PI MU EPSILON JOURNAL

VOLUME 9 FALL 1991 NUMBER 5

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Τὴν παίδευσιν καὶ τὰ μαθηματικὰ ἐπισπεύδειν

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Editorial correspondence, including books tor review, chapter reports, news Items and manuscripts (two copies) should be mailed to PI MU EPSILON JOURNAL, Richard L. Poss, EDITOR, St. Norbert College, De Pere, WI 54115. Students submitting mansucrlpts are to identify their college or university and their class or expected graduation date. Others are requested to provide their affiliation, academic, or otherwise.

...ms lor solution and solutions to problems should be mailed directly to the PROBLEM EDITOR. Puzzle proposals and puzzle ...ns should be mailed to the EDITOR.

PI MU EPSILON JOURNAL is published at St. Norbert College twice a year–Fall and Spring. One volume consists of five years beginning with the Fall 19x4 or Fall 19x9 issue, starting in 1949. For rates, see inside back cover.

PRELIMINARY ANNOUNCEMENT – 1992 NATIONAL MEETING

Pi Mu Epsilon has traditionally held its annual meeting in conjunction with the hummer meetings of the AMS and the MM. In 1992, however, the International Congress of Mathematics Educators (ICME) will hold its annual meeting in Quebec City, in Canada. It has been the policy of the AMS and M M to forego their summer meetings in years when an international mathematics meeting (e.g. ICME or the International Congress of Mathematicians) takes place in North America. Thus there will be no AMS-MM national meeting in the summer of 1992.

Pi Mu Epsilon will, however, have a national meeting in 1992. It will take place August 6 through August 8 at Miami University, in Oxford, Ohio. In addition to Pi Mu Epsilon student contributed paper sessions and invited faculty lectures, there will be the opportunity for members of student chapters of the M M to give presentations. Look for further information in the spring, 1992, Pi Mu Epsilon Journal and in an announcement to be sent to chapter advisors in the spring of 1992.

STUDENTPAPERS

In each year that at least five student papers have been received by the Editor, prizes of $200, $100. and $50, known as Richard V. Andree Awards, are given to student authors. All students who have not yet received a Master's Degree or higher are eligible for the prizes.

There are four student papers in this issue of the Journal. The first is 'Turning Triangles into Circles", by Judy Marie Kenney. Judy prepared this paper while she was a senior at the College of St. Benedict.

The second paper is "Computerized Segmentation of Liver Structures", by Heng Hak Ly. Heng prepared this paper, with the help of Dr. Maryellen Giger and Dr. Rose Carney, while he was a junior at Illinois Benedictine College.

The third paper is "Inversions and Adjacent Transpositions", by Amy Pinegar. Amy prepared this paper, under the supervision of David Sutherland, while she was a senior at Middle Tennessee State University.

The final student paper is "A Math Problem Within an Antique Clock Label" by Shannon L. Spittler. Shannon wrote this paper while she was a junior at Miami University.

TURNING TRIANGLES INTO CIRCLES

Judy Marie Kenney
College of St. Benedict

Imagine drawing any simple closed curve on a piece of paper, and then stretching or contracting the paper in several directions. If it were possible to do this without wrinkling or ripping the paper, we could create another simple closed curve In any shape we wanted. That is, there would be a mapping from the original closed curve to the transformed closed curve. But what happens to the points interior to the closed curves? The points interior to the original closed curve should remain interior to the new closed curve.

Suppose the original curve is an equilateral triangle, the created curve Is the unit circle, and the "paper" is the complex plane. We want to find a conformal mapping so that the interior of the equilateral triangle will be mapped onto the unit disk in the complex plane. Even though this mapping will take the interior of the equilateral triangle onto the unit disk, the edges of the triangle should transform into the unit circle. This mapping will take the vertices of the triangle to any three specified points on the unit circle. The Riemann Mapping Theorem ensures as a special case that a mapping from the interior of an equilateral triangle onto the unit disk exists.

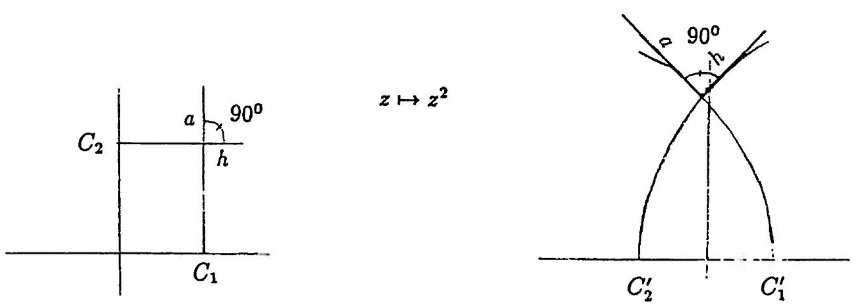
Riemann Mapping Theorem: Let A be a simply connected region such that $A \neq \mathbb{C}$. Then there exists a one-to-one and onto conformal map $f: A \mapsto D$, where D is the unit disk.

(This mapplng is analytic, which means that it is differentiable In the complex plane. It should also be noted that A is a region, which is an open set, and therefore the mapping is conformal inside the region A, not necessarily on the boundary of A.)

However, like many theorems in mathematics, the Rlemann Mapping Theorem is an existence theorem; that is, it does not state how to find this mapping. Before we show the details of this mapping, we need to define both conformal and a family of unfamiliar functions.

A mapping f, from a region A to a region B, is said to be conformal if, for each z_0 in A, f rotates tangent vectors to curves through z_0 by a definite angle θ. (The angle is locally preserved.) In other words, suppose two curves intersect at a point z_0, and the tangents to these curves at z_0 form an angle θ. Then, under the mapping f, the two mapped curves intersect at a point $f(z_0)$, and the tangents to these curves through $f(z_0)$ form the same angle θ (locally), in sense as well as size. An example should help make this definition clear.

The mapping $z \mapsto z^2$ maps the first quadrant onto the upper half plane. We can see this when we look at lines parallel to the y-axis: if $z = z + yi$, where a is constant, then under the transformation, z becomes $z' = (a + yi)^2$, yielding $z' = a^2 - y^2 + 2ayi$, the equation of a parabola. Similarly, lines parallel to the x-axis are transformed into parabolas. Thus the mapping is:



The tangents to the curves C_1 and C_2 form an angle of 90°. Under the conformal mapping $z \mapsto z^2$, the tangents to the transformed curves C_1 and C_2 again form an angle of 90°; so. the mapplng $z \mapsto z^2$ is conformal.

The family of unfamiliar functions are the Jacoblan Elliptic Functions, whlch are similar to the trigonometric functions. The Integrals representing trigonometric functions involve the equation for a circle; similarly, the integrals for the Elliptic Functions involve the equation for an ellipse. Arthur Cayley's An Elementary Treatise on Elliptic Functions provides a connection between the equation of an ellipse and the elliptic Integral.

If we look at the integrals

$$u = \int_0^x \frac{dt}{\sqrt{1-t^2}} \quad \text{and} \quad \frac{\pi}{2} = \int_0^1 \frac{dt}{\sqrt{1-t^2}}$$

we see that as x varies from 0 to 1, u varies from 0 to $\pi/2$. Also, from calculus, we see that

$$u = \int_0^x \frac{dt}{\sqrt{1-t^2}} = \arcsin x;$$

that is, $\sin u = x$. Now, if we look at the integrals

$$u = \int_0^x \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \quad \text{and} \quad m = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

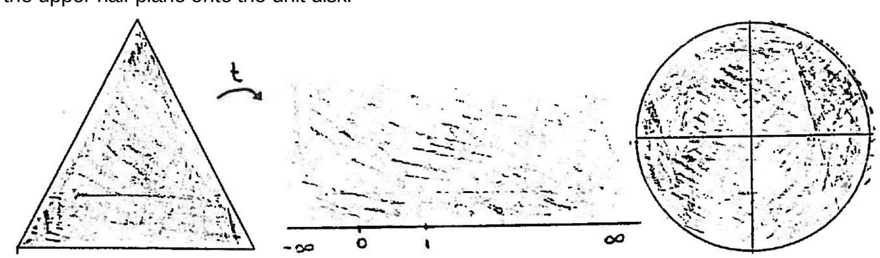
where $0 \le k \le 1$, we see that as x varies from 0 to 1, u varies from 0 to m. This relationship defines a function sn by the formula $\mathrm{sn}\,(u,k) = x$. We see that when $k = 0$, the integral becomes the arcsin Integral as above. Thus, $\mathrm{sn}\,(u,0) = \sin u$. Also, when $k = 1$, the Integral becomes

$$\int_0^x \frac{dt}{1-t^2}$$

and, from calculus, $\mathrm{sn}(u,1) = \tanh u$.

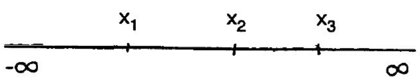
Trigonometry involves more than the sin function; similarly, there is more than one Jacobian Elliptic function. Two functions that we are interested in are $\mathrm{cn}\,(u,k)$ and $\mathrm{dn}\,(u,k)$. An explanation of the explicit forms of these functions can be found in Jacobian Elliptic Function Tables, by L.M. Milne-Thomson.

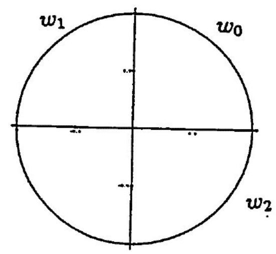
As mentioned above, the Riemann Mapping Theorem ensures that the mapping we want exists, but does not state how to find it. However, because the composition of two conformal mappings produces a conformal mapping, the mapping from the interior of an equilateral triangle onto the unit disk can be broken into two mappings, which we can then compose. First, a mapping from the interior of the equilateral triangle onto the upper half plane, and second, a mapping from the upper half plane onto the unit disk.



This second mapping from the upper half plane onto the unit disk can be found using a linear fractional transformation, which will take any three specified points on the real axis to three specified points on the unit circle. A linear fractional transformation is defined as

$$T(w) \mapsto \frac{aw+b}{cw+d}, \text{ where } ac - bd \neq 0.$$



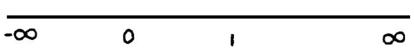


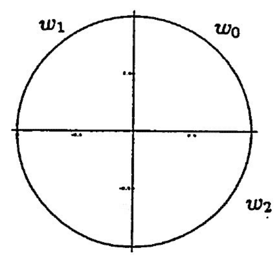
Thus, this mapping transforms the real number 0 to $T(0) = b/d = w_0$; the number 1 to $T(1) = (a+b)/(c+d) = w_1$; and if we take the limit as w becomes large, we see that $T(w) = T(\infty) = a/c = w_2$. Because we have four parameters (a,b,c,d), and only three arguments $T(0)$, $T(1)$, $T(\infty)$, we do not have a unique mapping. However, we can find a unique solution by arbitrarily assigning a value to one of the parameters. Setting $b = 1$, the other variables are:

$$d = \frac{1}{w_0}\ ;\ c = \frac{w_1 - w_0}{w_0(w_2 - w_1)}\ ;\ \text{and}\ \ a = \frac{w_2(w_1 - w_0)}{w_0(w_2 - w_1)}.$$

So. this mapping $T(w)$ will take the points 0, 1 and ∞ to any three specified points on the unit circle, w_0, w_1, and w_2:

$$T(w) \mapsto \frac{aw+b}{cw+d}, \text{ where } ac - bd \neq 0.$$





To verify that this mapping of $T(w)$ takes any point in the upper half plane to the unit disk, we choose a point, say $4 + 3i$, and show that its transformation is in the unit disk. We need to specify w_0, w_1, and w_2, so let $w_0 = -1 + 0i$, $w_1 = 0 - i$, and $w_2 = 1 + 0i$. Thus,

$$T(4+3i) = \frac{\frac{1(-i+1)}{-1(1+i)}(4+3i)+1}{\frac{-i+1}{-1(1+i)}(4+3i)+\frac{1}{-1}} = \frac{\frac{i-1}{1+i}(4+3i)+1}{\frac{i-1}{1+i}(4+3i)-1} = \frac{-6+2i}{-8} = \frac{3}{4} - \frac{1}{4}i,$$

which is in the unit disk.

To show that $T(w)$ Is a mapping from the upper half plane onto the unit disk. we need to verify that for any point, z, in the unit disk, there is a point, w, in the upper half plane such that $T(w) = z$.

Let $z \in$ the unit disk. Then, $$\frac{w_2(w_1 - w_0)w + w_0(w_2 - w_1)}{(w_1 - w_0)w + (w_2 - w_1)} = z$$

$$\Rightarrow \quad w_2(w_1 - w_0)w + w_0(w_2 - w_1) = z(w_1 - w_0)w + z(w_2 - w_1)$$

$$\Rightarrow \quad w_2(w_1 - w_0)w - z(w_1 - w_0)w = z(w_2 - w_1) - w_0(w_2 - w_1)$$

$$\Rightarrow \quad w = \frac{z(w_2 - w_1) - w_0(w_2 - w_1)}{w_2(w_1 - w_0) - z(w_1 - w_0)}.$$

So, for any point z in the unit disk, we can find a point w in the upper half plane such that $T(w) = z$; therefore, $T(w)$ is a mapping from the upper half plane onto the unit disk.

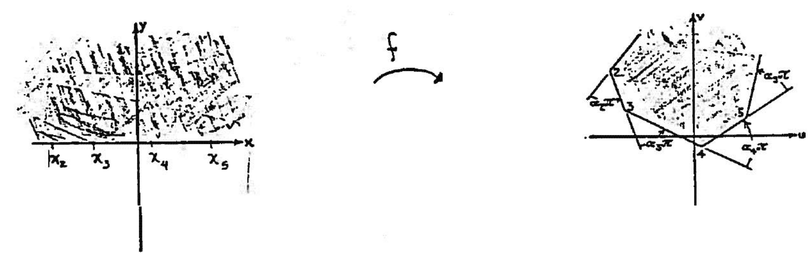
Now we have the mapping from the upper half plane onto the unit disk, but we still need to find the mapping from the interior of the equilateral triangle onto the upper half plane in order to compose the mappings. The **Schwarz-Christoffel** Formula gives an explicit mapping from the upper half plane onto any polygon:

Schwarz-Christoffel Formula: Let P be a polygon in the w-plane with vertices at $w_1, w_2, \ldots, w_n$ and exterior angles $\pi\alpha_i$, where $-1 \leq \alpha_i \leq 1$. Then

$$f(z) = a\int_0^z (z - x_1)^{-\alpha_1}(z - x_2)^{-\alpha_2}\cdots(z - x_{n-1})^{-\alpha_{n-1}}dz + b.$$

are conformal maps from the upper half plane onto the interior of P, where 'a' determines the size of the polygon and 'b' determines the position. Furthermore, $f(x_i) = w_i$, and three x_i's can be specified; if $x_n = \infty$, then the mapping becomes:

$$f(z) = a\int_0^z (z - x_1)^{-\alpha_1}(z - x_2)^{-\alpha_2}\cdots(z - x_{n-1})^{-\alpha_{n-1}}dz + b.$$



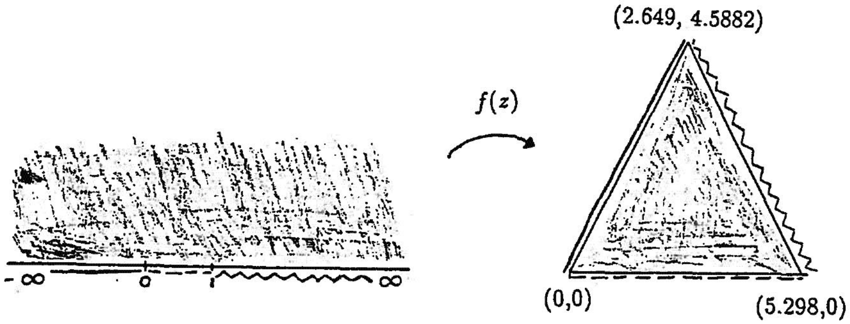
If we use this formula with the equilateral triangle as our **polygon**, with a = 1 and b = 0 for simplicity, and the three points $x_1 = 0, x_2 = 1$, and $x_3 = \infty$, we find the mapping is:

$$f(z) = \int_0^z \frac{dz}{(z-0)^{2/3}(z-1)^{2/3}}$$

Note that $$f(x_1) = f(0) = \int_0^0 \frac{dz}{(z^2-z)^{2/3}} = 0$$

and $$f(x_2) = f(1) = \int_0^1 \frac{dz}{(z^2-z)^{2/3}} \approx 5.298.$$

This gives the coordinates of two vertices of the triangle: **(0.0)** and **(5.298,0).** Thus, the third coordinate is **(2.649, 4.5882),** which corresponds to $f(\infty)$. Because $f(z)$ is continuous, points between 0 and 1 on the real axis will lie on the edge of the triangle with endpoints **(0,0)** and **(5.298,0);** points greater than 1 on the real axis will lie on the edge with endpoints **(2.649,4.5882)** and **(5.298,0);** points less than 0 on the real axis will lie on the edge with endpoints **(0,0)** and **(2.649,4.5882):**



Thus, f maps the upper half plane onto the interior of the equilateral triangle. However, because we want the mapping from the interior of the equilateral triangle onto the upper half plane, we need to find the inverse of $f(z)$. Dictionary of Conformal Representations, by **H.** Kober, gives this

inverse: $$t = \frac{1}{2} + \frac{(27)^{1/4}\, sn(\zeta,k)\, dn(\zeta,k)}{(1+cn(\zeta,k))^2}$$

$$k = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

where $$\zeta = \frac{2z - z_1}{(27)^{1/4}(2)^{1/3}}$$

$$z_1 = \int_0^1 \frac{dz}{(z^2-z)^{2/3}} \approx 5.298.$$

We now have the two mappings we need: the mapping from the interior of the equilateral triangle onto the upper half plane, and the mapping from the upper half plane onto the unit disk. Composing the mapping from the triangle onto the upper half plane

$$t = \frac{1}{2} + \frac{(27)^{1/4}\, sn(\zeta,k)\, dn(\zeta,k)}{(1+cn(\zeta,k))^2}$$

where k, ζ, and z_1 are given as above, with T, the mapping from the upper half plane onto the unit disk, the new mapping

$$g(z) = \frac{w_2(w_0 - w_1)t + w_0(w_1 - w_2)}{(w_0 - w_1)t + (w_1 - w_2)}$$

takes the equilateral triangle with vertices **(0,0), (5.298,0)** and **(2.649,4.5882)** onto the unit circle with specified points w_0, w_1, and w_2.

During the summer of 1990, I participated in a NSF Summer Research program at Washington University in St. Louis. I would like to thank the director of the program, Professor Clean *Yohe.* I *would* also like to thank my advisor, Professor Steven *Krantz,* who gave me this problem.

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[2] M. J. **Hoffman** and J. E. **Marsden,** Baste Complex Analysis, W. H. Freeman and Company, New York, **1987.**

[3] H. **Kober,** Dictionary of Conformal Representations, Dover Publications, Inc., U.S.A., **1952.**

[4] L. M. Milne-Thomson, Jacobian Elliptic Function *Tables*, **Dover** Publications, Inc., U.S.A., **1950.**

Judy Kenney prepared this paper while she was a senior at the College of St. Benedict.

There were **1635** initiates to Pi Mu Epsilon during the **1990-91** academic year.

COMPUTERIZED SEGMENTATION OF LIVER STRUCTURE FROM CT IMAGES

Heng Hak Ly
Illinois Benedictine College

Computed tomography (CT) produces cross sectional images of a 3-dimensional object such as a patient's abdomen. Once 2-dimensional cross sections are obtained, image processing and display need to be performed in order to visualize the original 3-dimensional object. The ability to view the 3-dimensional image of some organ of interest is a useful tool to physicians about to perform a related surgery.

In November of 1989, the first live donor liver transplant operation was successfully performed at the University of Chicago Medical Center. Without a liver transplant, liver disease in patients can prove to be fatal. The segmentation and 3-dimensional representation of the liver from CT scans is a desired step in the surgical planning of a living donor transplant operation. Currently, this visualization is done qualitatively by the physician as he views the series of cross-sectional Images (slices) of the abdomen obtained from a CT scan. A research project in the Kurt Rossmann Laboratories for radiologic Image Research in the Department of Radiology at the University of Chicago involves the computerized segmentation of the liver in the cross-sectional images of the abdomen and the subsequent 3-dimensional representation of the organ on a display monitor. I participated in the research by implementing the computerized scheme on an initial database of patients and by analyzing the effect of digital filters on the performance of the scheme as compared to that of a radiologist.

Before surgically removing a part of the liver, one needs to know the size, shape, and volume of the donor liver. To find the volume of the liver, computed tomography (using x-ray beams) was employed to generate cross-sectional images of the abdomen. The number of images can vary greatly. As few as 15 and as many as 38 images were used for different patients.

The order of the steps that the computer uses to segment the liver are as follows: obtain 2-D slices, detect abdomen boundary, detect liver boundary, smooth liver boundary, calculate liver area in each slice and estimate volume. To find the volume of the liver, the computer determines its contours and boundary. Then interpolation techniques are used to stack these contours and produce the 3-dimensional image.

The mathematical methods used in the extraction of the liver from the CT image of the abdomen use a priori information of iiver morphology as well as various image processing and computer-vision techniques. The segmentation algorithm is performed sequentially slice-by-slice, starting with a reference slice in which the liver occupies almost the entire right half of the abdomen cross section. The original image is converted to a binary image using what is called gray-level thresholding. Each gray level for each pixel in the image has a number value ranging from 0 to 256. Initially, the contour of the abdomen is located by the computer using information of its location and shape. The shape is represented by a measure of circularity obtained from the ratio of the square of the perimeter to the area of the structure in question.

In the reference slice the liver occupies a relatively large area in the abdomen and a region

of interest is created in this approproximate area. By readlng in the gray level values in this area and limiting the values, a binary Image can be generated by making pixels within this range "1" and making those above or below "0" (0 = black pixel and 1 = white pixel).

Most of the undesirable entities in the Image, such as the stomach, are readily eliminated because they are in the different part of the abdomen and do not have gray level values within the desired range. However, they may create certain disturbances in the binary Image that need to be eliminated. To detect the contour of the liver, the computer used a technique called region growing. First, the approximate center portion of the liver is found using a priori Information of liver anatomy. Liver gray level histogram analysis is performed within this portion. Gray-level thresholding is performed to yield an initial estimate of the liver structure by using the range of gray levels that correspond to possible iiver structure. This estimate is subjected to smoothing by a Gaussian filter and then region-growing techniques. The contour of the liver is then traced, using a process called the "8-point connectivity test."

