - 3(a + b) = 0,$$

and, without solving the equation, find the values of

(i) $$a^3 + b^3,$$

and (ii) $$\frac{a+2}{b+1} + \frac{b+2}{a+1}.$$

This problem was taken from the Pure Mathematics section of the Intermediate Examinations in Engineering, Mining and Metallurgy, given by the University of London, November 1946.

736. Proposed by Willie Yong, Singapore, Republic of Singapore.

Into a rectangle with sides 20 and 25 units, 120 squares are thrown, each with side 1. Show that inside the rectangle a unit circle may be drawn which does not intersect any of the squares. This is a 10th class problem from the 24th Mathematics Olympiad organized by Moscow State University, 1961.

737. Proposed by Timothy Sipka, Alma College, Alma, Michigan.

The California Lottery offers a daily card game called Decco, where a player selects 4 cards from a standard deck. one from each suit. It costs $1 to play, and prizes are awarded according to the number of cards that match the state's randomly selected set of four. One match gives a free replay ticket, two matches earn $5, three yield $50, and four matches produce $5000. Determine the avid player's expectation, the average profit or loss, for this game of chance.

738. Proposed by Man Wayne, Holiday, Florida.

If $[x]$ denotes the greatest integer less than or equal to x, prove that for any nonnegative integer n,

$$[n^{1/2} + (n+1)^{1/2}] = [(4n+1)^{1/2}].$$

739. Proposed by R. S. Luthar, University of Wisconsin Center, Janesville, Wisconsin.

Solve the equation

$$\sqrt{x^3 + 2x^2 - 11x + 12} - \sqrt{x^3 + x^2 - 13x + 11} = x + 1.$$

740. Proposed by J. S. Frame, Michigan State University, East Lansing, Michigan.

The Euler numbers E_j may be defined by the series

$$\sec x = \sum_{j=0}^{\infty} E_j \frac{x^{2j}}{(2j)!}.$$

The first few Euler numbers are

$$E_0 = E_1 = 1,\ E_2 = 5,\ E_3 = 61,\ E_4 = 1385, \text{ and } E_5 = 50521.$$

Prove that, for all $j > 0$, the E_j satisfy the congruences

$$E_{2k+1} \equiv 1 + 60k \pmod{1440} \quad\text{and}\quad E_{2k+2} \equiv 5 - 60k \pmod{1440}.$$

741. Proposed by John *M.* Howell, *Littlerock,* California.

*a) What numbers cannot be a leg of a Pythagorean triangle?

*b) What numbers cannot be a hypotenuse of a Pythagorean triangle?

c) What numbers can be neither a leg nor a hypotenuse of a Pythagorean triangle?

***742.** Proposed by Jack Garfunkel, Flushing, *New* York.

Construct squares outwardly on the sides of a triangle ABC. Prove or disprove that the centers A', **B'**, and C' of these squares form a triangle that is closer to being equilateral than is ABC. A proof would show that if the process were repeated on triangle **A'B'C'**, etc., that triangle $A^nB^nC^n$ would approach equilateral as n approached infinity.

743. Proposed by *R.* S. *Luthar,* University of Wisconsin Center, Janesville, Wisconsin.

Let A and B be the ends of the diameter of a semicircle of radius r and let P be any point on the semicircle. Let **I** be the **incenter** of triangle APB. Find the locus of **I** as P moves along the semicircle.

744. Proposed by Jack Garfunkel, Flushing, *New* York.

Let triangle ABC be inscribed in a circle. Draw a line through A to intersect side BC at D and the circle (again) at **E.** Without resorting to the calculus, prove that **AD/DE** is a minimum when AD bisects angle A.

Solutions

633. [Fall **1986,** Fall 1987, Fall **1988]** Proposed by *Dmitry* P. *Mavlo,* Moscow, USS. *R.*

Let a. b, c > 0, a + b + c = 1, and n $\in$ N. Prove that

$$\left[\frac{1}{a^n} - 1\right]\left[\frac{1}{b^n} - 1\right]\left[\frac{1}{c^n} - 1\right] \geq (3^n - 1)^3,$$

with equality if and only if a = b = c = **1/3.**

III. Further comment by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.

Employing the same method of solution as given by Chris Long in [Fall 1988, p. **603],** one can generalize the inequality to

$$[1/a_1^n - 1][1/a_2^n - 1]\cdots[1/a_m^n - 1] \geq (m^n - 1)^m,$$

where $a_1, a_2, \ldots, a_m > 0$, $a_1 + a_2 + \ldots + a_m = 1$, and **m,** n $\in$ N. As an open question, determine whether or not the latter inequality is valid for all real n > 1.

678. [Fall **1988,** Fall **1989]** Proposed by Brian Conrad, Centereach High School, Centereach, New York.

Find all solutions to this base ten multiplication alphameric in honor of my Soviet mathematician and theoretical physicist pen pal who also is a regular contributor to this department:

DMITRI = **P•MAVLO.**

I. Comment by Victor G. Feser, University of Mary, Bismarck, North Dakota.

In the published solution Alan Wayne claims "...to solve this problem on my small computer takes more than 500 hours..." **I** wrote the program below for a Kaypro PC with no fast

chip. It found both solutions in 3 minutes.

```
5 CLS: GOTO 20
10 CC = INT(MM/BB): DD=MM-CC*BB: RETURN
20 DEFINT A-Z
25 BB=10
30 FOR P=2 TO 9
40 FOR O = 2 TO 9:IF O=P THEN 390
45 PRINT "P, O =" P;O
50 MM=P*O: GOSUB 10: IF DD=P OR DD=O THEN 390 ELSE I=DD:C1=CC
60 FOR L=O TO 9:IF L=P OR L=O OR L=I THEN 380
70 MM=P*L+C1: GOSUB 10: IF DD=P OR DD=O OR DD=I OR DD=L THEN 380 ELSE
R=DD:          C2 = CC
80 FOR V=O TO 9: IF V=P OR V=O OR V=I OR V=L OR V=R THEN 370
90 MM=P*V+C2: GOSUB 10: IF DD=P OR DD=O OR DD=I OR DD=L OR DD=R OR
DD=V  THEN 370 ELSE T=DD: C3 = CC
100 FOR A=O TO 9: IF A=P OR A=O OR A=I OR A=L OR A=R OR A=V OR A=T THEN
360
110 MM=P*A+C3: GOSUB 10: IF DD<>I THEN 360 ELSE C4 = CC
120 FOR M=1 TO 9: IF M=P OR M=O OR M=I OR M=L OR M=R OR M=V OR M=T OR
M=A   THEN 350
130 MM=P*M+C4: GOSUB 10: IF DD<>M OR CC=0 THEN 350 ELSE D=CC
140 IF D=P OR D=O OR D=I OR D=L OR D=R OR D=V OR D=T OR D=A OR D=M
THEN  350
200 LPRINT M;A;V;L;O "x" P "=" D;M;I;T;R;I
350 NEXT M
360 NEXT A
370 NEXT V
380 NEXT L
390 NEXT O
400 NEXT P
```

Output of the program:

3 2 6 9 5 x 4 = 1 3 0 7 8 0

5 0 9 1 8 x 7 = 3 5 6 4 2 6

704. [Fall **1989]** Proposed by the late Charles W. Trigg, San *Diego,* California.

