PI MU EPSILON JOURNAL

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Τὴν παίδευσιν καὶ τὰ μαθηματικὰ ἐπισπεύδειν

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Τὴν παίδευσιν καὶ τὰ μαθηματικὰ ἐπισπεύδειν

Editorial correspondence, including books for review, chapter reports, news items and manuscripts (two copies) should be mailed to PI MU EPSILON JOURNAL. Richard L. Poss, EDITOR. St. Norbert College, 100 Grant Street. De Pere, WI 54115; E-Mail Address possrl@sncac.snc.edu; FAX # 414-337-4098. Students submitting manuscripts are requested to identify their college or university and their class or expected graduation date. Others are requested to provide their affiliation, academic or otherwise.

Problems for solution and solutions to problems should be mailed directly to the PROBLEM EDITOR. Puzzle proposals and puzzle solutions should be mailed to the EDITOR.

The PI MU EPSILON JOURNAL is published at St. Norbert College twice a year–Fall and Spring. One volume consists of five years (10 issues) beginning with the Fall 19x4 or Fall 19x9 issue, starting in 1949. For rates, see inside back cover.

THE RICHARD V. ANDREE AWARDS

Richard V. Andree was, until his death in 1987, Professor Emeritus at the University of Oklahoma. Professor Andree had served Pi Mu Epsilon in many capacities: as President, as Secretary-General, and as Editor of the Pi Mu *Epsilon Journal.* The Council has designated the prizes in the National Student Paper Competition as Richard V. Andree Awards.

First prize winner for 1992 is Nataniel Greene, for his paper "Fractorial!" which appeared in the fall issue of the *Journal.* Nataniel prepared this paper while he was a junior at Carmel High School in Carmel, NY. He is currently enrolled at Yeshiva University. Nataniel will receive $200.

Second prize winner is Michael Lin, for his paper "Rings of Small Order," which appeared in the spring issue. Michael prepared his paper while he was a senior at Moorhead Senior High School, in Moorhead, MN. He now attends Stanford University. Michael will receive $100.

Third prize winner is Mark Lancaster, for his paper "On the Number of Invertible Matrices Over $\mathbf{Z}_{p^e}$," which appeared in the fall issue. Mark prepared his paper while he was a senior at Hendrix College; his work on related topics continued into the following summer at the University of Tennessee (Knoxville) with Dr. David E. Dobbs as his advisor. Mark will receive $50.

There were three other student-written papers that appeared in 1992:

"Exploring Self-Duality in Graphs," the result of joint research between Concetta DePaolo and Russell Martin during the National Science Foundation's Research Experience for Undergraduates Program, which was held at Worcester Polytechnic Institute in the summer of 1991. At that time, Concetta had been a student at Worcester Polytechnic Institute and Russell a student at Syracuse University. Both authors are currently in graduate school: Concetta at Rutgers and Russell at Clemson.

'Change Ringing: Mathematical Music," by Heather DeSimone, of Youngstown State University. She is currently attending graduate school at the College of William and Mary.

"On Transpositions Over Finite Fields," by Beth Miller, of Pennsylvania State University – New Kensington Campus. Beth prepared the paper under the supervision of Professor Javier Gomez-Calderon.

The current issue of the *Journal* contains four papers with student authors:

"Some Operations on Matrix-Valued Expressions," by Carol Clifton of Middle Tennessee State University. Carol completed this paper during her senior year under the direction of Dr. Kevin Shirley.

"Outerplanar Graphs and Matroid Isomorphism," by Jeremy M. Dover while he was a student at Worcester Polytechnic Institute. He currently attends graduate school at the University of Delaware.

"Uniform Embeddings of Graphs," by James R. Murphy and Mohammed P. Shaikh while they were students at Western Michigan University.

"Intrinsic Reaction Coordinate Methodologies: Comparative Analyses," by Lisa Pederson (while a student at North Dakota State University) and Kim Baldridge (on the staff of the San Diego Supercomputer Center). Lisa is currently a graduate student in chemistry at Johns Hopkins University.