The screen Is searched for a "1" or a "0" that is part of the binary liver image. Then the computer searches for the next closest black pixel using a set of eight vectors, labeled 0-7, each 45^o apart, starting in the direction of the 0-vector. If the pixel in that direction is black, then the two pixels are connected, otherwise the next pixel directed by the next higher number is checked for its color. Two pixels are connected if their color is the same. The computer terminates the process when the original pixel is reached.

If certain "objects" or "dots" are close enough to be part of the liver but are not connected to the binary image, then they can be connected by applying the linear, shift-invariant Gaussian filters. The function is the liver image. The process of changlng and altering the function using another function is called convolution.

Up to this point most of the rough edges along the boundary of the liver have been eliminated through the convolution with the Gaussian filter, but artifacts can occur due to the presence of other anatomic structures that neighbor the liver. To eliminate the remnants of these projections, morphological filtering was performed. The area enclosed by the detected contour can be considered as a binary image. A morphological filter sequence of an erosion followed by a dilation is used to remove protruding projections along the boundary of the liver. Erosion is a non-linear operation that calculates the logical AND of pixels w'thin a kernel of given size and shape, whereas dilation determines the logical OR of such pixels. The sequence of an erosion followed by dilation is referred to as an "open" process.

In the research, the sizes of both the linear Gaussian kernel and the non-linear open filter were varied systematically and the effect on the computer-calculated area of the liver In each slice was analyzed. The calculated area of each slice was then compared to that as drawn by a radiologist. The variation in area between the computer and the radiologist was about 10% as determined from four clinical cases, each with approximately 20 slices to the case. The future aim of the research will be to display the detected liver contours in 3-dimensional format for physician viewing and for the electronic simulation of abdominal surgery.

Acknowledgement is made to the Kurt Rossmann Laboratory for the use of their facilities; the National Institutes of Health for funding this project (CA 48985); Dr. Maryellen Giger, assistant professor at the University of Chicago, for directing my work on this research project; Dr. Rose Carney, professor at Illinois Benedictine College, for her encouragement *and* support.

Heng Hak Ly prepared this paper while he was a junior at Illinois Benedictine College.

INVERSIONS AND ADJACENT TRANSPOSITIONS

Amy Pinegar
Middle Tennessee State University

We begin with a review of basic definitions. A permutation of a set A is a one-to-one function from A onto A. More simply, a permutation of the numbers $1,2,3,\ldots,n$ is a rearrangement of these numbers into a particular order. Two common notations are used for writing permutations:

column notation -- e.g., $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 5 & 2 \end{pmatrix}$

cycle notation -- e.g., $\pi = (1\ 3)(2\ 4\ 5)$

Permutation multiplication is defined as function composition on two permutations from left to right. For example,

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 5 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 1 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 4 & 5 \end{pmatrix}$$

and

$$[(13)(245)][(12543)] = (23)$$

The permutations on $1,2,3,\ldots,n$ with permutation multiplication form a group called S_n, the Symmetric Group on $1,2,3,\ldots,n$.

Definition 1. If π is a permutation written in column notation, an inversion $i{:}j$ in a permutation π is a pair of numbers i and j such that in π, the position of i comes before the position of j and $i > j$. $I(\pi)$ denotes the set of inversions for π and $|I(\pi)|$ denotes the number of inversions in π.

Example 1. If $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 5 & 2 \end{pmatrix}$, then $I(\pi) = \{4{:}1, 4{:}2, 3{:}1, 3{:}2, 5{:}2\}$ and $|I(\pi)| = 5$.

The goal of this paper is to describe the effect of permutation multiplication on inversions. We begin by examining simpler permutations. A transposition is defined as a cycle containing only two elements. An adjacent transposition is a transposition where the two elements are consecutive. For example (12) is an adjacent transposition, while (13) is not. We learn in group theory that every permutation can be written as a product of transpositions. For example,

$$\pi = (1346) = (13)(14)(16).$$

However, properties of adjacent transpositions are not as well known.

Theorem 1. If $\pi = (i\ j)$ is a transposition, then π can be expressed as a product of adjacent transpositions. In particular,

$$\pi = (i\ j) = (i\ i+1)(i+1\ i+2)\ldots(j-1\ j)(j-1\ j-2)(j-2\ j-3)\ldots(i+1\ i)$$

Example 2. The transposition $\pi = (25)$ can be written as

$$\pi = (25) = (23)(34)(45)(43)(32).$$

Proof. Assume $\pi = (i\ j)$ is a transposition. Without loss of generality assume $i < j$. Let σ denote the decomposition

$$(i\ i+1)(i+1\ i+2)\ldots(j-1\ j)(j-1\ j-2)(j-2\ j-3)\ldots(i+1\ i).$$

We will show that $\pi = \sigma$.

First consider all numbers x such that $x < i$. Thus $\pi(x) = x$. And since x is not involved in the decomposition, $\sigma(x) = x$. Similarly, $\sigma(x) = x = \pi(x)$ for numbers x such that $x > j$.

Next consider all x such that $i \le x \le j$. We know that $\pi(i) = j$, $\pi(j) = i$, and $\pi(x) = x$ for all $i < x < j$. Using permutation multiplication from left to right on the transpositions in σ, we compute $\sigma(i) = j$ and $\sigma(j) = i$. We also notice that for $i < x < j$, the first time that x appears in σ, x is sent to $x-1$. Then as we continue multiplying to the right. we find that $x-1$ does not appear again until the right side where $x-1$ is sent to x. Thus, $\sigma(x) = x$. Therefore, $\pi = \sigma$. ▫

Theorem 2. Every permutation can be written as a product of adjacent transpositions.

Example 3. If π is the permutation $(126)(45)$, then

$$\pi = (12)(16)(45) = (12)(12)(23)(34)(45)(56)(45)(34)(23)(12)(45).$$

Proof. Given a permutation, we can first express it as a product of transpositions. Then, by Theorem 1, we can take each one of these transpositions and express it as a product of adjacent transpositions. Therefore, it follows that any permutation can be written as a product of adjacent transpositions. ▫

Theorem 3. The expansion in Theorem 1 of the transposition $\pi = (i\ j)$ contains $2j - 2i - 1$ adjacent transpositions.

Example 4. The permutation $\pi = (14)$ can be decomposed into $(12)(23)(34)(23)(12)$. This decomposition has $2j - 2i - 1 = 2(4) - 2(1) - 1 = 5$ adjacent transpositions.

Proof. Let $\pi = (i\ j)$ be a transposition with $i < j$. In column notation,

$$\pi = \begin{pmatrix} 1 & 2 & 3 & \cdots & i-1 & i & i+1 & \cdots & j-1 & j & j+1 & \cdots & n \\ 1 & 2 & 3 & \cdots & i-1 & j & i+1 & \cdots & j-1 & i & j+1 & \cdots & n \end{pmatrix}$$

Let σ denote the decomposition of π into adjacent transpositions. Rewrite σ by dividing it into three parts. Call the three parts the left side. the middle, and the right side:

$$\sigma = [(i\ i+1)(i+1\ i+2)\ldots(j-2\ j-1)]\ (j-1\ j)\ [(j-1\ j-2)\ldots(i+2\ i+1)(i+1\ i)].$$

Notice that there are the same number of transpositions on the left side as there are on the right side, with $(j-1\ j)$ being the middle. Also notice that the left and right sides contain the same transpositions written in reverse order.

First consider $\{x : 1 < x < i\}$ and $\{x : j < x < n\}$. From the form of the expansion, no x in these sets Is Included in an adjacent transposition in the expanded expression.

Now consider $\{x : i \le x \le j - 1\}$. There are $j - 1$ elements in **this** set with x and $x + 1$ forming an adjacent transposition **in** the expression. If we look at the adjacent transpositions on the left side, we know that all numbers in the set except $x = i$ and $x = j - 1$ will appear In two adjacent transpositions. Therefore there will be $j - i - 1$ adjacent transpositions on the left side with $(i\ i+1)$ adjacent transpositions on the left **side** with $(i\ i+1)$ being the first and $(j-2\ j-1)$ being the last. Since we know that the right side has the same transpositions as the left side, we know that the right side also has $j - i - 1$ adjacent transpositions. Finally we count the middle adjacent transposition, $(j-1\ j)$, which we know only appears once In the expression.

Therefore adding up the number of adjacent transpositions from each part of a, we get $(j - i - 1) + (j - i - 1) + 1 = 2j - 2i - 1$ adjacent transpositions in a. ◻

Theorem 4. If $\pi = (i\ j)$ is a transposition, then *it* has $2j - 2i - 1$ inversions.

Example 5. If $\pi = (1\,4) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 1 \end{pmatrix}$, then $I(\pi) = \{4{:}2, 4{:}3, 4{:}1, 2{:}1, 3{:}1\}$ and $|I(\pi)| = 2(4) - 2(1) - 1 = 5$.

Proof. Let *it* $= (i\ j)$ be a transpositlon with $i < j$. Written in column notation

$$\pi = \begin{pmatrix} 1 & 2 & 3 & \cdots & i-1 & i & i+1 & \cdots & j-1 & j & j+1 & \cdots & n \\ 1 & 2 & 3 & \cdots & i-1 & j & i+1 & \cdots & j-1 & i & j+1 & \cdots & n \end{pmatrix}.$$

First consider $\{x : x < i \text{ or } x > j\}$. Since $\pi(x) = x$ and all numbers following x In *it* are larger than x and all numbers preceding x in *it* are smaller than x, there are no inversions **involving x**.

Now consider $\{x : i + 1 \le x \le j - 1\}$. There are $j - i - 1$ elements in this set and all are smaller than j. Thus $j : x$ is an inversion and there are $j - i - 1$ inversions. Similarly, there are $j - i - 1$ elements in this set that all are larger than i. Thus, $x : i$ **is** an inversion. Therefore there are $j - i - 1$ more inversions. Finally since $j : i$ is an Inversion we have $(j - i - 1) + (j - i - 1) + 1 = 2j - 2i - 1$ inversions. ◻

Now we return to the question of the effect of permutation multiplication on inversion. Given permutations *it* and a, what is the relationship between $I(\pi)$, $I(\alpha)$ and $I(\pi\alpha)$? The preceding theorems suggest we first consider the question when a is an adjacent transposition.

Lemma 1. The product of a permutation π and a transposition $(i\ j)$ is the same permutation as *it* except that i and j are switched in the product.

Example 6. Consider

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 1 & 2 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 3 & 2 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 1 & 4 & 2 \end{pmatrix}.$$

Notice that the product is the original permutation with 2 and 4 in switched positions.

Proof. Assume π is a permutation and $(i\,f)$ is a transposition. Let $a = (\pi)(i\ j)$, the product of *it* and $(i\,f)$. Choose x and y such that $\pi(x) = i$ and $\pi(y) = j$. We want to show that $\sigma(z) = \pi(z)$ for **all** numbers $z \ne x$ and $z \ne y$. When $\pi(z)$ is multiplied by $(i\ j)$, since $\pi(z)$ does not involve i or j, $\pi(z)$ will not change. Therefore $\sigma(z) = \pi(z)$.

Now consider x and y. By definitiion of the multiplication of permutations, $\sigma(x) = \pi(y)$ and $\sigma(y) = \pi(x)$. Therefore the product of any permutation and a transposition $(i\,f)$ is the same permutation except for i and j **being** switched in the product. ◻

Theorem 5. If *it* is a permutation and a is the product of *it* and the adjacent transpositlon $(i\ i+1)$, then $|I(\sigma)| = |I(\pi)| \pm 1$.

Example 7. Given the product

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 1 & 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 4 & 3 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 1 & 4 & 2 \end{pmatrix},$$

we find that $I(\pi) = \{5{:}4, 5{:}1, 5{:}3, 5{:}2, 4{:}1, 4{:}3, 4{:}2, 3{:}2\}$ and $I(\sigma) = \{5{:}3, 5{:}1, 5{:}4, 5{:}2, 3{:}1, 3{:}2, 4{:}2\}$. Thus $7 = |I(\sigma)| = |I(\pi)| - 1 = 8 - 1 = 7$.

Proof. Assume *it* is a permutation and a is the product of *it* and $(i\ i+1)$. First consider all inversions **in** *it* and a not Involving i or $i + 1$. These inversions will be of the form $x : y$ such that $x \ne i, i+1$ and $y \ne i, i+1$. We know by Lemma 1 that the product a of any permutation π and some adjacent transposition $(i\ i+1)$ is the same permutation as π except for i and $i + 1$ being switched In a. Since the positions of x and y are unchanged **in** *it* and a, **if** $x{:}y$ is an inversion in π, it wlll also be an Inversion in a.

Next we consider all inversions Involving i or $i + 1$. First consider the inversions using i or $i + 1$ and some number x whose position is before the positions of i and $i+1$. If $x{:}i$ or $x{:}i+1$ is an inverslon in *it*, it will also be an inverslon in a since the **positlon** of x is still before the positions of i **and** $i + 1$ in a.

Next consider the inversions using i or $i + 1$ and some number x whose position is after the positions of i and $i + 1$. If $i{:}x$ or $i+1{:}x$ is an inversion in π, it will also be an inversion in a since the position of x is still after the positions of i and $i + 1$ In σ.

Now we consider the inversions involving i or $i + 1$ and some number x such that the position x comes between the positions of i and $i + 1$. Without loss of generality, suppose i comes before $i + 1$ In *it*. There are four possible inversions to consider:

i) If $i{:}x$ Is an Inversion in π, $i+1{:}x$ will be an inversion in a.
ii) **If** $x{:}i+1$ is an inversion in *it*, $x{:}i$ will be an inversion In a.
iii) If $i{:}x$ is not an inversion in *it*, then $i+1{:}x$ will not be an inversion in a.
iv) If $x{:}i+1$ is not an inversion in *it*, then $x{:}i$ will not be an inversion in a.

Observe that x will either be smaller than i or larger than $i + 1$.

If $i{:}x$ is an inversion in π, then the position of i comes before the position of x and $i > x$. Since in a the positions of i and $i + 1$ are switched, then $i + 1$ comes before x and $i + 1 > x$. Thus, $i+1{:}x \in I(\sigma)$ and i) Is true. Similar reasoning proves ii), iii), and iv). Therefore the number of Inversions involving i or $i + 1$ and some number in a position between the positions of i and $i + 1$ In *it* will be the same number of inversions in a.

Now finally we consider the inversion invoking i and $i + 1$. If $i+1{:}i$ is an inversion in π, Lemma 1 implies that it wlll not be an inversion in a. If $i{:}i+1$ is not an inversion in π, it will be an inversion in a by similar reasoning.

Therefore, depending on the positions of i and $i+1$ in π, $|I(\sigma)| = |I(\pi)| \pm 1$. ◻

The next example shows that Theorem 5 is not always true if the transposition is not adjacent.

Example 8. Let $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 6 & 2 & 1 & 4 \end{pmatrix}$ and a = (25). Then $|I(\pi)| = 10$ and

$$\sigma = \pi\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 6 & 5 & 1 & 4 \end{pmatrix}. \quad \text{Thus } |I(\sigma)| = 7 \neq |I(\pi)| \pm 1.$$

Next we turn to the question of the effect on the Inversions when permutation π is multiplied by any other permutation a. Theorem 2 implies that we can restrict attention to multiplication of π by a finite number of adjacent transpositions. We first consider the special case when the adjacent transpositions are disjoint transpositions.

Lemma 2 The product σ of any permutation π and n disjoint transpositions is the same permutation as π except that each pair of numbers in each transposition has switched positions in *a*.

Example 9 The product of $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 5 & 1 & 6 & 3 & 4 \end{pmatrix}$ with the disjoint transpositions (12), (35), and (46) is found by switching the positions in π of 1 and 2, 3 and 5, and finally 4 and 6. Thus

$$\pi(12)(35)(46) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 2 & 4 & 5 & 6 \end{pmatrix}.$$

Proof. Since the transpositions are disjoint, the result follows from Lemma 1 and induction on *n*. ◻

Theorem 6 If π is a permutation and a is the product of π and n disjoint adjacent transpositions a,, $\alpha_2,\ldots,\alpha_n$, then

$$|I(\sigma)| = \sum_{i=1}^{n} \left[|I(\pi\alpha_i)| - |I(\pi)|\right] + |I(\pi)|$$
$$= \sum_{i=1}^{n} \left[|I(\pi\alpha_i)|\right] + (-n+1)\,|I(\pi)|.$$

Example 10 The result of multiplying $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 5 & 1 & 6 & 3 & 4 \end{pmatrix}$ by the disjoint adjacent transpositions (12), (34) and (56) is found by Lemma 2. Thus

$$\sigma = \pi(12)(34)(56) = \begin{pmatrix} 1 & 2 & & & & 6 \\ 1 & 6 & & & & 3 \end{pmatrix}$$ Thus, $|I(\pi)| = 6$ and $|I(\sigma)| = 7$. We

can also calculate $|I(\pi\alpha_1)| = 5$, $|I(\pi\alpha_2)| = 7$ and $|I(\pi\alpha_3)| = 7$. Therefore,

$$7 = |I(\sigma)| = \sum_{i=1}^{3} |I(\pi\alpha_i)| + (-2)(6)$$
$$= (5+7+7) - 12$$
$$= 7$$

Proof. Let π be a permutation and a be the product of π and n disjoint adjacent transpositions a,, $\alpha_2,\ldots,\alpha_n$. The proof is by induction on n.

Suppose n = 1. Then $a = \pi\alpha_1$ where α_1 Is an adjacent transposition. The formula says $|I(\sigma)| = |I(\pi\alpha_1)| + 0(|I(\pi)|) = |I(\pi\alpha_1)|$ which is true.

Now suppose the formula holds for $a = \pi\alpha_1\alpha_2\ldots\alpha_n$ where the a_l's are disjoint adjacent transpositions for $i = 1,2\ldots,n$.

Suppose $a = \pi\alpha_1\alpha_2\ldots\alpha_{n+1}$ where the α_i's are disjoint adjacent transpositions for $i = 1,2,\ldots,n+1$. Then $\sigma = [\pi\alpha_1\alpha_2\ldots\alpha_n]\alpha_{n+1}$ by associativity of permutation multiplication. From Theorem 5,

$$|I(\sigma)| = |I[(\pi\alpha_1\alpha_2\ldots\alpha_n)\alpha_{n+1}]| = |I(\alpha_1\alpha_2\ldots\alpha_n)| \pm 1.$$

(1)

Applying the Induction hypothesis to $|I(\pi\alpha_1\alpha_2\ldots\alpha_n)|$, we obtain

$$|I(\sigma)| = \sum_{i=1}^{n} |I(\pi\alpha_i)| + (-n+1)\,|I(\pi)| \pm 1$$
$$= \sum_{i=1}^{n} |I(\pi\alpha_i)| + (-n)\,|I(\pi)| \pm 1. \qquad (2)$$

Appiying Theorem 5 again, we find that

$$|I(\pi\alpha_{n+1})| = |I(\pi)| \pm 1. \qquad (3)$$

At this point we would like to be able to substitute $|I(\pi\alpha_{n+1})|$ for $|I(\pi)| \pm 1$ in Equation.2 However, in Equation 2 the $|I(\pi)| \pm 1$ came from Equation 1. So in order to make a valid substitution we need to verify that the ± 1 in Equations 1 and 3 have the same value.

Suppose that α_{n+1} = (If). Since the transpositions are adjacent, $j = I + 1$ and since the transpositions are disjoint, Lemma 2 implies that the positions of *i* and *j* are the same in $\pi\alpha_1\alpha_2\ldots\alpha_{n+1}$ (Equation 1) as in $\pi\alpha_{n+1}$ (Equation 3). Also recall that from the proof of Theorem 5, the inversion that is lost or gained in Equations 1 and 3, is the inversion $i+1{:}i$ in $\pi\alpha_1\alpha_2\ldots\alpha_{n+1}$. Thus the ± 1 in these two equations must have the same value. Thus

$$|I(\sigma)| = \sum_{i=1}^{n} |I(\pi\alpha_i)| + (-n)\,|I(\pi)| + |I(\pi\alpha_{n+1})|$$
$$= \sum_{i=1}^{n+1} |I(\pi\alpha_i)| + (-n)\,|I(\pi)|.$$

Therefore the formula holds true by induction. ◻

We were not able to generalize Theorem 6 completely. However the final theorem describes limits on how much $|I(\pi\alpha)|$ can differ from $|I(\pi)|$.

Theorem 2 If π Is a permutation and σ Is the product of π and n adjacent transpositions, α_1, $\alpha_2,\dots,\alpha_n$, then

i) $|I(\pi)| - n \le |I(\sigma)| \le |I(\pi)| + n$.

ii) There are $\binom{2+n-1}{n} = 4$ possible values for $|I(\sigma)|$.

iii) These values are $|I(\sigma)| = |I(\pi)| + (n - 2k)$

$\forall k$ such that $0 \le k \le \binom{n+1}{n} - 1$

Example 10. Let π = (145362) and multiply on the right by α_1 = (23), α_2 = (34), and α_3 = (12). In column notation, we obtain

$$\begin{pmatrix}1&2&3&4&5&6\\4&1&6&5&3&2\end{pmatrix}\begin{pmatrix}1&2&3&4&5&6\\1&3&2&4&5&6\end{pmatrix}\begin{pmatrix}1&2&3&4&5&6\\1&2&4&3&5&6\end{pmatrix}\begin{pmatrix}1&2&3&4&5&6\\2&1&3&4&5&6\end{pmatrix}=\begin{pmatrix}1&2&3&4&5&6\\3&2&6&5&1&4\end{pmatrix}$$

Thus $I(\pi)$ = {4:1, 4:3, 4:2, 6:5, 6:3, 6:2, 5:3, 5:2, 3:2} and $I(\sigma)$ = {3:2, 3:1, 2:1, 6:5, 6:1, 6:4, 5:1, 5:4}. Thus we see that $|I(\pi)| = 9$ and $|I(\sigma)| = 8$.
Condition i) says $|I(\pi)| - 3 \le |I(\sigma)| \le |I(\pi)| + 3$. So $6 \le 8 \le 12$.

Condition ii) says there are $\binom{2+3-1}{3} = 4$ possible values for $|I(\sigma)|$.