Find the least HEAT necessary to BOIL the H_2O:

HEAT + HHO = BOIL

Solution by ALMA COLLEGE PROBLEM SOLVING GROUP, Alma College, Alma, Michigan.

We see that **H, E,** A, **T,** O, and B must all be nonzero; if there is a carry anywhere, it must be 1; and since H $\neq$ **B,** then there is a carry to H. Since we want to minimize HEAT, we **try** H = 1. **Then E must be 8 or 9, so E** + H +carry = **10 or 11, and O** = **0 or 1. If O** = 0, **then T** = L, and if O = 1, then O = **H.** Hence we cannot have H = 1.

So we try H = 2. **Then B** = **3 and E** = **7 or 8 or 9. Now E** $\neq$ **7 because O cannot be** 0. If **E** = 8, we must have a carry to the hundreds column and O = 1. Then A = 7 or 9. If A = 7, then **I** = 0 and there is a carry to the tens column. Hence T = 9 and L = 0, impossible **since**

then I = L. If I = 9, then there is no carry to the tens column and 1 = 1, another contradiction since I = O.

Thus we try H = 2 and E = 9. There cannot be a carry to the hundreds column since then O = H = 2. Hence O = 1. To minimize HEAT choose A = 4, so I = 6. Now T = 7 and L = 8. We obtain

$$2947 + 221 = 3168.$$

Also solved by CHARLES ASHBACHER, *Hiawatha*, IA, MARK EVANS, Louisville, *KY*, VICTOR G. FESER, University of Mary, Bismarck, *ND*, RICHARD I. HESS, Rancho Palos Verdes, *CA*, CARL LIBIS, Granada Hills, *CA*, MIKE PINTER, Belmont College, Nashville, TN, ST. OLAF PROBLEM SOLVING GROUP, St. Olaf College, Northfield, *MN*, L. J. UPTON, Mississauga, Ontario, Canada, KENNETH M. WILKE, Topeka, KS, and the PROPOSER.

705. [Fall **1989**] Proposed by the late Charles W. *Trigg*, San *Diego*, California.

In this **Ovls** group, the EWES and every LAMB are in prime condition. Find the two solutions:

$$\text{RAM} + \text{EWES} + \text{LAMB} + \text{LAMB} = \text{SHEEP}.$$

Comment by the Editor.

Although several correspondents submitted answers, no one submitted a solution to this problem. The unique answer is

$$390 + 4241 + 6907 + 6907 = 18445.$$

The editor deserves 50 lashes with a wet lamb's **tail** for changing the proposer's problem by inserting the statement "Find the two solutions." There is just one solution, as the proposer stated. The editor had checked the solution with a computer program that overlooked one slight detail: that E and L had to be distinct.

Also solved by CHARLES ASHBACHER, Hiawatha, IA, VICTOR G. FESER, University of Mary. Bismarck, *ND*, RICHARD I. HESS, Rancho Palos Verdes, CA, L. J. UPTON, Mississauga, Ontario, Canada, and the PROPOSER.

It was Feser who insisted that there was only one solution, showing that the editor's second answer was incorrect. He also raised the question as to what is the best method using BASIC to test that a new variable is distinct from all previously evaluated variables. His tests using interpreted BASIC showed that and and or are slightly faster than Boolean statements; "if A=B or **A=C,** then ..." is faster than "if (A=B)+**(A=C)**, then ..." The editor's corresponding tests using compiled basic (Borland's Turbo Basic) showed no difference in the time needed to run the statement types, including the above two forms and also the form "if **(a-b)*(a-c)** = 0, then ..."

706. [Fall **1989**] Proposed by John Dalbec, Ohio Xi Chapter, Youngstown State University, *Youngstown*, Ohio.

This alphametric is too "compact" to have a unique solution. If, however, one **CECHs** for primality, then there is just one conclusion:

$$\text{STONE} + \text{CECH} = \text{LECAR}.$$

Solution by KENNETH M. WILKE, Topeka, Kansas.

Let c_i denote the carry resulting from the addition in the **ith** column counting from the right. Note that c_4 must be 1 so that L = S + 1. Then the third and fourth columns yield

(1) $$c_2 + O + E = C + 10c_3$$
and
(2) $$c_3 + T + C = E + 10c_4 = E + 10.$$
These produce
(3) $$T + O + c_2 = 10 + 9c_3.$$

Since c_2 is at most 1, then $c_3 = 0$. Furthermore, none of **S, L, C, E,** and H can be zero.

Case **1,** $c_2 = 1$, so T + O = 9. Given a choice of T and O, **ralations** (1) and (2) determine C and E since C - E = O + 1. Since CECH is prime, then H = **1,** 3, 7, or 9. **For** each combination of **T,** O, **C,** E, and **H,** a unique set of unused digits is determined. Then E and H determine all possible choices for **R.** Then the remaining digits **can** be checked for consecutive digits for S and L and for possible values for N and A. I found the following solutions:

$$65413 + 8389 = 73802, \quad 24581 + 7179 = 31760$$
$$\text{and} \quad 57281 + 4149 = 61430.$$

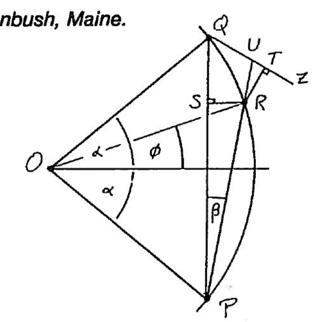
Now CECH is prime only for 8389, so the first solution is the correct one.

Also solved by CHARLES ASHBACHER, Hiawatha, *IA*, VICTOR G. FESER, University of Mary, Bismarck, ND, RICHARD I. HESS, Rancho Palos Verdes, CA, L. J. UPTON, Mississauga, Ontario, Canada, and the PROPOSER.

707. [Fall **1989**] Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.

From a point **R** taken on any circular arc **PQ** of less than a quadrant, two segments are drawn, one to an extremity P of the arc and the other RS perpendicular to the chord **PQ** of the arc and terminated by it. Determine the maximum of the sum PR + RS of the lengths of these two segments. This problem without solution is given in Todhunter's Trigonometry.

I. Solution by OLE *ANDERS, Greenbush, Maine.*



Let O be the center of the circle, let $2\alpha = \sphericalangle POQ$, and $\beta = \sphericalangle RPQ$. Then $\alpha \le 45°$, $\beta \le 45°$ since β is inscribed in arc **PQ,** and $\sphericalangle RQS = \alpha - \beta \le 45°$ since it is inscribed in arc PR. As shown in the figure, draw **QZ** so that $\sphericalangle RQZ = \sphericalangle RQS$ and drop a perpendicular **RT** from **R** to line **QZ.** Extend PR to cut **QZ** at U. Now $\sphericalangle SQU = 2\alpha - 2\beta < 90°$, so $\sphericalangle PUQ = 180° - 2\alpha + \beta \ge 90°$ and **PQ** > PU. Because $\sphericalangle SQR = \sphericalangle TQR$, then **RS** = **RT,** which is < RU. Hence

$$PR + RS = PR + RT < PR + RU = PU < PQ.$$

Clearly, as **R** approaches **Q,** PR + **RS** approaches PQ. Hence **PQ,** the length of the chord, is the upper limit for PR + RS.

