PI MU EPSILON JOURNAL

VOLUME 9 SPRING 1991 NUMBER 4

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PI MU EPSILON JOURNAL

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for solution and solutions to problems should be mailed directly to the PROBLEM EDITOR. Puzzle proposals and puzzle solutions should be milled to A* EDITOR

The PI MU EPSILON JOURNAL is published at St. Norbert College twice a year—Fall and Spring. One volume consists of five years (10 issues) beginning with the Fall 19x4 or Fall 19x9 issue, starting in 1949. For rates, see inside back cover.

THE RICHARD V. ANDREE AWARDS

Richard V. Andree, Professor Emeritus of the University of Oklahoma, died on May 8, 1987, at the age of 67. Professor Andree was a Past-President of Pi Mu Epsilon. He had also served the society as Secretary-General and as Editor of the Pi Mu *Epsilon* Journal. The Society Council has designated the prizes in the National Student Paper Competition as Richard V. Andree Awards.

First prize winners for 1990 are Amy Dykstra and Michelle Schultz for their paper "A Generalization of Odd and Even Vertices in a Graph," which appeared in the Spring, 1990, issue of the Journal. They prepared their paper while undergraduates at Western Michigan University under the supervision of Professor Gary Chartrand. They presented the paper in August, 1989, at the national Pi Mu Epsilon meeting in Boulder, Colorado. They will share the $200 first prize.

Second prize winner is Eric Berg for his paper "A Family of Fields," which appeared in the Fall, 1990, issue of the Journal. Eric prepared this paper while still a student in high school. Eric will receive $100.

Third prize winner is Joel Atkins for his paper "Regular Polygon Targets," which also appeared in the Fall, 1990, issue of the Journal. Joel prepared this paper while he was a student at Rose-Hulman Institute of Technology under the supervision of Professor Elton Graves. Joel will receive $50.

There were three other student-written papers that appeared in 1990:

"More Applications of Full Coverings," by Karen Klaimon, of James Madison University. Karen prepared this paper under the supervision of Professor John Marafino.

"An Approximation for the Number of Primes between K and K^2, When K Is Prime," by Randall J. Osteen. Randall prepared this paper while he was an undergraduate at the University of Central Florida.

"Convergent Ratios of Parallel Recursive Functions," by David Richter. David prepared this paper while he was a freshman at St. Cloud State University.

The current issue of the Journal contains two papers with student authors:

"A Pre-Calculus Method for Deriving Simpson's Rule" was written by John White, who is an undergraduate at Marshall University.

"A Note on a Paper of S. H. Friedberg" was co-written by Janet Valasek, a sophomore at Penn State University - New Kensington Campus, and Professor Javier Gomez-Calderon.

A PRE-CALCULUS METHOD FOR DERIVING SIMPSON'S RULE

John G. White
Marshall University

Simpson's Rule is one of a class of numerical methods, known as Newton-Cotes formulas, used to calculate definite integrals. This formula is credited to Thomas Simpson, a self-taught genius, who published it in his Mathematical Dissertations on Physical and Analytical Subjects in 1743. However, James Gregory presented the same results earlier in a different form in his Exercitationes Geometricae [1]. Its usefulness is in calculating definite Integrals of functions that are otherwise difficult or Impossible to integrate, such as

$$\int_{x_0}^{x_2} e^{x^3}dx.$$

There are several standard ways to derive Simpson's Rule using calculus. In one method, three equally spaced points, the endpoints and the midpoint of the interval, are chosen. A parabola is constructed from these points (since a polynomial of degree at most two passing through three given points can always be found) and it is integrated. This yields Sirnpson's Rule:

$$\frac{h}{3}(f(x_0)+4f(x_1)+f(x_2)) \approx \int_{x_0}^{x_2} f(x)\,dx$$

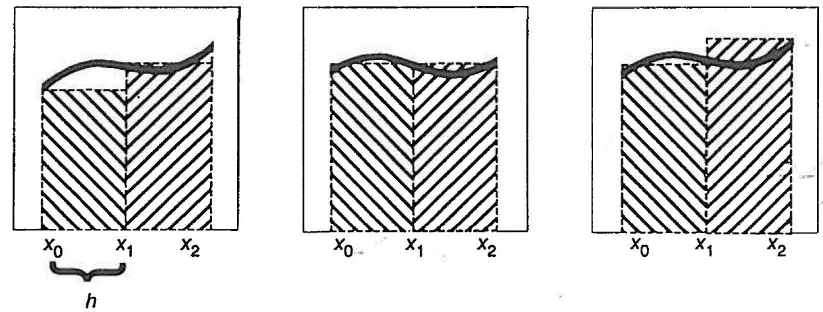
where $h = (x_2 - x_0)/2$. (See [3] for an example of this derivation.)

Another method takes three points and uses them to construct a Lagrange interpolating polynomial of degree two:

$$P(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}f(x_0)+\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}f(x_1)+\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}f(x_2)$$

This is then integrated, and the final result is once again Simpson's Rule. (See [2].) A third method integrates the Taylor series expansion of $f(x)$ to derive Simpson's Rule [2].

Here is one method of deriving Sirnpson's Rule that does not rely on integration. Rather, piecewise approximations are used to find three different values for the integral. The average is then taken to approximate the definite integral, and the end result is once again Simpson's Rule. For simplification, the following illustrations use only nonnegative functions, even though the derivation is the same for functions with negative values as well.



$h\,f(x_0) + h\,f(x_1)$ $\qquad$ $h\,f(x_1) + h\,f(x_1)$ $\qquad$ $h\,f(x_1) + h\,f(x_2)$

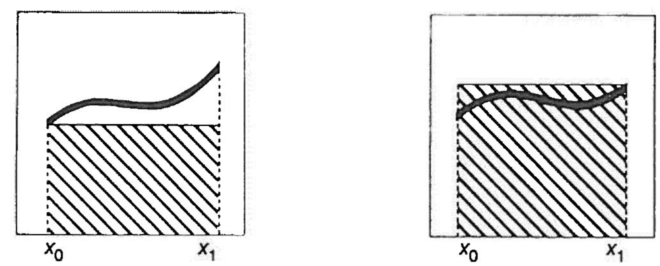
$$\frac{(hf(x_0)+hf(x_1))+(hf(x_1)+hf(x_1))+(hf(x_1)+hf(x_2))}{3}$$

$$= \frac{h}{3}(f(x_0)+f(x_1)+f(x_1)+f(x_1)+f(x_2))$$

$$= \frac{h}{3}(f(x_0)+4f(x_1)+f(x_2)).$$

This pre-calculus method of derivation also yields two other Newton-Cotes formulas: the Trapezoidal Rule and Simpson's Three-Eighths Rule.

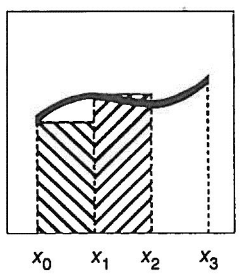
Trapezoidal Rule:



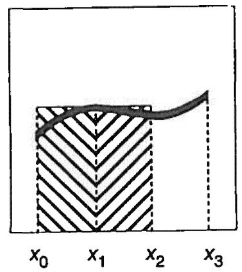
$h\,f(x_0)$ $\qquad$ $h\,f(x_1)$

$$\frac{hf(x_0)+hf(x_1)}{2} = \frac{h}{2}(f(x_0)+f(x_1))$$

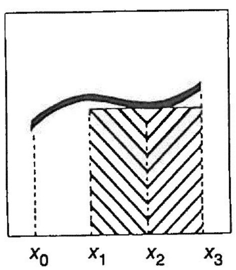
Simpson's Three-Eighths Rule:



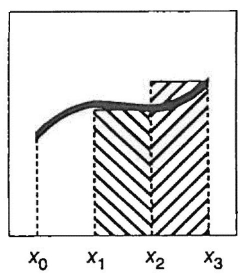
$hf(x_0) + hf(x_1)$



$hf(x_1) + hf(x_1)$



$hf(x_2) + hf(x_2)$



$hf(x_2) + hf(x_3)$

With this derivation, each section is approximately two-thirds the total integral, thus the integral is about three-eighths the sum of the four areas.

$$\frac{3(hf(x_0)+hf(x_1))+3(hf(x_1)+hf(x_1))+3(hf(x_2)+hf(x_2))+3(hf(x_2)+hf(x_3))}{8}$$

$$= \frac{3h}{8}\left(f(x_0)+3f(x_1)+3f(x_2)+f(x_3)\right)$$

References:

[1] C. B. Boyer, A History of Mathematics, John Wiley and Sons, inc., **1968.**

[2] R. L. Burden and J. D. Faires, Numerical Analysis, Fourth Edition, PWS, **1989.**

[3] E. W. Swokowski, Calculus with Analytic Geometry, Third Edition, PWS, **1984.**

John **White** prepared this paper while he was a senior at Marshall University.

A NOTE ON A PAPER OF S. H. FRIEDBERG

Javier Gomez-Calderon & Janet Valasek
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Recently in **[1], S.** H. Friedberg showed that the principal axis theorem, a very important theorem in linear algebra, does not extend to any finite field. He proved, using a simple counting argument, the following:

THEOREM: Let F be a finite field. Then there exists a 2x2 symmetric matrix (over F) that possesses no eigenvalues.

The purpose of this note is to point out that Friedberg's results can easily be generalized for a n x n symmetric matrix. We will prove the following two corollaries.

COROLLARY 1 (to Friedberg's Theorem): Let F be a finite field. Then for each $n > 1$, there exists a $(2n)\times(2n)$ matrix (over F) that possesses no eigenvalues.
PROOF: By Friedberg's Theorem, let A denote a 2x2 matrix over F such that $f_A(x)$, the characteristic polynomial of A, has no roots in F. Then the characteristic polynomial of the $(2n)\times(2n)$ block diagonal matrix C = $\mathrm{diag}(A,A,\ldots,A)$ is $f_C(x) = (f_A(x))^n$. Therefore, C possesses no eigenvalues.

COROLLARY 2: Let F be a finite field. Then for each $n \geq 3$, there exists a $n\times n$ **non-diagonalizable** symmetric matrix over F.
PROOF: With notation as in Corollary 1, let D denote the $n\times n$ block diagonal matrix

$$D = \begin{pmatrix} A & \cdot & \cdot & \cdot & 0 \\ \cdot & 0 & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ \cdot & & & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix}_{n\times n} \qquad (n \geq 3)$$

Then the characteristic polynomial of D is $f_D(x) = f_A(x)x^{n-2}$. Thus, the only eigenvalue of D is 0. Therefore, since $D \neq 0$, D is not **diagonalizable.**

References:

[1] S. H. Friedberg, "Extending the Principal Axis Theorem to Fields Other Than **R**," American Mathematical Monthly, **97 (1990), 147-149.**

Janet Valasek is currently a sophomore at the New Kensington Campus of Penn State University.

THE FIRST CENTURY

Richard *L.* Francis
Southeast Missouri State ***University***

An abundance of primes meets the eye in examining the first one hundred positive integers. Not quite so many emerge in the second century, and even fewer in the third. However, the frequency of prime encounters in these initial groupings suggests no scarcity. Actually, the first century of positive integers proves a veritable field of abundance in its containment of major number types. It likewise prompts the question of other collections of positive integers with a plentiful supply of numbers in a select category. The pursuit of primes by centuries is an intriguing part of this basic question.

Primeless Centuries

Centuries denote groupings by hundreds and begin with the first 100 positive integers. These may also be called aggregates of order two (whereas decades suggest aggregates of order one). Finding primes within the various centuries touches on the subject of the distribution of the primes. Such a distribution within an infinite set is, even today, highly perplexing. Similarly elusive is a formula for finding the n^{th} prime - or for generating a prime larger than a designated one. Of interest in this context of the infinitude of the primes is the fact that there exists, for example, a **one-** trillionth prime, but no one can say what it is.

Some centuries contain no primes whatever. Consider the century which begins with 1001 + **1** and ends with ***100!*** + 100. Each number in this set is composite as **100!** + n is divisible by n for $0 < n \leq 100$. Moreover, **100!** + **1** is divisible by 101 by Wilson's Theorem. It is easy to show that there are infinitely many centuries entirely devoid of primes by a similar factorial construction. For example, the century from **1,000,000!** + **101** to **1,000,000!** + 200 consists of composites. Or from **1,000,000!** + 201 to **1,000,000!** + 300. Infinitely many primeless centuries are implied by the generalized interval extending from $10^n! + 101$ to $10^n! + 200$ where n is greater than or equal to 3.

A Prime-Rich Century

More primes appear in the first century than in any other. All primes beyond the first century must "end" in **1**, 3, 7, or 9. This allows for a maximum of forty primes within the century. But a least three numbers in each terminal digit case must be multiples of 3. Accordingly, 40 - 12 or 28 denotes a more impressive maximum number of primes within the century. To lessen the maximum even more, note that centuries can begin in 21 ways based on the 21 possibilities in which the century's first number yields a remainder when divided by 3 and by 7. For example, the first number **100n** + 1 can be of the form 3r and **7k,** 3r and 7k + **1,** 3r and 7k + 2, etc. In each case, striking out the multiples of 3 and of 7 (and in one case, multiples of 11) establishes that no century beyond the first contains more than 24 primes. Of course, the first century contains 25 primes. It is thus the maximal century of primes.

Upper Limit on Number of Primes
[Based on Numbers Ending in 1, 3, 7, or 9)

Form of Century's First Number	x(Number of Sure Composites)	Upper Limit of Number of Primes (40-x)	First Number of Sample Century
3r,7k	18	22	8001
3r,7k + 1	17	23	14001
3r,7k + 2	18	22	20001
3r,7k + 3	17	23	26001
3r,7k + 4	18	22	32001
3r,7k + 5	18	22	38001
3r,7k + 6	18	22	44001
3r + 1, 7k	18	22	15001
3r + 1, 7k + 1	18	22	21001
3r + 1, 7k + 2	18	22	6001
3r + 1, 7k + 3	18	22	12001
3r + 1, 7k + 4	17	23	18001
3r + 1, 7k + 5	18	22	24001
3r + 1, 7k + 6	17	23	30001
3r + 2, 7k	16	24	1001
3r + 2, 7k + 1	17	23	7001
3r + 2, 7k + 2	16	24	13001
3r + 2, 7k + 3	15	25	19001
3r + 2, 7k + 4	16	24	25001
3r + 2, 7k + 5	16	24	31001
3r + 2, 7k + 6	16	24	37001

Note that the upper limit on the number of primes is 25 in the case for leading numbers of centuries which are of the form 3r + 2 and 7k + 3. In this case, an additional sure composite can be established by considering all possibilities of remainders In dividing the leading number of the century by 11. These forms are 11) + **1, 11j** + 2, ..., **11j** + 10.

Decades in Passing

As stated earlier, centuries denote groupings by hundreds and begin with the first 100 positive integers. These were called aggregates of order two based on the exponent appearing in 10^2 (where 10^2 is of course the number of elements in a century). Millennia thus denote aggregates of order three. The case for decades, where the order of aggregate is **1,** proves interesting. Actually, the first decade contains only four primes; this is obviously the maximum number of primes possible within a decade. Other decades may contain the same maximum number of primes. These include, for example, the second decade (with primes 11, 13, 17, and 19) as well as the eleventh (with primes 101, 103, 107, and 109). Were it not for the contrivance that **1** is not a prime, then the first decade would emphatically be the maximum decade in terms of primes possessed. (The arguments of convenience whereby **1** is excluded from the list of primes are **well** known and will not be pursued here.)

The least decade containing no primes is the one beginning with 201. Following this as the next primeless decade is the one which begins with the number 321. The first encounter with **two primeless** decades in succession has 1131 for its leading element. Three primeless decades in succession can be found by beginning with **1331.** Such a fascinating pattern continues. Infinitely many decades of various orders of succession may be found.

Least Century with No Primes

Although there are infinitely many primeless centuries (as shown earlier), there must also be a least such century. It is not necessarily the century whose first element is **100!** + 1. Note the magnitude of **100!.** The number of terminal zeros alone, namely, twenty-four, classifies 100! + 1 as gargantuan.

Some relatively early centuries come close to meeting the **"primeless"** standard. For example, the century beginning with **31401** contains only four primes. These are **31469, 31477, 31481,** and **31489.** Even more impressive is the century beginning with **58801.** Only three primes appear; they are **58831, 58889,** and **58897.** Likewise, only three primes can be found in the century beginning with **69501.**

The least century containing no primes whatever lies somewhere between 1 million and 2 million. It Is the century whose first element **is 1671801** and **is** shown below. As each of the elements **in** the listing is composite, the reader may wish to find the factors of some. For example, the number **1671813** yields **$(3^3)(11)(13)(433)$** when written **in** factored form. This prime factorization is, of course, unique (Fundamental Theorem of Arithmetic).