Condition iii) says $|I(\sigma)| = |I(\pi)| + (3-2k)$, fork = 0,1,2,3.
For: $k = 0$: $|I(\sigma)| = 9 + 3 = 12$,
$k = 1$: $|I(\sigma)| = 9 + 1 = 10$,
$k = 2$: $|I(\sigma)| = 9 + (-1) = 8$,
$k = 3$: $|I(\sigma)| = 9 + (-3) = 6$.

Proof. Assume π is a permutation and a is the product of π and n adjacent transpositions α_1, $\alpha_2,\dots,\alpha_n$. We first show $|I(\sigma)| \le |I(\pi)| + n$. Consider $\pi\alpha_1 = \pi_2$. By Theorem 5 we will either gain or lose an inversion. Therefore, at most $|I(\pi_2)| = |I(\pi)| + 1$.

Now we look at $(\pi\alpha_1)\alpha_2 = \pi_2\alpha_2 = \pi_3$. Again by Theorem 5, at most we will gain one inversion. Therefore, at most $|I(\pi_3)| = [\,|I(\pi)| + 1\,] + 1 = |I(\pi)| + 2$. Notice that each time we multiply by an adjacent transposition, we can gain at most one more inversion. Therefore If we multiply by an adjacent transposition n times, we can gain at most n Inversions. Therefore $|I(\sigma)| \le |I(\pi)| + n$. Similar reasoning proves that $|I(\pi)| - n \le |I(\sigma)|$.

ii) Each multiplication by an adjacent transposition correponds to either a gain (+) or loss (-) of an Inversion. Thus multiplying each of the n adjacent transpositions will correspond to either gaining (+) or losing (-) an Inversion. The number of possible combinations of gains and losses corresponds to choosing a n-element multiset from a 2-element set $P = \{+, -\}$. From combinatorics, we know this formula is $\binom{2+n-1}{n} = \binom{n+1}{n}$.

iii) The proof is by induction on the number of adjacent transpositions.
Let $n = 1$. Then $|I(\sigma)| = |I(\pi)| + (1 - 2k)$, fork = 0,1. Thus $|I(\sigma)| = |I(\pi)| \pm 1$ which Is confirmed by Theorem 5.
Now suppose $n = m$ works and a = $\pi\alpha_1\alpha_2\dots\alpha_m$. Then

$$|I(\sigma)| = |I(\pi)| + (m - 2k) \quad \text{fork} = 0, 1, 2,\dots \binom{m+1}{m} - 1 .$$

Suppose $\sigma = \pi\alpha_1\alpha_2\dots\alpha_{m+1}$. Let A = $\pi\alpha_1\alpha_2\dots\alpha_m$. Then $\sigma = \lambda\alpha_{m+1}$. By Theorem 5, $|I(\sigma)| = |I(\lambda)| \pm 1$. But by the induction hypothesis,

$$|I(\lambda)| = |I(\pi)| + (m - 2k), \text{ for } 0, 1, 2,\dots, \binom{m+1}{m} - 1 .$$

Now we show that the set A = $|I(\sigma)| = |I(\pi)| + (m + 1 - 2k)$, fork = 0,1,2,..., $\binom{m+2}{m+1} - 1$

is the same as B = $|I(\sigma)| = |I(\pi)| + (m - 2k) \pm 1$, fork = 0, 1, 2,..., $\binom{m+1}{m} - 1$. We need to show that

$$\{m + 1 - 2k \mid k = 0, 1, 2,\dots, m + 1\} = \{m - 2k \pm 1 \mid k = 0, 1, 2,\dots, m\}$$

To show that these are the same set. we will show that each set contains the same numbers. in the first set observe that as k takes on the values from 0 to $m + 1$, the numbers obtained range from $m + 1$ down to $-m - 1$ and increment by two. In the second set, observe that when we choose +1 and let k range from 0 to m, the numbers obtained range from $m + 1$ down to $-m + 1$ and increment by two. When the -1 Is chosen, the numbers obtained range from $m - 1$ down to $-m - 1$ and also increment by two. Thus. the numbers In the second set range from $m + 1$ down to $-m - 1$ and increment by two. Therefore, the two sets are equal.

Therefore, $|I(\sigma)| = |I(\pi)| + (m + 1 - 2k)$, fork = 0, 1, 2..., $\binom{m+2}{m+1} - 1$ and. by induction,

$$|I(\sigma)| = |I(\pi)| + (n - 2k), \text{ fork} = 0, 1, 2,\dots, \binom{n+1}{n} - 1 . \ \square$$

This paper was written under the supervision of David Sutherland while the author was a senior mathematics major at Middle Tennessee State University.

A MATH PROBLEM WITHIN AN ANTIQUE CLOCK LABEL

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inside an antique grandfather **clock** in my house is an interesting label that sets forth a 'question within a solution." The question is how. before the advent of rapid communications such as telegraph and radio, does one know what time it is and how much time has elapsed to make sure one's dock is keeping accurate time? The label is dated **1789** and it sets forth the solution to the question but, in doing so it brings to **life** another interesting fact about time. the solution of which is the concentration of this discussion of sidereal time.

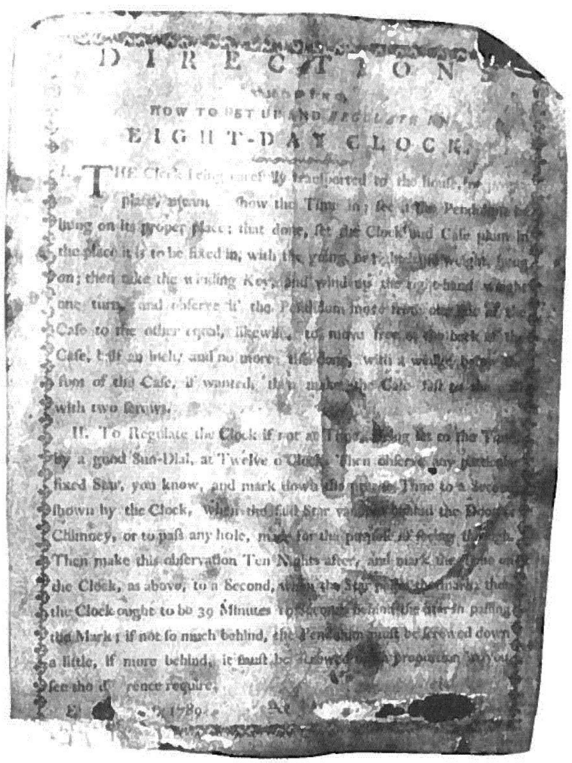

Figure **1:** Label in **1789** Grandfather Clock that contains a Math Problem

The label (Figure **1)** is located inside the door of a Scottish grandfather clock. The label is yellow from age and worn through in places. It reads:

DIRECTIONS

SHEWING

HOW TO SET UP AND REGULATE AN

EIGHT-DAY-CLOCK

1 The Clock being carefully transported to the house or place meant to show the Time in; see if the Pendulum be hung in **its** proper place; that done, set the Ciock and Case plum In the place it is to be fixed in with the going or right hand weight hung on; then take the winding Key, and wind up the right hand weight one turn, and observe if the Pendulum moves from one side of the Case to the other equal, likewise to move free of the back of the Case, half an inch. and no more, this done, with a wedge below the front of the Case if wanted, then make the Case fast to the wall with two screws.

II. To Regulate the Clock if not at Time, being set to Time with a good Sun-Dial at Twelve **o'** Ciock. Then observe any particular fixed Star, you know, and mark down the precise Time to a second shown by the Clock, when the fixed Star vanishes behind the Door or Chimney, or to pass any **hole,** made for the purpose of seeing through. Then make this observation Ten Nights after, and mark the ***time*** on the **Clock,** as above, to a Second, when the Star passes the mark; then the Clock ought to be **39** Minutes, **19** Seconds behind the Star In passing the Mark; if not so much behind, the Pendulum must be screwed down a little, if more behind, it must be screwed up in proportion as you see the difference require.
April 1789

It Is paragraph **II,** specifically the procedure explaining how to regulate the clock by timing the passing of a known fixed star over a ten-day period, that is the subject of this article. The question that needs to be answered is why does the star pass precisely **39** minutes and **19** seconds before the clock shows ten **full** days? This question involves some mathematics and some understanding of sidereal time.

it will be helpful to first mention that the only period of time that can be measured accurately with simple Instruments is the day. The exact instant of noon can be determined using a stick on a sunny day. The point of noon is when the sun is at its zenith, and the stick's shadow is the shortest, pointing exactly at true north in the northern latitudes. Our common day is the period between two noons. Every other period – a month, second, year, etc. -- can only be approximated without instruments or must be calculated by accurately dividing a day or adding days. Thus, the day is the basis of time and most time-keeping philosophies.

So, what exactly is a day? is it the time it takes the Earth to make 1 rotation on its axis? No, it is the time it takes the Earth to make **1** and **1/365.25** rotations. The Earth revolves around the sun once every **365** and **1/4** days (the **1/4** due to leap years). This is an approximation off by only **+11** minutes a year. In terms of orientation to the sun, the Earth makes **365** and **1/4** revolutions on its axis in a year. However, in terms of true rotations on **its** axis, it makes **366** and **1/4** rotations per year. This figure is determined in terms of any fixed object in space such as a star other than the sun or a planet. This is because the Earth must orient itself to point exactly at the sun at noon each day and thus in the course of **365** and **1/4** days, one extra true rotation on its axis is needed.

Now, how long does It take the Earth to make one true rotation on its axis?

Let X = the time it takes for the Earth to complete one rotation on its axis

$$1X + (1/365.25)\,X = 24 \text{ hours}.$$

The extra **1/365.25** rotation is the added rotation from above.
By substitution,

$$(365.25/365.25)X + (1/365.25)X = 24 \text{ hours}$$

$$365.25 + 11365.25 = 24 \text{ hours}/X$$

$$366.25/365.25 = 24 \text{ hours}/X$$

Solving, $$X = (365.25)\,(24)\,/\,366.25$$

$$X = 23.93447 \text{ hours}$$

$$\text{or } X = 23 \text{ hours. } 56.0682 \text{ minutes}$$

$$\text{or } X = 23 \text{ hours, } 56 \text{ minutes, } 4.092 \text{ seconds}$$

The Earth rotates once about Its axis every 23 hours, 56 minutes, and 4.092 seconds. **This** period is known as a sidereal day. One true rotation according to a star is 3 minutes and 55.908 seconds short of our 24 hour day. In ten rotations relative to a fixed star, the amount of time short of a ten (sun) day period Is 30 minutes and 559.08 seconds or 39 minutes and 19.08 seconds.

The label on the Robert Hon clock stated this time as 39 minutes and 19 seconds -- quite accurate for **1789!** The .08 seconds was probably known but dropped as no one could accurately measure that amount of time anyway. For the same reason a **10-day** period was probably selected to make the observation. Any shorter period and errors made in calculating the instant of passing of a fixed star are magnified and any greater period make the task of regulating the clock very time consuming as the entire procedure must be repeated each time the pendulum is adjusted.

Shannon ***Spittler*** was a junior English major (mathematics minor) at Miami University when she prepared this paper.

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REGULAR N-GONS ON THE SIDES OF A TRIANGLE

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A well known theorem in plane geometry, attributed to Napoleon Bonaparte, says: If equilateral triangles are erected externally on the sides of a triangle, then their centers form an equilateral triangle [1; theorem **3.36**]. For squares erected on the sides of a triangle, we have the following theorem [1; theorem 4.811: If squares, with centers O_1, O_2, O_3, are erected externally on the sides BC, CA, AB of triangle ABC, then O_1, O_2, and O, are equal and perpendicular. So we may ask what we can say about regular pentagons, regular hexagons and, more generally, regular n-gons erected on the sides of a triangle. To answer this question we need one more regular n-**gon.** We shall agree that in the notation $A, A, A, \ldots A_n$ denoting a regular n-**gon,** the vertices $A,, A,, A, ,\ldots, A_n$ lie counterclockwise.

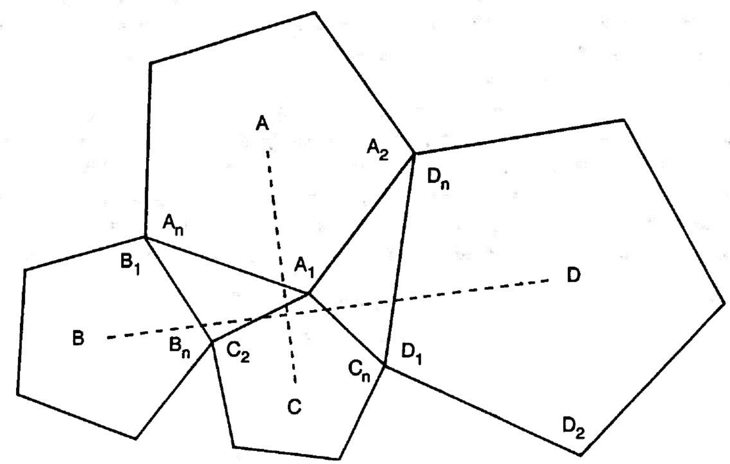


FIGURE 1

Theorem 1. Let $A_1 A_2 A, \ldots A,,$ $B_1 B_2 B_3 \ldots B_n$, $C_1 C_2 C_3 \ldots C_n$, $D_1 D_2 D_3 \ldots D_n$ be regular n-**gons** in the plane with the centers A, B, C, D such that $C_1 = A,,$ $A_n = B_1$, $B_n = C_2$, $C_n = D_1$, and $D_n = A_1$. Then $A = C$ and $B = D$, or AC and BD are perpendicular with $BD/AC = 2\sin\theta$, where θ is the interior angle of the n-**gons.** Moreover if we exclude the trivial case in which the figure is symmetric about the point A_1, AC and BD bisect each other If and only if n = 3 *and* A, is the mid-point of BD if and only if n = 4.

Proof. We denote points of the plane by complex numbers In the Argand diagram with the origin $A_1 = C_1$ and identify the points and their corresponding complex numbers. (See Figure 1.) Let $A_2 = r$, $C_2 = s$ (r and s are complex numbers); then, since the image of a point w by the dilative rotation with an angle ϕ about a point z and a ratio of magnification k is $(w - z)ke^{\phi i} + z$, we get $A_n = rt^2$ and $C_n = st^2$, where $t = e^{\theta i/2}$. From $A_1A / A_1A_2 = (2\cos(\theta/2))^{-1} = (t + t^{-1})^{-1}$, letting $u = (t + t^{-1})^{-1}$ we also have:

$$A = rtu,\quad B = (rt^3 + st^{-1})u,\quad C = stu,\quad D = (st^3 + rt^{-1})u$$

Therefore we obtain

$$AC = (s - r)tu \quad \text{and} \quad BD = (t^2 - t^{-2})(s - r)tu$$

Hence it is clear that $A = C$ implies $B = D$, and conversely. Now we suppose $A \neq C$ or $r - s \neq 0$. Then we have

$$BD / AC = t^2 - t^{-2} = 2i \sin\theta.$$

This shows that $BD/AC = 2 \sin\theta$ and AC is perpendicular to BD.

If our figure is non-symmetric about A_1 we have $s + r \neq 0$. The mid-points M and N of AC and BD are obtained by simple manipulations:

$$M = (1/2)(r + s)tu \quad \text{and} \quad N = (1/2)(t^2 + t^{-2})(r + s)tu.$$

Hence, $M = N$ if and only if $t^2 + t^{-2} = 1$ and this equation is equivalent to having $e^{2\theta i} - e^{\theta i} + 1 = 0$, or $e^{\theta i}$ be a primitive third root of 1. Therefore $M = N$ if and only if $n = 3$. Similarly N is the origin if and only if $t^2 + t^{-2} = 0$. The last equation is equivalent to that $e^{\theta i}$ be a primitive 4-th root of 1 or $n = 4$. The proof is now complete.

Since $\theta = (1 - 2/n)\pi$, we have $\sin\theta = 1/2$ and $AC = BD$ when $n = 12$. So we get the same conclusion as Von Aubel's quadrilateral theorem for regular dodecagons. In the case $n = 3$, AC and BD bisect each other and $AC : BD = 1 : \sqrt{3}$. Therefore ABC and DAC are equilateral triangles. This is Napoleon's theorem. Similarly we can deduce the theorem involving three squares erected on the sides of a triangle mentioned above. The figure in our theorem was considered by Japanese mathematicians in the case of $n = 4$ In the nineteenth century [2; p.47].

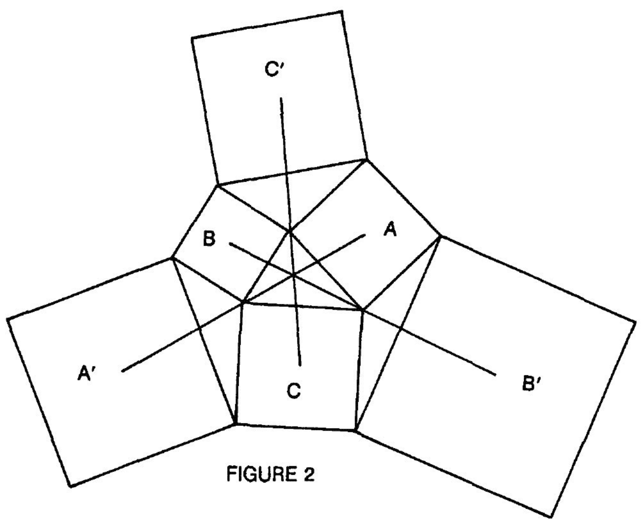


FIGURE 2

The proof of the following theorem is a simple application of Theorem 1 (see Figure 2). So we leave it to the interested reader.

Theorem 2. Let $A_1 A_2 A_3 \ldots A_n$, $B_1 B_2 B_3 \ldots B_n$, $C_1 C_2 C_3 \ldots C_n$ be regular n-gons with the centers A, B, C erected on the sides of a triangle XYZ externally such that $A_1 = Y = C_n$, $B_1 = 2 = 4$, $C_1 = X = B_n$ and $A'_1 A'_2 A'_3 \ldots A'_n$, $B'_1 B'_2 B'_3 \ldots B'_n$, $C'_1 C'_2 C'_3 \ldots C'_n$ regular n-gons with the centers A', B', C' such that $A'_n = B_{n-1}$, $A'_1 = C_2$, $B'_1 = C_{n-1}$, $B'_n = A_2$, $C'_1 = A_{n-1}$, $C'_n = B_2$, then AA', BB', CC' are concurrent.

References:

[1] H. S. M. Coxeter and S.L Greizer, Geometry Revisited, Mathematical Association of America, Washington DC, 1967.

[2] H. Fukagawa and D. Pedoe, Japanese Temple Geometry Problems, Charles Babbage Research Center, Winnipeg, Canada, 1989.

It is impossible for the human Intellect to grasp the idea of absolute continuity of motion. Laws of motion of any kind only become comprehensible to man when he can examine arbitrarily selected units of that motion. But at the same time it Is this arbitrary division of continuous motion into discontinuous units which gives rise of a large proportion of human error.

By adopting smaller and smaller units of motion we only approach the solution of the problem but never reach it.... A new branch of mathematics, having attained the art of reckoning with infinitesimals, can now yield solutions in other more complex problems of motion which before seemed insoluble.

This new branch of mathematics, which was unknown to the ancients, by admitting the conception, when dealing with problems of motion, of the infinitely small and thus conforming to the chief condition of motion (absolute continuity), corrects the inevitable error which the human intellect cannot but make if it considers separate units of motion Instead of continuous motion.

In the investigation of the laws of historical movement precisely the same principle operates. The march of humanity, springing as it does from an infinite multitude of individual wins, Is continuous. The discovery of the laws of this continuous movement Is the aim of history.

Only by assuming an infinitesimally small unit for observation – a differential of history (that is, the common tendencies of men) -- and arriving at the art of integration (finding the sum of the infinitesimals) can we hope to discover the laws of history.

L. Tolstoy, War and Peace, (first published in 1869). Translation by R. Edmonds, Penguin Books, inc., Baltimore, 1957, pp. 974-5.

SOME MATRIX IDENTITIES

Russell Euler
Northwest Missouri State University

In general, algebraic identities such as $(a+b)^2 = a^2 + 2ab + b^2$ and $a^2 - b^2 = (a-b)(a+b)$ are not valid for n x n matrices with respect to the usual matrix multiplication. The reason for this is that matrix multiplication is not commutative. The purpose of this paper is to define a **nontrivial** matrix multiplication (in terms of the usual matrix multiplication) in such a way that some common algebraic identities will still be valid for n x n matrices with respect to the defined matrix multiplication.

All matrices **in** this paper are of order n x n and have entries from the field of complex numbers. The usual matrix multiplication of A and B will be written using the standard juxtaposition notation, namely, AB. Define the matrix product $A \cdot B$ by

$$A \cdot B = (AB + BA)/2$$

where the addition of AB and BA is assumed to be the usual matrix addition. Some properties of this product **will** now be discussed.

First, notice that

$$A \cdot A = (AA + AA)/2 = (A^2 + A^2)/2 = A^2.$$

Similarly, $A \cdot A^{n-1} = A^n$ **for all integers** $n \geq 2$. **Also,** since matrix addition is commutative,

$$A \cdot B = (AB + BA)/2 = (BA + AB)/2 = B \cdot A.$$

If A is invertible, then

$$A \cdot A^{-1} = (AA^{-1} + A^{-1}A)/2 = (I + I)/2 = I.$$

There is a distributive property since

$$\begin{aligned} A \cdot (B + C) &= [A(B+C) + (B+C)A]/2, \\ &= (AB + AC + BA + CA)/2, \\ &= [(AB + BA) + (AC + CA)]/2, \\ &= (AB + BA)/2 + (AC + CA)/2, \\ &= A \cdot B + A \cdot C. \end{aligned}$$

Also, it can be shown that for any complex number α,

$$(\alpha A) \cdot B = A \cdot (\alpha B) = \alpha(A \cdot B).$$

Some algebraic properties will now be derived based upon some of the above properties.

$$\begin{aligned} (A + B)^2 &= (A + B) \cdot (A + B), \\ &= A \cdot A + A \cdot B + B \cdot B + B \cdot A + B \cdot B, \\ &= A^2 + A \cdot B + A \cdot B + B^2, \\ &= A^2 + 2A \cdot B + B^2. \end{aligned}$$

Similarly, $(A - B)^2 = A^2 - 2A \cdot B + B^2$. These properties are, of course, analogous to $(a \pm b)^2 = a^2 \pm 2ab + b^2$.