II. Solution by RICHARD I. HESS, Rancho Palos Verdes, California.

Place the center of the circle at the origin so that the x-axis bisects the arc PQ. Then there is an **angle** $\alpha \le 45°$ such that **P**$(\cos\alpha, -\sin\alpha)$ and **Q**$(\cos\alpha, \sin\alpha)$. Let **R**$(\cos\phi, \sin\phi)$. Then $-\alpha < \phi < \alpha$ and $S = (\cos\alpha, \sin\phi)$. Now

$$PR = 2\sin(\alpha + \phi)/2 \quad \text{and} \quad RS = \cos\phi - \cos\alpha,$$

so we define

$$f(\alpha,\phi) = PR + RS = \cos\phi - \cos\alpha + 2\sin(\alpha + \phi)/2.$$

Now

$$\frac{\partial f}{\partial \phi} = -\sin\phi + \cos\frac{\phi + \alpha}{2} = 0$$

implies

$$\cos(\alpha + \phi)/2 = \sin\phi = \cos(90° - \phi).$$

Since the involved angles are all acute, then we must have

$$(\alpha + \phi)/2 = 90° - \phi,$$

so

$$\phi = 60° - \alpha/3 > \alpha \quad \text{for } \alpha < 45°.$$

Therefore,

$$\frac{\partial f}{\partial \phi} \ne 0 \quad \text{for} \quad \phi < 45°.$$

It follows that $PR + RS$ is maximized for $\phi = \alpha$, which implies that $R = Q$ and $PR + RS \le PQ$.

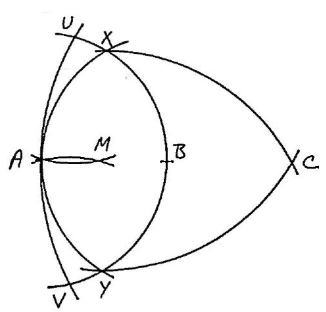
Also solved by **SEUNG-JIN** BANG, Seoul, Korea, HENRY S. **LIEBERMAN**, ***Waban,*** MA, PROBLEM SOLVING GROUP, University of Arizona, Tucson, ***and*** the PROPOSER.

708. [Fall **1989**] Proposed by JACK GARFUNKEL, Flushing, New York.

Find a Mascheroni construction (a construction using only compasses – no straightedge allowed) for the orthic triangle of an acute triangle ABC.

Solution by the Proposer.

The following construction locates the midpoint of a given segment AB. Draw the circle **A(B),** the circle with center A and passing through **B,** and the circle **B(A)** to intersect at X and Y. Draw the circles X(Y) and Y(X) to meet at **C,** the **intersection** nearer B. Then B is the midpoint of AC. Draw circle **C(A)** to meet circle **A(B)** at U and V. Draw circles **U(A)** and **V(A)** to meet again at **M,** the desired midpoint of AB. The proof is left for the reader to supply. See Eves, A Survey of Geometry, rev. ed., **Allyn** and Bacon, 1972, pp. 172, 173, 407, especially Exercise 3.



Using the **above** construction, find the midpoints of the three sides of the triangle and then draw the three circles whose diameters are the sides of the triangle. Let the circle on **BC** as diameter cut AC at **Q** and AB at **R.** Since angles BQC and CRB are each inscribed in **a** semicircle, they are right angles. Similarly locate the foot P of the altitude from A to BC. Then **PQR** is the orthic triangle.

709. [Fall 1989] Proposed by Norman ***Schaumberger,*** Bronx Community College, Bronx, New York.

If a, **b,** and c are the lengths of the sides of a triangle and if K and P are the area and perimeter, **respectively,** then prove that

$$a^2b^2 + b^2c^2 + c^2a^2 \ge 12K^2 + \frac{P^4}{108}$$

with equality if and only if the triangle is equilateral.

Solution by JACK GARFUNKEL, Flushing, New York.

From Heron's formula for the area of a triangle

$$16K^2 = 2\sum a^2b^2 - \sum a^4 \quad \text{or} \quad \sum a^2b^2 = 8K^2 + \frac{1}{2}\sum a^4.$$

We will prove the equivalent inequality

$$8K^2 + \frac{1}{2}\sum a^2 \ge 12K^2 + \frac{P^4}{108},$$

or

$$4K^2 \le \frac{1}{2}\sum a^4 - \frac{(a + b + c)^4}{108}.$$

item 4.10 of O. Bottema. Geometric inequalities. states that

$$\frac{a^4 + b^4 + c^4}{4} \ge 4K^2.$$

So, a sharper **inequality** to prove is

$$\frac{a^4 + b^4 + c^4}{4} \le \frac{1}{2}\sum a^4 - \frac{(a + b + c)^4}{108},$$

which reduces to

$$a^4 + b^4 + c^4 \ge \frac{(a + b + c)^4}{27},$$

which is true by the power mean inequality.

Also solved by MURRAY S. **KLAMKIN,** University of Alberta, Canada, DAVID E. MANES, ***SUNY*** at ***Oneonta,*** **YOSHINOBU MURAYOSHI,** Eugene, OR, BOB **PRIELIPP,** University of ***Wisconsin-Oshkosh,*** and the PROPOSER.

710. [Fall **1989**] Proposed by Thomas E. Moore, Bridgewater State College, Bridgewater, Massachusetts.

Under what conditions on the positive integers a and b will the sides of a **nondegenerate** triangle be formed by

a) a, b, and gcd(a,b)?
b) a, b, and lcm[a,b]?

Solution by DEREK LEDBETTER, University of Florida, Gainesville, Florida.

Let $A = a/\gcd(a,b)$ and $B = b/\gcd(a,b)$.

a) Then A, B, and $\gcd(A,B) = 1$ are the sides of an integral-sided triangle similar to the given one. Suppose $A \geq B$. For a nondegenerate triangle we must have $B + 1 > A$, so $B \geq A$. Hence $A = B$. The given triangle, then has $a = b$ and is equilateral.

b) Then A, B, and AB are the sides of an integral-sided triangle similar to the given one. Suppose $A \geq B$. To have $A + B > AB$ we must have $B = 1$. This implies that b divides a. Hence we have an isosceles triangle with equal sides a and lcm[a,b] and base b, where b divides a.

Also *solved* by ALMA COLLEGE PROBLEM SOLVING GROUP, *MI,* MARK EVANS, Louisville, *KY,* VICTOR G. FESER, University of Mary, Bismarck, ND, RICHARD I. HESS, Rancho Palos Verdes, *CA,* HENRY S. LIEBERMAN, Waban, MA, DAVID E. MANES, *SUNY* at *Oneonta,* MIKE PINTER, Belmont College, *Nashville, TN,* ST. OLAF PROBLEM SOLVING GROUP, St. Olaf College, Northfield, MN, KENNETH M. WILKE, Topeka, KS, and the PROPOSER. One incorrect solution was received.

711. [Fall 1989] Proposed by James N. Boyd, St. Christopher's School, Richmond, Virginia.

A pentagon is constructed with five segments of lengths 1, 1, 1, 1, and w. Find w so that the pentagon will have the greatest area.