First of the Primeless Centuries

1671801	1671811	1671821	...	1671881	1671891
1671802	1671812	1671822	...	1671882	1671892
1671803	1671813	1671823	...	1671883	1671893
...	...	...	...	...	...
1671809	1671819	1671829	...	1671889	1671899
1671810	1671820	1671830	...	1671890	1671900

The largest prime preceding this primeless century is **1671781.** The smallest which follows is **1671907.**

By some logic, all numbers can be considered "interesting." Hence, **it** is with reluctance that the above century is labeled "mathematically barren." Although **it** contains the exact square **1,671,849,** there are no prlmes of any kind. Nor are there cubes, fourth and higher powers, or factorials. Perfect numbers (even or odd), triangular numbers, palindromes, and odd abundant numbers likewise fail to appear. But, and Interestingly so, **it** Is the first of the primeless centuries. The next of the primeless centuries begins with **2,637,801** and extends through **2,637,900.** One must venture rather far in the sequence of positive Integers before two consecutlve primeless centuries emerge. **This** first happens with the century whose leading element is **191,912,801.**

The Earliest Encounter with Two Consecutive Primeless Centuries

191912801	191912811	191912821	...	191912881	191912891
191912802	191912812	191912822	...	191912882	191912892
...	...	...	...	...	...
191912901	191912911	191912921	...	191912981	191912991
191912902	191912912	191912922	...	191912982	191912992
...	...	...	...	...	...
191912909	191912919	191912929	...	191912989	191912999
191912910	191912920	191912930	...	191912990	191913000

The largest prime which precedes **this** prlmeless pair of consecutlve centuries **is 191,912,783.** The smallest which follows is **191,913,031.** No squares or cubes appear in the above long **interval** of two-hundred positive integers. Nor do higher powers, factorials, or perfect numbers, be they even or odd. Interestingly, only one odd abundant number surfaces. It **is 191,912,805.**

Extended questions concerning the first of the prlmeless millennia or other major groupings are not pursued here. But, and emphatically, such primeless groupings do exist, and there most be a first In each case.

The Remarkable First Century

The first century contains a remarkable assortment of notable number types. Included **in** this impressive assortment are:

25 primes	**4 factorials**
10 squares	**2 even perfect numbers**
4 cubes	**3 Mersenne primes**
3 fourth powers	**3 Fermat primes**
2 fifth powers	

Moreover, this leading century possesses more of the number of types here named than any other century; it stands out as a veritable gold **mine** of number encounters.

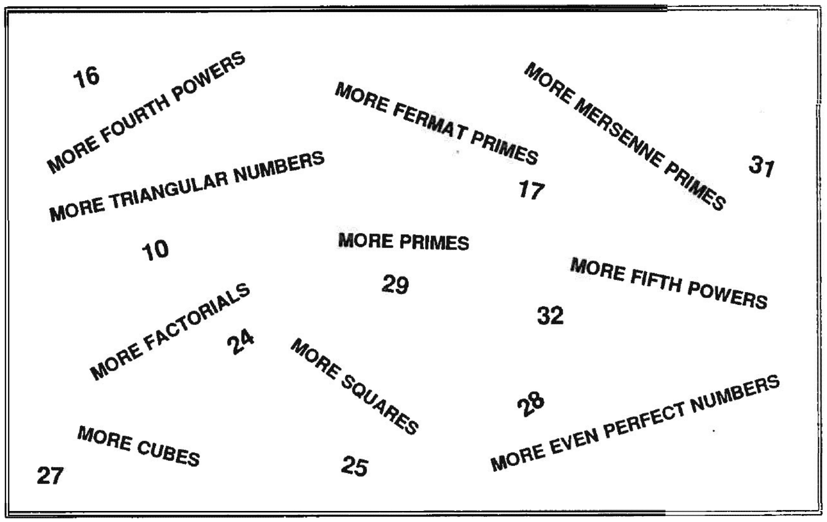
One should not infer that the earlier the century, the greater the number of primes. For example, the fourth century contains **16** primes whereas the fifth century has more **(17).** Otherwise, such erroneous logic would lead to the belief that any century following a primeless century must also be primeless. This contradicts the fact that the set of primes is infinite.

The earliest century with no squares begins at **2501,** with no cubes at **401,** and with no factorials at **201.** Careful checking also reveals that the earliest century with no fourth powers, fifth powers, sixth powers, as well as no perfect numbers is the one beginning at **101.** In fairness, **it** should be noted that certain significant number types avoid the first century altogether. For example, no **pseudoprimes,** no odd abundant numbers, and no amicable pairs appear.

Does the first century contain more of a given number type than any other century? So frequently, the answer is YES. Sometimes, responses are easily given as **in** the case for wen primes. Or for superpowers, namely, numbers of the form x^x where x **is** a positive integer **(e.g., $1^1 = 1$, $2^2 = 4$, 33 = 27).** other number classifications demand greater analysis. Such types as Pythagorean Triples or palindromic primes **(e.g., 2,** 3, **5, 7, 11)** fall into this last category.

The century definition requires the greatest element to be a multiple of **100.** Such an element thus "ends" in two zeros. If other groupings are allowed, various modifications of results stand out. For example, the one hundred consecutive integers **2** through **101** contain **26** prlmes. Or the ten consecutive integers **2** through **11** contain five primes. Definitions here **included** of decades, centuries, millennia, etc. preclude groupings which begin randomly.

THE FIRST CENTURY



THE FIRST CENTURY
contains more of the number types shown above than any other century.

Millennia and More

Groupings according to powers of ten lend themselves nicely to easy packaging and convenient compartments. This is due to our system of counting which is based on ten. Obviously, aggregates could be chosen so as to be of very unusual **size** (for example, primes within the first **169** positive integers, etc.). Nothing suggesting a mysterious intermingling of base ten notions and the concept of primality is implied.

Acknowledging the above, let us skip momentarily from decades and centuries and look at millennia. In particular, the first millennium contains exactly **168** primes. Counting further, such results as the following are noted:

Millennium	Number of Primes
1st	168
2nd	135
3rd	127
4th	120
5th	119
6th	114
...	...
60th	91
81st	88
..	...

Infinitely many millennia can be found. It is here conjectured that the first millennium contains more primes than any other millennium.

The examination of still larger powers of ten leads to additional **conjecturing.**

10^n	Number of Primes Less Than 10^n
10^1	4
10^2	25
10^3	168
10^4	1229
10^5	9592
10^6	78498
10^7	664579
10^8	5761455
10^9	50847534
10^{10}	455052512
10^{11}	4118054813
10^{12}	37607912018
...	...

Note that the first million positive integers contain **78498** primes. Will the following groupings of a million possess fewer than **78498** primes? More impressively, the first grouping of ten billion positive integers contains **455,052,512** primes whereas the second grouping contains only **427,154,204** primes. Will the succeeding groupings of ten billion positive integers contain fewer primes also than that of the first? All of this leads to what I have called the TOP HEAVY CONJECTURE, namely,

THE FIRST AGGREGATE OF ORDER N **(N≥2)** CONTAINS
MORE PRIMES THAN ANY OTHER AGGREGATE OF ORDER N.

Analytic number theory gives some insight on the subject of the occurrence of primes over vast intervals. Such results are approximative In nature and do not permit a meticulous look at select groupings of the positive integers. In particular, if **g(x)** denotes the number of primes not greater than **x,** then the ratio of **g(x)** to **x/ln** x approaches the number **1** as x becomes large without bound. Such a proof was completed in the late nineteenth century and was the work of Hadamard and de la **Vallee** Poussin.

$$\lim_{x\to\infty} \frac{g(x)}{x/\ln x} = 1$$

PRIME NUMBER THEOREM

This limiting relationship provides a look at prime occurrences in an average manner. It does not permit an exact disposition concerning the number of primes in a given aggregate. For example, the first grouping of ten trillion positive integers contains **346,065,535,898** primes. Yet a certain later grouping of ten trillion positive integers will contain no primes. Still later groupings will again contain primes. Note that the number of primes per century (within the first ten trillion positive integers) is roughly **3.46** on the average.

Explorations

Some centuries contain decidedly more primes than others. Accordingly, a century will be considered "crowded if it possesses at least ten primes. Crowded centuries stand out in the earlier encounters with the positive integers. Intriguing questions quickly come to mind in the context of loneliness and crowdedness. Among these, we find the inquiry "Is the set of crowded centuries finite, and, if so, what is the last century?" Generally, an aggregate of order n ($n \geq 2$) will be considered crowded if it contains at least 10^{n-1} primes.

To place greater focus on the first century as a numerically prominent century, the few additional explorations below are also offered.

1. Show that the first century contains more triangular numbers than any other.
2. Show that no century beyond the first can contain two even perfect numbers.
3. Prime triplets are triples of primes which differ consecutively by 2. The first century contains, for example, the triplet 3, 5, and 7. Show that no century contains more prime triplets than the first.
4. The first century contains seven primes "ending" in 3. Does any century contain more than seven such primes?
5. The next to the last element of a century "ends' in 99. Consider a century "special" if it next to the last element is of the form 199999...999 (all nines except for an initial one). Show that infinitely many special centuries have a next to the last element which is composite.
6. Note that the last decade of the first century contains exactly one prime (97) and is thus a lone-prime decade. A century containing exactly one prime is called a lone-prime century. An example of such is the century beginning with 13,200,001; its only prime is the number 13,200,001. Find another lone century. Does there exist a millennium with exactly one prime?
7. Are there infinitely many lone-prime centuries? If so, is It possible that all centuries will prove to be lone-prime centuries from a certain number on?
8. Show that infinitely many centuries "begin" with a prime number. Show that infinitely many also "begin" with a composite number.
9. The second decade is perfectly balanced as there are as many primes in the first half as in the second half. Does there exist a perfectly balanced (non-primeless) century? The tenth decade is extremely unbalanced as all of its primes are in one of the halves. Does there exist an extremely unbalanced century, that is, one with all its primes in either the first or second half?
10. Twin primes are primes differing by two. Eight such pairs appear in the first century. Does any century contain a greater number of twin primes?

The last mentioned exploration is a venture into a general area of many unsolved problems. It includes the cardinality of the set of prime twins. Although the first century contains eight such twins, the tenth century contains none whatever. The pattern of their unpredictable occurrence by centuries continues. For example, the entire millennium beginning with 956,001 contains only one such pair whereas the single century beginning with 1,006,301 remarkably contains five sets of prime twins.

Prime-placed primes likewise lead to additional conjecturing. Suppose p_k denotes the k^{th} prime. If k is also prime, then p_k is called a prime-placed prime. Such numbers as 5, 11, 67, and 83 fall into this category. Actually, the first century contains nine prime-placed primes, but the second century only five. All of this is to suggest still another venture. That is, does the first century contain more prime-placed primes than any other?

And more! Does the first century contain more Pythagorean primes (of the form $x^2 + y^2$) than the others? Or more absolute primes (those which are prime regardless of the arrangement of digits such as 17 or 31 or 73)? Or star primes (those with a prime number of digits such as 23 or 89)? Explorations appear numerous and branch out in varied directions.

Intuitively speaking, none of the results above concerning the first century abundance should prove shocking. Fewer divisors are available in the first century with which factoring attempts can be made. Likely suggested is a fruitful supply of primes in this earlier grouping. Increasing differences among squares and cubes likewise lead one to conjecture a more frequent encounter with such numbers in the smaller setting of the first century. Factorials, small at the outset, lead to the same conclusion. Of course, some numbers behave more mysteriously and superficially erratically than others. Highly intuitive notions often present the greatest of challenges in the many attempts at proof and rigorization. Here, the primes prove no exception. Highlighted in this and similar settings is the first century, an abundant field of golden pebbles called numbers.

Appreciation is expressed to Johnny *Lai*, Southeast Missouri State University, for his assistance in the computer verification of certain of the results of this paper.

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CHANGES OF ADDRESS

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USING THE MVT TO COMPLETE THE BASIC INTEGRATION FORMULA

Norman Schaumberger
Hofstra University

When considering the formula

$$\int_a^b x^n\,dx = \frac{1}{n+1}\left(b^{n+1} - a^{n+1}\right) \qquad (1)$$

we are obliged to exclude the case $n = -1$. The usual properties of the logarithmic function along with the formula $d(\ln x)/dx = 1/x$ are consequences of the definition

$$\ln x = \int_1^x t^{-1}\,dt\,, \quad x > 0. \qquad (2)$$

Furthermore, the relation

$$\ln\left(\frac{b}{a}\right) = \int_a^b x^{-1}\,dx\,, \quad b > a > 0 \qquad (3)$$

that it is reasonable to expect that the expression

$$\frac{1}{n+1}\left(b^{n+1} - a^{n+1}\right)$$

can readily be derived from (2). Equation (1) is still meaningless when $n = -1$, but (3) does suggest approaches $\ln(b/a)$ as n tends to -1. This point, although rarely discussed in standard texts, can be made plausible by considering values of n close to -1. Thus, for example,

$$\int_2^3 x^{-.999}\,dx = \frac{1}{.001}\left(3^{.001} - 2^{.001}\right) = .4058\ldots$$

and $\ln(3/2) = .4054\ldots$.

We offer a simple proof that

$$\lim_{n\to -1} \frac{1}{n+1}\left(b^{n+1} - a^{n+1}\right) = \ln\left(\frac{b}{a}\right). \qquad (4)$$

Using the Mean Value Theorem with $f(x) = \ln x$ gives

$$\frac{\ln\left(b^{n+1}\right) - \ln\left(a^{n+1}\right)}{b^{n+1} - a^{n+1}} = \frac{1}{c}$$

where $c \in (a^{n+1}, b^{n+1})$. Since $b > a > 0$, it follows that

$$\frac{1}{b^{n+1}} < \frac{\ln(b^{n+1}) - \ln(a^{n+1})}{b^{n+1} - a^{n+1}} < \frac{1}{a^{n+1}}.$$

This can be written as

$$a^{n+1}\ln\left(\frac{b}{a}\right) < \frac{b^{n+1} - a^{n+1}}{n+1} < b^{n+1}\ln\left(\frac{b}{a}\right).$$

If we let $n \to -1$, then the two outer terms tend to $\ln(b/a)$ and we get (4).

THE WEIGHTED JENSEN INEQUALITY

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If $x_1, x_2, \ldots x_n$, are angles satisfying $0 \le x_i \le \pi$ (i = 1, 2, ..., n), then

$$\sin\left(\frac{x_1 + x_2 + \ldots + x_n}{n}\right) \ge \frac{1}{n}\left(\sin x_1 + \sin x_2 + \ldots + \sin x_n\right) \qquad (1)$$

with equality iff $x_1 = x_2 = \ldots = x_n$. Furthermore,

$$\cos\left(\frac{x_1}{2} + \frac{x_2}{3} + \frac{x_3}{6}\right) \ge \frac{1}{2}\cos x_1 + \frac{1}{3}\cos x_2 + \frac{1}{6}\cos x_n \qquad (2)$$

holds If the x's satisfy $-\pi/2 \le x_i \le \pi/2$, with equality iff $x_1 = x_2 = x_3$.

Inequality (1) Is a special case of Jensen's inequality which states that if $f(x)$ has a second derivative $f''(x) < 0$ in the interval $a < x < b$ then for $a < x_i < b$ ($i = 1,2,\ldots,n$)

$$f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \ge \frac{1}{n}\sum_{i=1}^{n} f(x_i) \qquad (3)$$

with equality iff $x_1 = x_2 = \ldots = x$

The standard derivation of (3) follows Cauchy's method of proof of the AM-GM inequality. (See, for example, [3].) A proof of (3) using elementary properties of the derivative was given by the authors in [2]. Inequality (2), on the other hand, is a special case of Jensen's weighted inequality. This states that if $f(x)$ and x_i are as in (3) and $p_i > 0$ ($i = 1,2,\dots n$) are real numbers such that

$$\sum_{i=1}^{n} p_i = 1,$$

then

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \ge \sum_{i=1}^{n} p_i f(x_i) \tag{4}$$

with equality iff $x_1 = x_2 = \dots = x_n$.

A not particularly simple non-calculus proof of (4) where the p_i are restricted to rational numbers can be found in [1]. We offer a simple calculus proof of the weighted Jensen inequality which is valid for all real p_i and which is based on an extension of the argument in [2].

If $a < x < b$ and $w = p_1x_1 + p_2x_2 + \dots + p_nx_n$ where $a < x_i < b$, then

$$f(w) - wf'(w) \ge f(x) - xf'(w) \tag{5}$$

with equality iff $x = w$. (5) follows from the observation that $f''(x) < 0$ on (a, b) and thus $g(x) = f(x) - xf'(w)$ takes its maximum in (a, b) at $x = w$, because $g'(x) = f'(x) - f'(w)$ is monotone decreasing on this interval and thus vanishes iff $x = w$. Substituting $x = x_1$, $x = x_2$, ... $x = x_n$ into (5) gives the inequalities

$$f(w) - wf'(w) \ge f(x_i) - x_i f'(w), \quad (i = 1,2,\dots,n) \tag{6}$$

Multiplying (6), in turn, by $p_1, p_2, \dots p_n$ and adding, we get

$$f(w)\sum_{i=1}^{n} p_i - wf'(w)\sum_{i=1}^{n} p_i \ge \sum_{i=1}^{n} p_i f(x_i) - f'(w)\sum_{i=1}^{n} p_i x_i \tag{7}$$

Since

$$\sum_{i=1}^{n} p_i = 1$$

and

$$w = \sum_{i=1}^{n} p_i x_i,$$

we can use (7) to establish (4).

is, iff $x_1 = x_2 = \dots = x_n = w$. If we put $p_1 = p_2 = \dots = p_n = 1/n$ then (4) becomes (3). Also, if $f(x) = \ln x$, $f''(x) = -1/x^2 < 0$ for $x > 0$. Hence

$$\ln(p_1x_1 + p_2x_2 + \dots + p_nx_n) \ge p_1\ln x_1 + p_2\ln x_2 + \dots + p_n\ln x_n$$

or

$$p_1x_1 + p_2x_2 + \dots + p_nx_n \ge x_1^{p_1}x_2^{p_2}\dots x_n^{p_n}. \tag{8}$$

Equality holds iff $x_1 = x_2 = \dots = x_n$. Inequality (8) is the weighted AM-GM Inequality. Putting $p_1 = p_2 = \dots = p_n = 1/n$ gives the AM-GM Inequality.