Furthermore.

$$\begin{aligned} (A - B) \cdot (A + B) &= (A - B) \cdot A + (A - B) \cdot B, \\ &= A \cdot A - B \cdot A + A \cdot B - B \cdot B, \\ &= A^2 - A \cdot B + A \cdot B - B^2, \\ &= A^2 - B^2. \end{aligned}$$

Hence, $A^2 - B^2$ can be factored as the difference of two squares with respect to the defined matrix multiplication. As a result, one can obtain results such as

$$\begin{aligned} A^4 - B^4 &= (A^2 - B^2) \cdot (A^2 + B^2), \\ &= (A - B) \cdot (A + B) \cdot (A^2 + B^2). \end{aligned}$$

Unfortunately, the multiplication defined in this paper is **nonassociative.** That is, in general, $A \cdot (B \cdot C) \neq (A \cdot B) \cdot C$. As a result, anticipated results such as

$$(A + B)^3 = A^3 + 3A^2 \cdot B + 3A \cdot B^2 + B^3$$

are not valid in general. It can be shown that

$$(A + B)^3 = A^3 + 2A \cdot (A \cdot B) + A \cdot B^2 + A^2 \cdot B + 2(A \cdot B) \cdot B + B^3,$$

and

$$(A + B)^4 = A^4 + 4(A \cdot B) \cdot A^2 + 2A^2 \cdot B^2 + 4(A \cdot B)^2 + 4(A \cdot B) \cdot B^2 + B^4.$$

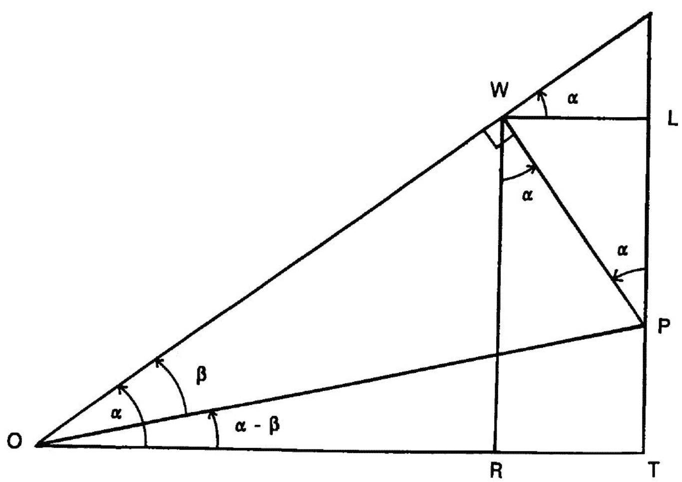
Undoubtedly, the interested reader can find other interesting properties **involving** the matrix product given in **this** article.

ON THE COS(α - β) AND SIN($\alpha \pm \beta$) IDENTITIES

Joseph *M.Moser*
San *Diego* State University

Trigonometry books generally Include a construction proof of the cos (a + β) identity. (See [I], p. 137, for example.) In **[1],** p. 116, one **will** find an Interesting proof of the cos (a - β) identity, which Is done by comparing chord lengths of a circle. To the best of our knowledge, no construction proof of the cos (a - β) identity has been given in texts on trigonometry. So we present one.

In Figure 1, one sees the following:



$$\cos(\alpha - \beta) = \frac{OT}{OP} = \frac{OR + RT}{OP} = \frac{OR}{OP} + \frac{RT}{OP}$$

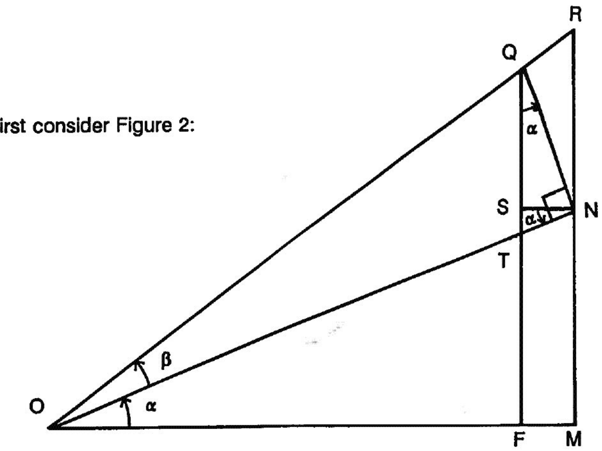
$$= \frac{OR}{OW}\cdot\frac{OW}{OP} + \frac{RT}{WP}\cdot\frac{WP}{OP}$$

$$= \frac{OR}{OW}\cdot\frac{OW}{OP} + \frac{WL}{WP}\cdot\frac{WP}{OP}$$

$$= \text{cosa cosfl} + \sin\alpha \sin\beta.$$

Then, in **[2],** Guetter presents formulas for the **sin($\alpha \pm \beta$)** identities, using areas of triangles. We will present a construction derivation of the **sin($\alpha \pm \beta$)** identities.

First consider Figure 2:

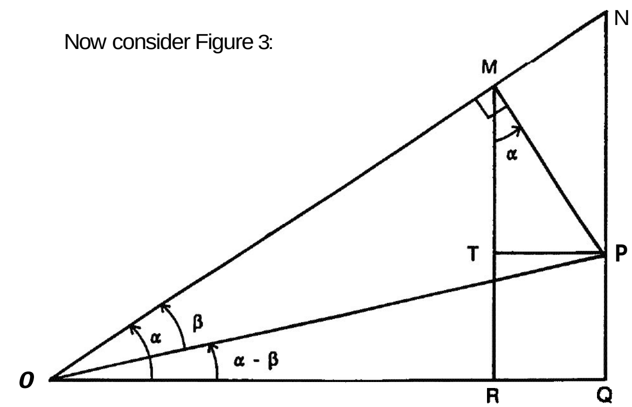


$$\sin(a + \beta) = \frac{FQ}{OQ} - \frac{FS + SQ}{00} = \frac{MN + SQ}{OQ}$$

$$= \frac{MN}{ON}\cdot\frac{ON}{OQ} + \frac{SQ}{QN}\cdot\frac{QN}{OQ}$$

$$= \sin\alpha \cos\beta + \text{cosa sinfl}.$$

Now consider Figure 3:



$$\sin(\text{a-fl}) = \frac{PQ}{OP} - \frac{MR - MT}{OP}$$

$$= \frac{MR}{OM}\cdot\frac{OM}{OP} - \frac{MT}{MP}\cdot\frac{MP}{OP}$$

$$= \text{sina cosfl} - \text{cosa}\ \sin\beta.$$

References:

[1] K. J. Smith, Trigonometry for College Students, Fifth Edition, **Brooks/Cole,** 1991, Pacific Grove, CA.

[2] A. Guetter, Pi Mu Epsilon Journal, ***Vol 9,*** No. 1, Fall, 1989, pp. 30-31.

YETANOTHER DAYATWERACES

John C. Fay
St. Paul's College

in preparing a presentation of the betting method described in "A Day at the Races" by William Tomcsanyi (Pi Mu Epsilon Journal, Spring 1982) for the Georgia Epsilon chapter of Pi Mu Epsilon, there occurred to me some considerations not touched upon by Mr. Tomcsanyi or Mr. Edward Anderson ("Another Day at the Races," Pi Mu Epsilon Journal, Fall 1983).

In Mr. Tomcsanyi's system, he chooses to make a profit, P, by betting on a number of horses in the same race. He settles on three as the best number. Say these horses are paying a : 1, b : 1, and c : 1 odds, respectively. The profit will be made if one of the three horses selected wins. Determining how to make this profit requires finding the amounts to bet on each of the three horses, say x_1, x_2, and x_3. To find these amounts he solves the simultaneous equations

$$\begin{aligned} 2ax_1 - 2x_2 - 2x_3 &= P \\ -2x_1 + 2bx_2 - 2x_3 &= P \\ -2x_1 - 2x_2 + 2cx_3 &= P. \end{aligned}$$

The factors of 2 are because $2 is the minimum bet.

As Mr. Anderson noted, the method fails to work if we try to bet on all the horses in a race, which would ensure that we bet on the winner. The amounts to bet are sure to come out negative. (it would also be a much bigger system to solve.) In fact, the method may fail to work even when betting on a subset of the field of horses. Mr. Anderson then simplifies the computations for cases where fewer than the total number of horses are bet on.

What is neglected are the probabilistic aspects of the situation. Mr. Tomcsanyi claims that the three-by-three (betting on the three horses with the shortest cdds) produces a winner 70% of the time. He gives an example of betting on three horses paying odds of 3 : 1, 3 : 2, and 4 : 1. In this situation, he eams $50 on $283 bet. However, if we assume a 70% probability of winning (as he does) and say we bet on ten such races, winning 7 and losing 3, we will win $350 (7 · 50) and lose $849 (3 · 282). This is a net loss of $499. More formally, the expected retum is .7(50) + .3 (-238) = -49.90, a loss of nearly $50.

if we assume that odds are fairly set (i.e., they correspond to probabilities which are proportional to the true probabilities of winning), then we can replace Mr. Tomcsanyi's 70% by the probability for the given situation. We need only note that the constant of proportionality will be $1 - s$, where s is the percentage which the track skims off the top of the handle (betting pool). Thus a horse at 3 : 1 odds has probability $(1-s)/4$ of winning. (3 : 1 odds imply a $1/4$ chance of winning if there is no skim.) in general:

A horse paying a : 1 odds has probabiiity $(1-s)/(a+1)$ of winning.

Working with the above example and letting $s = .2$ (as Mr. Anderson suggests), the probabilities of the three horses winning are $0.8/4 = 0.2$, $0.8/2.5 = 0.32$, and $0.815 = 0.16$. Thus the probabiiity of winning the bet is $0.2 + 0.32 + 0.16 = 0.68$ and the expected return is $0.68\,(50) + 0.32\,(-283) = -56.56$.

If the odds are shorter so that there is a higher probability of winning, there will be a higher bet required to achieve the $50 win. Thus the expected return remains negative. In fact, under these fair game assumptions:

The expected loss will be the product of the skim percentage and the total wager.

Thus the way to maximize expected value is to eliminate the track's percentage. This is not likely to happen since the tracks are run to make a profit.

How, then, do we make a profit? The answer may lie in knowing (or perhaps ensuring) that a certain horse will not win. Say, for instance a horse, D, going off at d : 1 odds is eliminated (from winning). Then (by the definition of conditional probability)

$$\begin{aligned} P(\text{we win} \mid D \text{ loses}) &= P(\text{we win and } D \text{ loses})/P(D \text{ loses}) \\ &= P(\text{we win})/[1 - P(D \text{ wins})] \\ &= P(\text{we win})/[1 - (1-s)/(d+1)] \\ &= P(\text{we win})[(d+1)/(d+s)]. \end{aligned}$$

To make our expected value positive, we need to have the factor

$$(d+1)/(d+s) > 1/(1-s).$$

This is to counteract the proportionality constant introduced by the skimming. Then we see that

$$d < (1-2s)/s.$$

For $s = .2$, we must have $d < 3$. So we need to eliminate a horse going off at odds no longer than 3 : 1.

Similarly, we can compute how to choose to eliminate a larger number of lower probability (hence lower profile) horses.

So, unfortunately, the way to win betting on horses is to know what others do not, possibly by engineering it. This is not something that law enforcement officials look kindly upon. Perhaps the track does not free us from making an honest living after all!

CHANGES OF ADDRESS

Subscribers to the Journal should keep the Editor informed of changes in mailing address. Journals are mailed at bulk rate and are not forwarded by the postal system. The cost of sending replacement copies by first class mail is prohibitive.

INQUIRIES

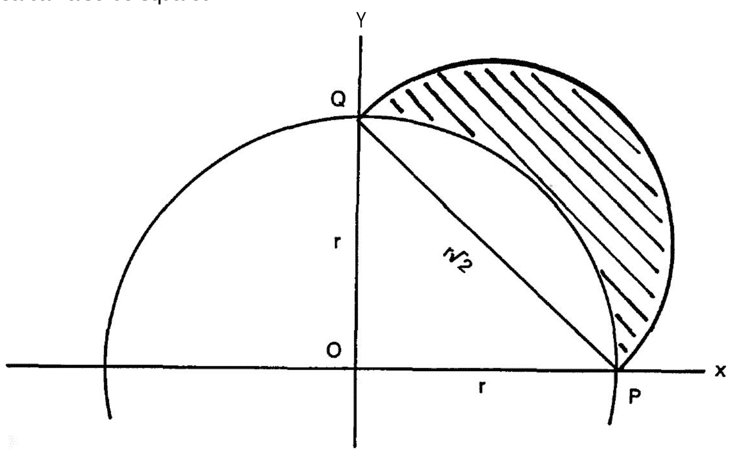
Inquiries about certificates, pins, posters, matching prize funds, support for regional meetings, and travel support for national meetings should be directed to the Secretary-Treasurer, Robert M. Woodside, Department of Mathematics, East Carolina University, Greenviile, NC 27858, 919-757-6414.

HIPPOCRATES REVISITED

Richard L. Francis
Southeast Missouri State University

Ancient attempts to square the circle were **motivated** in part by long-ago success in squaring certain contoured areas. That **is,** it was possible in some cases to construct squares having areas equal to those of **curved** regions. Notable among these successes is the figure often called "the **lune** of Hippocrates." Such a figure suggests a modern day **look** in the form of a generalization.

in the **figure** below, two circles, one with center at O and the other with chord QP as a diameter, form a shaded region which **is** lunar in shape. Such a **lunar** region has an area equal **exactly** to that of triangle POQ. As any polygon can be squared, it follows that this distinctly **curved, lunar** area can also be squared.



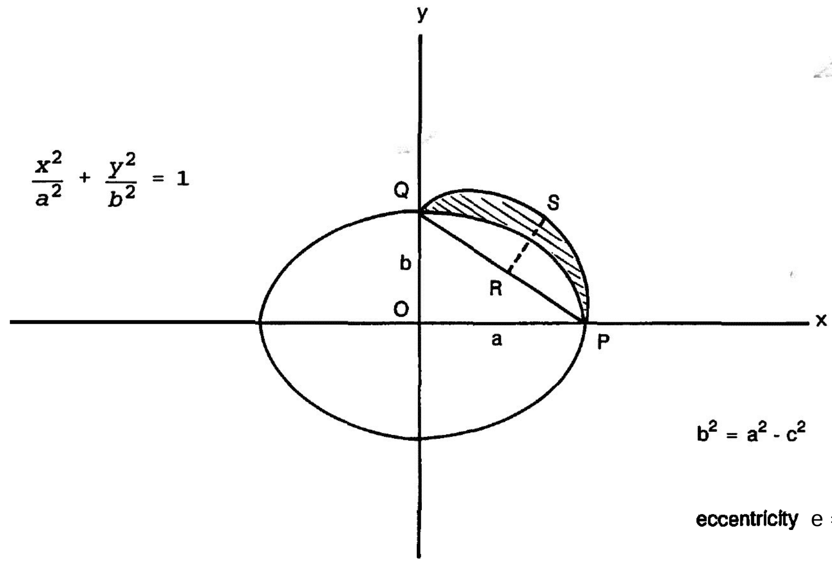
A LUNE OF **HIPPOCRATES**

The squaring problem associated with the lune of **Hippocrates gives** rise to an intriguing pursuit in the case for ellipses in general. Suppose thus that the two **circles** in the drawing **above** are replaced by **ellipses** of equal eccentricity. in particular, consider the ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the upper semi-ellipse on QP with the same eccentricity. **If** ΔPOQ

and the shaded lunar region ***PQ*** have the same area, is it true that each ellipse **is** a circle? The **powerful** tools of analytic geometry suffice in answering the question.

in the smaller **ellipse** of the figure **below,** let a', which is RP, be the **semi-major** axis and let b', which is RS, be the semi-minor axis.



if c is a focal segment of the larger **ellipse,** then

$$\frac{b^2}{a^2} = \frac{a^2 - c^2}{a^2} = 1 - \frac{c^2}{a^2} = 1 - e^2.$$

If c' is a **focal** segment of the smaller ellipse, then

$$\frac{(b')^2}{(a')^2} = \frac{(a')^2 - (c')^2}{(a')^2} = 1 - \frac{(c')^2}{(a')^2} = 1 - e^2$$

So $b/a = b'/a'$, or $a'b = ab'$.
The area of $\Delta POQ = (1/2)ab$. The area of the lune is:

$$\frac{\pi}{2} a' b' - \left[\frac{\pi}{4} ab - \frac{1}{2} ab\right].$$

Suppose that the two areas are **equal; i.e.,**

$$\frac{\pi}{2} a' b' - \left[\frac{\pi}{4} ab - \frac{1}{2} ab\right] = \frac{1}{2} ab$$

Then $(\pi/2)a'b' = (\pi/4)ab$, or $2a'b' = ab$. Note that:

$$\frac{2a'b'}{a'b} = \frac{ab}{ab'} \quad or \quad \frac{2b'}{b} = \frac{b}{b'}.$$

Thus,
$$\frac{b^2}{(b')^2} = 2 = \frac{a^2}{(a')^2}$$

or,
$$b = b'\sqrt{2}, \quad and \quad a = a'\sqrt{2}.$$

Accordingly,
$$\cos(\angle OPQ) = \frac{a}{\sqrt{a^2 + b^2}} = \frac{a'\sqrt{2}}{2a'} = \frac{\sqrt{2}}{2},$$

meaning that right **triangle** *POQ* **is** isosceles. Hence, b = a and *b'* = a' and the **ellipses** are circles. The eccentricity is of course zero in each case.

it is only in the circular case that lunar-triangular area equality hdds among ellipses. The resolution invdved an **easily** visualized relationship as well as the application of readily available formulas. Such a generaliiation and many more quite **like** it pose **challenging** and instructive activities in the **teaching-learning** situation of mathematics. Students may find it fascinating that the formula for the area of an ellipse reduces nicely to that for the area of a circle simply by letting a **equal** b.

The three famous problems of antiquity concern angle trisection, cube duplication, and circle squaring. Varied but unsuccessful attempts to sdve these problems contain an assortment of configurations and relationships which pose interesting generalizations. The **two-dimensional** generallzation involving iunes and conic sections fails into such a category. Spherical and eliipsoidai counterparts are areas of further exploration which the reader may wish to pursue.

References:

[1] H. W. Eves, An Introduction to the *History* of Mathematics, Saunders College Publishing, Philadelphia, **1990.**

[2] **R. L** Francis, "Just How Impossible **Is It?"** Journal of Recreational Mathematics, Voiume 20, Number 4, **1988.**

[3] **R.** L Francis, "Long-Ago Problems for the Student of **Today,"** *Missouri* Journal of *Mathematical* Sciences, Vdume 2, Number 2, **1990.**

WHAT ST. AUGUSTINE DIDN'T SAY ABOUT MATHEMATICIANS

Ralph P. Boas

Northwestern *University*

At about the time when I was becoming seriously interested in mathematics, I had a friend who was more interested in theology. He once confronted me with a warning by St. Augustine, who wrote, "A good Christian must beware of mathematicians and those **soothsayers** who make predictions by unholy methods, especially when their predictions come true, lest they ensnare the **soul** through association with demons.' (The original was, of course, in Latin **[1]**, and I have given a rather free translation; *the* original dates from around A.D. 400.) The same sentiment was repeated, in a somewhat different form, in Augustine's Confessions **[2]**, and it still gets quoted from **time** to time. (See, for **example, [3], p.167.)** If you happen to come across it, you should be aware that, in Augustine's day, **"mathematicians" meant** what we now **call "astrologers." [3].** The **old** usage seems to have occurred **occasionally** as recently as the **1700's,** although the modern meaning goes back to around 1400.

Notes:

[1] ***Quapropter*** bono christiano sive tnathematici sive ***quilibet inpie divinantium, maxime dicentes vera,*** cavendi sunt, ne consort10 ***daemoniorum animam deceptam pacto quodam societas*** inretiant. (De genesi ad litteram, Book 11, chap. **xvii,** in **vol.** 3, part 1 of Augustine's works, edited by J. **Zycha,** Prague, Vienna, and **Leipzig,** 1894, pp. 61-62.

[2] Book 4, chap. 1

[3] M. Greenberg, Euclidean and Non-Euclidean Geometries, *2nd ed.*, W. H. Freeman & Co., **1980.**

[4] Compare the modern **slang** use of "mathematician" to mean "card sharp."

Remember: the 1992 Pi Mu Epsilon **National** Meeting **will** be held August 6-8 at **Miami** University in Oxford, **Ohio.** See the Spring, **1992,** issue of this Journal for further information.

PUZZLE SECTION

Edited by Joseph D. E. Konhauser
Macalester College

The PUZZLE SECTION is for the enjoyment of those readers who are addicted to working doublecrostics or who find an occasional mathematical puzzle or word puzzle attractive. We consider mathematical puzzles to be problems whose solutions consist of answers immediately recognizable as correct by simple observation and requiring little formal proof. Material submitted and not used here will be sent to the Problem Editor if deemed suitable for the PROBLEM DEPARTMENT.

Address all proposed puzzles and puzzle solutions to Professor Joseph D. E. Konhauser, Mathematics and Computer Science Department, Macalester College, St. Paul, MN 55105. Deadlines for puzzles appearing in the Fall Issue will be the next March 1, and for the puzzles in the Spring issue will be the next September 1.

PUZZLES FOR SOLUTION

1. *Proposed by the Editor of the Puzzle Section.*

What is a property shared by all triangles which have numerically equal perimeters and areas?

2. *Proposed by Christian Bronzebach, St. Paul, MN.*

Can every positive integer greater than 6 be written as a sum of two relatively prime integers each greater than or equal to 2?

3. *Proposed by the Editor of the Puzzle Section.*

It is easy to square LINE:

L I N E
I D E A
N E A R
E A R N.

Are you able to square CIRCLE?

4. *Proposed by P. O. T. Week, Macalester College, St. Paul, MN.*

There are several ways of arranging the positive integers 1 through 15 into five sets of three members each so that the sums of the numbers in each of the five sets are equal. In all possible such arrangements, is there one in which one of the five sets is {7, 8, 9}?

5. *Proposed by P. O. T. Week, Macalester College, St. Paul, MN.*

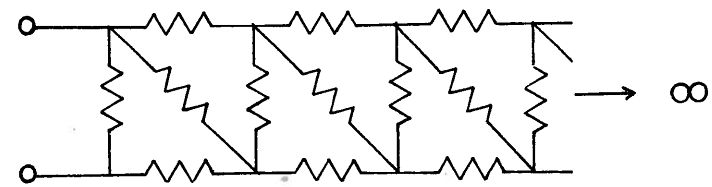
A rectangular candy bar is thinly coated with chocolate on all six faces. Cut the candy bar into three pieces so that each piece has the same volume and so that each piece contains the same amount of chocolate. Neglect the thickness of the chocolate.