Solution by MURRAY S. *KLAMKIN, University* of Alberta, Edmonton, Alberta, Canada.

More generally, assume there are n segments of length 1 with $n > 1$ and one of length, w. Then by reflection of the polygon across the segment of length w, the problem reduces to finding the maximum area polygon consisting of 2n sides of unit length. As well known, the polygon will have to be regular and w will then be a diameter of the circumcircle. Hence

$$w = 2R = \csc\frac{\pi}{2n}.$$

Also the maximum area of the (n+1)-gon is

$$\frac{1}{2}nR^2\sin\frac{\pi}{n} = \frac{1}{4}n\cot\frac{\pi}{2n}.$$

Also *solved* by MARK EVANS, Louisville, *KY,* JACK GARFUNKEL, Flushing, *NY,* RICHARD I. HESS, Rancho Palos Verdes, CA, UNIVERSITY OF ARIZONA PROBLEM SOLVING GROUP, Tucson, and the PROPOSER.

712. [Fall 1989] Proposed by Robert C. *Gebhardt,* Hopatcong, New Jersey.

A cube 4 inches on a side is painted. Then it is cut into 64 one-inch cubes. A cube is chosen at random and tossed. Find the probability that none of the five faces that are showing is painted.

Amalgam of independent solutions submitted by FRANK P. BATTLES and LAURA L. *KELLEHER (jointly),* Massachusetts Maritime Academy, Buzzards Bay. Massachusetts, CYNTHIA *COYLE,* Trenton State College, Laurel Springs, New Jersey, and MARK *EVANS,* Louisville, Kentucky.

More generally we consider a cube n inches on a side. Of the n^3 one-inch cubes, the 8 corner cubes will have three faces painted. Each of its 12 edges will have $n - 2$ cubes (all except the two end cubes) with exactly two faces painted, for a total of $12(n - 2)$ such cubes. On each of its 6 faces there is an $(n - 2)$ by $(n - 2)$ square of cubes having just 1 face painted, so there is a total of $6(n - 2)^2$ such cubes. Finally, we observe that all the cubes on the outside have at least one face painted and all the interior cubes are unpainted. That is. there are $(n - 2)^3$ cubes having no faces painted.

No painted face will show if a cube with 1 painted face is tossed and falls on that face (with probability 1/6) or if a cube with no painted faces is tossed (probability 1). The probability, then, that no painted face shows when a cube is selected at random and tossed is equal to

$$P = \frac{6(n-2)^2}{n^3}\cdot\frac{1}{6} + \frac{(n-2)^3}{n^3} = \frac{(n-2)^2(n-1)}{n^3}.$$

For $n = 4$ we get $P = 3/16$.

Also solved by ALMA COLLEGE PROBLEM SOLVING GROUP, *MI,* CHARLES ASHBACHER, Hiawatha, IA, VICTOR G. FESER, University of Mary, Bismarck, ND, DICK GIBBS, Fort Lewis College, *Durango, CO,* RICHARD I. HESS. Rancho Palos Verdes, *CA,* DEREK LEDBETTER, University of Florida, Gainesville, HENRY S. LIEBERMAN, Waban, MA, MICHAEL MINIC, Middle Tennessee State University, *Murfreesboro,* MIKE PINTER, Belmont College, Nashville, *TN,* ST. OLAF PROBLEM SOLVING GROUP, St. Olaf College, *Northfield,* MN, WADE H. SHERARD, *Furman* University, Greenville, *SC,* UNIVERSITY OF ARIZONA PROBLEM SOLVING GROUP, Tucson, KENNETH M. WILKE, Topeka, KS, and the PROPOSER.

713. [Fall 1989] Proposed by *R. S. Luthar,* University of Wisconsin Center, Janesville, Wisconsin.

Evaluate $$\int_{\pi/60}^{\pi/30} \tan 5x \tan 3x \tan 2x \, dx.$$

I. Solution by WADE *H.* SHERARD, *Furman* University. Greenville. South Carolina.

From the identity

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

we obtain $\tan(x+y)\tan x \tan y = \tan(x+y) - \tan x - \tan y.$

Therefore,

$$\int_{\pi/60}^{\pi/30} \tan 5x \tan 3x \tan 2x \, dx$$

$$= \int_{\pi/60}^{\pi/30} (\tan 5x - \tan 3x - \tan 2x)\, dx$$

$$= \left[-\frac{1}{5}\ln\cos 5x + \frac{1}{3}\ln\cos 3x + \frac{1}{2}\ln\cos 2x\right]_{\pi/60}^{\pi/30}$$

$$= \frac{1}{5}\ln\frac{\cos\frac{\pi}{12}}{\cos\frac{\pi}{6}} + \frac{1}{3}\ln\frac{\cos\frac{\pi}{10}}{\cos\frac{\pi}{20}} + \frac{1}{2}\ln\frac{\cos\frac{\pi}{15}}{\cos\frac{\pi}{30}}$$

$$= 0.00093589.$$

1 Evaluation by ***SEUNG-JIN*** BANG, Seoul, Korea.
The integral is equal to

$$-\frac{1}{10}\ln(3(2-\sqrt{3}))+\frac{1}{6}\ln\left(\frac{5+\sqrt{5}}{4+\sqrt{10+2\sqrt{5}}}\right)+\frac{1}{4}\ln\left(\frac{\sqrt{5}+9+\sqrt{3}\sqrt{10-2\sqrt{5}}}{\sqrt{5}+7+\sqrt{3}\sqrt{10+2\sqrt{5}}}\right)$$

Also solved by ALMA COLLEGE PROBLEM SOLVING GROUP, ***MI,*** MOHAMMAD K. **AZARIAN,** University of Evansville, IN, SEUNG-JIN BANG, Seoul, Korea, ROBERT **I.** EGBERT, The Wichita State University, KS, GEORGE P. NANOVICH, Saint Peter's College. Jersey City, NJ, JACK GARFUNKEL, Flushing, NY, IEM HENG, Providence College, RI, RICHARD **I.** HESS, Rancho Palos ***Verdes,*** CA, MURRAY S. **KLAMKIN,** University of Alberta, Canada, HENRY S. LIEBERMAN, Waban, MA, DAVID E. MANES, SUNY at Oneonta, and the PROPOSER. Manes noted that the integral Is not Improper since none of the functions in the Integrand vanishes on the interval of integration.

714. [Fall **1989]** Proposed by Sam ***Pearsall, Loyola Marymount*** University, Los Angeles, California.

A flea crawls at the constant rate r = 1 foot per minute along a uniformly stretched elastic band, starting at one end. The band is initially L = 1 yard in length and is instantaneously and uniformly stretched L = 1 yard at the end of each minute while the flea maintains his grip on the band at the instant of each stretch. It is well known that the flea will reach the other end of the band in under 11 minutes. Find all lengths L such that the flea will reach the other end of the band in finite time.

Solution by HARRY SEDINGER, St. Bonaventure University, St. Bonaventure, New York.