Finally, we note that if $f''(x) > 0$ then the inequality in (4) is reversed. If, for example, $f(x) = \tan x$ then $f''(x) = 2\sec^2 x \tan x > 0$ for $0 < x < \pi/2$ and by Jensen's weighted inequality,

$$p_1\tan x_1 + p_2\tan x_2 + \dots + p_n\tan x_n \ge \tan(p_1x_1 + p_2x_2 + \dots + p_nx_n) \tag{9}$$

for any set of n positive acute angles $x_1, x_2, \dots, x_n$, with equality iff $x_1 = x_2 = \dots = x_n$. If $n = 3$, x_1, x_2, x_3 are angles of an acute triangle, and

$$p_1 = \frac{x_1}{x_1 + x_2 + x_3}, \quad p_2 = \frac{x_2}{x_1 + x_2 + x_3}, \quad p_3 = \frac{x_3}{x_1 + x_2 + x_3}$$

then (9) becomes

$$x_1\tan x_1 + x_2\tan x_2 + x_3\tan x_3 \ge \pi\tan\left(\frac{x_1^2 + x_2^2 + x_3^2}{\pi}\right).$$

Equality holds iff the triangle is equilateral.

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$$\cos(\tau) = \cos\left(\frac{\theta+\frac{\pi}{2}}{4}\right) = \sqrt{\frac{1+\cos\left(\frac{\theta+\frac{\pi}{2}}{2}\right)}{2}}$$

$$= \sqrt{\frac{1+\sqrt{\frac{1+\cos\left(\theta+\frac{\pi}{2}\right)}{2}}}{2}}$$

Now, $\cos(\theta + \pi/2) = \cos(\theta)\cos(\pi/2) - \sin(\theta)\sin(\pi/2) = -\sin(\theta) = -3/5$. So

$$\cos(\tau) = \sqrt{\frac{1+\sqrt{\frac{1+(-3/5)}{2}}}{2}} = \sqrt{\frac{1+\sqrt{1/5}}{2}}$$

We also can see that:

$$\frac{1}{x} = \frac{1}{\sqrt{3-\phi}} = \frac{1}{\sqrt{3-\frac{1+\sqrt{5}}{2}}} = \frac{1}{\sqrt{\frac{5-\sqrt{5}}{2}}} \quad \frac{\sqrt{2}}{\sqrt{5-\sqrt{5}}}$$

Now consider the following identity:

$$2 = 2$$

$$2 = \sqrt{4}$$

$$2 = \sqrt{5+\frac{5}{\sqrt{5}}-\sqrt{5}-1}$$

$$2 = \sqrt{5-\sqrt{5}} \cdot \sqrt{1+\frac{1}{\sqrt{5}}}$$

$$\frac{\sqrt{2}}{\sqrt{5-\sqrt{5}}} = \sqrt{\frac{1+\frac{1}{\sqrt{5}}}{2}}$$

and, therefore, $\cos(\tau) = 1/x$.

I have shown before (*Pi Mu* Epsilon Journal, volume 9, number 2) that $\tan(\tau) = \tan((\theta + \pi/2)/4) = 1/\phi$. Therefore,

$$\sin(\tau) = \tan(\tau)\cdot\cos(\tau) = \frac{1}{\phi}\cdot\frac{1}{\sqrt{3-\phi}} = \frac{1}{\phi\sqrt{3-\phi}} = \frac{1}{y},$$

So. $\cos(\tau) = 1/x$, $\sin(\tau) = 1/y$, and $\tan(\tau) = 1/\phi$, which is what we were trying to prove.

A NOTE ON $(1 + k/n)^n$

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A standard textbook technique used to prove that the limit

$$(*) \qquad \lim_{n\to\infty} (1+1/n)^n$$

exists is to show that the sequence $\{(1 + 1/n)^n\}$ is increasing and bounded above by 3. This is sometimes followed with an exercise to show that limit (*) exists for some particular positive integer k [1, p. 115-116; 2, p. 33-38]. The purpose of this paper Is to prove that the sequence defined by $x_n = (1 + k/n)^n$ converges for every positive integer k by the completeness property of the real number system.

To prove that $\{x_n\}$ is increasing, the following result will be used. For positive real numbers $y_1, y_2, \ldots, y_{n+1}$, the arithmetic mean (M) and the geometric mean (G) are defined by $M = (y_1 + \ldots + y_{n+1})/(n + 1)$ and $G = (y_1 \ldots y_{n+1})^{1/(n+1)}$, respectively. It is well known that $M \geq G$, with equality holding only when $y_1 = \ldots = y_{n+1}$.

In particular, let $y_1 = 1$ and $y_i = 1 + k/n$, for $i = 2, 3, \ldots, n + 1$. Then it is easy to show that $M = 1 + k/(n + 1)$ and $G = (1 + k/n)^{n/(n+1)}$. Hence, since $M > G$,

$$1 + k/(n+1) > (1 + k/n)^{n/(n+1)}.$$

So,

$$x_{n+1} = [1 + k/(n+1)]^{n+1} > (1 + k/n)^n = x_n$$

and $\{x_n\}$ is an increasing sequence.

Using the fact that $(1 + 1/n)^n < 3$, it will now be shown that $x_n < 3^k$.

$$\begin{aligned} x_n = (1 + k/n)^n &\leq (1 + k/n + k(k-1)/2n^2 + \ldots + 1/n^k)^n \\ &= [(1 + 1/n)^k]^n \\ &= [(1 + 1/n)^n]^k \\ &< 3^k \end{aligned}$$

Since $\{x_n\}$ is increasing and bounded, the sequence converges by the completeness property.

References:

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A NAPOLEON TRIANGLE REVISITED

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[Jack Garfunkel submitted this paper shortly before his death. (See In Memoriam, on page 272.) Clayton Dodge was kind enough to complete the preparation of this paper.]

Some theorems in geometry come and go, but a few catch our fancy and remain popular and exciting. These theorems have a certain elegance and charm, and perhaps an unexpected result. One such theorem is credited to Napoleon Bonaparte. It states that if equilateral triangles are constructed on the three sides of any given triangle, all constructed externally or all internally, then their centroids form an equilateral triangle. The areas of these two centroid equilateral triangles differ by the area of the given triangle. Furthermore, the three lines formed by joining the third vertex of each equilateral triangle to the opposite vertex of the given triangle concur. The point of concurrence of the lines from the centroids of the equilateral triangles drawn outwardly **subtends** equal **120°** angles at the sides of the given triangle. If no angle of the given triangle exceeds **120°**, then **this** point of concurrence is the point from which the sum of the distances to the vertices of the given triangle is a minimum.

We shall prove that the centroids form equilateral triangles and also the area relationship as part of our proof of certain other inequalities. Later in the paper we shall prove the concurrence of the lines in a more general setting. The sizes of the angles and the minimum distance property will be left for the reader to investigate. See [3, pp. 63-65] and [5, p. **72]**.

It **is** convenient for us to use the following equivalent form of Napoleon's theorem in this **paper.**

Napoleon's Theorem. If on the middle **third** of each side of a given triangle ABC an equilateral triangle is constructed, all constructed externally or all internally, then their third vertices form an equilateral triangle.

In Figure 1 triangle A'B'C' is called the outer Napoleon triangle and triangle **A"B"C"** is the Inner Napoleon triangle. In this paper we shall prove some **additional** properties of the outer Napoleon triangle, and develop some interesting (and perhaps unexpected and surprising) extensions. To that end we shall assume the notation and terminology of Figure 1.

For convenience we shall use the notation Sa = a + b + c. Also we let $Q = S(b-c)^2 = (b-c)^2 + (c-a)^2 + (a-b)^2$, which is, of course, nonnegative. Then we prove the following lemma.

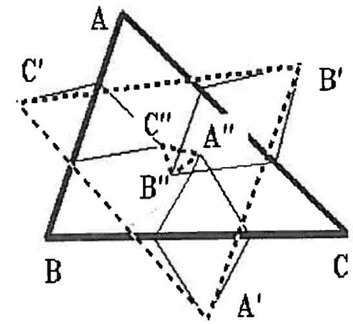


Figure 1

Lemma 1. If **s** is the semiperimeter of triangle ABC, then

$$4s^2 - 3\Sigma a^2 + Q = 0.$$

We have that

$$4s^2 + Q = (\Sigma a^2 + 2\Sigma ab) + (2\Sigma a^2 - 2\Sigma ab) = 3\Sigma a^2. \square$$

Now we are ready to prove our first theorem, In whose proof we shall make use of the result [1, p. 42, Item **4.3]**

$$s^2 \geq 3F\sqrt{3} + Q/2, \quad \text{whence} \quad 2s^2 - 6F\sqrt{3} \geq Q.$$

Theorem 1. The perimeter 2s of a given triangle ABC is not less than the perimeter 2s' of its outer Napoleon triangle A'B'C'.

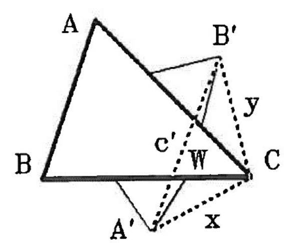


Figure 2

Let W be **1/3** of the way from C to B, let $x = A'C$, $y = B'C$, and $c' = A'B'$. See Figure 2. Then $WC = WA' = a/3$ and $\angle A'WC = 120°$, so $x = a/\sqrt{3}$. Similarly, $y = b/\sqrt{3}$. Then, using the law of cosines In triangle A'B'C, we have

$$(c')^2 = \frac{a^2+b^2}{3} - \frac{2ab}{3}\cos(60° + C)$$

$$= \frac{a^2+b^2}{3} - \frac{2ab}{3}(\cos 60° \cos C - \sin 60° \sin C).$$

Because cos C = $(a^2 + b^2 - c^2)/2ab$ and the area F of triangle ABC is given by F = (ab/2) sin C, we get that

$$(c')^2 = \frac{a^2+b^2+c^2}{6} + \frac{2F}{\sqrt{3}}.$$

Since side **c"** of the inner Napoleon triangle **subtends** an angle of **|60° - C|**, the corresponding relation is

$$(c'')^2 = \frac{a^2+b^2+c^2}{6} - \frac{2F}{\sqrt{3}}.$$

Because the expressions for **c'** and **c"** are symmetric In a, **b,** and **c,** it follows that a' = b' = **c'** and a" = b" = c", proving that the two Napoleon triangles are equilateral.

To show that 2s ≥ **2s',** we show that $(2s)^2 - (2s')^2$ a 0. Thus we have

$$(2s)^2 - (2s')^2 = 4s^2 - (3a')^2 = 4s^2 - 9(a')^2$$

$$= 4s^2 - (3/2)\Sigma a^2 - 6F\sqrt{3}$$

$$\geq 2s^2 - (3/2)\Sigma a^2 + Q$$

$$= Q \geq 0. \square$$

It Is easy now to prove the Napoleon theorem area relationship. Let **F, F',** and **F"** denote the areas of the triangles ABC, A'B'C', and **A"B"C"** respectively. Since the altitude **of** the equilateral triangle of side a' Is equal to **a'√3/2,** then its area is **F'** = $(a')^2\sqrt{3}/4$. Similarly, F" = $(a'')^2\sqrt{3}/4$. Thus, the difference between the areas of the outer and inner Napoleon triangles is given by

$$F' - F'' = \frac{\sqrt{3}}{4}\left(\frac{a^2+b^2+c^2}{6} + \frac{2F}{\sqrt{3}}\right) - \frac{\sqrt{3}}{4}\left(\frac{a^2+b^2+c^2}{6} - \frac{2F}{\sqrt{3}}\right) = F,$$

which is the desired result. □

Theorem 2. The **inradius** r' of the outer Napoleon triangle **A'B'C'** is not less than the **inradius** r of the given triangle ABC.

Since F = rs and **F'** = **r's'** and we have just shown that **F'** ≥ **F,** then **r's'** ≥ rs. Since also **s'** ≤ **s** by Theorem 1, then we must have that **r'** ≥ r. □

We have seen that **F'** a F and **r'** a **r,** but **s'** ≤ **s.** Let us see just what relationship exists between **R** and **R',** the circumradil. This result is not quite so obvious as that of Theorem 2. In It we shall use the results **[1,** p. 18, Item **2.3]** $\Sigma \sin^2 A \leq 9/4$ and **[1,** p. 20, Item **2.8]** $\Pi \sin A \leq 3\sqrt{3}/8$, and the known relations **[4,** p. **31]** F = **abc/4R** and **[4,** p. 33, Exercise **22]** a = 2R sin A, etc.

Theorem 3. The circumradius **R** of a given triangle **ABC** is not less than the circumradius **R'** of its outer Napoleon triangle A'B'C'.

Since the circumradius of an equilateral triangle is equal to **2/3** of its altitude, then

$$R' = \frac{2}{3}h'_a = \frac{2}{3}\left(\frac{a'}{2}\sqrt{3}\right) = \frac{a'}{\sqrt{3}}.$$

Now we have

$$(R')^2 = \frac{1}{3}\left(\frac{1}{6}\sum a^2 + \frac{2F}{\sqrt{3}}\right)$$

$$= \frac{1}{18}\sum a^2 + \frac{2F}{3\sqrt{3}}$$

$$= \frac{1}{18}\sum a^2 + \frac{abc}{6R\sqrt{3}}$$

$$= \frac{4R^2}{18}\sum \sin^2 A + \frac{4R^3\prod \sin A}{3R\sqrt{3}}$$

$$= \frac{2R^2}{18}\sum \sin^2 A + \frac{4R^2\prod \sin A}{3\sqrt{3}}.$$

To show that **R** ≥ **R',** we must prove that

$$1 \geq \frac{2}{9}\sum \sin^2 A + \frac{4}{3\sqrt{3}}\prod \sin A.$$

Thus

$$\frac{2}{9}\sum \sin^2 A + \frac{4}{3\sqrt{3}}\prod \sin A \leq \frac{2}{9}\left(\frac{9}{4}\right) + \frac{4}{3\sqrt{3}}\left(\frac{3\sqrt{3}}{8}\right)$$

$$= \frac{1}{2} + \frac{1}{2} = 1.$$

We have proved Theorem 3. □

Erecting equilateral triangles on the middle **third** of each side of a triangle to determine the points A', **B',** and C' is a rather special and arbitrary choice. The question arises as to what would happen if, as a generalization of the Napoleon figure, we erected arbitrary isosceles triangles instead. Equivalently, let us erect perpendiculars at the midpoints of the sides and extend them to lengths proportional to the sides.

Theorem 4. At the midpoints of the sides of a triangle ABC, perpendiculars are drawn, all outwardly or all Inwardly, and extended to lengths proportional to their respective **sides.** If the endpoints of these perpendiculars are denoted by A', B', and **C',** then triangles ABC and A'B'C' are in perspective.

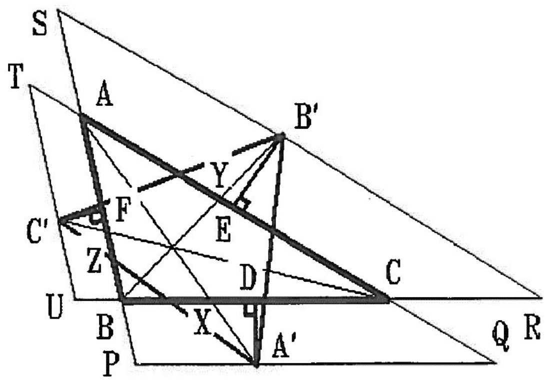


Figure 3

Refer to Figure 3. Let **D, E,** and F be the midpoints of the sides BC, CA, and AB of triangle ABC, and erect all outward or all inward perpendiculars A'D, **B'E,** and C'F to the sides such that **DA'/BC** = **EB'/CA** = **FC'/AB** = k for a given real **k.** Now draw a line through A' parallel to BC and meeting AB at P and AC at Q, a line through B' parallel to CA and meeting BC at R and BA at S, and a line through C' parallel to AB and meeting CA at T and CB at U. Let AA' meet BC at X, BB' meet CA at Y, and CC' meet AB at Z.

By **Ceva's** theorem, **it** suffices to show that

$$\frac{AZ}{ZB}\cdot\frac{BX}{XC}\cdot\frac{CY}{YA} = 1.$$

Because of the similar triangles CAB and CTU, etc., we have

$$\frac{AZ}{ZB} = \frac{TC'}{C'U}, \quad \frac{BX}{XC} = \frac{PA'}{A'Q}, \quad \text{and} \quad \frac{CY}{YA} = \frac{RB'}{B'S}.$$

Hence we need to show that

$$\frac{TC'}{C'U}\cdot\frac{PA'}{A'Q}\cdot\frac{RB'}{B'S} = \frac{PA'}{C'U}\cdot\frac{RB'}{A'Q}\cdot\frac{TC'}{B'S} = 1.$$

By the similarity of quadrilaterals BFC'U and BDA'P, etc., we get

$$\frac{PA'}{C'U} = \frac{BD}{FB}, \quad \frac{RB'}{A'Q} = \frac{CE}{DC}, \quad \text{and} \quad \frac{TC'}{B'S} = \frac{AF}{EA}.$$

Hence we find that

$$\frac{PA'}{C'U}\cdot\frac{RB'}{A'Q}\cdot\frac{TC'}{B'S} = \frac{BD}{FB}\cdot\frac{CE}{DC}\cdot\frac{AF}{EA} = \frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB} = 1. \ \square$$

We shall call the triangle **A'B'C'** of Theorem 4 a ***Garfunkel*** triangle for the given triangle ABC.