6. *Proposed by Clark Kinnaird, Flemington, N. J.*

The vertices and the points of intersection of the sides of two overlapping triangles comprise a set of twelve points with four points on each of six lines. Are you able to find an essentially different configuration of twelve points with the same property?

7. *An Oldie Which Resists Permanent Banishment.*

An infinite number of unit resistors are connected in a network as shown below:



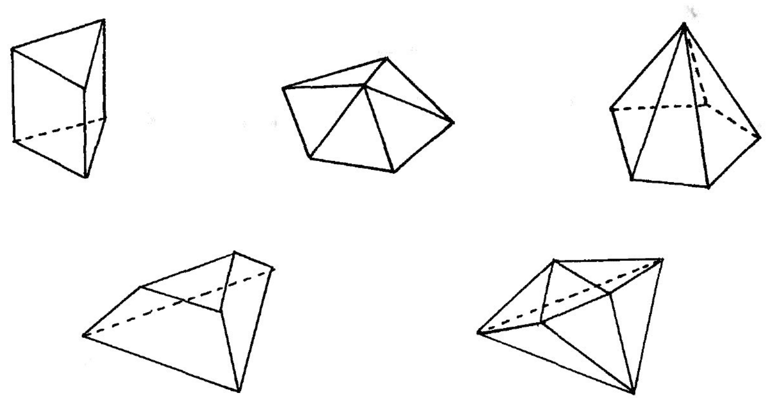
Calculate the resistance of the network.

COMMENTS ON PUZZLES 1-7, SPRING 1991

For Puzzle #1, a solution easily obtained by inspection is $a = 2^8$, $b = 2^6$ and $c = 2^5$. MARK EVANS and CHARLES ASHBACHER provided the family of solutions $a = 2^m$, $b = 2^n$, $c = 2^p$, where $3m = 4n = 5p - 1$, $p = 5 + 12k$, $k = 0, 1, 2, 3, \ldots$. RICHARD I. HESS supplied a different family of solutions. He said, "Pick x and y so that $z = x^5 - y^4$ is positive. Then $x^5 = y^4 + z$ implies $x^5z^{20} = y^4z^{20} + z^{21}$ whence $c^5 = a^3 + b^4$, where $c = xz^4$, $b = yz^5$ and $a = z^7$. Example, $3^5 = 2^4 + 227$ yields $[3\bullet 227^4]^5 = [2\bullet 227^5]^4 + [227^7]^3$." One respondent claimed that there were no solutions. For Puzzle #2, EMIL SLOWINSKI and the proposer, BASIL RENNIE, gave $\pi/2$ for the expected area, arguing that "each angle of the random spherical triangle is uniformly distributed in $(0,\pi)$ and, therefore, has expectation $\pi/2$, so

$$\text{area of triangle} = \text{sum of angles} - \pi = 3\pi/2 - \pi = \pi/2."$$

Proposer RENNIE gave the following second solution: "Take three points A, B and C and the diametrically opposite points A', B' and C'. Then the triangles ABC, $A'BC$, $AB'C$, ABC', $AB'C'$, $A'BC'$, $A'B'C$ and $A'B'C'$ just cover the sphere and so have total area 4π. Each is a random triangle, so 8 x (expectation of area) = 4π." CHARLES ASHBACHER provided a computer program that simulated the problem. EMIL SLOWINSKI and RICHARD I. HESS submitted solutions for Puzzle #3. In addition to the regular octahedron there are five other arrangements of six points in 3-space such that the distances between pairs of points fall just into two classes. These arrangements are sketched below.



The first is square-faced triangular prism. The remaining four are obtained by making judicious selections of six vertices of a **regular Icosahedron.** The Einhorn-Schoenberg paper in which the result of Puzzle **#3** appears as a special result is from the Proceedings, Series A, 69, No. 4 and Indag. Math., 28, No. 4, 1966 of the Koninkl. Nederl. Akademie van Wetenschappen - Amsterdam. For Puzzle #4, the correct response that the area of the octagon is one-sixth that of the square was submitted by EMIL SLOWINSKI, CHARLES ASHBACHER, ROBERT C. GEBHARDT and MARK EVANS. Contrary to the claims of some respondents, the octagon is <u>not</u> regular. The solution by ASHBACHER involved a little trig, those of EVANS and GEBHARDT some easy analytical geometry. The result generalizes to a parallelogram. A complete discussion of the generalization appears in Mathematical Problems and Puzzles by S. Straszewicz, Pergamon Press, 1965, where a synthetic treatment is given. Puzzle **#5** drew the following response from EMIL SLOWINSKI, CHARLES ASHBACHER, RICHARD I. HESS and MARK EVANS:

$$S = \{1, 4, 6, 7, 10, 11, 13, 16\} \text{ and } T = \{2, 3, 5, 8, 9, 12, 14, 15\}.$$

Sets S and T have identical **pairwise** sums. The result is a special case of a more general result, established by the method of generating functions, in 'On Some Two-way Classifications of Integers" by J. Lambek and Leo Moser, Can. Math. Bull., Vol. 2, No. 2, May 1959. No finite set $\{1, 2, 3, \ldots, n\}$ can be split into two classes such that the sets of **pairwise** products are the same. For Puzzle #6, EMIL SLOWINSKI provided an Initial arrangement of 31524. HESS provided that solution and four others. MARK EVANS and CHARLES ASHBACHER produced that solution and five others, namely 31452, 41532, 51423, 31542 and 41235. EVANS gave a kind of recipe for finding solutions beginning with the final arrangment. ASHBACHER provided a computer program which examined all 120 possible initial arrangements. Only HUGH L. PACKER and RICHARD I. HESS responded to Puzzle #7. PACKER'S analysis is too lengthy to reproduce here but his final result is "the number of triangles whose vertices are vertices of a regular polygon of $2k + 1$ sides and contain the center of the polygon is

$$1^2 + 2^2 + 3^2 + \ldots + k^2 = k(k + 1)(2k + 1)/6."$$

Editorial note. BOB PRIELIPP has pointed out that Puzzle #5, Fall, 1990, appeared as Problem 1339 in Mathematics Magazine and that a complete solution appeared on page **63** of the February, 1991, Vol. 64, No.1 issue of that journal. The problem is much older. This Editor used it as a Problem of the Week in March 1977. Unfortunately, he cannot remember the source.

Solution to Mathacrostic No. 32 (Spring 1991)

WORDS:

A.	Ideate	J.	Randomness	S.	Dymaxion
B.	Van Dyke	K.	Sweet potato	T.	Stella octangula
C.	Alpha helix	L.	One-eyes	U.	Olympus Mons
D.	Rift-sawed	M.	Newton	V.	Feigenvalue
E.	**Soma**	N.	**Iapetus**	W.	The Dot and the Line
F.	Primordial soup	O.	Snow job	X.	Rattailed
G.	Etruscan Venus	P.	**Loveknot**	Y.	Upshot
H.	Table of chords	Q.	Affined	Z.	The Scottish Book
I.	Epenthesis	R.	Natty	a.	Hatchetfish

AUTHOR AND TITLE: **IVARS** PETERSON ISLANDS OF TRUTH

QUOTATION: Studies show that chess experts look at only a handful of moves and evaluate deeply just a few of them. They tend to rely on an instantaneous perception of a chess posiiton as a whole. And the human mind's remarkable agility enables it to respond to expected situations. Computers don't have this kind of global view.

SOLVERS: THOMAS **F.** BANCHOFF, Brown University, Providence, **RI;** JEANETTE BICKLEY, St. Louis Community College at Meramec, MO; CHARLES **R. DIMINNIE,** St. Bonaventure University, NY; ROBERT FORSBERG, Lexington, MA; ROBERT C. **GEBHARDT, County** College of Morris, Randolph, NJ; MICHELE HEIBERG, Herman, MN; DR. **THEODOR** KAUFMAN, Brooklyn, NY; HENRY S. LIEBERMAN, Waban, MA; CHARLOTTE MAINES, Rochester, NY; and STEPHANIE SLOYAN, Georgian Court College, Lakewood, NJ.

Mathacrostic No. 33

Proposed by Joseph D. E. Konhauser

The 273 letters to be entered in the numbered spaces in the grid will be identical to those in the 30 keyed words at the matching numbers. The key numbers have been entered in the diagram to assist in constructing the solution. When completed, the initial letters of the Words will give the name and an author and the title of a book; the completed grid will be a quotation from that book.

Definitions	Words
A. a non-intrinsic property of a **set**	164 208 36 203 118 25 109 228
B. Raymond **Postgate's 1940 classic** of **its** genre **(3** wds.)	4 216 68 105 130 269 241 170 143 16 193 137 57 31 236
C. whitewash	218 263 240 131 162 44 206 180 192
D. curves with polar equation $r = a\sin n\Theta$ or $r = a\cos n\Theta$ **(3 wds.)**	136 250 231 155 50 72 88 24 9 272 129 146 256
E. spots of **light** that appear to encircle the moon Immediately before and after totality during a **solar** eclipse **(2 wds.)**	43 55 223 65 273 15 234 8 117 141 83
F. idle chatter **(slang; 2 wds.)**	34 85 175 125 221 186 148 254 96 144 237 69
G. magnitude; measure	181 13 1 211 52 152
H. a substance such as clay or cement for packing a **joint**	182 133 53 80
I. where hit recordings are found **(3 wds.)**	190 174 266 54 140 19 124 213 116 47 232
J. the absolute limit (slang)	224 233 95 12 132
K. attribute objective existence to **(rare)**	151 198 26 51 73 38
L. oblique **(comp.)**	235 77 11 39 2 103 134 168 108 212
M. in Scandinavian mythology, the progenitor of the giants	258 127 46 33
N. be irresolute **(3 wds.)**	158 91 179 268 168 194 22 226
O. That point equally near to heaven and to the infinite." Henry F. **Amiel**	3 188 81 106 242
P. pseudonym Director's Guild attaches to a film from which original director wants **his name** removed **(2 wds.)**	259 61 138 29 102 76 202 45 123 5 17
Q. devoid of bends or curves	171 264 119 40 257
R. what Broadway in the Times Square area is often **called (4 wds.)**	27 222 262 252 189 209 100 56 78 201 154 74 167 120 239 183
S. a shift **from** one to another	244 21 41 64 178 107
T. implication	97 35 82 176 122 10 187 66
U. **Viviani's** problem of perforating a hemispherical dome with four **equal** windows so **that the** residual surface **can** be squared **(2** wds.)	163 271 142 126 112 42 60 14 6 195 243 210 75 30 150 90
V. the Ether Wave Music instrument, played without being touched, invented in **1928**	249 205 114 153 161 139 28 214
W. field which offers methods of "seeing' what has been considered non-visualizable	185 220 110 32 196 204 253 49 229 238 121 70 23
X. tied **(2 wds.;** sometimes **comp.)**	[illegible] 255 [illegible] 104 156 [illegible] [illegible] 217 92 145
Y. move spirally	86 128 247 71 191 98 113 7 48
Z. a word of approximation used with a **whole** number and preceded by a hyphen	101 93 135
a. a continuous assemblage of points	147 84 37 270 58 215
b. virtually silent satire on the mechanical age and individual's plight therein, **1936 (2 wds.)**	149 230 261 87 159 18 200 62 169 245 115
c. first musical awarded the **Pulitzer** Prize for drama **(4 wds.)**	199 251 184 79 197 160 177 265 94 260 225
d. whimsical name for problem-solving approach which involves the stripping away of unnecessary detail (2 **wds.)**	59 207 111 267 219 173 246 227 165 67 89 157

1 G	2 L	3 O		4 B	5 P	6 U	7 Y	8 E	9 D		10 T	11 L		12 J
13 G	14 U	15 E	16 B	17 P	18 b	19 I	20 X		21 S	22 N	23 W		24 D	25 A
26 K	27 R	28 V	29 P	30 U		31 B	32 W	33 M	34 F		35 T	36 A	37 a	38 K
	39 L	40 Q	41 S	42 U		43 E	44 C	45 P		46 M	47 I		48 Y	49 W
50 D		51 K	52 G		53 H	54 I	55 E	56 R		57 B	58 a	59 d	60 U	
61 P	62 b	63 X	64 S	65 E	66 T		67 d	68 B	69 F	70 W	71 Y		72 D	73 K
	74 R	75 U	76 P	77 L		78 R	79 c	80 H	81 O	82 T		83 E	84 a	85 F
86 Y	87 b		88 D	89 d	90 U	91 N	92 X	93 Z		94 c	95 J	96 F	97 T	
98 Y	99 N	100 R	101 Z	102 P		103 L	104 X	105 B		106 O	107 S	108 L		109 A
110 W	111 d	112 U	113 Y	114 V	115 b		116 I	117 E	118 A	119 Q		120 R	121 W	122 T
123 P		124 I	125 F	126 U	127 M	128 Y	129 D	130 B	131 C	132 J		133 H	134 L	135 Z
136 D	137 B	138 P	139 V	140 I	141 E		142 U	143 B		144 F	145 X	146 D		147 a
148 F	149 b	150 U	151 K	152 G	153 V	154 R	155 D	156 X		157 d	158 N	159 b	160 c	
161 V	162 C	163 U	164 A	165 d	166 N	167 R	168 L		169 b	170 B	171 Q	172 X		173 d
174 I	175 F	176 T	177 c	178 S	179 N	180 C	181 G	182 H	183 R		184 c	185 W	186 F	187 T
	188 O		189 R	190 I	191 Y	192 C		193 B	194 N	195 U	196 W	197 c		198 K
199 c	200 b	201 R	202 P	203 A	204 W		205 V	206 C	207 d	208 A	209 R	210 U	211 G	212 L
	213 I	214 V	215 a		216 B	217 X	218 C	219 d	220 W	221 F	222 R	223 E	224 J	225 c
	226 N	227 d	228 A		229 W	230 b	231 D	232 I	233 J	234 E	235 L	236 B		237 F
238 W	239 R	240 C		241 B	242 O	243 U		244 S	245 b	246 d	247 Y	248 X	249 V	
250 D	250 c		252 R	253 W	254 F	255 X	256 D	257 Q	258 M		259 P	260 c	261 b	
262 R	263 C	264 Q	265 c	266 I	267 d	268 N	269 B	270 a		271 U	272 D	273 E		

PROBLEM DEPARTMENT

Edited by Clayton W Dodge
University of Maine

This department welcomes problems believed to be new and at a level appropriate for the readers of this **Journal.** Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that **will** assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

All communications should be addressed to C. W. Dodge, Math. **Dept.,** University of Maine, Orono, ME **04469.** Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly identified with name and address. Solutions to problems in this issue should be mailed by July 1, 1992.

PROBLEMS FOR SOLUTION

758. Proposed by Charles ***Ashbacher,*** Hiawatha, Iowa.

Solve this base ten alphametric which celebrates Leonhard **Euler's** contributions to graph theory:

$$\text{E} + \text{V} \bullet \text{GRAPH} = \text{EULER}.$$

759. Proposed **by** John ***E.*** Wetzel, University of Illinois, Urbana, Illinois.

Cali a plane arc ***speclal*** if it has length 1 and lies on one side of a line through its end points. Show that any speclal arc can be contained in an isosceles right triangle of hypotenuse 1.

760. Proposed by John E. ***Wetzel,*** University of Illinois, Urbana, Illinols.

Napoleon's theorem is concerned with erecting equilateral triangles outwardly on the sides of a given triangle ABC. Then DEF is the triangle formed by the third vertices of these equilateral triangles BCD, CAE, and ABF. **Lemoine** asked in 1868 if one can reconstruct triangle ABC when only triangle DEF is given. Shortly afterward, Keipert showed that the construction is to erect outward equilateral triangles ***EFX, FDY,*** and DEZ on triangle DEF, and then A, ***B,*** and C are the midpoints of the segments DX, *EY,* and FZ. His proof was quite tedious. Find a simple proof of Keipert's construction.

761. Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.

Determine all functions ***f(x)*** such that

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad \frac{1}{f(x)} = \sum_{n=0}^{\infty} (-1)^{n+1} a_n x^n.$$

762.** Proposed by Hao-Nhien ***Qui Vu, ***Purdue*** University, ***Lafayette,*** Indiana.

Following Cantor, we assume a list of the rationals in **[0,1)** can be made. Each rational is listed as a terminating decimal if possible, or as a repeating decimal. Thus numerals ending in nonterminating repeating 9's are not permitted. Define a new number x such that the kth place of x is **5** if the kth place in the kth number in the list is not **5,** and is **4** otherwise. So, for example, if the list starts with **0.5, 0.32,** 0.666666. then x = **0.455....** Show that the number x must be **irrational, and** therefore this process does not prove the rationals are not denumerable. Saying that x is **irrational** because the ratlonals are countable, however amusing, is not sufficient.

763. Proposed **by** Russell ***Euler,*** Northwest Missouri State University, ***Maryville,*** Missouri.

Find all real solutions to the equation $(x^2 - 7x + 11)^{x^2-11x+30} = 1.$

764. Proposed by William K. Delaney, S.J., ***Loyola Marymount*** University, Los Angeles, California.

Evaluate the indefinite integral $\int (x + 1) e^x \ln x \, dx.$

765. Proposed **by** the late Charles W Trigg, San ***Diego,*** California.

Find a square integer in base 4 that is a concatenation of two like integers.

766. Proposed by Murray ***S. Klamkin,*** University of Alberta, Edmonton, Alberta, Canada.

Determine $\dfrac{d^n}{dx^n}(x^n \ln^2 x)$ at $x = e$.

767. Proposed **by** J. L. ***Brenner, Palo*** Alto, California.

Let a_0 and a_1 be positive integers, and for $n \geq 2$, define

$$a_n = \frac{a_{n-1}^2}{a_{n-2}}.$$

For what choices of a_0 and a_1 will all the a_n be integers?

768. Proposed by the late Jack ***Garfunkel,*** Flushing, New York.

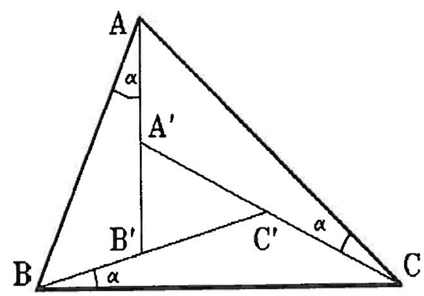
Given a triangle *ABC*, draw rays inwardly from each vertex to form a triangle $A'B'C'$ such that B', C', A' lie on rays AA', BB', CC', respectively, and

$$\angle BAB' = \angle ACA' = \angle CBC' = \alpha,$$

as shown in the figure. Prove that:

a) Triangle $A'B'C'$ is similar to triangle ABC.

b) The ratio of similitude is $\cos\alpha - \sin\alpha \cot\omega$, where ω is the Brocard angle of triangle *ABC*.



769. Proposed by ***R. S. Luthar,*** University of Wisconsin Center, ***Janesville,*** Wisconsin.

If ABC is a triangle in which $c^2 = 4ab \cos A \cos B$, Prove that the triangle Is isosceles.

770.** Proposed by Robert C. ***Gebhardt, Hopatcong, New Jersey.

A deck of cards, numbered from 1 to n, is dealt at random to n persons. Then a second similar deck Is dealt to the same n persons. What is the probability that at least one of the n persons received two cards with the same number?

SOLUTIONS

419. [Spring 1978, Spring 1979, Fall 1983] Proposed by Michael W Ecker, ***City*** University of New York.

Seventy-five bails are numbered 1 to 75 and are partitioned into sets of 15 elements each, as follows: $B = \{1, \ldots, 15\}$, $I = \{16, \ldots, 30\}$, $N = \{31, \ldots, 45\}$, $G = \{46, \ldots, 60\}$, and $O = \{61, \ldots, 75\}$, as in Bingo.

Balls are chosen at random, one at a time. **until** one of the following occurs: At **least** one from each of the sets B, I, G, O has been chosen, or four of the chosen numbers are from the set N, or **five** of the numbers are from one of the sets B, I, G, O.

Problem: Find the probability that. of these possible results, four N's are chosen first. (Comment: The result will be approximated by the situation of a very crowded bingo hall and will give the likelihood of what bingo players **call** "an N game," that is, bingo won with the winning line being the middle column N.)

Solution RICHARD 1 HESS, Rancho ***Palos*** Verdes, California.

Let Q_k denote the probability that from the first k draws exactly 4 are N's. Then we have

$$Q_1 = Q_2 = Q_3 = 0,$$

$$Q_4 = \frac{15\cdot 14\cdot 13\cdot 12}{75\cdot 74\cdot 73\cdot 72} = \frac{7\cdot 13}{5\cdot 37\cdot 73\cdot 6} = 0.001123040849,$$

$$Q_5 = \frac{5}{1}\cdot\frac{60}{71}\, Q_4 = 0.004745243024,$$

$$Q_6 = \frac{6}{2}\cdot\frac{59}{70}\, Q_5 = 0.01199868593,$$

$$Q_7 = \frac{7}{3}\cdot\frac{58}{69}\, Q_6 = 0.02353365453,$$

$$Q_8 = \frac{8}{4}\cdot\frac{57}{68}\, Q_7 = 0.03945347966,$$

$$Q_9 = \frac{9}{5}\cdot\frac{56}{67}\, Q_8 = 0.05935687686,$$

$$Q_{10} = \frac{10}{6}\cdot\frac{55}{66}\, Q_9 = 0.08244010675,$$

$$Q_{11} = \frac{11}{7}\cdot\frac{54}{65}\, Q_{10} = 0.1076251064,$$

$$Q_{12} = \frac{12}{8}\cdot\frac{53}{64}\, Q_{11} = 0.1336905619,$$

$$Q_{13} = \frac{13}{9}\cdot\frac{52}{63}\, Q_{12} = 0.1593912166,$$

$$Q_{14} = \frac{14}{10}\cdot\frac{51}{62}\, Q_{13} = 0.1835569817,$$

$$Q_{15} = \frac{15}{11}\cdot\frac{50}{61}\, Q_{14} = 0.2051680123,$$

$$Q_{16} = \frac{16}{12}\cdot\frac{49}{60}\, Q_{15} = 0.2234051690,$$

$$Q_{17} = \frac{17}{13}\cdot\frac{48}{59}\, Q_{16} = 0.2376774679.$$

Next we let P_k be the probability that the fourth N is drawn on the kth draw. Then $P_k = Q_k - Q_{k-1}$ and we have

$$P_1 = P_2 = P_3 = 0,\quad P_4 = 0.001123040849,$$
$$P_5 = 0.003622202175,\quad P_6 = 0.007253442908,$$
$$P_7 = 0.01153496860,\quad P_8 = 0.01591982513,$$
$$P_9 = 0.01990339720,\quad P_{10} = 0.02308322989,$$
$$P_{11} = 0.02518499965,\quad P_{12} = 0.02606545546,$$
$$P_{13} = 0.02570065475,\quad P_{14} = 0.02416576510,$$
$$P_{15} = 0.02161103063,\quad P_{16} = 0.01823715665.$$

Now let R_k designate the probabdity that another bingo does not occur before the kth draw, given that the fourth N does occur on the kth draw. Then

$$R_4 = R_5 = R_6 = R_7 = 1.$$

For R_8 we can have 3 N's and 4 other letters before the last N. There are

$$60\cdot 59\cdot 58\cdot 57$$

ways to pick the four other letters. The table **below** sorts these ways, where (n_1, n_2, n_3, n_4) represents the numbers of elements taken from the sets B, I, G, O in all possible orders.