SOME OPERATIONS ON MATRIX-VALUED EXPRESSIONS

Carol Clifton
Middle Tennessee State University

Consider the matrix equation $aX + bI = O$, where I is the identity matrix, O is the zero matrix, and X has four variable entries, x_{ij}, for $i, j = 1, 2$. We can solve for x_{ij} in the following manner:

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \frac{1}{a}\begin{pmatrix} -b & 0 \\ 0 & -b \end{pmatrix}. \tag{1}$$

By (1), we obtain $x_{11} = -b/a, x_{12} = 0, x_{21} = 0, x_{22} = -b/a$. As we will see, solving equations with matrix-valued expressions will involve performing operations on these expressions. To solve the linear equation above, for example, we apply the operation $\frac{1}{a}(X - bI)$ to both sides of the equation. However, that method will only work when the operation is defined for suitable matrix-valued expressions. To see where some difficulty might occur, we need only try to solve a quadratic equation. We can begin to solve the second degree matrix equation $aX^2 + bX + cI = O$, for x_{ij}, $i, j = 1,2$ by completing the square.

$$X^2 + \frac{b}{a}X = \frac{-c}{a}I$$

$$X^2 + \frac{b}{a}X + \frac{b^2}{4a^2}I = \frac{-c}{a}I + \frac{b^2}{4a^2}I$$

$$\left(X + \frac{b}{2a}I\right)^2 = \frac{b^2 - 4ac}{4a^2}I \tag{2}$$

It would now be desirable to take the square root of each term in (2). However, if $b^2 - 4ac \# 0$, we first need to investigate the square root of I. If $b^2 - 4ac = 0$, we need to investigate the square root of the zero matrix.

The square root of I should be a matrix, A, such that $A^2 = I$. If

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then

$$\begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We obtain the following equations from the above equality:

$$a^2 + bc = 1 \tag{3}$$

$$ac + cd = 0 \tag{4}$$

$$ab + bd = 0 \tag{5}$$

$$bc + d^2 = 1 \tag{6}$$

We may consider two distinct cases. First, assume that $c = 0$. From (3), $a = \pm 1$. From (6), $d = \pm 1$. From (5), $a = -d$ when $b \# 0$, and the solutions are

$$\begin{pmatrix} 1 & b \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & b \\ 0 & 1 \end{pmatrix}. \tag{7}$$

Now assume that $b = 0$. The solutions become:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The last two of these solutions are contained in the solutions in *(7)*.

Now we may assume that $c \# 0$. From (4), $a = -d$. From (3), $a = \pm\sqrt{1 - bc}$. So, the solutions are

$$\begin{pmatrix} \sqrt{1-bc} & b \\ c & -\sqrt{1-bc} \end{pmatrix}, \quad \begin{pmatrix} -\sqrt{1-bc} & b \\ c & \sqrt{1-bc} \end{pmatrix}. \tag{8}$$

Notice that the solutions in (7) are contained in *(8)*.

So, the solutions of the square root of I are the following:

$$\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} \sqrt{1-bc} & b \\ c & -\sqrt{1-bc} \end{pmatrix}, \begin{pmatrix} -\sqrt{1-bc} & b \\ c & \sqrt{1-bc} \end{pmatrix}\right\}. \tag{9}$$

Thus, the square root of I has an infinite number of solutions, where I is the principal root.