A special case of theorem 4 proves the concurrence of the three lines joining the third vertices of either Napoleon triangle to the corresponding vertices of the given triangle.

At **this** point we remind the reader of two delightful **special** points in a triangle, which enter into our final theorems. If a **point** is chosen on each **side** of a triangle and If three circles are drawn, each through a vertex and the chosen points on the two adjacent sides, then these three circles concur at a point called the Miguel point for the triangle and the three selected points. See Figure 4.

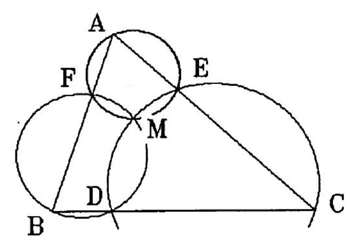


Figure 4

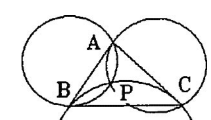


Figure 5

In triangle ABC draw a circle through vertex A and tangent to side BC at **B,** a circle through B and tangent to CA at **C,** and a circle through C and tangent to AB at A. Then these three circles concur at a point called a Brocard point for the triangle. See Figure 5. By symmetry there are two Brocard points for a triangle. By considering inscribed angles, **it** is easy to show that angles CBP, ACP, and BAP are equal. In fact, the converse is also true. If those three angles are equal, then point P is a Brocard point for triangle ABC.

Theorem 5. Construct a Garfunkel triangle A'B'C' for a given triangle ABC. Let the lines C'A and A'B meet at **P,** lines A'B and B'C meet at **Q,** and B'C and C'A meet at **R.** Then the Miguel point for triangle **PQR** associated with the three points A', B', and C' is the circumcenter of triangle ABC.

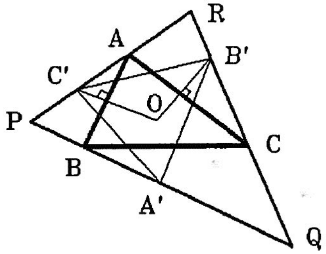


Figure 6

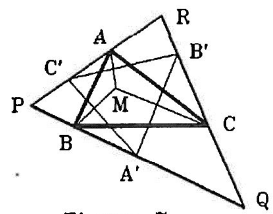


Figure 7

In Figure 6 let O be the circumcenter of triangle ABC. Now we have that angles CBQ, ACR, and BAP are **equal** because triangles CBA', ACB', and **BAC'** are similar by construction. Then

$$\angle A = \angle BAC = 180° - \angle BAP - \angle RAC = 180° - \angle ACR - \angle RAC = \angle ARC = \angle R.$$

Since $\angle A + \angle C'OB' = 180°$, then $\angle R + \angle C'OB' = 180°$. Therefore, the circle through **B', R,** and C' passes through O. Similarly, the circles through A', **Q,** and **B'** and through C', **P,** and A' both pass through **O,** so O is the desired Miguel point. □

We conclude our list of theorems with an interesting relation between a Miguel point, a Brocard point, and a Garfunkel triangie.

Theorem 6. Under the hypothesis of Theorem 5, the Miguel point for triangle PQR associated with the three points A, **B,** and C is a Brocard point of triangle ABC.

Let M be the Miguel point for triangle **PQR** and points A, **B,** and C. See Figure **7.** From the proof of Theorem 5, we know that $\angle A = \angle R$. Because AMCR is a cyclic quadrilateral, then

$$\angle AMC = 180° - \angle R = 180° - \angle A.$$

Therefore we have

$$180° = \angle MAC + \angle AMC + \angle MCA = \angle MAC + 180° - \angle A + \angle MCA,$$

so that

$$\angle MAC + \angle MCA = \angle A = \angle MAC + \angle MAB.$$

Now $\angle MCA = \angle MAB$, which in **turn** = $\angle MBC$ by symmetry. Hence M is a Brocard point for triangle ABC. □

References

[1] O. Bottema, et **al,** Geometric Inequalities, **Wolters-Noordhoff** Publishing, The Netherlands, **1968.**

[2] N.A. Court, College Geometry, Johnson Publishing Company, Virginia, **1925.**

[3] H.S.M. **Coxeter,** and **S.** L. **Greitzer,** Geometry Revisited, The Mathematical **Association of** America, Washington, D.C., **1967.**

[4] C.W. Dodge, Euclidean Geometry and Transformations, Addison-Wesley Publishing Company, Massachusetts, **1972.**

[5] H.W. Eves, A Survey of ***Geometry,*** rev ed, **Allyn &** Bacon, Massachusetts, **1972.**

LETTER TO THE EDITOR

Dear Editor:

In the Fall, 1990, issue of the Journal, there was a letter to the editor from me concerning the article 'The AM-GM Inequality: A Calculus Quickie," by Norman Schaumberger, which had appeared in Spring, **1990.** In my letter **I** stated that an equality condition given by Schaumberger was Incorrect. The equality condition was actually correct as stated in Schaumberger's article.

Sincerely,

Murray Klamkin
Mathematics Department
University of Alberta
Edmonton, Alberta, Canada T6G **2G1**

INQUIRIES

Inquiries about certificates, pins, posters, matching prize funds, support for regional meetings, and travel support for national meetings should be directed to the Secretary-Treasurer, Robert M. **Woodside,** Department of Mathematics, East Carolina University, Greenville, **NC 27858; 919-757-6414.**

FAIR FARE FUNCTIONS

J.N. Boyd and P.N. Raychowdhury
Virginia Commonwealth University

INTRODUCTION

The Acme Bus Corporation (ABC for short) was created to meet the needs of the good citizens of towns $x_1, x_2, x_3, \dots x_{n-1}$. The essential geographical feature explaining these transportation needs is Bear Mountain as indicated on the map below.

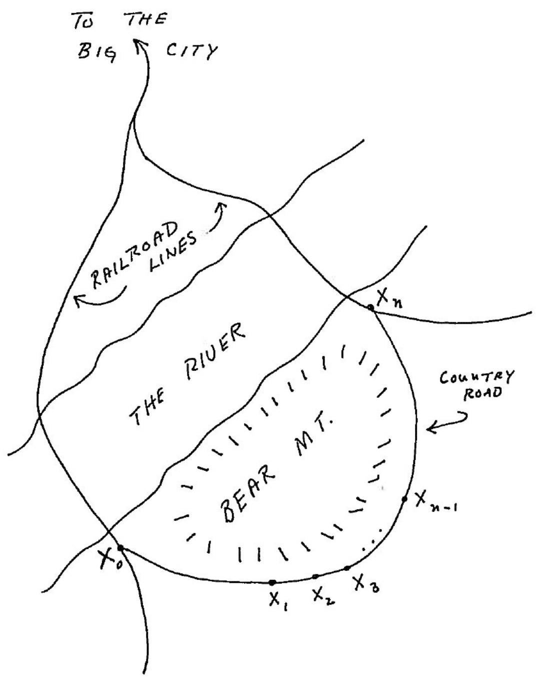


Figure 1. The Geography of Towns $x_1, x_2, x_3, \dots x_{n-1}$.

The towns are connected by a country road which runs over level ground at the foot of the mountain. The read also links the towns with villages x_0 and x_n which lie on the main railroad line to the big city. Many of the citizens of $x_1, x_2, x_3, \dots x_{n-1}$ work In the big city; and, from both x_0 and x_n, commuter tralns travel to the city with convenient regularity. Eventually, the ABC was established to run buses back and forth along the country road between x_0 and x_n, picking up and letting off passengers along the way.

The distances between any two towns, x_i, x_j ($i \neq j$; $i, j \bullet \{1, 2, 3, \dots, n-1\}$), are relatively short when compared to the distance from any of the towns to either x_0 or x_n. Consequently, commuters do not care whether they catch a bus headed for x_0 or one headed for x_n since either bus will carry them to a station where the wait for the next train is never long. Therefore, they simply take the first bus that comes along.

The round trip fair $f(i)$, from town x_i to either railroad station in the morning and back again in the afternoon was established by the board of directors of ABC. It so happened that the Chairman of the Board had been a mathematician in his youth with a particular interest in discrete harmonic functions. [1] He persuaded the board that the average value property of harmonic functions represented the fairest model for establishing the round trip fares from the different towns.

Unfortunately, since $f(0)$ and $f(n)$ both had to be zero, the harmonic rule

$$f(i) = [f(i-1) + f(i+1)]/2$$

would have implied that $f(i) = 0$ for all i, thereby quickly putting ABC out of business. So, the board, acting upon the advice of the Chairman, added a surcharge of one dollar to each fare (as indicated in Rule 3 below). The board then set the fare as a function of I by the following rules:

1.) $f(0) = f(n) = 0$.
2.) $f(i) = f(n-i)$ for $i \in \{0, 1, 2, \dots n\}$ to reflect the obvious symmetry resulting from the citizens' willingness to catch their tralns at either x_0 or x_n.
3.) $f(i) = [f(i-1) + f(i+1)]/2$ for $i \in \{1, 2, 3, \dots, n-1\}$.

The extra one dollar (in Rule 3) was justified as consistent with the policy of charging one dollar for a round trip over the relatively short distances between any two towns x_i and x_j ($i \neq j$ and neither i nor j $\bullet \{0, n\}$). There had always been a modest amount of travel among the various towns in addition to the primary traffic to and from x_0 and x_n.

The Chairman was quite pleased with the properties of his fare function $f(i)$ and it is the intent of this paper to investigate some of those properties.

THE FIRST SEVERAL CASES

If there are n-1 towns with stations x_0 and x_n at the ends of the country road, we will denote the fare function by $f_n(i)$ for $n \geq 0$ and $I = 0, 1, 2, \dots, n$.

By definition, we simply say that $f_0(0) = 0$ and $f_1(0) = f_1(1) = 0$.

For $n = 2$, we have $f_2(0) = f_2(2) = 0$ by Rule 1 and $f_2(1) = [0 + 0]/2 + 1 = 1$ by Rule 3.

For $n = 3$, we have $f_3(0) = f_3(3) = 0$ and $2f_3(1) = [0 + f_3(2)] + 2$. By Rule 2, $f_3(2) = f_3(1)$. Therefore, $f_3(1) = f_3(2) = 2$.

For $n = 4$, we find $f_4(0) = 0$, $f_4(1) = 3$, $f_4(2) = 4$, $f_4(3) = 3$, $f_4(4) = 0$.

If the results of these and further computations are displayed in a triangular array, interesting relationships become apparent.

$$
\begin{array}{ccccccccc}
 & & & & f_0(0) & & & & \\
 & & & f_1(0) & & f_1(1) & & & \\
 & & f_2(0) & & f_2(1) & & f_2(2) & & \\
 & f_3(0) & & f_3(1) & & f_3(2) & & f_3(3) & \\
f_4(0) & & f_4(1) & & f_4(2) & & f_4(3) & & f_4(4) \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}
$$

becomes

$$
\begin{array}{ccccccccccccccccc}
 & & & & & & & & 0 & & & & & & & & \\
 & & & & & & & 0 & & 0 & & & & & & & \\
 & & & & & & 0 & & 1 & & 0 & & & & & & \\
 & & & & & 0 & & 2 & & 2 & & 0 & & & & & \\
 & & & & 0 & & 3 & & 4 & & 3 & & 0 & & & & \\
 & & & 0 & & 4 & & 6 & & 6 & & 4 & & 0 & & & \\
 & & 0 & & 5 & & 8 & & 9 & & 8 & & 5 & & 0 & & \\
 & 0 & & 6 & & 10 & & 12 & & 12 & & 10 & & 6 & & 0 & \\
0 & & 7 & & 12 & & 15 & & 16 & & 15 & & 12 & & 7 & & 0 \\
 & & & & & & & & \vdots & & & & & & & &
\end{array}
$$

Most of the patterns which arise along various lines through the triangle are so obvious that no comment on those patterns seems required. They suggest that the triangle should serve as a useful source for Inductive statements and proofs.

MORE GENERAL RESULTS

To make more general sense out of the triangular array, let us take first and then second differences across the horizontal rows of numbers. By so doing, we find that, for each row shown above (except those with all zeros), the second difference has the constant value of -2. This result leads us to suspect that f_n can be written as a quadratic function of i. That is, $f_n(i) = a + bi + ci^2$.

For example, if $n = 8$ (across the last row shown in our triangle of function values), our calculations yield

	0		7		12		15		16		15		12		7		0
first difference:		7		5		3		1		-1		-3		-5		-7	
second difference:			-2		-2		-2		-2		-2		-2		-2		

It Is then easy (e.g., [2]) to find the coefficients a, b, c and to show that

$$f_8(i) = 8i - i^2.$$

Thereafter, a bit more work suggests that

$$f_n(i) = ni - i^2. \qquad (1)$$

Checking our result against our three rules, we find that $f_n(0) = f_n(n) = 0$ implying that Rule 1 Is satisfied. Since $f_n(n-1) = n^2 - ni - (n^2 - 2ni + i^2) = ni - i^2$, Rule 2 is satisfied. And, since $[f_n(i-1) + f_n(i+1)]/2 + 1 = ni - i^2 = f_n(i)$, Rule 3 is also satisfied.

Furthermore, we can show that $f_n(i)$ from Equation 1 uniquely satisfies all three rules. Suppose, to the contrary, both $f_n(i)$ and $g_n(i)$ satisfy the three rules. Then

$$f_n(i) - g_n(i) = \{[f_n(i-1) + f_n(i+1)]/2 + 1\} - \{[g_n(i-1) + g_n(i+1)]/2 + 1\}$$

$$= [(f_n(i-1) - g_n(i-1) + f_n(i+1) - g_n(i+1))]/2$$

implying that the function $h_n(i) = f_n(i) - g_n(i)$ Is harmonic. Since $h_n(0) = h_n(n) = 0$, it follows that $h_n(i) = 0$ for every **i** by the uniqueness of discrete harmonic functions having identical boundary conditions. Therefore, $f_n(i) = g_n(i)$ for every i. It follows **also** that Rule 2 Is implied by **Rules** 1 and 3.

OBSERVATIONS

We leave it to our readers to decide whether or not Rules 1, 2, and 3 lead to fair fares in our scenario and to generalize the fare functions by making changes in Rule 3.

The Chief Engineer for ABC was not to be outdone. After the Chairman had explained the reasoning behind the definition of his fare function, the Chief Engineer recalled that, for each harmonic function, there ought to be an electrical network for which the harmonic function describes the potentials at the branch points of the network. He claimed that he **could** design a circuit for resistors for which the n-th fare function defined the potentials at the branch points.

Eventually, he submitted the design below.

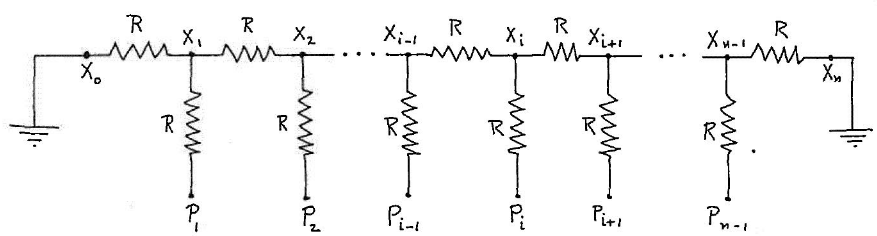


Figure 2. The Chief Engineer's Circuit.

All resistors are identical with resistance **R** ohms. Point P_i Is maintained at a potential of 2 volts above the potential V_i at branch point x_i for $i = 1, 2, 3, \ldots, n-1$. The potentials V_0 and V_n (at x_0 and x_n) are both set at zero volts.

By **Kirchhoff's** Rule for currents at any branch point, we have

$$(V_i - V_{i-1})/R = (V_{i+1} - V_i)/R + 2/R$$

where current along the chain $x_0, x_1, x_2, ..., x_n$ Is taken to be positive in the direction from left to right. After a bit of simplification, the **last** equation becomes

$$V_i = (V_{i-1} + V_{i+1})/2 + 1$$

in accord with Rule 3. Rule 1 is satisfied by $V_0 = V_n = 0$; and, as we have noted. Rule 2 is automatically satisfied whenever Rules 1 and 3 hold true.

REFERENCES

[1] J.N. Boyd and P.N. Raychowdhury, "Discrete Dirichlet Problems, Convex Coordinates, and a Random Walk on a Triangle," College Mathematics Journal 20 (1989), pp. 385-391.

[2] P.F. Dierker and W.L. Voxman, Discrete Mathematics, Harcourt Brace Jovanovich, 1986.

A rebus is a kind of puzzle whose meaning is indicated by things rather than by words. The following rebus was submitted by *Florentin* Smarandache, of Phoenix, AZ.

$$\begin{pmatrix} J & 1 & 0 & 0 & 0 & 0 \\ 0 & O & 1 & 0 & 0 & 0 \\ 0 & 0 & R & 1 & 0 & 0 \\ 0 & 0 & 0 & D & 1 & 0 \\ 0 & 0 & 0 & 0 & A & 1 \\ 0 & 0 & 0 & 0 & 0 & N \end{pmatrix}$$

MATCHING PRIZE FUND

If your chapter presents awards for Outstanding Mathematical Papers or for Student Achievement in Mathematics, you may apply to the National Office for an amount equal to that spent by your Chapter up to a maximum of fifty dollars. Contact Professor Robert Woodside, Secretary-Treasurer.