	Frequencies (F)	Sample combination	Ways to pick letter types (M_1)	Permutations of letters (M_2)	Ways to pick letters (M_3)
(a)	(1,1,1,1)	BIGO	1	24	15^4
(b)	(2,1,1,0)	BBIG	12	12	$15^3\cdot 14$
(c)	(2,2,0,0)	BBII	6	6	$15^2\cdot 14^2$
(d)	(3,1,0,0)	BBBI	12	4	$15^2\cdot 14\cdot 13$
(e)	(4,0,0,0)	BBBB	4	1	$15\cdot 14\cdot 13\cdot 12$

The five row products add to $60 \cdot 59 \cdot 58 \cdot 57$. Rows (b) through (e) contribute to

$$R_8 = \frac{2 \cdot 7 \cdot 2081}{19 \cdot 29 \cdot 59} = 0.8961825959.$$

More generally, for R_k the frequencies (n_1,n_2,n_3,n_4) are all the cases where $n_1 + n_2 + n_3 + n_4 = k - 4$, each $n_i \le 4$, and $n_4 = 0$. The "Ways to pick letter types" column M_1 is $24/(f_0!f_1!f_2!f_3!f_4!)$, where f_i is the number of times the number i appears among the numbers n_1, n_2, n_3, n_4. The "Permutations of letters" column M_2 is $(k-4)!/(n_1!n_2!n_3!n_4!)$. Finally the "Ways to pick letters" column M_3 is given by

$$15^{f_4+f_3+f_2+f_1} \cdot 14^{f_4+f_3+f_2} \cdot 13^{f_4+f_3} \cdot 12^{f_4}.$$

For R_9 we can have 3 N's and 5 other letters before the last N. There are $60 \cdot 59 \cdot 58 \cdot 57 \cdot 56$ ways to pick the five other letters. The R_9 table is:

F	M_1	M_2	M_3
(2,2,1,0)	12	30	$15^3 \cdot 14^2$
(3,1,1,0)	12	20	$15^3 \cdot 14 \cdot 13$
(3,2,0,0)	12	10	$15^2 \cdot 14^2 \cdot 13$
(4,1,0,0)	12	5	$15^2 \cdot 14 \cdot 13 \cdot 12$

We get that $R_9 = 0.7382570980$.

For R_{10} we have 3 N's and 6 other letters before the last N. There are $60 \cdot 59 \cdot 58 \cdot 57 \cdot 56 \cdot 55$ ways to pick the six other letters. The R_{10} table is:

F	M_1	M_2	M_3
(2,2,2,0)	4	90	$15^3 \cdot 14^3$
(3,2,1,0)	24	60	$15^3 \cdot 14^2 \cdot 13$
(3,3,0,0)	6	20	$15^2 \cdot 14^2 \cdot 13^2$
(4,1,1,0)	12	30	$15^3 \cdot 14 \cdot 13 \cdot 12$
(4,2,0,0)	12	15	$15^2 \cdot 14^2 \cdot 13 \cdot 12$

We get that $R_{10} = 0.5688145101$.

For R_{11} there are 3 N's, 7 other letters, and $60 \cdot 59 \cdot 58 \cdot 57 \cdot 56 \cdot 55 \cdot 54$ ways. The R_{11} table is:

F	M_1	M_2	M_3
(3,2,2,0)	12	210	$15^3 \cdot 14^3 \cdot 13$
(3,3,1,0)	12	140	$15^3 \cdot 14^2 \cdot 13^2$
(4,2,1,0)	24	105	$15^3 \cdot 14^2 \cdot 13 \cdot 12$
(4,3,0,0)	12	35	$15^2 \cdot 14^2 \cdot 13^2 \cdot 12$

We get that $R_{11} = 0.4052514129$.

For R_{12} there are 3 N's, 8 other letters, and $60 \cdot 59 \cdots 53$ ways. The table is:

F	M_1	M_2	M_3
(3,3,2,0)	12	560	$15^3 \cdot 14^3 \cdot 13^2$
(4,2,2,0)	12	420	$15^3 \cdot 14^3 \cdot 13 \cdot 12$
(4,3,1,0)	24	280	$15^3 \cdot 14^2 \cdot 13^2 \cdot 12$
(4,4,0,0)	6	70	$15^2 \cdot 14^2 \cdot 13^2 \cdot 12^2$

We get that $R_{12} = 0.2642858755$.

For R_{13} there are 3 N's, 9 other letters, and $60 \cdot 59 \cdots 52$ ways. The table is:

F	M_1	M_2	M_3
(3,3,3,0)	4	1680	$15^3 \cdot 14^3 \cdot 13^3$
(4,3,2,0)	24	1260	$15^3 \cdot 14^3 \cdot 13^2 \cdot 12$
(4,4,1,0)	12	630	$15^3 \cdot 14^2 \cdot 13^2 \cdot 12^2$

We get that $R_{13} = 0.1540453916$.

Now R_{14} has 3 N's, 10 other letters, and $60 \cdot 59 \cdots 51$ ways. The table is:

F	M_1	M_2	M_3
(4,3,3,0)	12	4200	$15^3 \cdot 14^3 \cdot 13^3 \cdot 12$
(4,4,2,0)	12	3150	$15^3 \cdot 14^3 \cdot 13^2 \cdot 12^2$

We get that $R_{14} = 0.07611654643$.

Now R_{15} has 3 N's, 11 other letters, and $60 \cdot 59 \cdots 50$ ways. The table is:

F	M_1	M_2	M_3
(4,4,3,0)	12	11550	$15^3 \cdot 14^3 \cdot 13^3 \cdot 12^2$

We get that $R_{15} = 0.02968545311$.

Also R_{16} has 3 N's, 12 other letters, and $60 \cdot 59 \cdots 49$ ways. The table is:

F	M_1	M_2	M_3
(4,4,4,0)	4	24650	$15^3 \cdot 14^3 \cdot 13^3 \cdot 12^3$

We get that $R_{16} = 0.007269906884$. Finally, $R_k = 0$ for all $k > 17$.

Let $T_k = P_k R_k$ symbolize the probability that a fourth N is drawn on the kth draw and no other bingo has come earlier. Then we have

$$\begin{array}{ll} T_1 = T_2 = T_3 = 0, & T_4 = 0.001123040849, \\ T_5 = 0.003622202175, & T_6 = 0.007253442908, \\ T_7 = 0.01153496860, & T_8 = 0.01426707020, \\ T_9 = 0.01469382426, & T_{10} = 0.01313007610, \\ T_{11} = 0.01020625669, & T_{12} = 0.006888731715, \\ T_{13} = 0.003959067425, & T_{14} = 0.001839414581, \\ T_{15} = 0.0006415332365, & T_{16} = 0.0001325824307, \end{array}$$

so the probability of an N bingo is $\Sigma T_k = 0.08929221117$.

Editorial comment. In the Spring 1979 Issue it was stated that no solution to this problem had been received. The solution printed here was sent in March 1980. Somehow it became lost in the pile of material shipped to me when I assumed this column from Leon Bankoff. In cleaning out my old correspondence files earlier this year I ran across the letter containing this submission. So here, eleven years late and with the deepest apologies from the editor to Mr. Hess, is the solution to the bingo problem.

719 [Spring 1990, Fall 1990]. *Corrected.* Proposed by the late John M. Howell, Littlerock, California.

Professor E. P. B. Umbuglo translated Problem 626 [Fall 1986, Fall 1987] into Spanish, as shown below. Since he didn't like zeros because they reminded him of his score on an IQ test, he used only the nine nonzero digits. He found solutions in which **2** divides DOS. **3** divides TRES, and **6** divides SEIS. Find that solution in which also **7** divides SEIS and 9 divides DOS.

UNO + DOS + TRES = SEIS.

I. *Solution* by JEFFREY BOATS ***and*** MICHAEL A. VITALE, Saint Bonaventure University, Saint Bonaventure, New York.

For the addition to hold in the units column, the sum **of** O and ***S*** must equal **10.** Since DOS is divisible by **2,** then both O and ***S*** are even. There are then only two possibilities in which ***DOS*** Is divisible by **9: 846** and **864.**

To evaluate SEIS we note that any multiple of **1001** is divisible by 7, so the number EI must be divisible by **7.** Since ***D,*** O and ***S*** are **8.4** and **6,** then EI must **be** taken from **21, 35,** and **91.** Then SEIS is divisible by **3** only for **4914** and **6216.**

If DOS = 846 and SEIS = **6216,** then N = **4** and **4** is already taken. So we have DOS = 864 and SEIS = **4914.** Then N = 5, T = **3,** and finally ***U*** = **7** and ***R*** = **2.** The solution is

756 + 864 + 3294 = 4914.

II. Comment by Elizabeth Andy, Limerick, Maine.
To the editor:

My counsel to you is to shun
A change in a problem or function.
Your action precocious
Makes problems atrocious.
And then they don't have no solution.

Also solved ***by*** ALMA COLLEGE PROBLEM SOLVING GROUP, ***MI,*** CHARLES ASHBACHER, Hiawatha IA. SEUNGJIN BANG. Seoul, Korea, FRANK P. BATTLES. Massachusetts Maritime Academy. Buzzards Bay, GREGORY A. CANNON, Texas A and ***M*** University, College Station, **CAVELAND** MATH GROUP, Western Kentucky ***University,*** Bowling Green, KAREN L. COOK, Royal Palm Beach, FL, MARK EVANS, Louisville, ***KY,*** VICTOR G. FESER, University of Mary, Bismarck, ND, HOWARD **FORMAN,** Parsippany, NJ, S. **GENDLER,** Clarion University of Pennsylvania, RICHARD **I. HESS,** Rancho Palos Verdes, CA, NATHAN JASPEN, Stevens Institute of Technology, ***Hoboken,*** NJ, CARL **LIBIS,** Granada Hills, CA, MIKE **PINTER,** Belmont College, Nashville, TN, L. J. **UPTON,** ***Mississauga,*** Ontario, Canada. KENNETH **M.** WILKE, Topeka, KS, and the PROPOSER.

Many solvers sent proofs or comments that the original problem had no solution and then solved the corrected problem, too. As originally proposed, the problem had four solutions, so the editor made a 'slight change' to make the solution unique. Unfortunately, the editor's 'unique solution' used the digit 0, so all went for nought.

732. [Fall **1990]** Proposed by Alan Wayne, Holiday, Florida.
The following is a partially enciphered multiplication:

(*AY*)(*HARD*) = 21340.

Restore the digits. Of whom might it have been said that his mathematics was "AY HARD?"

I. Solution by KENNETH ***M.*** WILKE, Topeka, Kansas.
Since **21340** = $2^2 \cdot 5 \cdot 1197$, its only two-digit divisors that produce a four digit quotient are **10,** 11, and **20,** which produce the quotients **2134, 1940,** and **1067** respectively. The only possibility is AY = **10,** HARD = **2134,** and the mathematician is **21430** = HARDY.

II. Solution by the PROPOSER.
Let b **be** the base of numeration and let x denote the number HARD. Then

$$(Ab + Y)x = 2134 b.$$

It follows that ***Y*** = **0** and A = **1,** so x = **2134** in any base b **> 4** and **10·2134** = HARDY.

Also solved by ALMA COLLEGE PROBLEM SOLVING GROUP, ***MI,*** CHARLES ASHBACHER, Hiawatha, ***IA,*** SEUNGJIN BANG, Seoul, Korea, JEFFERY JOHN BOATS and MICHAEL A. **VITALE,** St. Bonaventure University, ***NY,*** MARTIN J. BROWN, Jefferson Community College, Louisville, ***KY,*** SCOTT H. BROWN, Stuart Middle School, ***FL,*** MARK EVANS, Loulsville, ***KY,*** VICTOR G. FESER, University of Mary, Bismarck, ND, HOWARD **FORMAN,** ***Parsippany,*** NJ, RICHARD **I.** HESS, Rancho Palos Verdes, CA, CARL **LIBIS,** Granada ***Hills,*** CA, MOHAMMAD P. SHAIKH, Western Michigan University, ***Kalamazoo,*** and DAVID S. SHOBE, New Haven, CT.

733. [Fall **1990]** Proposed by Roger ***Pinkham,*** Stevens Institute of Technology, ***Hoboken,*** New Jersey.

If $p(x)$ is a polynomial and $p(x) \geq 0$ for all x, then

$$p + p' + p'' + \dots \geq 0$$

for all x.

I. *Solution* by MURRAY ***S. KLAMKIN,*** University of Alberta, Edmonton, Alberta, Canada. Letting

$$f(x,a) = p/a + p'/a^2 + p''/a^3 + \cdots,$$

where a Is an arbitrary positive constant, it follows by integration by parts that

$$f(x,a) = e^{ax}\int_x^{\infty} e^{-at}p(t)\,dt = \int_0^{\infty} e^{-at}p(x+t)\,dt.$$

It now follows more generally that $f(x,a) \geq 0$ for all $a > 0$ and for all x. Also it follows that If $p(x) \geq 0$ for all $X \geq k$, then $f(x,a) \geq 0$ for all $X \geq k$.

Similarly, If $p(x) \leq 0$ for all $X \geq k$, then by considering

$$e^{-ax}\int_{-\infty}^{x} e^{at}p(t)\,dt = \int_{-\infty}^{0} e^{at}p(x+t)\,dt,$$

it follows that $p/a - p'/a^2 + p''/a^3 - \cdots \leq 0$ for all $x \geq k$. Furthermore, since $G(x) = f(x,1) \geq 0$ for all x, then $G + G' + G'' + \cdots \geq 0$, or

$$p + 2p' + 3p'' + 4p''' + \cdots \geq 0 \text{ for all } x.$$

In similar fashion we have

$$p + 3p' + 5p'' + 7p''' + \cdots \geq 0, \quad p + 4p' + 8p'' + 12p''' + \cdots \geq 0,$$

etc., for all x.

II. Solution by *ALMA* COLLEGE PROBLEM SOLVING GROUP, Alma College, Alma, Michigan.

Let $S(x) = p(x) + p'(x) + p''(x) + \cdots$, so that $S'(x) = p'(x) + p''(x) + p'''(x) + \cdots$. Then

$$p(x) = S(x) - S'(x) \geq 0, \quad \text{so that} \quad S(x) \geq S'(x) \text{ for all } x.$$

Since $p(x) \geq 0$, then its leading term must be of even degree and have a positive coefficient. Since this term is also the leading term of $S(x)$, then $\lim_{x\to\pm\infty} S(x) = +\infty$. Therefore, if there is any point x_0 such that $S(x_0) < 0$, the intermediate value theorem guarantees that $S(x)$ has at least two real roots a and b. Then, by **Rolle's** theorem, there is c $\in$ (a,b) such that $S'(c) = 0$ and $S(c) < 0$, which violates the statement that $S(x) \geq S'(x)$. The contradiction proves that $S(x) \geq 0$ for all x.

III. Comment submitted Independently by *Seung-Jin* Bang, Seoul, Republic of Korea, and Bob *Prielipp,* University of *Wisconsin-Oshkosh,* Oshkosh, Wisconsin.

This problem and two solutions, similar to those above but without the generalization, appeared in Mathematical Spectrum 1(**1968-1969)60.**

Also solved by **SEUNG-JIN** BANG, Seoul, Korea, G. G. BILODEAU, Boston College, Chestnut Hill, AM, HENRY S. **LIEBERMAN,** Waban, MA, and the PROPOSER.

734. [Fall **1990]** Proposed by Mohammad *K. Azarian,* University of *Evansville,* Evansville, Indiana.

Let f and g be two real-valued functions defined on the set of positive Integers with the following properties:

a) $f(1) = g(1)$ and $f(2) = g(2)$;

b) $f(n) > g(n)$ for $n \geq 3$;

c) there are Infinitely many pairs (m,n) such that $f(m) = g(n)$ and $m > n > 2$; and

d) $\lim_{n\to\infty} f(n) = \lim_{n\to\infty} g(n) = L$, a finite real number.

Show that there are **infinitely** many functions f and g satisfying these conditions and find formulas for them.

I. Solution by CHARLES ASHBACHER, Hiawatha, Iowa.

Let a, b, c be any real constants with $c > 0$. Define $f(1) = g(1) = a$, $f(2) = g(2) = b$, and for $n > 2$, let $f(n) = c/(n - j)$ and $g(n) = c/(n - i)$, where $0 \leq i < j \leq 2$ are integers.

II. Solution by RICHARD **I.** HESS, Rancho Palos Verdes, California.

Let a and b be any real numbers. Let $k = 1$ or 2 and choose any real number r such that $0 < r < 3 - k$. Define $f(1) = g(1) = a$, $f(2) = g(2) = b$, and for $n > 2$, let $f(n) = (n - k)/(n - k - r)$ and $g(n) = n/(n - r)$.

III. Solution by DAVID S. *SHOBE,* New Haven, Connecticut.

Let $F(x)$ be any monotone decreasing real function with finite limit L as $x \to \infty$. (Examples would include $F(x) = L + Bx^{-m}$ form, $B > 0$.) Let G be a function from the positive integers to the positive integers with $G(1) = 1$, $G(2) = 2$, and for $x > 2$, $G(x) > x$. Take $f(n) = F(n)$ and $g(n) = F(G(n))$.

IV. Solution by the PROPOSER.

Clearly, for any real number k, $f(n) = k + n^{1/n}$ and $g(n) = k + (n!)^{1/n!}$ satisfy the **given** conditions.

One other near ***solution*** was submitted. ***This*** editor tries to catch ***faulty*** solutions as they arrive and note any problems on a reply card. ***Unfortunately,*** when It arrived **I** ***did*** not check this one carefully enough to find its error. ***Sorry*** about that! Minor oversights are generally corrected without comment.

*735. [Fall **1990]** Proposed by Robert *C. Gebhardt,* Hopatcong, New Jersey.

If a and b are roots of the equation $x^2 + 7x - 3 = 0$, prove that

$$a^3 + b^3 + 7(a^2 + b^2) - 3(a + b) = 0,$$

and, without solving the equation, find the values of

(i) $$a^3 + b^3,$$

and

(ii) $$\frac{a+2}{b+1} + \frac{b+2}{a+1}.$$

This problem was taken from the Pure Mathematics section of the Intermediate Examinations in Engineering, Mining and Metallurgy, given by the University of London, November **1946.**

Solution *by* HOWARD ***FORMAN, Parsippany,*** New Jersey.

Since a and b are roots of the given equation, then we have

$$a^2 + 7a - 3 = 0 \quad \text{and} \quad b^2 + 7b - 3 = 0,$$

whence

$$a^3 + 7a^2 - 3a = 0 \quad \text{and} \quad b^3 + 7b^2 - 3b = 0,$$

and finally

$$a^3 + b^3 + 7(a^2 + b^2) - 3(a + b) = 0. \tag{1}$$

Since a and b are roots of the given equation, then we have

$$(x - a)(x - b) = x^2 - (a + b)x + ab = x^2 + 7x - 3,$$

which Implies that

$$a + b = -7 \quad \text{and} \quad ab = -3,$$

and also

$$a^2 + b^2 + 7(a + b) - 6 = 0, \quad \text{whence} \quad a^2 + b^2 = 55.$$

Substituting into Equation (1) above we find that

$$a^3 + b^3 = -406.$$

For Expression **(ii)** we get that

$$\frac{a+2}{b+1} + \frac{b+2}{a+1} = \frac{(a+2)(a+1) + (b+2)(b+1)}{(a+1)(b+1)}$$

$$= \frac{(a^2 + b^2) + 3(a+b) + 4}{ab + (a+b) + 1} + \frac{55 + 3(-7) + 4}{-3 - 7 + 1} + -\frac{38}{9}.$$

Also solved by ALMA COLLEGE PROBLEM SOLVING GROUP, *MI,* CHARLES ASHBACHER, Hiawatha, *IA,* SEUNGJIN BANG, Seoul, Korea, FRANK P. BATTLES, Massachusetts Maritime Academy, Buzzards Bay, MARTIN J. BROWN, Jefferson Community College, Louisville, *KY,* SCOTT H. BROWN. Stuart Middle School, *FL,* KAREN L. COOK. Royal Palm Beach, *FL,* RUSSELL EULER, Northwest Missouri State University, *Maryville,* GEORGE P. EVANOVICH, Saint Peter's College, Jersey City, *NJ,* VICTOR G. FESER, University of Mary, Bismarck, *ND,* SEAN FORBES, Drake University. Des *Moines,* IA, S. GENDLER, Clarion University of Pennsylvania, RICHARD I. HESS, Rancho Palos Verdes, CA, CARL LIBIS, Granada Hills, CA, HENRY S. UEBERMAN, *Waban,* AM, PETER A. LINDSTROM, North Lake College. living, *TX,* G. MAVRIGIAN, *Youngstown* State University, OH, YOSHINOBU MURAYOSHI, Eugene, OR, OXFORD RUNNING CLUB, University of Mississippi, University, MIKE PINTER, *Belmont* College, Nashville, TN, BOB PRIELIPP, University of Wisconsin-Oshkosh, GEORGE W. RAINEY, Los Angeles, CA, MOHAMMAD P. SHAIKH, Western Michigan University, *Kalamazoo,* WADE H. SHERARD, *Furman* University, *Greenville,* SC, DAVID S. SHOBE, New Haven, *CT,* and KENNETH M. WILKE, Topeka, KS.