The flea reaches the other end in finite time for all L > 0. Let B_k be the length of the band and F_k the position of the flea immediately after the **kth** stretch, where all units are in feet Then B_k = (k + l) L and

$$F_k = \frac{B_k}{B_{k-1}}(1 + F_{k-1}) = \frac{k+1}{k}(1 + F_{k-1})$$

where $F_1 = 2$. By induction we then have that

$$F_k = \frac{k+1}{k} + \frac{k+1}{k-1} + \cdots + \frac{k+1}{2}(1 + F_1)$$

$$= (k+1)\left(\frac{1}{k} + \frac{1}{k-1} + \cdots + \frac{1}{2} + 1\right).$$

The flea crawls to the other end after the **kth** stretch if $1 + F_k \geq B_k$, or equivalently, if

$$\frac{1}{k+1} + \frac{1}{k} + \cdots + \frac{1}{2} + 1 \geq L.$$

Since the harmonic series on the left **diverges,** such a k exists for any given **L** > 0.

Also solved by ALMA COLLEGE PROBLEM SOLVING GROUP, ***MI,*** FRANK P. BATTLES, Massachusetts Maritime Academy, Buzzards Bay, RICHARD **I.** HESS, Rancho ***Palos*** Verdes, ***CA,*** DEREK **LEDBETTER,** University of Florida, Gainesville, UNIVERSITY OF ARIZONA PROBLEM SOLVING GROUP, Tucson, and the PROPOSER.

715. [Fall **1989]** Proposed by Christopher ***Stuart,*** New Mexico State University, University Park, New ***Mexico.***

Euler's constant γ is defined by the equation

$$\gamma = \lim_{N\to\infty}\left(\sum_{k=1}^{N}\frac{1}{k} - \ln N\right).$$

Show that

$$\gamma = \sum_{k=2}^{\infty}\sum_{j=1}^{\infty}\frac{(-1)^k}{kj^k}.$$

Solution by DAVID E. MANES, SUNY at Oneonta, Oneonta. New York.
Since

$$\lim_{N\to\infty}\ln\frac{N+1}{N} = 0,$$

then

$$\gamma = \lim_{N\to\infty}\left(\sum_{k=1}^{N}\frac{1}{k} - \ln(N+1)\right).$$

Also

$$\sum_{j=1}^{N}\ln\left(1+\frac{1}{j}\right) = \sum_{j=1}^{N}[\ln(j+1) - \ln j] = \ln(N+1).$$

From the **Maclaurin** series expansion for **ln(1 + x),** we have

$$\ln(1+x) = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}x^k}{k}, \quad \text{so} \quad x - \ln(1+x) = \sum_{k=2}^{\infty}\frac{(-1)^k x^k}{k}$$

for all real numbers x in the interval (-1,1]. Hence.

$$\gamma = \lim_{N\to\infty}\left[\sum_{k=1}^{N}\frac{1}{k} - \sum_{j=1}^{N}\ln\left(1+\frac{1}{j}\right)\right]$$

$$= \lim_{N\to\infty}\left[\sum_{j=1}^{N}\left(\frac{1}{j} - \ln\left[1+\frac{1}{j}\right]\right)\right]$$

$$= \lim_{N\to\infty}\left(\sum_{j=1}^{N}\sum_{k=2}^{\infty}\frac{(-1)^k}{kj^k}\right) = \sum_{j=1}^{\infty}\sum_{k=2}^{\infty}\frac{(-1)^k}{kj^k}.$$

Reversing the order of the double sum yields the desired result.

Also solved by SEUNG-JIN BANG, Seoul, Korea, DEREK **LEDBETTER,** University of Florida, Galnesville, and the PROPOSER.

716. [Fall 1989] Proposed by Jack Garfunkel, Flushing, New York.
It Is known that, for **x, y,** $z > 0$,

$$\sqrt{xy} + \sqrt{yz} + \sqrt{zx} \le \sqrt{3}\sqrt{xy + yz + zx}.$$

Prove the "other side" of this inequality, namely,

$$\sqrt{xy} + \sqrt{yz} + \sqrt{zx} \ge 3\sqrt{3}\sqrt{\frac{xyz}{x+y+z}}.$$

I. ***Solution*** by DICK GIBBS, Fort Lewis College, Durango, Colorado.
By the arithmetic mean-geometric mean inequality we have

$$x + y + z \ge 3\sqrt[3]{xyz} \quad\text{and}\quad \sqrt{xy} + \sqrt{yz} + \sqrt{zx} \ge 3\sqrt[3]{xyz}.$$

Now take square roots of each side of the first inequality and multiply side for side by the second inequality to get

$$\sqrt{x+y+z}\,(\sqrt{xy} + \sqrt{yz} + \sqrt{zx}) \ge 3\sqrt{3}\sqrt{xyz}$$

and the desired result follows.

II. Solution by HENRY S. LIEBERMAN, ***Waban,*** Massachusetts.
We will prove, in fact, that

$$\sqrt{xy} + \sqrt{yz} + \sqrt{zx} \ge 3\sqrt[3]{xyz} \ge 3\sqrt{3}\sqrt{\frac{xyz}{x+y+z}}.$$

The left Inequality **is** just the A.M.-G.M. inequality applied to the three radicals. To prove the right side, apply the A.M.-G.M. Inequality to **x, y,** and **z**, divide both sides by **(x + y + z),** **multiply** by 3, and then take square roots of each side to get

$$\frac{x+y+z}{3} \ge \sqrt[3]{xyz} \quad\text{and}\quad 1 \ge \sqrt{\frac{3\sqrt[3]{xyz}}{x+y+z}}.$$

Now multiply each side by $3\sqrt[3]{xyz}$ to get the desired inequality.

III. Solution ***by*** DAVID E. MANES, SUNY at Oneonta, Oneonta, New York.
The harmonic mean of three positive numbers a, **b,** and c is less then or equal to their root-mean-square:

$$\frac{3}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}} \le \left(\frac{a^2+b^2+c^2}{3}\right)^{1/2}.$$

Let $a = 1/\sqrt{xy}$, $b = 1/\sqrt{yz}$, and $c = 1/\sqrt{zx}$. Then

$$\frac{3}{\sqrt{xy} + \sqrt{yz} + \sqrt{zx}} \le \left(\frac{\frac{1}{xy} + \frac{1}{yz} + \frac{1}{zx}}{3}\right)^{1/2} = \left(\frac{z+x+y}{3xyz}\right)^{1/2},$$

from which the desired inequality follows by taking reciprocals of each **side** and then multiplying by 3. Equality occurs In each Inequality if and only if $x = y = z$.

IV. Comment by the Editor.
To prove the "known" inequality, apply the A.M. $\le$ R.M.S. inequality to the three radicals $\sqrt{xy}$, $\sqrt{yz}$, and $\sqrt{zx}$.

Also solved by SEUNG-JIN BANG, Seoul, Korea. MURRAY S. KLAMKIN, University of Alberta, Canada, YOSHINOBU MURAYOSHI, Eugene, OR, BOB PRIELIPP, University of Wisconsin-Oshkosh, and the PROPOSER.

717. [Fall **1989]** Proposed by Russell ***Euler,*** Northwest Missouri State University, ***Maryville,*** Missouri.
Find all positive integers n for which

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\frac{1}{k}$$

is an integer.