Two New Numbers Aid Mathematicians

James *Metz*

Maryknoll High School, Honolulu, HI

For many years mathematicians, and perhaps a few students of mathematics, have enjoyed rationalizing the denominators of expressions such as $7/\sqrt{2}$ and $y/\sqrt{3}$, and wen more complicated ones **including** $6/(\sqrt{2} - 5)$ and $(3 + 7i)/i$. Until now they have been forced to live with such nasty expressions as $9/\pi$ and $3/e$ with their irrational denominators. Two new numbers now solve this problem and allow expressions with denominators it, e, or a non-zero multiple of **either,** tko be changed to a form which has a rational denominator.

The two numbers have always existed in the set of real numbers, but they were never given names, since they seemed rather useless except for filling a couple of holes on the number line. (The situation is something akin to "new" asteroids.) The decimal names of these new numbers are impossible to pronounce because you can never finish trying to say them.

The first number is called **TINAPAY**, after the Tagalog word for "bread." It is pronounced teen - a - pi. Written $\widetilde{10}$, it is defined as $\widetilde{10} = 10/\pi$. As an example of the usefulness of this number, consider the expression $7/\pi$ which has the irrational denominator π.
$7/\pi = 7/\pi \cdot (\widetilde{10}/\widetilde{10}) = (7\,\widetilde{10})/10$. Notice the rational denominator. As a bonus, this expression also has the convenient decimal representation $.7\,\widetilde{10}$. For converting radian measure to degree measure, just multiply by $18\,\widetilde{10}$.

The second new discovery is the number EATEN, pronounced e -ten, and written ex. The symbol is the juxtaposition of e and the Roman numeral for 10, thus giving the number a classical flavor. EATEN is defined as $ex = 10/e$, and it functions with expressions with denominators e in much the same way as 10 does with expressions with denominators π. As an example, we see that $9/e = 9/e \cdot (ex/ex) = (9ex)/10$ or $.9ex$.

The reader should notice immediately that ex will confuse students who will interpret it as the product of e and x, or worse as "example." This is nothing new in mathematics. We use "x" as a variable, to indicate multiplication, and as a numeral for 10. We use a dot for a decimal and to indicate multiplication. The choice of symbol is in keeping with the tradition of math symbols.

Problems for mathematicians to solve in the future will include the rationalization of the denominators of expressions such as $6/(2 + \pi)$, $7y/(e + \pi)$, and $8/(\sqrt{2}$ - it). The reader can appreciate that the mathematics community, up to now, has not advanced far in the rationalization of denominators. $\widetilde{10}$ and ex are two numbers that help.

Gleanings from the Chapter Reports

GEORGIA EPSILON (Valdosta State College) The speaker at the fall, 1989, meeting was Dr. John Fay, from the Department of Mathematics and Computer Science. The title of his talk was "How to Win Betting on Horse Racing." During the winter quarter, the chapter held its second annual mathematics contest. The contest was open to all students enrolled at Valdosta State College. Steve Hoffman won the contest. The speaker at the spring quarter meeting was Dr. George Meghabghab. The title of his talk was "Inductive Learning." The talk was followed by the initiation ceremony for eight students. Afterwards, the election of new officers was held.

ILLINOIS IOTA (Elmhurst College) The Mathematics and Computer Science Club and the Pi Mu Epsilon Chapter sponsored a barbecue at the beginning of the year, participated in field trips to Argonne National Laboratory, and, along with the Mathematics Department, sponsored a weekly seminar at Elmhurst College. The president of the chapter, Dieter Kunas, inducted new members at the fall meeting of the Associated Colleges of the Chicago Area (ACCA), Mathematics Division. The speaker was Prof. Richard G. Cornell, Department of Biostatlstics, U. of Michigan. He spoke on "Careers in Biostatistics" and "Some Statistical Issues in the Evaluation of the Sweetener Aspartame." At the ACCA Student Spring Symposium, five members presented papers and members were inducted. From this group of students, one presented his work at the Illinois M.A.A. Sectional Meeting and one presented her work at the national Pi Mu Epsilon meeting in Columbus, Ohio.

ATTENTION FACULTY ADVISORS

To have your chapter's report published, send copies to Robert M. Woodside, Secretary-Treasurer, Department of Mathematics, East Carolina University, Greenville, NC 27858 and to Richard L. Poss, Editor, St. Norbert College, De Pere, WI 54115.

Message from the Secretary-Treasurer

Copies of the new, revised Constitution and Bylaws are now available. The prices are: $1.50 for each of the first four copies and $1 for each copy thereafter. I.e., $(1.50 n) for n < 4 and $(n + 2) for n a 4.

The videotape of Professor Joseph A. Gallian's AMS-MAA-PME Invited Address, "The Mathematics of Identification Numbers," given as part of PME's 75th Anniversary Celebration at Boulder, CO, in August, 1989, is also now available. The tape may be borrowed free of charge by PME chapters, and by others upon an advance payment of $10. Please contact my office if you desire to borrow the tape, telling me the date on which you would like to use it. I prefer to mail the tape directly to faculty advisors, and expect them to take responsibility for returning it to my office. Please submit your request in writing and include a phone number and a time that I might reach you if there are problems. Robert M. Woodside, Secretary-Treasurer, Department of Mathematics, East Carolina University, Greenville, NC 27858.

PUZZLE SECTION

Edited by Joseph D. E. Konhauser
Macalester College

The PUZZLE SECTION is for the enjoyment of those readers who are addicted to working doublecrostics or who find an occasional mathematical puzzle or word puzzle attractive. We consider mathematical puzzles to be problems whose solutions consist of answers immediately recognizable as correct by simple observation and requiring little formal proof. Material submitted and not used here will be sent to the Problem Editor if deemed suitable for the PROBLEM DEPARTMENT.

Address all proposed puzzles and puzzle solutions to Professor Joseph D. E. Konhauser, Mathematics and Computer Science Department, Macalester College, St. Paul, MN 55105. Deadlines for puzzles appearing in the Fall Issue will be the next March 15, and for the puzzles in the Spring issue will be the next September 15.

PUZZLES FOR SOLUTION

1. A Teaser from the legacy of Leo Moser, first Problem Department Editor of the Pi Mu Epsilon Journal.

Find positive integers a, b and c such that $a^3 + b^4 = c^5$.

2. Proposed by Basil Rennie, Burnside, South Australia.

Take three points at random on the unit sphere. What is the expected value of the area of the triangle that they form?

3. From a 1966 paper by S. J. Einhorn and I. J. Schoenberg.

The vertices of a regular octahedron are such that the fifteen distances between pairs of vertices assume just two values. There are five other arrangements of six points in 3-space such that the distances between pairs of points fall into just two classes. How many of them are you able to find?

4. Proposed by the Editor of the Puzzle Section.

Given a unit square, what is the area of the octagonal region bounded by the eight lines joining the four side midpoints to the endpoints of the opposite sides?

5. From a 1959 paper by J. Lambek and Leo Moser.

Separate the integers 1 through 16 into two disjoint eight-member sets S and T such that the **28** sums of pairs of elements of S are identical with the **28** sums of pairs of elements of T.

6. Contributed by E. N. Igma.

Cards labelled 1 through k, without duplication, are shuffled and held face up. If the number on the top card is m then the mth card counting from the top is moved to the bottom of the k-card pile. Next, the number now on the second card is noted. If the number is n then the nth card from the top is moved to the bottom of the pile. The process is repeated for the 3rd, 4th, 5th cards and so on. If the card to be moved is already on the bottom the pile remains unchanged. For example, for $k = 4$ if the initial arrangement is 2143 then the final arrangement is 2431. If the final arrangement for five cards is 12345 what was the initial arrangement? Is the solution unique?

7. Proposed by the Editor of the Puzzle Section.

To how many triangles whose vertices are vertices of a regular polygon of $2k + 1$ sides is the center of the polygon interior?

COMMENTS ON PUZZLES 1-7, FALL 1990

For Puzzle #1, RICHARD I. HESS wrote $\overline{xx...x}/xx...x$, where $\overline{xx...x}$ consists of one or more 0's, 1's, 6's, 8's and 9's subject to the conditions (1) there are no leading or trailing 0's and (2) there is at least one 6 or 9. $xx...x$ is $\overline{xx...x}$ turned upside down. Examples: 619, 916, 89168, 9600811180096. Similar responses were received from MARK EVANS and EMIL SLOWINSKI. CHARLES ASHBACHER, MARK EVANS, RICHARD I. HESS, BOB PRIELIPP and EMIL SLOWINSKI responded to Puzzle #2. Most submitted a solution consisting of linear equations in the amounts bet on each horse with results $33 on horse A, $22 on horse B and $6 on horse C. For Puzzle #3, MOHAMMAD PARVEZ SHAIKH (freshman at Western Michigan University) gave a complete analytical geometry solution showing that the area of the three-pointed "star" equals 215 that of the given triangle. RICHARD I. HESS solved the problem by projecting the given triangle into an equilateral triangle using a transformation which preserves ratios of areas. Then, using elementary trigonometry, he obtained the result 2/5. EMIL SLOWINSKI did not reveal his method but supplied the correct answer. Only RICHARD I. HESS responded to Puzzle #4. The scheme used by the proposer was to start with a first row of 0, 1, -2, 3. The elements of the following rows, from left to right, were obtained, respectively, as the sum of the first two elements in the row above, the sum of the last two, the first minus the second and the third minus the fourth. It is easy to show that the elements in the k + 4th row equal four times those in the kth row, so that the elements of the 100th row are those of the 4th row multiplied by 4 to the power 24. In Puzzle #5, the three-member set {2, 3, 5) has the property that the product of any two members leaves a remainder of 1 when divided by the third. Are there any other triplets of distinct positive integers with the same property? EMIL SLOWINSKI and RICHARD I. HESS both said "No," but only HESS supplied a proof. Only RICHARD I. HESS and EMIL SLOWINSKI gave analyses for a winning strategy for the second player in the square-marking game in Puzzle #6. Very briefly put, these strategies are to leave the first player with only two squares empty but not in the same row or column, or to leave the first player with four empty squares which are the vertices of a rectangle. The correct response to Puzzle #7 is 17 bishop moves to move a bishop from the upper left corner (white) of an 8x8 board to the lower right corner so that each of the white squares is occupied at least one time. Solutions and/or answers were supplied by RICHARD I. HESS, EMIL SLOWINSKI and MARK EVANS. Here is the solution of MARK EVANS. From left to right, let the first (top) row of squares be labelled 11, 12, ... , 18; the second row 21, 22, ... , 28; and so on, then, in order, the bishop moves from square 11 to 55, 28, 17, 71, 82, 64, 86, 68, 13, 31, 42, 51, 84, 48, 15, 33, 88.

Solution to Mathacrostic No. 31 (Fall 1990)

WORDS:

A. Butterfly effect
B. Relativism
C. Invariance
D. Gingerbread man
E. Gardens of Eden
F. Stimp meter
G. Athbash
H. Naupathia
I. Dissipative
J. Phase space
K. Easy
L. Antichthon
M. Totemism
N. Tesla coil
O. Umbilic
P. Roach
Q. Busy beaver
R. Unknots
S. Limit cycle
T. El Nino
U. Navaho
V. Tachylyte
W. Metathesis
X. Immortals
Y. Ratiocinate
Z. Rataplan
a. Olive
b. Revolute

AUTHOR AND TITLE: BRIGGS AND PEAT TURBULENT MIRROR

QUOTATION: (Thus) the dynamics of bifurcations reveal that time is irreversible yet recapitulant. They also reveal that time's movement is immeasurable. Each decision made at a branch point involves an amplification of something small. Though causality operates at every instant, branching takes place unpredictably.

SOLVERS: THOMAS F. BANCHOFF, Brown University, Providence, RI; JEANETTE BICKLEY, St. Louis Community College at Meramec, MO; CHARLES R. DIMINNIE, St. Bonaventure University, NY; MICHELE HEIBERG, Herman, MN; DR. THEODOR KAUFMAN, Brooklyn, NY; HENRY S. LIEBERMAN, Waban, MA; CHARLOTTE MAINES, Rochester, NY; STEPHANIE SLOYAN, Georgian Court College, Lakewood, NJ.

LATE SOLUTIONS: Solutions for Mathacrostic No. 30 (Spring 1990) were received from MICHAEL TAYLOR, Indianapolis Power and Light Company, IN and from JOAN and DICK JORDAN, Indianapolis, IN.

Mathacrostic No. 32

Proposed by Joseph D. E. Konhauser

The 256 letters to be entered in the numbered spaces in the grid will be identical to those in the 27 keyed words at the matching numbers. The key numbers have been entered in the diagram to assist in constructing the solution. When completed, the initial letters of the Words will give the name and an author and the title of a book; the completed grid will be a quotation from that book.

Definitions	Words
A. conceive	138 229 119 61 33 166
B. shape with deep indentations	235 106 35 4 97 170 82
C. corkscrew-like structure formed by linked amino acids (2 wds.)	184 252 133 155 109 129 6 28 177 203
D. said of lumber cut radially so that annual rings are perpendicular to the face (comp.)	227 136 246 152 164 127 256 51 199
E. tradename of Piet Hein's seven polycube puzzle	219 201 83 143
F. a mix of randomly construcled small proteins and fatty acids and a variety of active, energy-rich nucleotide units (2 wds.)	223 25 179 167 10 193 91 239 100 36 131 230 60 204
G. three-dimensional shadow of a four-dimensional Klein bottle (2 wds.)	18 15 94 3 27 220 149 159 50 76 124 44 104
H. Hipparchus developed bask of Greek trigonometry (3 wds.)	62 234 185 173 205 217 47 16 145 78 117 64 228
I. insertion or development of a sound or letter in the body of a word	174 196 187 41 108 17 63 52 161 8
J. kind of order different from the deterministic one	165 74 231 42 134 222 101 56 19 114
K. edible tuberous plant of the morning glory variety (2 wds.)	195 144 24 65 232 120 46 105 14 73 34
L. Jack of Spades, Jack of Heads and King of Diamonds (comp.)	245 107 182 86 69 202 20
M. formerly known as a large dyne	150 116 77 237 192 54
N. third largest natural satellite of Saturn	211 38 67 236 84 113 240
O. a conman's patter (slang; 2 wds.)	142 162 249 11 70 197 172
P. "We have adroitly defined the infinite in arithmetic by a ____, in this manner ∞; but we possess not therefore the clearer notion of it." Voltaire	178 30 57 255 241 90 125 2
Q. connected	168 75 43 216 99 111 55
R. trig	243 214 180 88 37
S. H. Buckminster Fuller trademark copyrighted in his name in 1926 by Marshall Field	209 181 157 171 22 254 98 140
T. compound polyhedron formed by two intersecting regular tetrahedra in a cube (2 wds.)	1 121 95 68 147 32 49 128 26 251 218 176 156 59 40
U. huge shield volcano on Mars (2 wds.)	139 45 87 48 115 71 188 160 221 198 210
V. pun-lover's name for 4.6692016090	126 148 5 247 194 110 253 158 96 213 208
W. Norton Juster's delightful romance in lower mathematics published in 1963 (5 wds.)	12 153 21 163 29 200 53 103 151 191 85 226 248 122 183 89
X. round, slender and tapering	169 58 80 92 141 102 186 154 244
Y. final result	224 23 72 13 112 137
Z. informal collection of problems in mathematics begun in Lwow, Poland in 1935 (3 wds.)	212 238 66 7 206 123 190 225 242 132 39 250 146 93 31
a. capable of making shod flights out of the water and of 'flying. with a propulsive force while in the air	81 175 207 118 9 130 215 79 189 135 233

1 T	2 P	3 G	4 B	5 V	6 C	7 Z	■	8 I	9 a	10 F	11 O	■	12 W	13 Y
14 K	15 G	■	16 H	17 I	18 G	19 J	20 L	■	21 W	22 S	23 Y	24 K	25 F	26 T
27 G	■	28 C	29 W	30 P	31 Z	■	32 T	33 A	■	34 K	35 B	36 F	37 R	■
38 N	■	39 Z	40 T	41 I	42 J	43 Q	44 G	45 U	■	46 K	47 H	■	48 U	49 T
50 G	51 D	52 I	■	53 W	54 M	55 Q	■	56 J	57 P	58 X	59 T	60 F	61 A	62 H
63 I	■	64 H	65 K	66 Z	67 N	68 T	69 L	■	70 O	71 U	72 Y	73 K	■	74 J
■	75 Q	76 G	77 M	■	78 H	79 a	■	80 X	81 a	82 B	83 E	■	84 N	85 W
86 L	87 U	■	88 R	89 W	90 P	91 F	■	92 X	93 Z	■	94 G	95 T	96 V	97 B
■	98 S	99 Q	■	100 F	101 J	■	102 X	103 W	104 G	105 K	106 B	107 L	108 I	109 C
110 V	111 Q	112 Y	113 N	114 J	■	115 U	116 M	117 H	118 a	119 A	120 K	121 T	122 W	123 Z
124 G	■	125 P	126 V	■	127 D	■	128 T	129 C	130 a	131 F	132 Z	■	133 C	134 J
135 a	136 D	137 Y	138 A	139 U	140 S	■	141 X	142 O	■	143 E	■	144 K	145 H	146 Z
147 T	148 V	■	149 G	150 M	151 W	■	152 D	153 W	154 X	■	155 C	156 T	157 S	158 V
159 G	■	160 U	161 I	162 O	163 W	164 D	■	165 J	166 A	167 F	168 Q	169 X	170 B	171 S
172 O	173 H	174 I	■	175 a	176 T	177 C	178 P	179 F	180 R	181 S	■	182 L	183 W	184 C
185 H	186 X	187 I	188 U	■	189 a	190 Z	■	191 W	192 M	■	193 F	194 V	195 K	196 I
197 O	198 U	199 D	■	200 W	201 E	■	202 L	203 C	204 F	205 H	206 Z	207 a	208 V	209 S
■	210 U	211 N	212 Z	213 V	214 R	215 a	216 Q	217 H	218 T	219 E	■	220 G	221 U	222 J
223 F	224 Y	225 Z	226 W	227 D	228 H	■	229 A	230 F	231 J	232 K	■	233 a	234 H	235 B
236 N	■	237 M	238 Z	239 F	240 N	■	241 P	242 Z	243 R	244 X	■	245 L	246 D	■
247 V	248 W	249 O	250 Z	251 T	252 C	■	253 V	254 S	255 P	256 D	■	■	■	■

PROBLEM DEPARTMENT

Edited by Clayton Dodge
University of Maine

*This department welcomes **problems** believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.*

*All communications should be addressed to C. W. Dodge, Math. Dept., University of Maine, Orono, ME 04469. Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly identified with name and address. Solutions to problems in this issue should be mailed by December **15, 1991.***

*We generally publish **13** problems per issue, one alphametric followed by one to three problems from each of the areas listed below. To aid you in submitting problems for solution, each area is followed by the number of proposals currently in its file. Please notice that four folders are utterly empty. The areas are algebra* **(21),** *alphametrics* **(6),** *geometry* **(6),** *trigonometry* **(5),** *analysis* **(2),** *logic and combinatorics* **(0),** *number theory* **(0),** *probability and statistics* **(0),** *and miscellaneous* **(0).**

PROBLEMS FOR SOLUTION

745. *Proposed by Alan Wayne, Holiday, Florida.*

Find all solutions to

ENID
+ DID
DINE.