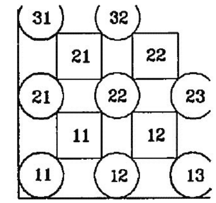
Lindstrom pointed out the caution that to obtain $a + b = -7$ *and* $ab = -3$, one must not solve the general quadratic, in violation of the conditions of the problem. The published solution answers this objection by showing how these equations are obtained without finding any solution.

***736.** [Fall **1990]** Proposed by Willie Yong, Singapore, Republic of Singapore.

Into a rectangle with sides 20 and 25 units, 120 squares are thrown, each with side 1. Show that inside the rectangle a circle of unit diameter may be drawn **which** does not intersect any of the squares. This is a 10th class problem from the 24th Mathematics Olympiad organized by Moscow State University, 1961.

Solution by MARK EVANS, Louisville, Kentucky.

Place the rectangle on the Cartesian plane with comers at **(0,0), (0,25), (20,25),** and **(20,0).** Then consider the region near the origin as shown **In** the accompanying figure. Intuitively, the arrangement shown, If continued throughout the rectangle, would provide the optimum arrangement of unit squares.



The lower left corner of square 11 is at $((2 + \sqrt{2})/4, (2 + \sqrt{2})/4)$ and Its lower right corner Is at $((6 + \sqrt{2})/4, (2 + \sqrt{2})/4)$, so the center of circle 12 is at $((3 + \sqrt{2})/2, 1/2)$. The distance between the centers of circles 11 and 12 is $1 + \sqrt{2}/2$. The number of circles that could fit along the x-axis Is given by n, where

$$25 = \frac{1}{2} + (n-1)\left(1 + \frac{\sqrt{2}}{2}\right) + \frac{1}{2}, \quad \text{so} \quad n = 15.1.$$

Hence we need more than 14 squares to separate these circles. Similarly we need more than 11 squares in the ydirection. Since more than 154 squares are required to separate the indicated circdes, 120 squares will not suffice.

737. [Fall **1990]** Proposed by Timothy Sipka, Alma College, Alma, Michigan.

The California lottery offers a daily card game called ***Decco,*** where a player selects 4 cards from a standard deck, one from each suit. It costs $1 to play, and prizes are awarded according to the number of cards that match the state's randomly selected set of four. One match gives a free replay ticket, two matches earn $5, three yield $50, and four matches produce $5000. Determine the expectation, the average profit or loss, for this game of chance.

Solution by S. GENDLER, Clarion University, Clarion, Pennsylvania.

The value v of the ticket is the sum of the products of the payoffs and their probabilities

$$v = 5000\cdot\left(\frac{1}{13}\right)^4 + 50\cdot 4\left(\frac{1}{13}\right)^3\left(\frac{12}{13}\right) + 5\cdot 6\left(\frac{1}{13}\right)^2\left(\frac{12}{13}\right)^2 + v\cdot 4\left(\frac{1}{13}\right)\left(\frac{12}{13}\right)^3,$$

which gives rise to

$$v = \frac{11720}{21649} = 0.54136$$

so the expectation $E = \$1 - v = -\$0.45864$ for each play of the game.

Also solved by ALMA COLLEGE PROBLEM SOLVING GROUP, *MI,* CHARLES ASHBACHER, Hiawatha, *IA,* MARK EVANS, Louisville, *KY,* BOB PRIEUPP (two solutions), University of Wisconsin-Oshkosh, and the PROPOSER.

By assuming the value of a ***replay*** ticket was Its cost of $1, instead of *$v,* the ***following*** solvers arrived at an expectation of *-$0.3476:* SCOTT H. BROWN, Stuart Middle School, *FL,* GEORGE P. EVANOVICH, Saint Peter's College, Jersey City, NJ, HOWARD **FORMAN,** Parsippany, NJ, RICHARD **I.** HESS, Rancho **Palos** Verdes, CA, MIKE PINTER, Belmont College, Nashville, TN, WADE H. SHERARD, **Furman** University, **Greenville,** SC, DAVID **S.** SHOBE, New Haven, **CT, and** TIMOTHY SIPKA, Alma College, **MI.**

MARTIN **BAZANT,** Tucson, AZ, arrived at an expectation of **-$0.2487** by assuming that the prizes are cumulative. That is, for example, if you get 3 matches, then you receive $50 plus $5 **plus** a free ticket.

738. [Fall **1990**] Proposed by Alan Wayne, Holiday, Florida.

If $[x]$ denotes the greatest integer less than or equal to x, prove that for any **nonnegative** integer n,

$$[n^{1/2} + (n + 1)^{1/2}] = [(4n + 1)^{1/2}].$$

Solution by BOB PRIELIPP, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin.

We shall show that, for every nonnegative integer n,

$$(*)\quad [n^{1/2} + (n + 1)^{1/2}] = [(4n + 1)^{1/2}] = [(4n + 2)^{1/2}] = [(4n + 3)^{1/2}].$$

Clearly (*) holds for $n = 0$. If a is an even integer, then a^2 is of the form $4j$, and if a is odd, then a^2 is of the form $4j + 1$. Thus there are no squares of the forms $4j + 2$ and $4j + 3$. Therefore,

$$(**)\quad [(4n + 1)^{1/2}] = [(4n + 2)^{1/2}] = [(4n + 3)^{1/2}].$$

Now we have, for $n > 0$, that

$$n^2 < n^2 + n < n^2 + 2n + 1,$$

$$n < (n^2 + n)^{1/2} < n + 1,$$

$$4n + 1 < n + 2(n^2 + n)^{1/2} + n + 1 < 4n + 3,$$

so

$$(4n + 1)^{1/2} < n^{1/2} + (n + 1)^{1/2} < (4n + 3)^{1/2}.$$

The theorem follows from Equation (**).

Also solved by SEUNGJIN BANG, Seoul, Korea, SCOTT H. BROWN, Stuart **Middle** School, **FL,** RICHARD **I.** HESS, Rancho Palos Verdes, **CA,** HENRY **S.** LIEBERMAN, Waban, MA, DAVID S. SHOBE, New Haven, **CT,** and the PROPOSER.

Bang found this problem in the form proved by Prielipp as Problem 5 of the 19th Canadian Mathematical **Olympiad.** Brown referred to the similar problem **E3010** in the American Mathematical Monthly **95(1988)**133.

739. [Fall **1990**] Proposed by ***R.*** S. **Luthar,** University of Wisconsin Center, Janesville, Wisconsin.

Solve the equation

$$\sqrt{x^3 + 2x^2 - 11x + 12} - \sqrt{x^3 + x^2 - 13x + 11} = x + 1.$$

I. Solution by MARK EVANS, Louisville, Kentucky.

Let

$$y = x^3 + x^2 - 13x + 11 \quad \text{and} \quad z = x + 1.$$

Then the given equation becomes

$$\sqrt{y + z^2} - \sqrt{y} = z.$$

Now square both sides of this equation and then simplify to get

$$y = \sqrt{y}\sqrt{y + z^2}, \qquad y^2 = y^2 + yz^2,$$

and

$$0 = yz^2.$$

Hence either

$$y = x^3 + x^2 - 13x + 11 = (x - 1)(x^2 + 2x - 11) = 0 \quad \text{or} \quad z = x + 1 = 0,$$

whose roots are 1, $-1 + 2\sqrt{3}$, $-1 - 2\sqrt{3}$, and -1.

Thus there are four possible solutions. However, $x = -1 - 2\sqrt{3}$ yields $2/3$ on the left hand side of the equation and $-2\sqrt{3}$ on the right. The solution then becomes

$$x = -1, 1, \text{ or } -1 + 2\sqrt{3}.$$

II. Comment by ***Jeffery*** Boats and Michael A. **Vitale,** Saint Bonaventure University, Saint Bonaventure, New York.

The November **1989** version of Mathematica lists the three valid roots and the extraneous one, too. Derive lists only the valid roots. Perhaps Mathematica does not check its roots.

Also solved by ALMA COLLEGE PROBLEM SOLVING GROUP, **MI,** CHARLES ASHBACHER, **Hiawatha, IA,** SEUNGJIN BANG, Seoul, Korea, FRANK P. BATTLES, Massachusetts Maritime Academy, Buzzards Bay, JEFFERY JOHN BOATS and MICHAEL A. **VITALE,** St. Bonaventure University, NY, MARTIN J. BROWN, Jefferson Community College, Louisville, KY, SCOTT H. BROWN, Stuart Middle School, FL, KAREN L. COOK, Royal Palm Beach, FL, RUSSELL EULER, Northwest Missouri State University, **Maryville,** GEORGE P. EVANOVICH, Saint Peter's College, Jersey City, NJ, VICTOR G. FESER, University of Mary, Bismarck, **ND,** SEAN FORBES, Drake University, Des Moines, IA, HOWARD **FORMAN,** Parsippany, NJ, S. GENDER, Clarion University of **Pennsylvania,** RICHARD **I.** HESS, Rancho Palos Verdes, CA, HENRY S. LIEBERMAN, Waban, **MA,** PETER A. LINDSTROM, North Lake College, Irving, TX, G. **MAVRIGIAN,** ***Youngstown*** State University, OH, YOSHINOBU MURAYOSHI, Eugene, OR, OXFORD RUNNING CLUB, University of

Mississippi, University, WILLIAM H. PEIRCE, *Stonington,* CT, BOB PRIEUPP, University of *Wisconsin-Oshkosh,* **MOHAMMAD** P. SHAIKH, Western Michigan *University,* Kalamazoo, WADE H. **SHERARD,** *Furman* University, Greenville, SC, DAVID **S.** SHOBE, New Haven, CT, KENNETH **M.** WILKE, Topeka, KS, and the PROPOSER.

Fifty lashes with a wet noodle go to 40% of the solvers, including the proposer, for not checking their solutions and listing *all* four values as roots. Most of the wayward solvers corrected that omission when it was pointed out to them.

740. [Fall **1990]** Proposed by *J. S.* Frame, Michigan State University. East *Lansing,* Michigan.

The Euler numbers E_j may be defined by the series

$$\sec x = \sum_{j=0}^{\infty} E_j \frac{x^{2j}}{(2j)!}.$$

The first few Euler numbers are

$E_0 = E_1 = 1$, $E_2 = 5$, $E_3 = 61$, $E_4 = 1385$, and $E_5 = 50521$.

Prove that, for all $j > 0$, the E_j satisfy the congruences

$$E_{2k+1} \equiv 1 + 60k \pmod{1440} \quad \text{and} \quad E_{2k+2} \equiv 5 - 60k \pmod{1440}.$$

Solution by the PROPOSER.

A set of generating functions for the E_j, namely

$$\sum_{j=1}^{n} (-1)^j \binom{2n}{2j} E_j = 0 \quad \text{for } n > 0; \qquad E_0 = 1,$$

Is obtained by multiplying the two **MacLaurin** series for sec x and cos x, and equating coefficients of $x^{2n}/(2n)!$ In the identity sec x cos x = **1.** We combine the two congruences into one as follows:

$$E_j \equiv 18 - 64\cdot\delta_{j0} + (-1)^j(47 - 30j) \pmod{1440}$$

and prove this congruence by mathematical induction on n by proving that $f(n) \equiv 0 \pmod{2^5 3^2 5}$ if we define

$$f(n) = \sum_{j=0}^{n} \binom{2n}{2j} [(-1)^j (18 - 64\cdot\delta_{j0}) + 47 - 30j],$$

$$f(n) = 9[(1+i)^{2n} + (1-i)^{2n}] - 64 + 94\cdot 4^{n-1} - 30n\sum_{j=1}^{n}\binom{2n-1}{2j-1},$$

$$f(n) = 9\cdot 2^n\cdot[i^n + (-i)^n] + 64(4^{n-1} - 1) + 30(1-n)\cdot 4^{n-1}.$$

We compute the values $f(1) = f(2) = f(3) = 0$, $f(4) = -1440 = -2^5 3^2 5$. **Clearly** 32 divides $f(n)$ when $n > 4$. We also have

$$f(n) \equiv (1+3)^{n-1} - 1 - 3(n-1)(1+3)^{n-1} \equiv 0 \pmod 9.$$

We note that $4^{n-1} \equiv 1 \pmod 5$ for odd n, and check that

$$f(m) \equiv -4^m \cdot 2(-1)^m + 1 - 4^{2m-1} \equiv -2 + 1 - 4 \equiv 0 \pmod 5.$$

Thus $2^5 3^2 5 = 1440$ divides $f(n)$, and the congruence is proved.

741. [Fall **1990]** Proposed by the late John *M. Howell,* Littlerock, California.

*a) What numbers cannot be a leg of a Pythagorean triangle?

*b) What numbers cannot be a hypotenuse of a Pythagorean triangle?

c) What numbers can be neither a leg nor a hypotenuse of a Pythagorean triangle?

Solution by KENNETH *M.* WILKE, Topeka, Kansas.

We write (a, b, c) to denote that a, b, c are positive integers and $a^2 + b^2 = c^2$. For part (a) we note that, when n is an odd integer greater than **1,** we have $(n, (n^2-1)/2, (n^2+1)/2)$ and, when $n > 2$ is even, then $(n, n^2/4 - 1, n^2/4 + 1)$. Hence any integer greater than 2 can serve as a leg of a Pythagorean triangle.

Suppose now that $x^2 = z^2 - y^2 = (z-y)(z+y)$ **with** $x = 1$ or 2. Since we are dealing with Integers, if $x = 1$, then $z - y = 1$ and $z + y = 1$, so $z = 1$ and $y = 0$. Therefore, there is no Pythagorean triple with a leg of length 1. If $x = 2$, then $4 = (z-y)(z+y)$. Now either $z - y = 1$ and $z + y = 4$ or $z - y = 2$ and $z + y = 2$. Neither of these sets of equations has a solution in positive Integers, so 2 cannot be a leg, either. For part (a), then, each integer greater than 2 and only such an integer can serve as a leg of a Pythagorean triangle.

Since the general form for all Pythagorean triples Is $(2pqt, t(p^2 - q^2), t(p^2 + q^2))$, where $p > q$ are relatively prime integers of opposite parity and t is a positive integer, then the hypotenuse must be a multiple of an odd positive integer that is expressible as a sum of two positive integral squares. An odd prime is so expressible if and only if it is of the form $4k + 1$ (see, for example, Burton, Elementary Number Theory, **Allyn** & Bacon, 1976, **p.264).** Hence, for part (b), if an integer z Is to be the hypotenuse of a Pythagorean triangle. **it** must contain at least one odd prime factor of the form $4k + 1$, the smallest of which are 5, 13, and 17.

For part (c) we combine the results of parts (a) and (b), finding that the only natural numbers that can serve as neither a leg nor a hypotenuse in some Pythagorean triangle are 1 and 2.

Also solved by CHARLES ASHBACHER, Hiawatha, *IA,* SEUNG JIN BANG, Seoul, Korea, and BOB PRIEUPP, University of Wisconsin-Oshkosh. Partial solutions were submitted by ALMA COLLEGE PROBLEM SOLVING GROUP. *MI,* and the PROPOSER.

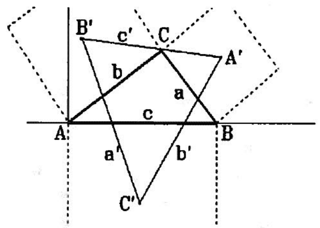
Prielipp found part (b) as a corollary in Sierpinski, Elementary Theory of Numbers, *Hafner,* 1964, *p.* 361.

***742.** [Fall **1990]** Proposed by the late Jack *Garfunkel,* Flushing, New York.

Construct squares outwardly on the sides of a triangle ABC. Prove or disprove that the centers A', B', and C' of these squares form a triangle that is closer to being equilateral than Is ABC. A proof would show that if the process were repeated on triangle A'B'C', etc., that triangle $A^n B^n C^n$ would approach equilateral as n approached infinity.

I. Solution by RICHARD 1 HESS, Rancho *Palos Verdes,* California.

Label the triangle ***ABC*** so that $a \leq b \leq c$ and place Cartesian coordlnates so that $A(0,0)$, $B(1,0)$, and $C(x,y)$ with $x \geq 1/2$. Then also $x^2 + y^2 \leq 1$. See the figure. In the square on side ***BC*** let the vertex opposite B have the coordlnates (u,v). Then $v - y = 1 - x$ and $u - x = y - 0$, from which it follows that A' has coordinates $((x + y + 1)/2, (-x + y + 1)/2)$. Slmllatiy, $B'((x - y)/2, (x + y)/2)$ and $C'(1/2, -1/2)$.



Now $c^2 - a^2 = 1 - (x - 1)^2 - y^2 = 2x - x^2 - y^2$. Also

$$4a'^2 = (x - y - 1)^2 + (x + y + 1)^2 = 2x^2 + 2y^2 + 2 + 4y,$$

$$4b'^2 = (x + y)^2 + (2 - x + y)^2 = 2x^2 + 2y^2 + 4 - 4x + 4y,$$

$$4c'^2 = (1 + 2y)^2 + (1 - 2x)^2 = 4x^2 + 4y^2 + 2 + 4y - 4x,$$

from whlch it follows that $a' \; a \; b' \; a \; c'$ since $2 - 4x \leq 0$ and $2 \; a \; 2x^2 + 2y^2$. Then

$$4(a'^2 - c'^2) = 4x - 2x^2 - 2y^2 = 2(c^2 - a^2),$$

so the difference between the squares of the longest and shortest sides Is halved with each formation of another triangle. Hence, as $n \to \infty$, that difference becomes zero and the triangle becomes equilateral.

II. Solution by MURRAY *S. KLAMKIN,* University of Alberta, Edmonton, Alberta, Canada.

Take the origin at the centroid of the given triangle and let a_0, b_0, c_0 be the complex numbers representing Its vertices. Let a^n, b^n, c^n be the affixes of the nth triangle formed by the process. We then have

$$a_{n+1} = \{b_n + c_n + i\lambda(b_n - c_n)\}/2$$

where here $\lambda = 1$. However, we **will** let λ be any positive number, whlch corresponds to constructing similar **isosceles** triangles instead of squares outwardly on the sides of the triangles and taking the three new vertices of these triangles as our next triangle. Then

$$a_{n+1} = r b_n + s c_n, \tag{1}$$

$$b_{n+1} = r c_n + s a_n, \tag{2}$$

$$c_{n+1} = r a_n + s b_n, \tag{3}$$

where $r = (1 + i\lambda)/2$ and $s = (1 - i\lambda)/2$. Slnce $r + s = 1$, it follows that

$$a_n + b_n + c_n = a_0 + b_0 + c_0 = 0,$$

so the new triangle has the same centroid as the original one. Next replace c_n by $-a_n - b_n$ In (1) and (2) to get

$$(r - s)b_n = a_{n+1} + s a_n, \tag{4}$$

$$(s - r)a_n = b_{n+1} + r b_n. \tag{5}$$

Eliminate b_n between these two equations to get

$$a_{n+2} + a_{n+1} + \{rs + (r - s)^2\}a_n = 0. \tag{6}$$

We search for a representation of the form $a_n = x^n$. Slnce $rs + (r - s)^2 = (1 - 3\lambda^2)/4$, the characteristic equation for (6) is

$$4x^2 + 4x + 1 - 3\lambda^2 = 0,$$

whose roots are $r_1 = (-1 + \lambda\sqrt{3})/2$ and $r_2 = (-1 - \lambda\sqrt{3})/2$. Hence

$$a_n = \alpha r_1^n + \beta r_2^n.$$

Because a_0 is known and $a_1 = r b_0 + s c_0$, we have

$$a_n = \{(r b_0 + s c_0 - r_2 a_0)r_1^n - (r b_0 + s c_0 - r_1 a_0)r_2^n\}/(r_1 - r_2).$$

Then

$$-a_n(r_1 - r_2)/(r b_0 + s c_0 - r_1 a_0)r_2^n - 1,$$

assuming the coefficient of $r_2^n \neq 0$. If it is, we then use the r_1^n term whlch cannot also vanish. Then from (4),

$$-b_n(r_1 - r_2)/(r b_0 + s c_0 - r_1 a_0)r_2^n \sim (r_2 + s)/(r - s) = (-1 + i\sqrt{3})/2$$

Independently of λ. Consequently, a_n, b_n, c_n are asymptotic to the vertices of an equilateral triangle.

Note that the isosceles triangles can be constructed inwardly instead of outwardly with the same result. Also, if $\lambda = 0$, the result Is false since each of the new triangles formed **is** similar to the original triangle. For a closely related result, see the proposer's problem 1179, Crux *Mathematicorum* 14 (1988) 22-24.

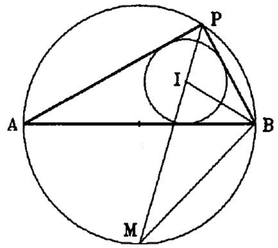
Also solved *by* HENRY S. LIEBERMAN, Waban, MA.

743. [Fall **1990**] ***Proposed*** by *R.* S. *Luthar,* University of Wisconsin Center, Janesville, Wisconsin.

Let A and B be the ends of the diameter of a semicircle of radius r and let P be any **point** on the semicircle. Let I be the **incenter** of triangle ***APB.*** Find the locus of I as P moves along the semicircle.

Solution by HENRY S. ***LIEBERMAN,*** Waban, Massachusetts.

Since PI bisects angle ***APB***, it passes through the midpoint ***M*** of the opposite semicircle ***AB***. See the figure.



Because angie ***MIB*** is an exterior angie to triangle ***IBP***, PI and ***BI*** are angle bisectors, and angles ***APM*** and ***ABM*** are inscribed in the same arc AM, we have that

$$\measuredangle MIB = \measuredangle IBP + \measuredangle IPB = \measuredangle IBA + \measuredangle API = \measuredangle IBA + \measuredangle APM = \measuredangle IBA + \measuredangle ABM = \measuredangle IBM.$$

Thus triangle ***MIB*** is isosceles and $MI = MB = r\sqrt{2}$. The locus of **I** is thus the arc *AB* of the circie with center ***M*** and radius $r\sqrt{2}$.

More generally, if *AB* is not a diameter of the **circumcircle,** the locus of 1 is still the arc AB of the circle with center ***M*** and radius of length MB.

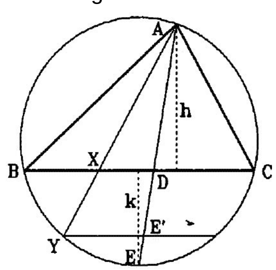
Also solved by CHARLES ASHBACHER, Hiawatha, ***IA,*** SEUNGJIN BANG, Seoul, Korea, RICHARD 1 HESS, Rancho ***Palos*** Verdes, CA, and the PROPOSER.