A square root of the zero matrix should be a matrix which satisfies the equation

$$A^2 = O. \tag{10}$$

From *(10)* and properties of determinants, it is clear that the $\det A = 0$. From (10), we obtain the following system of equations:

$$a^2 + bc = 0 \tag{11}$$

$$ac + cd = 0$$

$$ab + bd = 0$$

$$bc + d^2 = 0 \tag{12}$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Since $\det A = 0$, we know $ad = bc$. So, *(11)* and *(12)* become $a(a + d) = 0$ and $d(a + d) = 0$, respectively. By adding *(11)* and (12), we obtain $a = -d$, or $\operatorname{tr} A = 0$. Conversely, we can show that $\det A = 0$ and $\operatorname{tr} A = 0$ implies that $A^2 = O$. Thus, we can parametrize the solutions to *(10)* in the following way:

$$\left\{\begin{pmatrix} a & b \\ c & -a \end{pmatrix} : bc = -a^2\right\} \tag{13}$$

Now that we know the solutions for $A^2 = I$ and $A^2 = O$, we may solve the equation $aX^2 + bX + cI = O$. When $b^2 - 4ac \neq 0$, then

$$X = \frac{-b}{2a}I \pm \frac{\sqrt{b^2 - 4ac}}{2a}\sqrt{I},$$

where $\sqrt{I}$ is an element of the solution set in (9). When $b^2 - 4ac = 0$, then

$$X = \frac{-b}{2a}I \pm \sqrt{O},$$

where $\sqrt{O}$ is an element of the solution set in *(13)*. The fact that in complex algebra a polynomial equation of degree n in a single unknown x has exactly n solutions, therefore, does not hold true in" matrix algebra.

After studying the square root of I, one may now want to investigate the square root of A, where $A \neq I, O$. First notice that for a diagonal matrix D = diag$(d_1, d_2, \ldots, d_n)$, where diag$(d_1, d_2, \ldots, d_n)$ means that $d_1, d_2, \ldots, d_n$ are entries along the principal diagonal and only zeros are elsewhere, $D^{1/2}$ = diag$(\sqrt{d_1}, \sqrt{d_2}, \ldots, \sqrt{d_n})$. Recall that if $\lambda \in \mathbb{C}$ (the set of complex numbers) and there exists a nonzero vector x such that $Ax = \lambda x$, then λ is said to be an eigenvalue for the matrix A, and x is an eigenvector corresponding to λ. In certain cases, an $n \times n$ matrix can be factored by using its eigenvectors. That is, $A = PDP^{-1}$, where D is a diagonal matrix where the eigenvalues of A are placed along the principal diagonal each according to its multiplicity, and P is a matrix whose columns are eigenvectors appearing in the same order as their corresponding eigenvalues appear on the diagonal of D. If $A = PDP^{-1}$, then A is said to be diagonaliiable. If A is a matrix with complex entries, then the **adjoint** of A is the conjugate transpose of A, given by $A^* = \overline{A}^t$. Note that $P^* = P^{-1}$ for the matrix which diagonalizes a matrix A. A matrix U with the property that $U^* = U^{-1}$ is said to be unitary. It can be shown that every unitary matrix can be diagonalized. If $A = PDP^{-1}$, then we can define what is meant by the principal square root of A. Notice that if $A = PDP^{-1}$, then $A^n = PD^nP^{-1}$ for any $n \in \mathbb{Z}^+$. One can see that $A = (PD^{1/2}P^{-1})(PD^{1/2}P^{-1})$. It follows that $A^{1/2} = PD^{1/2}P^{-1}$. It is interesting to note that $A^{1/2}$ does not exist for just any matrix A. For example,

$$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

has no square root, as seen below.

Proposition **1.** If det A = 0 and tr$A$ = 0, then X^2 = A has no solution for a 2×2 matrix $A \neq O$.

Proof: Suppose there exists a matrix X such that $X^2 = A$ where not all $x_{ij} = 0$, $i, j = 1, 2$. We have

$$\begin{aligned} x_{11}^2 + x_{12}x_{21} &= a_{11} \\ x_{11}x_{12} + x_{12}x_{22} &= a_{12} \\ x_{11}x_{21} + x_{21}x_{22} &= a_{21} \\ x_{12}x_{21} + x_{22}^2 &= a_{22}. \end{aligned}$$