746. *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, Pennsylvania.*

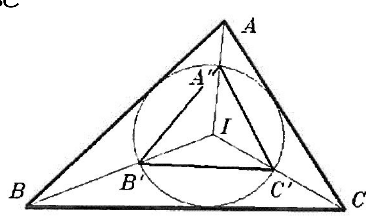
Find the least positive integer n that will have remainder **1** when divided by r, the quotient will have remainder **2** when divided by r, the new quotient will have remainder **3** when divided by r, and so forth through $r - 1$ divisions. That is, $n = q_0$, and $q_{k-1} = q_k r + k$ for $k = 1, 2, \ldots, r - 1$, r a positive integer greater than 1.

747. *Proposed by the late Jack Garfunkel, Flushing, New York.*

Let ABC be a triangle with inscribed circle (I) and let the line segments AI, BI, and CI cut the **incircle** at A', B', and C' respectively. Prove that

$$\sin A' + \sin B' + \sin C' \ge \cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2},$$

where A', B', and C' are the angles of triangle $A'B'C'$.



748. *Proposed by the late John Howell, Littlerock, California.*

a) An urn contains n balls numbered **1** to n. Algernon, Beauregard, and Chauncey draw a ball one after another with replacement. The game is terminated when two consecutive drawings produce the same ball. Find the pmbabilities of terminating on Algernon's draw, on Beauregard's draw, and on Chauncey's draw.

b) Repeat the problem for the case that the game terminates when three consecutive drawings produce the same ball.

749. *Proposer by R. S. Luthar, University of Wisconsin Center at Janesville, Janesville, Wisconsin.*

If $\sin x + \sin y + \sin z = 0$, then pmve that

$$|\sin 3x + \sin 3y + \sin 3z| \le 12|xyz|.$$

***750.** *Proposed by Dmitry P. Mavlo, Moscow, U.S.S.R.*

Solve the system of equations

$$2^x y + (3^x)\sqrt{1 - y^2} = \sqrt{3} \quad \text{and} \quad 3^x y - (2^x)\sqrt{1 - y^2} = \sqrt{2}.$$

This problem appeared in the SYMP-86 Entrance Exam Mathematical Problems.

751. *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.*

Determine all pairs of positive numbers x and y such that

$$9(x + y) + \frac{1}{x} + \frac{1}{y} \ge 10 + \frac{x}{y} + \frac{y}{x}.$$

752. *Proposed by the late Charles W. Trigg, San Diego, California.*

Martin Gardner ("Mathematical Games," *Scientific American,* April **1964,** page **135)** has shown that the minimum sum of three 3-digit primes that contain the nine non-zero digits is **999.** Find a set of three such primes that sums to another multiple of **37.**

753. *Proposed by R. S. Luthar, University of Wisconsin Center at Janesville, Janesville, Wisconsin.*

Solve simultaneously

$$e^{4x} + e^{4y} = 82 \quad \text{and} \quad e^x - e^y = 2.$$

754. *Proposed by Seung-Jin Bang, Seoul, Korea.*

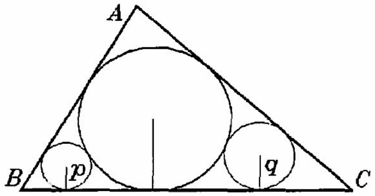
Let $a_1 = a_2 = 1$, $a_3 = 2$, and $a_{n+1} = a_n \cdot a_{n-1} + a_{n-2}$ for $n > 3$. Show that

$$a_{n+2}a_na_{n-2} - a_{n+2}a_{n-1}^2 - a_{n+1}^2a_{n-2} + 2a_{n+1}a_na_{n-1} - a_n^3 + 3 = 0.$$

755. *Proposed by Stanley Rabinowitz, Alliant Computer Systems Corp., Littleton, Massachusetts.*

In triangle ABC, a circle of radius p is inscribed in the wedge bounded by sides AB and

BC and the **incircle** *(I)* of the triangle. A circle of radius *q* is inscribed in the wedge bounded by sides AC **and** BC and the incircle. **If** p = *q*, prove that AB = AC.



756. ***Proposed*** by Basil ***Rennie, Burnside,*** South Australia.

Consider covering the unit **interval [0,1]** with *n* measurable subsets, under the **constraint** that **all** *n* subsets must have the same centroid. The centroid m of a set *E* may be defined by $\int_E (x - m)\,dx = 0$. How can you choose the *n* sets to minimize *m*?

For example, if $n = 4$, it is possible to make $m = 7/20$ by choosing the four sets **[0,2/5]∪[9/10,1], [0,1/5]∪[4/5,9/10], [1/20,1/4]∪[[7/10,4/5],** and **[0,7/10].**

757. Proposed by Paul Anthony ***Coartney,*** graduate student, San ***Diego*** State University, San ***Diego,*** California.

Find the overall height of the pyramid formed from four spherical balls of radius **r.** Student solutions are especially solicited.

SOLUTIONS

720. [Spring **1990]** Proposed by the late Charles W. ***Trigg,*** San ***Diego,*** California.

In base 4, find two repdyads, one the reverse of the other, whose squares are concatenations of two repdyads. A **repdyad** has the form abab...ab. For example, a base ten solution is

$$8989^2 = 80802121 \quad \text{and} \quad 9898^2 = 97970404.$$

Solution by WILLIAM H. *PEIRCE,* Stonington, Connecticut.

Let N = abab be a four-digit ***repdyad*** in base B. The square of N is an eight-digit number which must be of the form

$$N^2 = pqpqrsrs.$$

Then we must have that

$$(1) \quad N^2 = [(aB + b)(B^2 + 1)]^2 = (aB + b)^2(B^2 + 1)^2$$

$$= (pB + q)(B^2 + 1)B^4 + (rB + s)(B^2 + 1)$$

$$= (B^2 + 1)[(pB + q)(B^4 - 1) + (pB + q) + (rB + s)].$$

Now $(B^2 + 1)^2$ is a factor of the right side of the expression in the first displayed line, so it is a factor of the expression in the last line. Hence

(2) $(B^2 + 1)$ must divide $(pB + q) + (rB + s)$.

Since **p, q, r,** and s are digits in base **B** and not all zero, then $(pB + q) + (rB + s)$ can range from **1** to $2B^2 - 2$. Since $2B^2 - 2$ is more than $B^2 + 1$ but less than twice $B^2 + 1$, the only way for (2) to hold is to have

$$(3) \quad (pB + q) + (rB + s) = B^2 + 1$$

[It is at this point that the search for **repdyads** of three or more pairs would end, **since, for** example, when N = ababab, the expression $B^4 + B^2 + 1$ would have to divide $(pB + q) + (rB + s)$. This is not possible since $B^4 + B^2 + 1$ is greater than $2B^2 - 2$.**]**

Substituting (3) into **(1)** gives

$$(4) \quad N^2 = (B^2 + 1)^2[(B^2 - 1)(pB + q) + 1],$$

which will be considered the fundamental expression of the problem. **It** is **necessary** to find values of $pB + q$ that make the expression in brackets in (4) a square. That is,

(5) $(B^2 - 1)(pB + q) + 1$ is a perfect square.

When B is small, a direct search suffices. [General parametric methods for solving (5) are not included here.]

Two values of $pB + q$ that satisfy (5) are $pB + q = B^2 - 3$ and $pB + q = B^2 - 2B$.

If $pB + q = B^2 - 3$, then $a = B - 1$, $b = B - 2$, $p = B - 1$, $q = B - 3$, and $rB + s = 4$. If $B > 4$, then $r = 0$ and $s = 4$. If $B = 4$, then $r = 1$ and $s = 0$. If $B = 3$, then $r = s = 1$. This is not a solution for $B < 3$.

If $pB + q = B^2 - 2B$, then $a = B - 2$, $b = B - 1$, $p = B - 2$, $q = 0$, and $rB + s = 2B + 1$, so $r = 2$ and $s = 1$. This solution holds for all $B > 1$.

Hence, for **B** = 4, we have the two required solutions

$$N = 3232 \quad \text{and} \quad N^2 = 31311010,$$

$$N = 2323 \quad \text{and} \quad N^2 = 20202121.$$

There are no other base 4 solutions.

The illustrations given in the proposal are examples of these two solutions for base ten. Other bases can have additional solutions. For example, bases **5, 7,** and 9 have six solutions, and base 11 has fourteen solutions. Selected solutions appear in the table **below.**

Base	Repmonads	Repdyads	Repdyads	Reptriads
3		1212	2121	221221
4		2323	3232	332332
				313313
5	33	1212	2121	
		2323	3232	
		3434	4343	
6	44	4545	5454	554554
				443443
				112112

The method outlined above can be used to study repmonads (N = aa, $N^2 = ppqq$), reptriads ($N = abcabc$, $N^2 = pqrpqrstustu$), etc. There is always at least one solution.

Subjects for further study would be 1) showing the specific relation between the number of solutions and the prime factors of $B - 1$ for repmonads, of $B^2 - 1$ for repdyads, of $B^3 - 1$ for reptriads, etc., and 2) proving or dispmving that repdyads *are the only case where reversals of solutions are also solutions.*

Also solved by CHARLES ASHBACHER, Hiawatha, **IA,** *KAREN L. COOK, Lantana,* **FL,** *VICTOR* **G.** *FESER, University of Mary, Bismarck,* **ND,** *RICHARD* **I.** *HESS, Rancho* **Palos** *Verdes, CA, NATHAN JASPEN, Stevens Institute of Jechnology,* **Hoboken, NJ,** *DEREK* **LEDBETTER,** *University of Florida, Gainesville, HENRY S. LIEBERMAN, Waban, MA, KENNETH* **M. WILKE,** *Topeka,* **KS,** *and the PROPOSER.*

721. [Spring **1990]** *Proposed by Robed C. Gebhardt, Hopatcong, New Jersey.*
Evaluate the integral

$$\int \frac{b - \cot ax}{1 + b \cot ax}\, dx.$$

I. *Solution by the PROPOSER.*

Multiplying numerator and denominator by sin ax, *we get*

$$\int \frac{b \sin ax - \cos ax}{\sin ax + b \cos ax} dx = -\frac{1}{a} \int \frac{a \cos ax - ab \sin ax}{\sin ax + b \cos ax} dx$$
$$= -\frac{1}{a} \ln |\sin ax + b \cos bx| + C.$$

II. *Solution by GEORGE P. EVANOVICH, Saint Peter3 College, Jersey City, New Jersey.*

Let $t = \tan ax$, so that $x = \frac{1}{a} \arctan t$ and $dx = \frac{dt}{a(1 + t^2)}$. *Then we have that*

$$\int \frac{b - \cot ax}{1 + b \cot ax} dx = \int \frac{b \tan ax - 1}{\tan ax + b} dx$$
$$= \frac{1}{a} \int \frac{bt - 1}{(t + b)(1 + t^2)}\, dt$$
$$= -\frac{1}{a} \int \frac{dt}{t + b} + \frac{1}{a} \int \frac{t\, dt}{1 + t^2}$$
$$= -\frac{1}{a} \ln |t + b| + \frac{1}{2a} \ln |1 + t^2| + C$$
$$= \frac{1}{a} \ln |\sec ax| - \frac{1}{a} \ln |\tan ax + b| + C.$$

Also solved by JOHN **T. ANNULIS,** *University of Arkansas-Monticello, CHARLES ASHBACHER, Hiawatha,* **IA,** *MOHAMMAD K.* **AZARIAN,** *University of Evansville,* **IN, SEUNG-JIN** *BANG, Seoul, Korea, FRANK P BATTLES (two solutions),* **Massachusetts** *Maritime Academy, Buzzards Bay, MARTIN BAZANT, Tucson, AZ,* **J.** *D. BRASHER, Teledyne Brown Engineering, Huntsville, AL, MARTIN* **J.** *BROWN, Jefferson Community College,* **Louisville, KY, CAVELAND** *MATH GROUP, Western Kentucky University, Bowling Green, KAREN L. COOK, Lantana,* **FL,** *ROBERT* **I. EGBERT,** *The Wichita State University,* **KS,** *STEPHEN HALE, Drake University,-Des Moines,* **IA,** *IEM HENG, Providence College, RI, RICHARD* **I.** *HESS, Rancho* **Palos** *Verdes,* **CA,** *NATHAN JASPEN, Stevens Institute of Technology,* **Hoboken, NJ, R.** *N.* **KALIA,** *St. Cloud State University, MN, RALPH E. KING, St. Bonaventure University, NY, MURRAY S. KLAMKIN, University of* **Alberta,** *Edmonton, Canada, DEREK LEDBETTER, University of Florida, Gainesville, HENRY S. LIEBERMAN, Waban, MA, PETER A. LINDSTROM, Nodh Lake College,* **Irving,** *JX, DAVID* **E.** *MANES, SUNY at Oneonta, G.* **MAVRIGI***AN, Youngstown State University, OH, LEV S. NAKHAMCHIK, Willowdale,* **Ont.,** *Canada, OXFORD RUNNING CLUB, University of* **Mississippi,** *University,* **I.** *PHILIP* **SCALISI,** *Bridgewater State College, MA, HARRY SEDINGER, St. Bonaventure University,* **NY,** *WADE H. SHERARD,* **Furman** *University, Greenville,* **SC,** *SAHIB* **SINGH,** *Clarion University of* **Pennsylvania,** *TIMOTHY SIPKA, Alma College,* **MI,** *and KENNETH* **L.** *YOKOM, South Dakota State* **University, Brookings.**

722. [Spring **1990]** *Proposed by Robed C. Gebhardt,* **Hopatcong,** *New Jersey.*
On **Interstate** *84 in Connecticut a road sign, indicating a route number change, reads*

NOTICE
66
IS NOW
322.

This, of course, is startling news to mathematicians. But consider: in what base would the number 66 equal 322 in what other base?

Solution by S. GENDLER, Clarion University of Pennsylvania, Clarion, Pennsylvania.
Let x be the base of the number 66 and y be the base for 322. Then

$$6x + 6 = 2 + 2y + 3y^2 \quad \text{so} \quad y \equiv 0 \pmod 2.$$

Also 3 divides $2 + 2y$, *so that* $y \equiv 2 \pmod 3$.

By the Chinese remainder theorem, $y = 2 + 6n$ *for any integer* n, *so that*

$$6x + 6 = 2 + 2(2 + 6n) + 3(2 + 6n)^2$$

from which we get that

$$x = 2 + 14n + 18n^2 \quad \text{and} \quad y = 2 + 6n$$

for any integer $n > 0$ *(since* $x > 7$*). Some solutions* (x,y) *are* (34,8), (102,14), (206,20), *and (346,261.*

Full solutions were submitted by DAVID ASCHBRENNER *and* KENDALL BAILEY, *Drake University, Des Moines, IA,* SEUNG-JIN BANG, *Seoul, Korea,* FRANK P. BATTLES, *Massachusetts Maritime Academy, Buzzards Bay,* MARTIN BAZANT, *Tucson, AZ,* JEFFREY JOHN BOATS, *St. Bonaventure University, NY,* BARRY BRUNSON, *Western Kentucky University, Bowling Green,* CAVELAND MATH GROUP, *Western Kentucky University, Bowling Green,* JOE DeMAIO, *Emory University, Lenoir, NC,* GEORGE P. EVANOVICH, *Saint Peter's College, Jersey City, NJ,* MARK EVANS, *Louisville, KY,* VICTOR G. FESER, *University of Mary, Bismarck, ND,* RICHARD I. HESS, *Rancho Palos Verdes, CA, the late* JOHN M. HOWELL, *Littlerock, CA,* NATHAN JASPEN, *Stevens Institute of Technology, Hoboken, NJ,* DEREK LEDBETTER, *University of Florida, Gainesville,* CARL LIBIS, *Granada Hills, CA,* DAVID E. MANES, *SUNY at Oneonta,* LEV S. NAKHAMCHIK, *Willowdale, Ont., Canada,* WILLIAM H. PEIRCE, *Stonington, CT,* DAMEN PETERSON, *Alma College, MI,* WADE H. SHERARD, *Furman University, Greenville, SC,* SAHIB SINGH, *Clarion University of Pennsylvania,* TIMOTHY SIPKA, *Alma College, MI,* KENNETH M. WILKE, *Topeka, KS,* DAVID YAVENDITI, *Alma, MI,* KENNETH L. YOKOM, *South Dakota State University, Brookings, and the* PROPOSER.