I. ***Solution*** by DAVID E. MANES, SUNY at Oneonta, Oneonta, New York.
The sum is an **integer** if and only if $n = 1$. If $n = 1$, the sum Is **1.** To prove the converse, it **is** known that

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\frac{1}{k} = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

(see Riordan, Combinatorial Identities, Wiley, 1968, Example 3, pp. 4-5). If n > 1, the right hand side cannot be an integer (Sierpiński, Elementary Theory of Numbers. Hafner, 1964, Exercise 2, p. 139). Hence the result.

II. *Solution* by the PROPOSER.

Using the binomial theorem it is easy to show that

$$\frac{1-(1-x)^n}{x} = \sum_{k=1}^{n} (-1)^{k-1}\binom{n}{k} x^{k-1}.$$

Therefore,

$$\int_0^1 \frac{1-(1-x)^n}{x}\,dx = \sum_{k=1}^{n} (-1)^{k-1}\binom{n}{k}\frac{1}{k}.$$

Now, it is known (Whittaker and Watson. A Course of Modern Analysis, p. 236) that

$$\int_0^1 \frac{1-(1-x)^n}{x}\,dx = \sum_{k=1}^{n} \frac{1}{k}.$$

Since it is well known that the sum on the right is never an integer when n > 1, the given expression also is not an integer when n > 1.

Also solved by SEUNGJIN BANG, Seoul, Korea, RICHARD I. HESS, Rancho *Palos Verdes, CA,* MURRAY S. KLAMKIN, University of Alberta, Canada, and BOB PRIELIPP, University of Wisconsin-Oshkosh.

718. [Fall 1989] Proposed by David Petty, Eugene, Oregon.

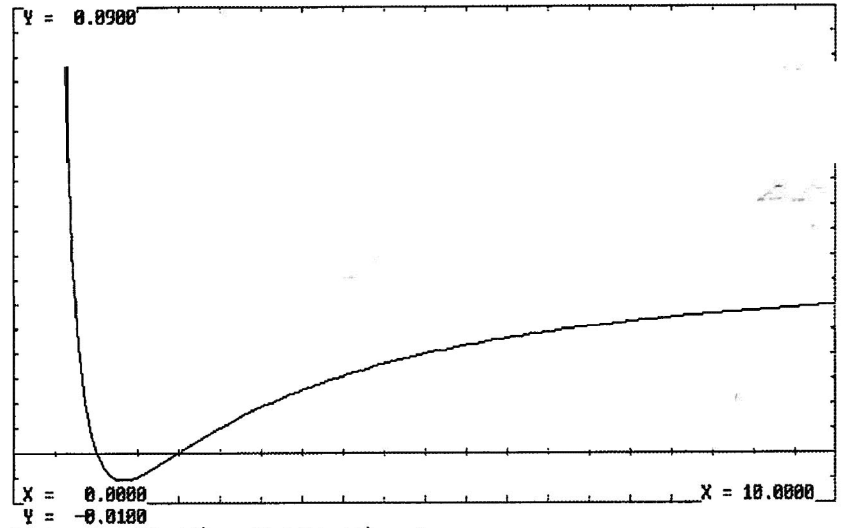
Prove or find a counterexample: If a, b, c, p are Integers such that $0 \le a < b < c \le 2p+1$, then $a^p + b^p \le c^p$.

Solution by MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta, Canada.

The worst case for the inequality is a = c - 2, b = c - 1, where $1 < c \le 2p+1$. Since the inequality now becomes

$$\left(1-\frac{2}{c}\right)^p + \left(1-\frac{1}{c}\right)^p \le 1,$$

it suffices to choose c as large as possible, i.e. c = 2p + 1.



f(x) = (1+1/(2*x))^x - (1-1/(2*x))^x - 1

Then the inequality becomes

$$(2p-1)^p + (2p)^p \le (2p+1)^p,$$

which Is equivalent to

$$\left(1+\frac{1}{2p}\right)^p - \left(1-\frac{1}{2p}\right)^p \ge 1. \qquad (1)$$

Expanding out the left hand side of (1) by the binomial theorem, we get

$$1 + \frac{p(p-1)(p-2)}{24p^3} + \frac{p(p-1)(p-2)(p-3)(p-4)}{1920p^5} + \ldots,$$

which Is larger than 1 for $p \ge 3$ (it equals 1 for p = 1 or 2). A graph of the left hand side of (1) minus 1 appears in the figure, showing the intersections with the x-axis at 1 and 2.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, HENRY S. LIEBERMAN, *Waban,* MA, KENNETH M. WILKE, Topeka, KS, and the PROPOSER.

CHANGES OF ADDRESS/INQUIRIES

Subscribers to the Journal should keep the Editor Informed of changes in mailing address. Journals are mailed at bulk rate and are not forwarded by the postal system. The cost of sending replacement copies by first class mail is prohibitive.

Inquiries about certificates, pins, posters, matching prize funds, support for regional meetings, and travel support for national meetings should be directed to the Secretary-Treasurer, Robert M. Woodside, Department of Mathematics, East Carolina University, Greenville, NC 27858, 919-757-6414.

Gleanings from the Chapter Reports

ARKANSAS BETA (Hendrix Cdlege) The chapter had fifteen meetings, including five joint meetings with the Central Arkansas MAA Student Chapter. Invited speakers included: Dr. Gaylor, National Center for Toxicological Research; Dr. Cholke, Oklahoma State University; Dr. Phil Parker, Wichita State Univers'ty; Dr. Jeffrey Cooper, University of Maryland; Dr. Darryl McCullough, University of Oklahoma; Dr. Diestel, Kent State Univers'ty; Dr. Jackie Gamer, Mississippi State University; Dr. Richard Redner, University of Tulsa; Dr. Paul Fjelstad, St. Olaf College; Dr. John Duncan, University of Arkansas at Fayetteville,; Dr. RG. Dean, Stephen F. Austin State University; and Dr. Tommy Levelle, John Brown University. Along with these meetings, Chapter members gave talks at the M M Oklahoma-Arkansas Section Meeting and the Hendrix-Rhodes-Sewanee Mathematics Symposium.

CONNECTICUT GAMMA (Fairfield Un'versity) During the fall semester the chapter sponsored a very successful Math Bowl Contest. Twelve teams of four students competed in a "GE College Bowl" type of competition. In which all the questions were mathematical. In the spring, members of Pi Mu Epsilon assisted the Mathematics Department in coordinating the activities for Math Counts, which is a mathematics contest for junior high school students. "Forbidden Symmetries" by Judith Fiagg Moran of Smith College was the title of the Pi Mu Epsilon Lecture during the ceremony. During the Annual Arts and Sciences Awards Ceremony, two members, Valerie A. Albano and Anna M. Contadino, received recognition for their outstanding performance in mathematics. Each was given a Pi Mu Epsilon certificate of achievement, a book each selected in an area of mathematics, and one-year memberships in the Mathematical Association of America.

GEORGIA BETA (Georgia Institute of Technology) At the 1990 Honors Program, outstanding graduates in mathematics were presented with a book award of their choice. The recipients were: Jeffrey Herrmann, Elaine Knight, and Mark LaDue. These students were majors in Applied Mathematics with grade point averages of at least 3.7 on a 4.0 point scale.