However, $(\det X)^2 = \det A = 0$, Thus, $x_{12}x_{21} = x_{11}x_{22}$. So, we have

$$x_{11}(x_{11} + x_{22}) = a_{11} \tag{14}$$
$$x_{12}(x_{11} + x_{22}) = a_{12}$$
$$x_{21}(x_{11} + x_{22}) = a_{21}$$
$$x_{22}(x_{11} + x_{22}) = a_{22}. \tag{15}$$

By adding (14) and (15), we see that $(x_{11} + x_{22})^2 = a_{11} + a_{22} = 0$. Therefore, tr$X$ = 0 and A is the zero matrix as seen from the four equations above, a contradiction. A characterization of a matrix A for which the equation $X^2 = A$ has a solution is given in texts, particularly [4] (Lancaster, p. 95). From the discussion above, it is sufficient for the equation $X^2 = A$ to have a solution if A is diagonalizable. The question arises: "Which matrices are diagonalizable?"

Besides unitary matrices, another category of diagonalizable matrices is the collection of **Hermitian** matrices, named after the French mathematician Charles **Hermite** (1822-1901). A square matrix A is called Hermitian provided that $A = A^*$. In the real case, a Hermitian matrix is said to be symmetric.

Theorem **1.** If A is **Hermitian,** then A is diagonalizable. (See Hohn, p. 472.)

The proof can be found in most texts of linear algebra. In proving this theorem, one finds that the eigenvalues of a Hermitian matrix are necessarily real. Also, eigenvectors corresponding to

different eigenvalues are orthogonal. If $A = A^*$, $(\lambda_1 \ldots \lambda_n)$ are the eigenvalues corresponding to A repeated as often as their multiplicity, and $(v_1, \ldots, v_n)$ is the corresponding set of eigenvectors, then we can recover A as follows:

$$A = \sum \lambda_k v_k \overline{v}_k^t = PDP^*.$$

A less common topic in elementary linear algebra is simultaneous diagonalization. Two $n \times n$ matrices A and B are said to be simultaneously diagonalizable if they have a **common set of** eigenvectors which diagonalize both A and B; (i.e., $A = PDP'$ and $B = PD'P^*$). The following theorem about the simultaneous diagonalization of two Hermitian matrices is useful.

Theorem 2. Let A and B be $n \times n$ Hermitian matrices. Then, AB = BA if and only if there exists a linearly independent set of vectors $\{v_k\}_{k=1}^n$ such that $Av_k = a_k v_k$ and $Bv_k = b_k v_k$ for $k = 1, \ldots, n$.

Any matrix A can be decomposed into its real and imaginary parts by defining

$$X = \frac{1}{2}(A + A^*) \quad \text{and} \quad Y = \frac{1}{2i}(A - A^*).$$

It is easy to see that

$$A = X + iY. \tag{16}$$

This is analogous to writing a complex number z as $z = x + iy$, where $x, y \in \mathbb{R}$. From (16) we compute

$$A^*A - AA^* = 2i(XY - YX). \tag{17}$$

The following theorem characterizes matrices which can be diagonalized,

Theorem 3. (Spectral Theorem) An $n \times n$ matrix A can be diagonalized if and only if $A^*A = AA^*$. Such a matrix is said to be normal. (See Hohn, p. 405.)

Proof: From (16), A can be diagonalized if and only if X and Y can be simultaneously diagonalized. Since $X = X^*$ and $Y = Y^*$, X and Y can be simultaneously diagonalized if and only if $XY = YX$, by Theorem 2. By (17), $XY = YX$ if and only if A is normal ■

For example, we may use the Spectral Theorem to show that M can be diagonalized, where

$$M = \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix}$$

Computing MM^* and M^*M, we obtain

$$M^*M = MM^* = \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix}.$$

By the Spectral Theorem, M can be diagonalized. However, note that M is neither Hermitian (because $M \neq M^*$) nor unitary (since $M^* \neq M^{-1}$).