At least one solution was submitted by CHARLES ASHBACHER, *Hiawatha, IA,* MARTIN J. BROWN, *Jefferson Community College, Louisville, KY,* BARBARA TON FERULLO, *Boylston, MA,* MICHAEL W. LANSTRUM, *Kent State University, OH,* HENRY S. LIEBERMAN, *Waban, MA,* LOWELL F. LYNDE, JR., *University of Arkansas at Monticello, and* MIKE PINTER, *Belmont College, Nashville, TN.*

One incorrect solution was received.

723. [Spring 1990] *Proposed by John L. Leonard, University of Arizona, Tucson, Arizona.*

Show that, for any positive integers n and k, the product

$$(1+n)\left(1+\frac{n}{2}\right)\left(1+\frac{n}{3}\right)\cdots\left(1+\frac{n}{k}\right)$$

is always an integer.

Solution by DAVID YAVENDITI, Alma, Michigan.

We have that

$$\begin{aligned}(1+n)\left(1+\frac{n}{2}\right)\left(1+\frac{n}{3}\right)\cdots\left(1+\frac{n}{k}\right)\\ &= \left(\frac{n+1}{1}\right)\left(\frac{n+2}{2}\right)\left(\frac{n+3}{3}\right)\cdots\left(\frac{n+k}{k}\right)\\ &= \frac{(n+k)!}{n!k!} = \binom{n+k}{n},\end{aligned}$$

which is a positive integer for all positive integers n and k.

Also solved by JOHN T. ANNULIS, *University of Arkansas-Monticello,* CHARLES ASHBACHER, *Hiawatha, IA,* KENDALL BAILEY *and* SEAN FORBES, *Drake University, Des Moines. IA,* SEUNG-JIN BANG, *Seoul, Korea,* FRANK P. BATTLES, *Massachusetts Maritime Academy, Buzzards Bay,* CAVELAND MATH GROUP, *Western Kentucky University, Bowling Green,* DAVID DELSESTO, *North Scituate, RI,* GEORGE P. EVANOVICH, *Saint Peter's College, Jersey City, NJ,* MARK EVANS, *Louisville, KY,* VICTOR G. FESER, *University of Mary, Bismarck, ND, the late* JACK GARFUNKEL, *Flushing, NY,* ROBERT C. GEBHARDT, *Hopatcong. NJ,* S. GENDLER, *Clarion University of Pennsylvania,* DICK GIBBS, *Fort Lewis College, Durango, CO,* RICHARD I. HESS. *Rancho Palos Verdes. CA,* NATHAN JASPEN, *Stevens Institute of Technology, Hoboken, NJ,* DEREK LEDBETTER, *University of Florida, Gainesville,* CARL LIBIS, *Granada Hills, CA,* HENRY S. LIEBERMAN, *Waban, MA,* PETER A. LINDSTROM, *North Lake College, living, TX,* DAVID E. MANES, *SUNY at Oneonta,* LEV S. NAKHAMCHIK, *Willowdale, Ont., Canada,* OXFORD RUNNING CLUB, *University of Mississippi, University,* WILLIAM H. PEIRCE, *Stonington, CT,* DAMEN PETERSON, *Alma College, MI,* BOB PRIELIPP, *University of Wisconsin-Oshkosh,* JOHN PUTZ, *Alma College, MI,* VIVEK RATAN, *Wesleyan University, Middletown, CT,* HARRY SEDINGER, *St. Bonaventure University, NY,* WADE H. SHERARD, *Furman University, Greenville, SC,* SAHIB SINGH, *Clarion University of Pennsylvania,* TIMOTHY SIPKA, *Alma College, MI,* UNIVERSITY OF ARIZONA PROBLEM SOLVING LAB. *Tucson.* KENNETH M. WILKE (2 *solutions), Topeka, KS,* KENNETH L. YOKOM, *South Dakota State University, Brookings, and the* PROPOSER.

724. [Spring 1990] *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta. Canada.*

Which of the following triangle inequalities, if any, are valid?

(1) $\max\{h_a,h_b,h_c\} \geq \min\{m_a,m_b,m_c\}$,

(2) $\max\{w_a,w_b,w_c\} \geq \min\{m_a,m_b,m_c\}$,

(3) $\min\{w_a,w_b,w_c\} \geq \min\{m_a,m_b,m_c\}$.

As usual, h_a, m_a, w_a, etc., denote the altitude, median, and angle bisector, respectively, to side a.

I. *Solution by RICHARD I. HESS, Rancho Palos Verdes, California.*

Consider the triangle with vertices at $A(0,0)$, $B(1,0)$, and $C(1000,1)$. Then $h_{max} = h_c = 1$, $w_{min} = w_b < 1$, and $m_{min} = m_b > 499$, so inequalities (1) and (3) are false.

Inequality (2) is true. Let $a \leq b \leq c$. Then $w_{max} = w_a$ and $m_{min} = m_c$. Then $w_a \geq h_a$ and $\cos C \leq 1/2$ with equality if and only if $a = b = c$. Recall that $c^2 = a^2 + b^2 - 2ab\cos C$ by the law of cosines and that

$$h_a^2 = b^2\sin^2 C = b^2(1-\cos^2 C) \quad\text{and}\quad 4m_c^2 = 2a^2+2b^2-c^2.$$

Now we have

$$\begin{aligned}4(h_a^2 - m_c^2) &= 4b^2 - 4b^2\cos^2 C - 2a^2 - 2b^2 + c^2\\ &= 2b^2 - 4b^2\cos^2 C - 2a^2 + a^2 + b^2 - 2ab\cos C\\ &= 3b^2 - a^2 - 2ab\cos C - 4b^2\cos^2 C\\ &\geq 3b^2 - a^2 - ab - b^2\\ &= (b-a)(2b+a) \geq 0\end{aligned}$$

from which equation (2) follows.

II. *Comment by the Editor.*

Unfortunately, somewhere between the proposal and the publication, one letter was changed. Inequality (3) should have read "mid" on the left. The correct proposed inequality is

(4) $\quad \text{mid}\{w_a, w_b, w_c\} \geq \min\{m_a, m_b, m_c\}.$

I. *Solution to Inequality (4) by the PROPOSER.*

By considering an isosceles triangle with small vertex angle it follows that (4) is invalid.

Also solved by the PROPOSER.

725. [Spring 1990] *Proposed by Seung-Jin Bang, Seoul, Korea.*

Let A, B, C be vectors, Let $\|A\|$ denote the usual norm of A, and let p and q be real numbers such that p + q = 1. Show that

$$\|(p^2+q^2)A + 2pqB + C\|^2 - (p^2+q^2)\|A + C\|^2 - 2pq\|B + C\|^2$$

is independent of *C*.

Solution by KENNETH L. YOKOM, South Dakota State University, Brookings, South Dakota.

Let $a = p^2 + q^2$ and $b = 2pq$, and note that $a + b = 1$. Then

$$\|(aA + bB) + C\|^2 - a\|A + C\|^2 - b\|B + C\|^2$$
$$= \|(aA+bB)\|^2 + 2a\langle A,C\rangle + 2b\langle B,C\rangle + \|C\|^2$$
$$- a\|A\|^2 - 2a\langle A,C\rangle - a\|C\|^2 - b\|B\|^2 - 2b\langle B,C\rangle - b\|C\|^2$$
$$= \|(aA+bB)\|^2 - a\|A\|^2 - b\|B\|^2,$$

which is independent of C.

Also solved by CHARLES ASHBACHER. *Hiawatha. IA,* KENDALL BAILEY, *Drake University, Des Moines, IA,* SUSAN BYE *and* LINDA RETTIG, *St. Cloud State University, MN,* CAVELAND MATH GROUP, *Western Kentucky University, Bowling Green,* CYNTHIA COYLE, *Trenton State College, Laurel Springs, NJ,* S. GENDLER *(solution for 2-dimensional vectors), Clarion University of Pennsylvania,* RICHARD I. HESS, *Rancho Palos Verdes, CA,* NGUYEN HOA, *St. Cloud State University, MN,* SANDRA KEITH, *St. Cloud State University, MN,* DEREK LEDBETTER, *University of Florida, Gainesville,* HENRY S. LIEBERMAN, *Waban, MA,* YOSHINOBU MURAYOSHI, *Eugene, OR,* LEV S. NAKHAMCHIK, *Willowdale, Ont., Canada,* WILLIAM H. PEIRCE, *Stonington, CT,* WADE H. SHERARD, *Furman University, Greenville, SC,* MICHAEL R. SIEGFRIED, *St. Cloud State University, MN,* SAHIB SINGH, *Clarion University of Pennsylvania,* TIMOTHY SIPKA, *Alma College, MI,* DAVID YAVENDITI, *Alma, MI, and the* PROPOSER.

726. [Spring 1990] *Proposed by the late Jack Garfunkel, Flushing, New York.*

Given that x, y, z > 0 and x + y + z = 1, prove that

$$\sqrt[3]{1+x} + \sqrt[3]{1+y} + \sqrt[3]{1+z} \leq \sqrt[3]{36}.$$

I. *Solution by HENRY S. LIEBERMAN, Waban, Massachusetts.*

Let $a = 1 + x$, $b = 1 + y$, and $c = 1 + z$. Then a, b, and c are positive and $a + b + c = 4$. It is known (cf. Hall and Knight, *Higher Algebra,* p. 216) that

$$\frac{\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}}{3} \leq \sqrt[3]{\frac{a+b+c}{3}}$$

Hence

$$\sqrt[3]{1+x} + \sqrt[3]{1+y} + \sqrt[3]{1+z} \leq 3\sqrt[3]{\left(\frac{4}{3}\right)}$$

and the theorem follows.

II. *Solution by CAVELAND MATH GROUP, Western Kentucky University, Bowling Green. Kentucky.*

Writing $z = 1 - x - y$, we will show that $\sqrt[3]{36}$ is the maximum value of

$$f(x,y) = \sqrt[3]{1+x} + \sqrt[3]{1+y} + \sqrt[3]{2-x-y}$$

over the closed rectangle $[0,1] \times [0,1]$. The desired result then follows immediately. Now

$$f_x(x,y) = \frac{1}{3}(1+x)^{-2/3} - \frac{1}{3}(2-x-y)^{-2/3}$$

which is zero when $y + 2x = 1$ or $y = 3$. We discard the latter value. By symmetry, $f_y = 0$ when $x + 2y = 1$. Solving this linear system gives $(x,y) = (113,113)$ as the only critical point in the domain.

To see that $f(1/3,1/3) = \sqrt[3]{36}$ is a maximum, we show that $f(x,y)$ is less than this value along the boundary of the square. if x = 0, then

$$f(0,y) = 1 + \sqrt[3]{1+y} + \sqrt[3]{2-y} = g(y)$$

and

$$g'(y) = \frac{1}{3}(1+y)^{-2/3} - \frac{1}{3}(2-y)^{-2/3}.$$

There is a critical value for g in $[0,1]$ at y = *112,* so we find

$$g(0) = g(1) = 2 + \sqrt[3]{2} \doteq 3.26 \quad \text{and} \quad g\left(\frac{1}{2}\right) = 1 + 2\sqrt[3]{\frac{3}{2}} \doteq 3.29,$$

both less than $\sqrt[3]{36} \doteq 3.30$. By the symmetry of f, the same values occur along the edge $y = 0$ of the square.

For the edge $x = 1$ we have

$$f(1,y) = \sqrt[3]{2} + \sqrt[3]{1+y} + \sqrt[3]{1-y} = h(y)$$

and

$$h'(y) = \frac{1}{3}(1+y)^{-2/3} - \frac{1}{3}(1-y)^{-2/3}.$$

Since h has a critical point at $y = 0$, we calculate

$$h(0) = 2 + \sqrt[3]{2} \doteq 3.26 \quad \text{and} \quad h(1) = 2\sqrt[3]{2} \doteq 2.52,$$

both less than $\sqrt[3]{36}$. By symmetry, this same situation exists along the edge $y = 1$, too, and the proof is complete.

III. *Solution and generalization by MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta, Canada.*

If $F(t)$ is a concave function and $x_1 + x_2 + \dots + x_n = s$, then by Jensen's inequality,

$$F(x_1) + F(x_2) + \dots + F(x_n) \le nF(s/n).$$

The given inequality corresponds to the special case $n = 3$, $F(Q= \sqrt[3]{1+t}$ and $x_i \ge -1$.

Also solved by MOHAMMAD K. AZARIAN, *University of Evansville, IN,* SEUNG-JIN BANG, *Seoul, Korea,* GEORGE P. EVANOVICH, *Saint Peter's College, Jersey City, NJ,* MARK EVANS, *Louisville. KY,* ROBERT C. GEBHARDT, *Hopatcong, NJ,* DICK GIBBS, *Fort Lewis College, Durango, CO,* RICHARD I. HESS, *Rancho Palos Verdes, CA,* YOSHINOBU MURAYOSHI, *Eugene, OR,* LEV S. NAKHAMCHIK, *Willowdale. Ont., Canada,* OXFORD RUNNING CLUB, *University of Mississippi, University,* BOB PRIELIPP, *University of Wisconsin-Oshkosh,* HARRY SEDINGER, *St. Bonaventure University, NY,* TIMOTHY SIPKA, *Alma College, MI,* KENNETH L. YOKOM, *South Dakota State University, Brookings, and the* PROPOSER.

727. [Spring **1990**] *Proposed by the late Jack Garfunkel, Flushing, New York.*

If A, B, C are the angles of a triangle ABC, prove that

$$2 + \prod \cos\frac{B-C}{2} \ge 2\sum \cos A.$$

Solution by MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta, Canada.
Since

$$\sum \cos A = 1 + 4\prod \sin\frac{A}{2},$$

the given inequality is equivalent to

$$\prod \cos\frac{B-C}{2} \ge 8\prod \sin\frac{A}{2}.$$

The latter inequality appeared by the proposer as Problem *585, Crux* ***Mathematicorum,*** **7(1981)p.303**. In the solution there I had shown that it was equivalent to the known elementary inequality

$$(b+c)(c+a)(a+b) \ge 8abc.$$

This follows from

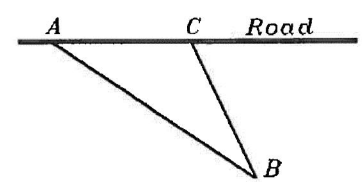
$$\frac{b+c}{a} = \frac{\sin B + \sin C}{\sin A} = \frac{2\cos\frac{A}{2}\cos\frac{B-C}{2}}{2\sin\frac{A}{2}\cos\frac{A}{2}} = \frac{\cos\frac{B-C}{2}}{\sin\frac{A}{2}},$$

etc.

Also solved by HENRY S. LIEBERMAN, *Waban, MA,* YOSHINOBU MURAYOSHI, *Eugene, OR,* BOB PRIELIPP, *University of Wisconsin-Oshkosh, and the* PROPOSER.

728. [Spring **1990**] *Proposed by* ***Dmitry*** *P. Mavlo, Moscow, U.S.S.R.*

The distance between towns A and B is 5 km. A straight road passes through town A and forms the angle a = arccos *(415)* with the line *AB.* Two hikers leave town A at the same time and arrive at town B simultaneously. The first hiker goes by the direct route at 4 kmlhr The second hiker first travels along the road at 6 **km/hr** and then turns off the road and goes directly to B at *4* **km/hr**. Find the distance traveled by the second hiker.



Solution by FRANK P. BATTLES, Massachusetts Maritime Academy, Buzzards Bay, Massachusetts.

More generally, let d be the distance between towns A and B, w the speed of the second hiker along the mad, v (with $w > v$) the speed of the first hiker and of the second hiker when he changes direction and heads directly to B, and α the angle between the road and the segment AB. Let C be the point on the mad at which the second hiker turns, and t the time the second hiker travels along the road. The total time of travel is d/v, so the second hiker travels from C to B in time $d/v - t$. Then the distance CB is given by $v(d/v - t) = d - tv$. From the law of cosines we have

$$(wt)^2 + d^2 - 2wtd\cos\alpha = (d - tv)^2.$$

Next we solve for t, obtaining

$$t = \frac{2d(w\cos\alpha - v)}{w^2 - v^2}.$$

Clearly we must have $w\cos\alpha > v$. Then the distance travelled by the second hiker is

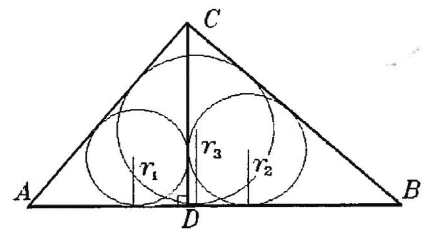
$$wt + (d - tv) = \frac{d}{v + w}(2w\cos\alpha + w - v).$$

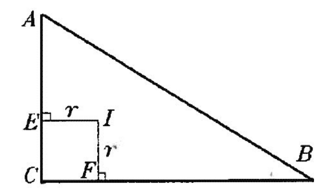
Substituting the specific numbers given, we find that the second hiker travels 5.8 miles.