744. **[Fall 1990]** Proposed by the late Jack ***Garfunkel,*** Flushing, New York.

Let triangle ABC be inscribed in a circle. Draw a **line** through A to intersect side ***BC*** at ***D*** and the circle (again) at E. Without resorting to the calculus, prove that AD/DE is a minimum when AD bisects angie A.

I. Solution by HENRY S. ***LIEBERMAN,*** Waban, Massachusetts.

Let the line ***ADE*** be the bisector of angle ***BAC,*** so that E is the midpoint of arc BC. Let X be any point other that ***D*** on side BC and **let** AX cut the circie again at ***Y***. Draw the chord through ***Y*** that is **parallel** to ***BC*** to cut *AE* at E'. See the figure.



From similar triangles *AXD* and AYE' we get that $AX/XY = AD/AE'$, which is a **minimum** when AE' is a maximum, that is. when $E' = E$. Hence the bisector of angle A **produces** the **minimum** ratio.

II. Solution by *AL* T. ***TUDE, Veazie Heights, Maine.***

As shown in the figure above, let h and k be the altitudes from A and from E to BC. By similar right triangles, $AD/DE = h/k$, which is a minimum when k is a **maximum** since h is fixed, that is, when E is the midpoint of arc BC. But then, AE bisects angle A.

Also solved by RICHARD 1 HESS, Rancho Pales Verdes, *CA,* MURRAY S. **KLAMKIN,** University of Alberta, Canada, **MOHAMMAD** P. **SHAIKH,** Western Michigan ***University, Kalamazoo,*** *and* the PROPOSER.

The Pi Mu Epsilon Journal was founded in 1949 and is dedicated to undergraduate and beginning graduate students interested in mathematics. Submitted articles, announcements, and contributions to the Puzzle Section and Problem Department of the Journal should be directed toward this group.

Undergraduates and beginning graduate students are urged to submit papers to the Journal for consideration and possible publication. Student papers are given top priority. Expository articles by professionals in all areas of mathematics are especially welcome. Some guidelines are:

1. Papers must be correct and honest.
2. Most readers of the Pi Mu Epsilon Journal are undergraduates; papers should be directed to them.
3. With rare exceptions, papers should be of **general** interest.
4. Assumed definitions, concepts, theorems, and notations should be part of the average undergraduate curriculum.
5. Papers should not exceed 10 pages in length.
6. Figures provided by the author should be camera-ready.
7. Papers should be submitted in duplicate to the Editor.

THE 1991 NATIONAL PI MU EPSILON MEETING

The Annual Meeting of the Pi Mu **Epsllon** National Honorary Mathematics Society was held at the University of Maine in Orono, August 7 through 9. The meeting was held in conjunction with the national meetings of the American Mathematical Society and the Mathematical Association of America.

The J. Sutherland Frame Lecturer was Henry **O. Pollak,** Bell Communications Research and Teacher's College of Columbia University. He presented "Some Mathematics of Baseball."

The Pi Mu Epsilon Council at its annual meeting, voted to held the 1992 Pi Mu Epsilon National Meeting from August 6 through August 8 at Miami University in Oxford, Ohio. (See note on page 277.) The Council also approved the refunding of registration fees and the cost of the Pi Mu Epsilon Banquet for student speakers. The Council made this decision In the hopes of encouraging even more students to speak at the future national meetings. There were 45 student presentations at the meeting.

PROGRAM – STUDENT PAPER SESSIONS

History: From Newton to Point Set Topology	Rebecca Adams Ohio Delta Miami University
Characterizing Finite Groups That Are The Union of a Few Subgroups	Jonathan Atkins Indiana Gamma Rose-Hulman Institute of Technology
En Route to Steiner Triple Systems	Jonathan Atkins Indiana Gamma
Approximating e^{At} Using Chebyshw Polynomials	Jim **Baglama** Ohio Xi Youngstown State University
Global Climate Modeling	James **M. Banoczi** Ohio Xi Youngstown State University
Metaphors for Mathematics	Carol **Brennan** New York Upsilon **Ithaca** College

Easy Calculations	Hester Brothag Ohio Xi Youngstown State University
Rearrangements of the Alternating Harmonic Series	James F. Burke, Jr. Illinois **Iota** Illinois Benedictine College
The **Allometry** of Richard **III**	Sharyn Campbell Ohio Xi Youngstown State Univers'ty
Zeno's Arrow Paradox: The Disproof	**Buffy Cashell** Ohio Delta Miami University
Dimitros Drives You Nuts	Dimitros Chalop Ohio Xi Youngstown State Univers'ty
On Maximizing Directed **k-cycles** in n-tournaments	John Davenport Ohio Delta Miami University
Numerical Simulations of Solvent Penetration into Glassy Polymers	Joseph M. **Deitzel** **E.Von Merrwall** Ohio Nu Akron University
A New Approach to the Feynman Path Integral: Vector-Valued Danieii Type Integrals	Anthony F. **DeLia** Florida Theta Univers'ty of Central Florida
Graphs with Isomorphic Cycle and Cocycle Matroids	**Concetta DePaolo** Massachusetts Alpha Worcester **Polytechnical** Institute
Change Ringing	Heather **DeSimone** Ohio Xi Youngstown State University
Some Combinatorial Results Arising From Complete Digestion of Proteins	Matthew Dalby Jennifer Miners Arkansas Beta Hendrix College

A Note on the Intermediate Value Theorem	Mark Dobner Illinois Iota Elmhurst Cdlege
On Kronecker's Theorem Concerning the Fastest Euclidean Algorithm	Paul **Dufresne,** Jr. Massachusetts Gamma Bridgewater State College
Statistical Process Control for the Department of Defense	Joseph Fousek Wisconsin Alpha Marquette University
Tricks and Traps in the Use of Pseudo-Random Number Generators	Mike Fuller Ohio Nu Akron University
A Survey of the Theory of Coxeter Groups	Francis Y.C. Fung Kansas Beta Kansas State University
Approximation for the Unknown Parameter of a Distribution Based on a Random Sample Taken from It	Anna **Georgieva** Ohio Iota **Denison** University
Bits **'n** Bytes	Sandra **Gestl** Wisconsin Delta St. Norbert College
Quality Control: A Necessary Revolution	Nancy Leigh Griffin Ohio Delta Miami University
Anatomy of a Perfect Pitch	Linda Hughes Ohio **Xl** Youngstown State University
Pushing the Pebbling Number Over the Edge	David **Jessup** Ohio Zeta University of Dayton
Hyperbolic Trigonometry	Barry E. Jones Ohio Delta Miami University
Just An Average Integral	Amy **Krebsbach** Wisconsin Delta St. Norbert College

Singular Value Decay in the Numerical Inversion of the **Weierstrass** Transform	Mark Kust Michigan Epsilon Western Michigan University
The **Schrödinger** Equation and the Hydrogen Atom	Michael **Lang** Wisconsin Delta St. Norbert College
Universal Binary Quadratic Forms	Albert J. Lee **California** Iota University of Southern California
Subgroups of Finite Abelian Groups and Hasse Diagrams of Representative Subgroups	Jae S. Lee Tennessee Gamma Middle Tennessee State University
The Data Encryption Standard: Description and Analysis	Ted J. **Mallo** Ohio Nu University of Akron
Numerical Approximation of the Solution to the One-Dimensional Thermoelastic Equation	Kathleen A. **McTavish** Minnesota Eta University of Minnesota-Duluth
Statistical Testing of Random Number Generators	Peter Morris South Dakota Beta South Dakota School of Mines and Technology
Problem Solving and Paper Towels	Linda **Mueller** Wisconsin Delta St. Norbert College
The Markov Process in Clinical Decision Making	Marguerite **Nedreberg** Ohio Xi Youngstown State University
The Relationships Between Game Theory and Pseudo-Boolean Functions	David J. **Rader,** Jr. Virginia Alpha University of Richmond
On the Chain Rule	Ursula **Sallinger** Louisiana Alpha Louisiana State University
Symmetries of Generalized Mandeibrot Sets and Their Julia Sets	Xiang Sheng North Carolina Delta East Carolina University

Chaos in Number Theory: Extending the 3n+1 Problem	Joshua **Tempkin** Virginia Beta Virginia Polytechnic Institute
On the Division Properties of the Fibonacci Numbers	Daniel Viar Arkansas **Alpha** University of Arkansas
Deterministic Fractals in Mathematics	Kirk Wallace New York Upsilon ithaca Coilege
Pearls, Sham Pearls and D.U.D.E.N.E.Y.	Marc Wallace **Missouri** Beta Washington University-St. Louis

ATTENTION FACULTY ADVISORS

To have your chapter's report published, send copies to Robert M. Woodside, Secretary-Treasurer, Department of Mathematics, East Carolina University, Greenville, NC 27858 and to Richard L. **Poss,** Editor, St. Norbert College, De Pere, **WI** 54115.

Message from the Secretary-Treasurer

Copies of the new, revised Constitution and Bylaws are now availabie. The prices are: $1.50 for each of the first four copies and $1 for each copy thereafter. **I.e., $(1.50** n) for **n** < 4 and $(n + 2) for **n** ≥ 4.

The videotape of Professor Joseph A. **Gallian's** AMS-MAA-PME Invited Address, The Mathematics of Identification Numbers," given as part of **PME's** 75th Anniversary Celebration at Boulder, CO, in August, 1989, is also now availabie. The tape may be borrowed free of charge by PME chapters, and by others upon an advance payment of $10. Please contact my office if you desire to borrow the tape, telling me the date on which you would **like** to use **it. I** prefer to mail the tape directly to faculty advisors, and expect them to take responsibility for returning **it** to my office. Please submit your request in **writing** and include a phone number and a **tlme** that **I** might reach you **if** there are problems. Robert M. Woodside, Secretary-Treasurer, Department of Mathematics, East Carolina University, Greenville, NC 27858.

For the third consecutive year, the American Mathematical Society has given Pi Mu Epsilon a grant to be used as prize money for excellent student presentations. There were many excellent presentations, and eight of them were singled out to receive prizes of $100 each. The winning speakers were:

Anthony F. **DeLia,** University of Central Florida
A New Approach to the **Feynman** Path Integral: Vector-Valued Daniel1 Type Integrals

Heather **DeSimone,** Youngstown State Univers'ty
Change Ringing

Mark Dobner, Elmhurst College
A Note on the Intermediate Value Theorem

Mike Fuller, Akron University
Tricks and Traps in the Use of Pseudo-Random Number Generators

Linda Hughes, Youngstown State **University**
Anatomy of a Perfect Pitch

Marguerite Nedreberg, Youngstown State University
The Markov Process in Clinical Decision Making

Joshua **T. Tempkin,** Virginia Polytechnic institute
Chaos in Number Theory: Extending the **3n+1** Problem

Marc Wallace, Washington University - St. Louis
Pearls, Sham Pearls and **D.U.D.E.N.E.Y**

PI Mu Epsilon wishes to thank the American **Mathematical** Society for its continued support for student speakers.

GLEANINGS FROM THE CHAPTER REPORTS

FLORIDA KAPPA (University of West Florida) This is one of the newest chapters In PI Mu Epsilon. (It was installed In early 1990). Rene Hawkins represented the chapter at the 1990 summer meeting in Columbus, Ohio. Most of the activities have focused on building membership. These efforts paid off with 13 new members being initiated at the April Induction banquet.

GEORGIA BETA (Georgia Institute of Technology) At the 1991 Honors Program at Georgia Tech, the chapter presented book awards to outstanding graduates in mathematics. Those receiving the awards were students earning the degree of B.S. in Applied Mathematics with a grade point average of at least 3.9 (out of 4) in all mathematics courses taken. The recipients were Cheri L. Gatland-Llightner, Steven F. Grondin, and David G. Grupta. The awards were mathematics books of the recipient's choice.

MICHIGAN ZETA (University of Michigan - Dearborn) We continued our Focus on Faculty series for the second year. We had three faculty members present lectures as part of this series. Lectures were presented on the following topics: intermediate differential equations, classical and statistical tolerance limits and the mathematics software MATHEMATICA. As a special presentation, one of our student members presented an informal discussion which was titled "Introduction to Quasi-Empiricism". Another project which we engaged in this year was our math advising session. We sponsored two of these sessions, one each semester. We had three professors from the mathematics department representing three areas including mathematics, computers, and statistics. These professors discussed possible careers in these fields, graduate school opportunities, and basic course advising for University of Michigan-Dearborn. They also held a question and answer period at the end of the session. This event was open to all students in the university. We sponsored two faculty/student mixers, in December and April. The April mixer was a faculty thank-you luncheon where the students of Pi Mu Epsilon were able to thank the faculty members for their support over the past year. On a social level, we had two game nights with pizza and refreshments. One of the nights included a mathematics quiz bowl. We also had a Christmas party for our members over winter break and sponsored a hayride and bon fire in the fall.

MINNESOTA DELTA (St. John's University and College of St Benedict) This year the Math Society had a variety of speakers and events, with speakers in fields ranging from actuaries to computer science, and events such as math volleyball and the science picnic. In the fall Ed Banach, Vice President and Chief Actuary of North American Life and Casualty Company, came and spoke about his actuarial profession. Professor Jerry Lenz of the math department at St. John's University also spoke about his field of interest, the history of women in mathematics, and Gary Brown, a math professor at the College of St. Benedict, spoke on the Axiom of Choice. Two students, Bob Hesse and Judy Kenney, gave talks about their summer research programs which they presented at the Pi Mu Epsilon Conference at St. Norbert College. Hesse studied numerical methods at the University of Colorado and Kenny studied conformal mapping in the complex plane at Washington University in St. Louis. Once again, the Math Society had a cookie bake sale in the fall to raise money for the Christmas Party, when we went bowling. The spring was filled with events also. We had two outside speakers, three faculty speakers, and two student speakers. Cheri Shakibahn from St. Thomas University spoke on the computer program Mathematics, and Mike Heroux from Cray Computers came to speak about numerical analysis and computers. Three math professors from the College of St. Benedict and St. John's University, Ben Collins, Dave Hartz, and Mike Tangredi, each gave talks in the spring. The two student speakers were Bob Hesse and Mike Witham, each preparing for the Pi Mu Epsilon Conference at St. John's University. Hesse spoke about the probability of winning in the game Jai-Alai, and Witham talked about fractals. The featured speaker at the Pi Mu Epsilon Conference was Raymond Smullyan, the logician and philosopher. He gave two talks entitled "Puzzles & Paradoxes" and "Logic and Infinity". For the fundraiser, Math Society sold T-shirts, which were a big success. For the second year all of the sciences had combined picnic; once again, the Math Society "womped" the other sciences at volleyball. Math Society had a very successful year in 1990-91!

NEW YORK PHI (Potsdam College of the SUNY) Paula Golding was elected the 1991 PI Mu Epsilon award winner by the members of the chapter. The election was based on contributions to the Mathematics Department. The award consists of $100 in mathematics books of her choice.

NEW YORK OMEGA (St. Bonaventure University) The Chapter held five meetings this year. These were devoted to planning activities, selecting new members, and electing officers for next year. We have been working in close cooperation with the St. Bonaventure University Student Chapter of the MAA in trying to offer a broad range of activities for mathematics students. This year's program consisted of a bi-weekly forum featuring speakers from our own faculty, our second annual Mathematics Awareness Week celebration, and a problem competition run in conjunction with this celebration. Once again, we were able to Invite a speaker from Syracuse University as part of their visiting lecturers program. Steven Diaz provided a stimulating "Points and Curves - An Introduction to Geometry." We were able to schedule five talks in the forum series: 'The Mystique of Mersenne Numbers", Charles R. Diminnie; "Inclusion-Exclusion", Douglas L. Cashing; 'Taxi-cab Geometry", Francis C. Leary; "Maximums and Minimums: A Variational Approach", Harry Sedinger; These talks were aimed at an undergraduate audience and were well received. We are looking forward to running the forum series again next year. Mathematics Awareness Week activities included a film on the work of M.C. Escher, a talk by Jeffrey Boats on the use of complex analysis in the evaluation of certain real integrals, Sedinger's talk and the Pi Mu Epsilon ceremony. The problem competition offered a cash prize of $25 to the solver of the following problem:

If $S = \{(x,y) : |x| \le 1 \text{ and } |y| \le 1\}$, evaluate $\iint_S e^{\max\{x,y\}}\, dA$

The winning solution belonged to sophomore, Rochelle Ellis.

TENNESSEE GAMMA (Middle Tennessee State University) Pi Mu Epsilon began the 1990-91 year with its semi-annual pizza party on October 1. We had another pizza party/initiation again in the spring semester. On October 22, Mike Pinter from Belmont College gave a presentation entitled "Graphs in the Plane and on a Torus." Dr. Tom Cheatham, chairperson of the MTSU Department of Computer Science, spoke on "Paradox Lost" on November 19. For our March 11 meeting, we had a panel discussion on graduate school. Dr. James Lea gave insight into the advantages and disadvantages of the Ph.D., Ed.D., and D.A. programs; Dr. Vatsala Krishnamani spoke about choosing an appropriate graduate school; Mr. Kevin Shirley talked about choosing an area of study and a related topic; and Ms. Dovie Kimmins discussed math education. On April 16, Ms. Lora Brewer from the MTSU Department of Mathematics and Statistics gave a presentation on the highlights of her dissertation. During February, Pi Mu Epsilon sold T-shirts with a math oriented design to students and faculty. Later, the shirts were available for sale at the Tennessee Mathematics Teachers Association high school math contest. For Mathematics Awareness Week, four students presented their undergraduate research projects on April 25. Amy Pinegar, last year's winner, presented a continuation of her paper entitled "Inversions and Adjacent Transpositions." For this year's Tom Vickery Mathematics Project competition, Jae Lee (first place) gave his paper entitled "Subgroups of Finite Abelian Groups and Hasse Diagrams of Representative Subgroups." Carol Clifton (second place) presented her project, "Some Operations on Matrix Valued Expressions." Michael Darrell discussed his paper entitled "An Introduction to Cryptography." The week (and year) ended with a combined BBQ picnic with members of the other two mathematics clubs and ACM.

Sixth Annual
MORAVIAN COLLEGE
STUDENT MATHEMATICS CONFERENCE
Bethlehem, Pennsylvania
Saturday, February 15,1992

We invite you to join us, whether to present a talk or just to listen and socialize. **The** conference will begin at **9:00** a.m. and continue into late afternoon. After an invited address, the remainder of the **day will** be devoted to **undergraduate student** talks. Talks may be fifteen or thirty minutes long. They may be on any **topic** related to mathematics, operations research, statistics or computing. **We** encourage students doing research or honors work to present their work **here.** We also welcome expository talks, talks about interesting problems or **applications** and talks about interships, field studies and summer **employment.** We need your title, time of presentation (15 or 30 minutes) and **a 50** word (approximate) abstract by February 7,1992.

Sponsored by the **Moravian College Chapter of Pi Mu Epsilon** and the **Lehigh Valley Association of Independent Colleges,**

Please contact: Alicia **Sevilla,** Department of Mathematics, Moravian College
1200 Main St. Bethlehem, PA 18018
(Telephone: (215) 867-1787)

NINTH ANNUAL ROSE-HULMAN CONFERENCE ON UNDERGRADUATE MATHEMATICS

Friday and Saturday
March 13 and 14, 1992

You are cordially invited to attend this event. Undergraduate students are encouraged to submit abstracts of papers, in any area of the mathematical sciences, for presentation.

Invited Speakers

William Dunham
Professor of Mathematics
Hanover College

Joseph **Gallian**
Professor of Mathematics
University of Minnesota - Duluth

William Dunham is widely recognized for his research in the history of mathematics and as an enthusiastic speaker and writer. He has conducted an NEH Summer Seminar on mathematics in historical context and is author of the book ***Journey Through Genius - The Great Theorems of Mathematics.***

Joseph Gallian is a distinguished expositor of mathematics. His publications include the textbook ***Contemporary Abstract Algebra,*** and he has directed a successful undergraduate research summer program at UMD for many years.

For more information contact: Steve Carlson
Department of Mathematics
Rose-Hulman Institute of Technology
Terre Haute, IN 47803
(812) 877-8458
CARLSON@ROSEVC.Rose-Hulman.edu

St. John's University/College of St. Benedict
Annual Pi Mu Epsilon Student Conference

JUDITH GRABINER

Professor of Mathematics
Pitzer College

"Descartes and Problem Solving"
Friday, April 3, 1992
8:00 p.m.

'How Did We Come to Live in a Non-Euclidean World"
Saturday, April 4, 1992
10:00 a.m.

The PI Mu **Epsilon** Conference serves **as** a forum for undergraduates to present original mathematics and/or papers of an expository nature. Student **talks** precede the guest speaker both days.

Judith Grabiner is a historian of mathematics. She has written two books on the history of calculus. Origins of *Cauchy's **Rigorous*** **Calculus** and ***Calculus as*** Algebra- ***J.L. Lagrange 1736-1813.***

For more information contact; Mike **Zielinski**, Shoba **Gulati**, or Jerry Lenz, Department of Mathematics, St. John's **University, Collegeville, MN 56321,** Phone **612-363-3094**

ERRATA

There were two errors in Volume 9, Number 4, of the Pi Mu Epsilon Journal (spring, 1991).

In Russell **Euler's** article "A Note on $(1 + k/n)^n$," on page 233, there is a reference to the parameter k **before** it is given a definition.

In Norman Schaumberger's article "Using the MVT to Complete the Basic Integration Formula," on page 226, several lines of print were permuted. The middle part of the article should have read:

Furthermore, the relation

$$\ln\left(\frac{b}{a}\right) = \int_a^b x^{-1}\,dx\,, \quad b > a > 0 \qquad (3)$$

can readily be derived from (2). Equation (1) is still meaningless when n = -1, but (3) does suggest that it is reasonable to expect that the expression

$$\frac{1}{n+1}\left(b^{n+1} - a^{n+1}\right)$$

approaches $\ln(b/a)$ as n tends to -1. This point, although rarely discussed in standard texts, can be made plausible by considering values of n close to -1. Thus, for example,

$$\int_2^3 x^{-.999}\,dx = \frac{1}{.001}\left(3^{.001} - 2^{.001}\right) = .4058\ldots.$$

and $\ln(3/2) = .4054\ldots$.

Both of the above errors were the sole responsibility of the Editor. The Editor apologizes for any confusion that the errors may have caused.

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