Similarly, we may use the Spectral Theorem to illustrate the Q cannot be diagonalized, where

$$Q = \begin{pmatrix} 3 & 1 \\ -2 & 2 \end{pmatrix}.$$

Computing QQ^* and Q^*Q, we obtain

$$QQ^* = \begin{pmatrix} 10 & 1 \\ 1 & 5 \end{pmatrix} \quad \text{and} \quad Q^*Q = \begin{pmatrix} 10 & -1 \\ -1 & 5 \end{pmatrix}.$$

By the Spectral Theorem, Q cannot be diagonaliied since $QQ^* \neq Q^*Q$.

Now that we know an easy test to determine if a matrix is diagonalizable or not, we may investigate applications of diagonalizable matrices. We have seen previously that if A is diagonalizable, then $X^2 = A$ has at least one solution, so that $A^{1/2}$ can be defined. We consider $f(z) = a_n z^n + a_{n-1}z^{n-1} + \ldots + a_0$ for a single unknown z. The question arises: "What about $f(A)$ where A is a matrix?" Substituting A into $f(z)$, we obtain

$$f(A) = a_n A^n + a_{n-1}A^{n-1} + \ldots + a_0 I. \tag{18}$$

Since we know that $A^n = PD^nP^{-1}$, we obtain

$$f(A) = a_n PD^nP^{-1} + a_{n-1}PD^{n-1}P^{-1} + \ldots + a_0PP^{-1}.$$

Factoring, we have

$$f(A) = P(a_nD^n + a_{n-1}D^{n-1} + \ldots + a_0)P^{-1}.$$

Thus,

$$f(A) = Pf(D)P^{-1}.$$

Notice that $f(A)$ is defined for any matrix A, as seen in (18). We have seen that if $f(z) = z^{1/2}$, then $f(A)$ cannot be defined for all 2 x 2 matrices A. However, recall that if $f(z)$ is analytic at a point λ_0, then it can be expanded in a Taylor series about λ_0 with a positive radius of convergence. If the eigenvalues of a matrix A are contained in this disk of convergence, then $f(A)$ can be defined using the Taylor's series expansion for $f(z)$ as seen in the following theorem.

Theorem 4. (See Lancaster, p. 183.) Let matrix $A \in C_{n\times n}$ have eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. If the function f has a Taylor series about λ_0,

$$f(\lambda) = \sum_{p=0}^{\infty} \alpha_p(\lambda - \lambda_o)^p$$

with circle of convergence $|\lambda - \lambda_0| = r$, and if $|\lambda_j - \lambda_0| < r$, $j = 1, 2, \ldots, n$, then $f(A)$ is defined and

$$f(A) = \sum_{p=0}^{\infty} \alpha_p(A - \lambda_0 I)^p.$$

Other techniques for defining the value of a function applied to a matrix can be found in texts, particularly [1] (Grantmacher, Chapter V). If $f(z)$ is analytic at every point in the complex plane (entire) and A is diagonalizable and $A = PDP^{-1}$, then $f(A) = Pf(D)P^{-1}$ provides an equivalent definition for the expression in Theorem 4. One class of entire functions are the trigonometric functions. Therefore, we can define

$$\sin(A) = P\sin(D)P^{-1}$$

where

$$\sin(D) = \operatorname{diag}(\sin\lambda_1, \ldots, \sin\lambda_n).$$

It is easy to see without justifying all the interchanges of limits that if A is normal, the two representations of the $\sin(A)$ are equivalent.

$$\begin{aligned}
\sin A &= \lim_{N\to\infty} \sum_{k=0}^{N} (-1)^k \frac{A^{2k+1}}{(2k+1)!} \\
&= \lim_{N\to\infty} \sum_{k=0}^{N} (-1)^k \frac{PD^{2k+1}P^{-1}}{(2k+1)!} \\
&= P\left(\lim_{N\to\infty} \sum_{k=0}^{N} (-1)^k \frac{D^{2k+1}}{(2k+1)!}\right) P^{-1} \\
&= P \lim_{N\to\infty} \left(\sum_{k=0}^{N} (-1)^k \frac{\lambda_1^{2k+1}}{(2k+1)!}, \ldots, \sum_{k=0}^{N} (-1)^k \frac{\lambda_n^{2k+1}}{(2k+1)!}\right) P^{-1} \\
&= P(\sin\lambda_1, \ldots, \sin\lambda_n)P^{-1}.
\end{aligned}$$