Also solved by SEUNG-JIN BANG, Seoul, Korea, MARTIN **BAZANT**, Tucson, *AZ*, MARTIN J. BROWN, Jefferson Community College, Louisville, **CAVELAND** MATH GROUP (two solutions), Western Kentucky University, Bowling Green, CYNTHIA COYLE, Trenton State College, Laurel *Springs*, NJ, GEORGE P. EVANOVICH, Saint Peter's College, Jersey City, NJ, MARK EVANS, Louisville, *KY*, ROBERT C. GEBHARDT, Hopatcong, NJ, S. GENDLER, Clarion University of Pennsylvania, STEPHEN A. HERR, Alma College, *MI*, RICHARD I. HESS, Rancho *Palos* Verdes, *CA*, NATHAN JASPEN, Stevens Institute of Technology, *Hoboken*, NJ, RALPH E. KING (two solutions), St. Bonaventure University, *NY*, CARL **LIBIS**, Granada Hills, CA, HENRY S. LIEBERMAN, Waban, MA, PETER A. LINDSTROM, North Lake College, Irving, TX, DAVID E. MANES, SUNY at *Oneonta*, G. MAVRIGIAN, Youngstown State University, OH, LEON MOSER, Hunter College, New York, *NY*, YOSHINOBU MURAYOSHI, Eugene, OR. LEV S. NAKHAMCHIK, Willowdale, *Ont.*, Canada, OXFORD RUNNING CLUB, University of Mississippi, University, MIKE PINTER, Belmont College, Nashville. *TN*, BOB PRIELIPP, University of Wisconsin-Oshkosh, JOHN **PUTZ**, Alma College, *MI*, VIVEK **RATAN**, Wesleyan University, *Middletown, CT*, HARRY SEDINGER, St. Bonaventure University, *NY*, WADE H. SHERARD, *Furman* University, Greenville, *SC*, SAHIB SINGH, Clarion University of Pennsylvania. TIMOTHY SIPKA, Alma College, *MI*, KENNETH M. WILKE, Topeka, KS, DAVID YAVENDITI, Alma, *MI*, KENNETH L. YOKOM, South Dakota State University, Brookings, and the PROPOSER.

729. [Spring **1990]** Proposed by the late Jack Garfunkel, Flushing, New York.

Given a non-obtuse triangle ABC with altitude CD = h drawn to side AB, denote the **inradii** of triangles ACD, BCD, and ABC by r_1, r_2, and r_3, respectively. Prove that if $r_1 + r_2 + r_3 = h$, then triangle ABC is a right triangle with right angle at C.





I. Solution by HENRY S. LIEBERMAN, Waban, Massachusetts.

We first prove the following lemma.

Lemma: Let ABC be a triangle with **inradius** r, semiperimeter s, and side lengths a, b, and c. Then ABC is a right triangle with right angle at C if and only if $r = s - c$.

Let I be the **incenter** and IE and IF **the inradii** to sides CA and BC, as shown in the figure. It is well-known (and easy to prove from the fact that the two tangents from an exterior point to a circle are equal in length) that $CE = CF = s - c$. If angle C is a right angle, then $CEIF$ **is** a square, so $r = s - c$. Conversely, if $r = s - c$, then CEIF is a rhombus with two right angles, therefore a square. So angle C is a right angle. The lemma is proved.

By the lemma,

$$r_1 = \frac{b + AD + h}{2} - b = \frac{AD + h - b}{2} \quad \text{and} \quad r_2 = \frac{BD + h - a}{2},$$

whence

$$r_1 + r_2 = h + \frac{c - b - a}{2}$$

because $AD + DB = c$ when neither angle A nor B is obtuse. Therefore,

$$r_1 + r_2 + r_3 = h \quad \text{Iff} \quad r_3 = \frac{a + b - c}{2}$$

Because this last condition is an "if and only if" statement, we have proved both the theorem and its converse, that if ABC is a right triangle with right angle at C, then $r_1 + r_2 + r_3 = h$.

II. *Comment by Murray S. Klamkin and Andy Liu, University of Alberta, Edmonton, Alberta, Canada.*

By using the general formula rs = area, we have that

$$r_1 = \frac{hb\cos A}{h + b(1+\cos A)},\quad r_2 = \frac{ha\cos B}{h + a(1+\cos B)},\quad r_3 = \frac{hc}{a+b+c},$$

and $h = b\sin A = a\sin B$. *Then* $r_1 + r_2 + r_3 = h$ *becomes*

$$(1)\quad \frac{\cos A}{1+\cos A+\sin A} + \frac{\cos B}{1+\cos B+\sin B} + \frac{\sin C}{\sin A+\sin B+\sin C} = 1$$

Equation (1) can independently be proved equivalent to the condition that ABC is a right triangle with right angle at C. First, we note that

$$\frac{\cos A}{1+\cos A+\sin A} = \frac{(1+\cos A-\sin A)\cos A}{(1+\cos A)^2-\sin^2 A}$$
$$= \frac{1}{2}\left(1 - \frac{\sin A}{1+\cos A}\right) = \frac{1}{2} - \frac{1}{2}\tan\frac{A}{2},$$

etc. Also

$$\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

Then Equation (1) reduces to

$$\frac{1}{2} - \frac{1}{2}\tan\frac{A}{2} + \frac{1}{2} - \frac{1}{2}\tan\frac{B}{2} + \frac{\sin\frac{C}{2}}{2\cos\frac{A}{2}\cos\frac{B}{2}} = 1.$$

Now use the relation

$$\tan\frac{A}{2} + \tan\frac{B}{2} = \tan\left(\frac{A}{2}+\frac{B}{2}\right)\left(1 - \tan\frac{A}{2}\tan\frac{B}{2}\right)$$

to simplify the equation to $\sin(A+B)/2 = \sin C/2$, *and finally to* $\tan C/2 = 1$, *which is equivalent to* $C = \pi/2$.

Also solved by GEORGE P. EVANOVICH, Saint Peter's College, Jersey City, NJ, NATHAN JASPEN, Stevens Institute of Technology, Hoboken, NJ, MURRAY S. KLAMKIN and ANDY LIU, University of Alberta. Canada, BOB PRIELIPP, University of Wisconsin-Oshkosh, TIMOTHY SIPKA, Alma College, MI, DAVID YAVENDITI, Alma, MI, and the PROPOSER.

730. [Spring 1990] Proposed by R. S. Luthar, University of Wisconsin Center, Janesville, Wisconsin.

Solve in integers the equation

$$2xy + 13x - 5y - 11 = 4x^3.$$

Solution by JOHN T. ANNULIS, University of Arkansas at Monticello, Monticello, Arkansas.

Solving the equation for y yields

$$y = \frac{4x^3 - 13x + 11}{2x - 5} = 2x^2 + 5x + 6 + \frac{41}{2x-5}.$$

The only integer solutions are those in which $2x - 5$ *is a factor of 41. Hence* $2x - 5$ *equals* ± 1 or ± 41, *yielding the solutions*

$$(x,y) = (2,-17),\ (3,80),\ (-18,563),\ \text{and}\ (23,1180).$$

Also solved by CHARLES ASHBACHER, Hiawatha, IA, STEVE ASCHER, McNeil Pharmaceutical, Spring House. PA, SEUNG-JIN BANG, Seoul, Korea, FRANK P. BATTLES, Massachusetts Maritime Academy, Buzzards Bay, MARTIN J. BROWN, Jefferson Community College, Louisville, CAVELAND MATH GROUP, Western Kentucky University, Bowling Green, GEORGE P. EVANOVICH, Saint Peter's College, Jersey City, NJ, MARK EVANS, Louisville, KY, VICTOR G. FESER, University of Mary, Bismarck, ND, the late JACK GARFUNKEL, Flushing, NY, ROBERT C. GEBHARDT, Hopatcong, NJ, S. GENDLER, Clarion University of Pennsylvania, RICHARD I. HESS, Rancho Palos Verdes. CA, NATHAN JASPEN, Stevens Institute of Technology, Hoboken, NJ, RALPH E. KING, St. Bonaventure University, NY, MURRAY S. KLAMKIN, University of Alberta, Canada, JAMIE KONRAD, Rockford College, IL, DEREK LEDBETTER, University of Florida. Gainesville. HENRY S. LIEBERMAN, Waban. MA. CARL LIBIS, Granada Hills, CA, G. MAVRIGIAN, Youngstown State University, OH, LEV S. NAKHAMCHIK, Willowdale, Ont., Canada, WILLIAM H. PEIRCE, Stonington, CT, DAMEN PETERSON, Alma College, MI, BOB PRIELIPP, University of Wisconsin-Oshkosh, SAHIB SINGH, Clarion University of Pennsylvania, TIMOTHY SIPKA, Alma College, MI, KENNETH M. WILKE, Topeka, KS, DAVID YAVENDITI, Alma, MI, KENNETH L. YOKOM, South Dakota State University, Brookings, and the PROPOSER. Occasional arithmetic errors on some of the submissions were overlooked, which is a general policy of this editor.

Partial solutions were submitted by MOHAMMAD K. AZARIAN, University of Evansville, IN, KAREN L. COOK, Lantana, FL, JOE DEMAIO, Emory University, Lenoir, NC, and WADE H. SHERARD, Furman University, Greenville, SC.

731. [Spring 1990] Proposed by Roger Pinkham, Stevens Institute of Technology, Hoboken, New Jersey.

a) Show that on the lattice points in the plane having integer coordinates one cannot have the vertices of an equilateral triangle.

***b)** What about a tetrahedron in 3-space?*

I. Solution to Part (a) by the late JACK GARFUNKEL, Flushing, New York.

Let a triangle have vertices (x_1,y_1), (x_2,y_2), and (x_3,y_3). The area of the triangle is

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

which is an integer whenever the coordinates are all integers. However, the area of an equilateral triangle is given by the well-known formula

$$A = \frac{s^2}{4}\sqrt{3} = \frac{(x_1 - x_2)^2 + (y_1 - y_2)^2}{4}\sqrt{3},$$

which is irrational when the coordinates are integers. Hence, a contradiction, **proving** Part (a).

II. Solution to Part (a) by *S.* GENDLER, Clarion University of Pennsylvania, Clarion, Pennsylvania.

Assume there is such a triangle. Translate it so its vertices are at $O(0,0)$, $P(a,b)$, $Q(c,d)$ with all coordinates integers. We assume that any common factor of a, b, c, and d has been divided out, so that the triangle is of smallest possible dimensions. Since the triangle is equilateral, we must have that $OP^2 = PQ^2 = OQ^2$, that is,

$$a^2 + b^2 = (a - c)^2 + (b - d)^2 = c^2 + d^2.$$

The left inequality simplifies to

$$2(ac + bd) = c^2 + d^2.$$

Since the left side is even, then c and d are both even or both odd. If both are odd, then

$$a^2 + b^2 = c^2 + d^2 \equiv 2 \pmod 4,$$

so both a and b are odd, too. But then $2(ac + bd) \equiv 0 \pmod 4$, which is impossible. If c and d are both even, then

$$a^2 + b^2 = c^2 + d^2 \equiv 0 \pmod 4,$$

and a and b must both be even, contradicting our hypothesis that triangle *OPQ* is smallest possible. Hence there are no solutions.

III. Comment by Seung-Jin Bang, Seoul, Republic of Korea.

Part (a) of this problem appeared in the mathematical competition of university students in Korea held in June 1989. The solution given there is essentially solution II above.

IV. Solution to Part (b) by ALLEN J. SCHWENK, Western Michigan University, *Kalamazoo*, Michigan.

In 3-space the situation is entirely different. Let us seek a tetrahedron of the form $O(0,0,0)$, $A(a,b,c)$, $B(b,c,a)$, $C(c,a,b)$ with a, b, and c integers. Clearly we already have $OA = OB = OC$ and $AB = BC = CA$. Thus we need only have $OA = AB$, that is,

$$a^2 + b^2 + c^2 = (a - b)^2 + (b - c)^2 + (c - a)^2.$$

Now use the quadratic formula to solve for c, obtaining

$$c = a + b \pm 2\sqrt{ab}.$$

Writing $a = m^2r$, where r is square-free, in order for c to be rational, then we must have $b = n^2r$. Thus a triple (a,b,c) will give us a regular tetrahedron of lattice points of the form above if and only if $(a,b,c) = (m^2r, n^2r, (m \pm n)^2r)$, where m, n, and r are integers. (Note that r need not be **square-**free.) For example, the smallest equilateral lattice tetrahedron of this form is $(0,0,0)$, $(0,1,1)$, $(1,1,0)$, and $(1,0,1)$.

V. Comment by Murray *S.* Klamkin, University of Alberta, Edmonton, Alberta, Canada.

First, the word "regular" should be inserted in the statement of Part (b). Also, it has been shown that the only regular polygon that can be imbedded in a square lattice is the square [1, p.4]. The only other regular polygons that can be imbedded in an n-dimensional cubic lattice are the triangle and the hexagon and n = 3 suffices [1, p.43]. It has been shown [2] that it is sometimes possible to imbed a regular n-simplex in an n-dimensional cubic lattice. In particular, if $n \equiv 3 \pmod 4$, that imbedding is always possible. Finally, a proof by Andy Liu and myself that the only regular polygons that can be imbedded in an equilateral triangular lattice are the triangle and the hexagon is to appear in Mathematics Magazine.

References

1. H. Hadwiger, H. Debrunner, V. **Klee**, Combinatorial geometry in the Plane. New York: Holt, Rinehart and Winston, 1964.

2. I. J. Schoenberg, "Regular Simplices and Quadratic Forms," Jour. London Math. *Soc.* 12(1937)48-55.

VI. Comment by the Editor.

Two solvers of Part (a) cleverly took two vertices of the triangle to be located on the x-axis. One used the points $(0,0)$, $(a,0)$ and $(a/2,b)$; the other used $(-a,0)$, $(a,0)$, and $(0,b)$. In either case, the computations are simplified. It is not obvious, however, that such a choice of coordinates can be made without loss of generality. Clearly, translations are possible, so there is no harm in placing one vertex at the origin. One must prove, then, that if $(0,0)$, (p,q), (r,s) are points with integral coordinates, then it is possible to find a similar triangle $(0,0)$, $(a,0)$, (b,c) with integral coordinates

To that end, suppose a rotation-homothety centered at the origin maps (p,q) to $(a,0)$, where a, p, and q are integers. In complex numbers the mapping can be represented by $u + vi$ and we have

$$(p+qi)(u+vi) = a,$$

which we solve for u and v to get

$$u = \frac{pa}{p^2+q^2} \quad \text{and} \quad v = \frac{-qa}{p^2+q^2}.$$

Hence u and v are rational. It follows that $(r+si)(u+vi) = b+ci$ yields rational coordinates b and c. Now multiply each of a, b, and c by the common denominator p^2+q^2 to get the desired integral coordinates.

Also solved by NATHAN JASPEN, Stevens Institute of Technology, Hoboken, NJ, DEREK LEDBETTER, University of Florida, Gainesville, and HENRY S. LIEBERMAN (Part (b) solution of the form of Solution IV above, found "while walking on a trail at the Audubon Society Sanctuary in Wellfleet"), Waban, MA. Most solvers of Part (b) found just the one solution given in the very last line of our Solution IV.

Part (a) solutions were submitted by CHARLES ASHBACHER, Hiawatha, IA, SEUNG-JIN BANG, Seoul, Korea, FRANK P. BATTLES, Massachusetts Maritime Academy, Buzzards Bay, MARK EVANS, Louisville, KY, RICHARD I. HESS, Rancho Palos Verdes, CA, LEV S. NAKHAMCHIK, Willowdale, Ont., Canada, DAMEN PETERSON, Alma College, MI, ALLEN J. SCHWENK, Western Michigan University, Kalamazoo, and the PROPOSER.

IN MEMORIAM
John M. Howell
Jack Garfunkel

John M. Howell taught mathematics, probability, statistics, and computer programming at Los Angeles City College for 23 years, retiring in **1969.** He was an active contributor to this department for many years, thoroughly enjoying his Commodore 64 computer. Number theory problems seemed to be his special interest. After retirement he became quite interested in stamp collecting, forming the Mailer's Postmark Permit Club. He was born February **21, 1910,** and died June **29, 1990.**

Jack Garfunkel taught at Queensboro Community College. Although retired several years, he returned to teaching this past fall semester because he was getting bored just sitting home. He and I met professionally when I was asked to review his article The *Equilic* Quadrilateral, which appeared in this JOURNAL in the Fall of **1981.** Jack's curious facility for ferreting out geometrical truths and my organizational skill complemented one another nicely and we collaborated on four more papers, the last one appearing last spring. Many of his proposals and solutions have appeared in this column over the years. Jack died December 31, **1990,** at age **80** after a brief illness.

Editor's Note

The Pi Mu Epsilon Journal was founded in 1949 and is dedicated to undergraduate and beginning graduate students interested in mathematics. Submitted articles, announcements, and contributions to the Puzzle Section and Problem Department of the Journal should be directed toward this group.

Undergraduates and beginning graduate students are urged to submit papers to the Journal for consideration and possible publication. Student papers are given top priority. Expository articles by professionals in all areas of mathematics are especially welcome. Some guidelines are:

1. Papers must be correct and honest.
2. Most readers of the Pi Mu Epsilon Journal are undergraduates: papers should be directed to them.
3. With rare exceptions, papers should be of general interest.
4. Assumed definitions, concepts, theorems, and notations should be part of the average undergraduate curriculum.
5. Papers should not exceed 10 pages in length.
6. Figures provided by the author should be camera-ready.
7. Papers should be submitted in duplicate to the Editor.

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