INDIANA GAMMA (Rose-Hulman Institute of Technology) At the beginning of the 1990 school year the chapter helped sponsor the First Annual Alfred R. Schmidt Mathematics Competition. This competition was introduced to encourage freshman students to become interested in mathematics at Rose-Hulman. Over $110 in book prizes were awarded. Eight students attended the Miami University Conference with Greg Gass, Aaron Wendelin, Mike Wilson, and Joel Atkins presenting papers. Our chapter helped administer the Rose-Hulman Institute of Technology-St. Mary of the Woods Mathematics Competition (for area high school students). Two students attended the St. Norbert Conference, with Chris Halioris and Joel Atkins presenting papers. Jeff Dierckman, Kevin O'Bryant, and Joel Atkins were chosen as SIAM winner in the Annual Mathematical Contest in Modeling. They presented their paper at the Indiana Section meeting of the M M. Dr. David Womble, of Sandia National Laboratories, was the invited speaker at our installation banquet. Dr. Womble is a 1981 graduate of Rose-Hulman and was a member of the Indiana Gamma Chapter during his undergraduate days. Our chapter helped our department stage the Annual Rose-Hulman Undergraduate Mathematics Conference, which was hosted in conjunction with the Journal of Undergraduate Mathematics this year. There were over 130 participants and 27 student papers. The featured speakers were Dr. Bruce Reznick of the University of Illinois, Dr. Gary Sherman of Rose-Hulman, and Dr. Marty Lewinter of SUNY-Purchase. Mike Wilson, Greg Gass, Keith Strauss, Jeff Dierckman, Kevin O'Bryant, John O'Bryan, Greg Ford, and Joel Atkins presented papers. In the spring we accepted a challenge from Upsilon Pi Epsilon, the computer science honorary society, to play ultimate frisbee.

KANSAS GAMMA (Wichita State University) The chapter sponsored an extensive program of speakers: Elizabeth Clarkson, "Evolutionary Evaluation of Risk Strategies"; Dr. Kirk Lancaster, "A Survey of Research in Mathematics"; Dr. Phillip Parker, "Hyperbolas and Fundamental Units'; Dr. Don Hommerzheim, "Artificial Neural Networks: An Introduction and How They Can Be Used to Solve Combinatorial Problems"; Paul Chawla, "Periodic Machine Calibration and Capability Verification (CAL/CAP) System"; Dr. Dan Fitzgerald, "Chords and Cosets: Remarks on Mathematics in Music Theory"; Dr. Shrikant Panwalker, "Some Combinatorial Optimization Problems"; Dr. Shahar Boneh, "Optimal Stopping in Applied Probability"; Dr. Prem Bajaj, "Choice of a Major: Some Case Studies'; Balaji Sudabattula, "A Fallacy in Solution of Differential Equations"; Dr. William Perel, Who's Afraid of the Big Bad Math?" Karen Taylor, president of the chapter, moderated a panel discussion focusing on the opportunities in mathematical sciences. The panel consisted of: Jeanne Daharsh. Assistant Vice President, Alliance Life Insurance, Wichita; Dr. Bill Hammers, Academic and Technical Affairs Assistant, Boeing Military Airplane, Wichita; Elaine Hillman, Vice President, Operations, First National Bank, Wichita; Dr. Denise Johnston, TQC/SPC Instructor, Boeing Military Airplane, Wichita; Phyllis McNickle, Assistant Director, Placement Office, The Wichita State University. Wichita. Balaji Sudabanula gave a talk entitled "Commutativity of Matrices in Ordinary Differential Equations" at the joint meetings of the Kansas Section of the M M and the Kansas Association of Teachers in Mathematics.

MASSACHUSETTS GAMMA (Bridgewater State College) On February 8.1990, Ms. Phyllis Warren, Silver Lake Regional High School (Kingston, MA) gave a workshop on "Escher-type Tessalations" and on May 11, 1990, at the annual induction ceremony, Prof. Walter Gleason, Bridgewater State College, gave a talk on "Zeller's Congruence'. Several members attended the June regional meeting of the M M at Roger Williams College (Bristd R.I.).

MICHIGAN EPSILON (Western Michigan University) Special Pi Mu Epsilon talks with guest speakers were "Mathematics in Iran: The Recent Years" by Dr. Mehdi Behzad, Visiting Professor at WMU from Iran; "Voting Theory: From Pizza Pies to Nobel Prize" by Professor Garry Johns, Saginaw Valley State University; The Mystery of Mathematics: Fact and Fallacy" by Professor Christina Mynhardt, University of Victoria and University of South Africa; "Let Newton Be!" by Professor Robin Wilson, Open University of England; and "Graph Theory with an Application to Computer Graphics' by Professor Joseph Gallian, University of Minnesota at Duluth. Student Pi Mu Epsilon talks were: "A Generalization of Odd and Even Vertices in Graphs" by Amy Dykstra and Michelle Schultz and "Physics, Biology and Traveling Salespeople by Bruce Tesar. The Pi Mu Epsilon Initiation Banquet and Program was held in October with the talk "Statistical Science: The Profession" being given by Cindy Grzegorczyk, the Upjohn Company. A spring banquet followed the talk by Professor Joseph Gallian; a second talk was given by him entitled "The Beatles". Students Mark Kust, Christopher Oliver and Bruce Tesar presented papers at the Annual Meeting of the Michigan Section of the Mathematical Association of America held in Flint, Michigan. In addition, a book sale was held in the Fall, the Annual Business Meeting was held in the Winter, and a picnic was held in the Spring.

MICHIGAN ZETA (University of Michigan-Dearborn) The chapter has had a successful first year. In October, three members attended the Annual Pi Mu Epsilon Conference at Miami University.

Later in the year, we corresponded with the chapters at Western Michigan University and Michigan State University and attended **MSU's** induction in May. Our main project this year was the Focus on Faculty Speaker Series. Six UM-D faculty members presented lectures, mainly on their research interests. The **FOF** topics included integral equations, computer-aided geometric design, difference equations and recurrence relations with spreadsheets, game theory, graph theory, and derivation of summation formulas. We thanked the faculty for their support over the past year with a Faculty Appreciation Luncheon in April. Also, in April, John Kelly, a student, gave a talk about the intuitionists and constructionists. On a social **level,** we had two evenings of pizza and games. and a Winter Break party.

NEW YORK EPSILON (St. Lawrence University) In April, the chapter sponsored the 46th Annual Pi Mu Epsilon Interscholastic Mathematics Contest for 57 high school students in teams representing 14 area schools. Ogdenburg Free Academy won this year's Pi Mu Epsilon Cup and Rajesh **Suryadevera** of **Potsdam** Central School won the gold medal for the highest individual score on the exam. Karen Kobasa and Mark Hays were recognized as Outstanding Mathematics Seniors at the university's awards ceremony, receiving AMS memberships and cash awards. For the second consecutive year, a member of the chapter has been the academic leader for the St. Lawrence **class** at graduation - George ***Ashline*** in ***1989*** and Karen **Kobasa** in 1990.

NEW YORK PHI **(Potsdam** College of the State University of New York) Ms. Heidi Learned was selected by the membership of the New York Phi Chapter of Pi Mu Epsilon for the **1990** senior award which consists of $100 in mathematics books. She was selected on the basis of her contributions to Pi Mu Epsilon, the Mathematics Department, and **Potsdam** College.