In the same way, we may define $\cos(A)$, where A is normal. We notice below that the tangent does not exist for certain normal matrices.

$$\begin{aligned}
\tan A &= P\operatorname{diag}(\tan\lambda_1, \ldots, \tan\lambda_n)P^* \\
&= P\operatorname{diag}(\sin\lambda_1/\cos\lambda_1, \ldots, \sin\lambda_n/\cos\lambda_n)P^* \\
&= P\operatorname{diag}(\sin\lambda_1, \ldots, \sin\lambda_n)P^*P\operatorname{diag}(1/\cos\lambda_1, \ldots, 1/\cos\lambda_n)P^*.
\end{aligned}$$

Thus $\tan A = \sin A/\cos A$ exists only if $\cos\lambda_i$ is nonzero for $i = 1, \ldots, n$.

The trigonometric identities can now be proven on matrices by using the trigonometric identities from trigonometric functions defined on complex variables. For example,

$$\begin{aligned}
\sin^2 A + \cos^2 A &= P\operatorname{diag}(\sin^2\lambda_1, \ldots, \sin^2\lambda_n)P^* + P\operatorname{diag}(\cos^2\lambda_1, \ldots, \cos^2\lambda_n)P^* \\
&= P[\operatorname{diag}(\sin^2\lambda_1, \ldots, \sin^2\lambda_n) + \operatorname{diag}(\cos^2\lambda_1, \ldots, \cos^2\lambda_n)]P^* \\
&= P\operatorname{diag}(\sin^2\lambda_1 + \cos^2\lambda_1, \ldots, \sin^2\lambda_n + \cos^2\lambda_n)P^* \\
&= P\operatorname{diag}(1, \ldots, 1)P^* \\
&= PIP' \\
&= I.
\end{aligned}$$

Also, an example of a cofunction identity is given by:

$$\begin{aligned}
\sin(A + \tfrac{\pi}{2}I) &= \sin(PDP^{-1} + \tfrac{\pi}{2}I) \\
&= \sin(PDP^{-1} + \tfrac{\pi}{2}PIP^{-1}) \\
&= \sin P(D + \tfrac{\pi}{2}I)P^{-1} \\
&= P\sin(\operatorname{diag}(\lambda_1 + \tfrac{\pi}{2}, \ldots, \lambda_n + \tfrac{\pi}{2}))P^{-1} \\
&= P\operatorname{diag}(\sin(\lambda_1 + \tfrac{\pi}{2}), \ldots, \sin(\lambda_n + \tfrac{\pi}{2}))P^{-1} \\
&= P\operatorname{diag}(\cos\lambda_1, \ldots, \cos\lambda_n)P^{-1} \\
&= \cos A.
\end{aligned}$$

In summary, my work investigates the solutions to equations and identities containing matrix-valued expressions. This investigation leads to a study of unitary, Hermitian, and normal matrices. Last, the implications of replacing the complex arguments of polynomial and trigonometric functions with matrix-valued arguments are explored.

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This project was directed by Dr. Kevin Shirley and was completed while the author was a senior at Middle Tennessee State University.

OUTERPLANAR GRAPHS AND MATROID ISOMORPHISM

Jeremy M. Dover
Worcester Polytechnic Institute and The University of Delaware

The goal of this paper is to present a method for counting the number of matroids which are the cycle matroids of outerplanar graphs. For those who are not familiar with matroids, we begin with a brief introduction to some of the basic concepts and definitions of matroid theory. From there, we introduce the fractured dual and discuss some of its properties. Finally, we use this fractured dual to answer some questions about the number of matroids and matroid isomorphism.

What is a Matroid?

Consider the group of real numbers under addition. When considering the reals in this light, we completely blind ourselves to their multiplicative properties, but we can apply all of the results of group theory to the addition of reals. So, by restricting our attention, we have gained some knowledge.