NEW YORK OMEGA (St. Bonaventure University) Chapter activities included the presentation "Mathematical Models of Heat Flow in a Solid Body" by Dr. Gregory Verchota, Syracuse University. The chapter celebrated National Mathematics Awareness Week in April with a series of three events: a panel discussion on preparing for the actuarial exams, with Dr. Albert White, SBU, as moderator; a showing of the movie "Stand and Deliver"; and the talk **"*The*** Golden Section," by Dr. Charles Diminnie, SBU. The week's events were co-sponsored with the St. Bonaventure **MAA** Student Chapter and the Computer Science Club. The Myra J. Reed Award was presented to Karen **M. Tolve** at the University Honor's Banquet.

OHIO ZETA (University of Dayton) At the Pi Mu Epsilon National Meeting in Boulder, CO, in August, Tim **Bahmer,** Colleen Galligher, **Chikako Mese,** and Marla Prenger presented the results of the research they conducted In an undergraduate research program at Dayton. At the Pi Mu Epsllon Regional Conference at Miami U. in October these four students again presented their talks. Also speaking were David Delle, "On Self-Complementary Graphs," and Lisa **Tsui,** "Perpendiculars to a Parabola." At the joint meetings of the **AMS-MAA** at Louisville in January, David Delle presented 'How to Please Most of the People Most of the Time,' which was based on a solution to the air traffic controller's problem in the **1989** Modeling Competition. The team, which also included Matt Davidson and David **Jessup,** won an Honorable Mention for their solution. David Delle won both a Barry M. Gddwater Scholarship and **UD's** Faculty Award for Excellence in Mathematics. David **Jessup** received the Pi Mu Epsilon Sophomore of the Year Award. Invited speakers were Prof. David Miller, **(Wright** State University) and Prof. Richard Schoen (Stanford University). Members of the chapter also went to Wittenberg University to hear Dr. Ronald Graham of AT & T speak on the shortest network problem and on computers and mathematics.

PENNSYLVANIA BETA **(Bucknell** University) The chapter sponsored the 18th Professor John Steiner Gold Mathematical Competition for high school students. The winning team was State Cdlege Area High School followed by Lewisburg Area High School and **Selinsgrove** Area High School. The first five places Individually went to Allen Hunt, David **Gerber** (both of State College) Jason Schweinsberg and Jon Confer (both of Lewisburg) and Mike Minnich (of Line Mountain) in this order. Professor Laurence Sigler enriched the annual initiation banquet with a talk on **"Leonardo *Pisano*** and The Book of Squares". Besides several talks by local faculty the Chapter co-sponsored, together with the MAA Student Chapter, two lectures by visitors: Professor Barry **Tesman** of Dickinson Cdlege spoke on 'Graph Colorings and their Applications"; Professor George **Rosenstein** of Franklin & Marshall College spoke on The Discovery of Wallis's Formula for PI". Another interesting talk was presented by Joel Mercer, a graduate student and chapter member. He spoke on The Orchard Problem, a Fruitful Apple-cation". In the fall semester a social gathering for students and faculty was arranged.

TENNESSEE GAMMA (Middle Tennessee State University) The chapter sponsors an annual Pi Mu Epsilon Mathematics Project Award. The purpose of the cash award to promote the mathematical and scholarly development of **MTSU** mathematics students by encouraging independent study projects culminating in ***oral*** presentations to Pi Mu Epsilon. The presentations are made during the annual National Mathematics Awareness Week in April.

WEST VIRGINIA BETA (Marshall University) The chapter held **elght** meetings; several of these meetings featured talks by members of the mathematics faculty. The chapter provided proctors for Marshall University's SCORES high school competition and for the mathematics department's high school competition. The major fund raiser was the sale of **old** math finals. Some of the money raised was used to replace the David Hilbert display in the math department with a display of the Mandelbrot Set.

WISCONSIN DELTA (St. Norbert College) In April, **1990,** 7 students. Chris Ferriter, Amy Gerrits, Sandy **Gestl,** Amy **Krebsbach,** Mike **Lang**, Linda **Mueller,** and Tim **Strnad** attended the Undergraduate Math Conference at Rose-Hulman Institute of Technology, with Chris and Tim presenting papers. Speakers on campus were Dr. John Frohliger (St Norbert College) speaking on " The American High School Mathematics Examination"; Dr. Norb **Kuenzi, (UW-Oshkosh)** on **"The** Old Tower of Hanoi **Ploy";** Dr. **Brauch** Fugate, (University of Kentucky) on "How Do We Know That Geometry is True?" and Mike **O'Callaghan** (SNC and Schnelder National) on **"Object-Orlented** Programming". There were several highlights for the 1989-90 school year. The chapter hosted the 4th Annual St. Norbert College Pi Mu Epsllon Regional Math Conference; the invited speaker was Dr. J. Sutherland Frame from Michigan State University. In conjunction with Sigma Nu Delta (SNC Math Club), the chapter held the 8th Annual High School Math Competition. The combined **Pi** Mu Epsilon-Sigma Nu **Delta** math organization was named "Volunteer Organization of the Year" by the regional chapter of the American Red Cross for the organization's work in recruiting blood donors.

ATTENTION FACULTY ADVISORS

To have your chapter's report published, send copies to Robert **M.** Woodside, Secretary-Treasurer, Department of Mathematics, East Carolina University, Greenville, NC 27858 and to Richard **L. Poss,** Editor. **St.** Norbert College, De Pere, **WI** 54115.

ST. JOHNS UNIVERSITY/COLLEGE OF ST. BENEDICT
ANNUAL PI MU EPSILON STUDENT CONFERENCE

Raymond Smullyan

Professor Emeritus of Mathematics
City University of New York
Herbert H. Lehman College

"Puzzles and Paradoxes"
Friday, April 12, 1990
8:00 p.m.

"Logic of Infinity"
Saturday, April 13, 1990
10:30 a.m.

The PI Mu Epsilon Conference serves as a forum for undergraduates to present original **mathematics** and synthesis of other **mathematics.** Student talks precede the guest speaker both days.

Professor Smullyan Is a logician and philosopher. Author of What Is the Name of This Book? on **Godel's** Incompleteness Theorem that Is both fun and Instructive, This Book Needs No Title, The Chess Mysteries of ***Sherlock Holmes*** and The Tao is Silent, he Is also an accomplished classical pianist and professional magician.

For more information contact: Shoba Gulati, Mike Zielinski, or Phil Byrne, Department of Mathematics, St. John's University, Collegeville, MN 56321, Phone 612-363-3087.

8th Annual Rose-Hulman Conference on
Undergraduate Mathematics
March 15-16, **1991**

Speaker: Steve Maurer, **Swarthmore** College

Titles of Addresses: "Proof by **Algorithm,** Parts **I & II"**

Professor Maurer Is the author, together with Tony **Ralston,** of the book, 'Discrete Algorithmic Mathematics," which Is soon to appear.

For information contact: **George Berzsenyi,** Department of Mathematics, **Rose-Hulman Institute** of Technology, Terre Haute, IN **47803, (812) 877-8474.**

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