Now, in matroid theory, we attempt to do the same thing. With a matroid, the concept we focus on is "independence," in one form or another, of subsets of a given universal set.

The concept of a matroid was introduced in 1935 in a paper by Hassler Whitney ("On the Abstract Properties of Linear Dependence," *Amer. J. Math.*, *57* **(1935)**, pp 509-533). In this paper, he looked at the set of columns of a matrix. A given subset of these columns, when considered as vectors, is either linearly independent of linearly dependent. Now, Whitney noticed that the sets of columns which are linearly independent satisfy the following properties:

a) The empty set is linearly independent,

b) Subsets of linearly independent sets are independent, and

c) Given two linearly independent sets, one smaller than the other, then some element of the larger may be added to the smaller such that the resulting set will be linearly independent.

This motivates the following definition:

Definition. A *matroid* is a pair (E, I), where E is a finite, non-empty set, and I is a collection of subsets of E which satisfy the above three properties. The sets in I are called *independent sets.*

Now we can define several important matroid concepts. A *base* of a matroid is a maximal independent subset of E; **i.e.,** an independent set which is not properly a subset of another independent set. A *circuit* of a matroid is a minimal dependent subset of E; **i.e.,** the removal of any element from a circuit yields an independent set. If A is a subset of E, then the *rank* of A is the cardinality of the largest independent set contained in A. And, finally, the *closure* of A is the largest set containing A such that the rank of A equals the rank of the closure.

Now, what have we gained in going from the vector space to the matroid? It may not be immediately clear, but we have made our definitions of "independence"-related concepts much simpler. For example, a set is linearly independent if and only if it is in the set I. Also, the operation of closure is entirely analogous to taking spans in the vector space. However, we have lost the operations of addition and scalar multiplication; but, if one is interested in the independence properties of vector spaces, matroids are a useful tool.

We have chosen to define a matroid in terms of independent sets. We could, however, have chosen differently. Every matroid concept defined above, along with appropriate axioms, can be used to define a matroid. In graph theory, it is most useful to define a matroid in terms of its circuits:

A *matroid* is a pair $(E;C)$, where E is a finite, non-empty set, and C is a collection of subsets of E, called circuits, which satisfy:

i) no circuit contains another circuit (except itself), and

ii) if c_1 and c_2 are distinct circuits such that $x \in c_1$ and $x \in c_2$, then there is a circuit in $c_1 \cup c_2$ which does not contain x.

This definition of a matroid is equivalent to the previously given definition. For more details on this and other definitions of a matroid, see [4] or [2].

Cycle Matroids

Consider a graph G. A subset of its edges is, in some sense, dependent if it contains a cycle. This motivates the following definition. The *cycle matroid* of a graph, denoted $M(G)$, is a matroid $(E;C)$, where E is the set of edges of G and C is the set of all cycles of G, which are the circuits of the matroid.

Why make this definition? One use of cycle matroids is to get a better handle on dualization processes. The geometric dual of a *plane* graph is given by the following process:

1. Place a vertex in each face (including the infinite face) of the graph.
2. Across each edge of the graph, draw a new edge between the vertices of the two faces to which the edge of the graph is adjacent.

The graph given by these new vertices and edges is the geometric dual. Now, the geometric dual of a graph can change as one changes its embedding, so it does not make sense to talk about *the* geometric dual. However, the dual of a matroid is defined only if the graph is a plane, while the dual of a matroid is defined for all matroids. To make a connection between these dualization processes, we note that if a graph G has a geometric dual H, then the dual matroid of $M(G)$ is $M(H)$. Anyone interested in a thorough introduction to matroid theory and its applications should consult the very readable article by Wilson [3].

The Fractured Dual

(Note: to avoid questions about existence and connectivity of dual graphs, we now restrict our attention to graphs which are both planar and 2-connected.)

While cycle matroids are a useful tool, they have the unfortunate property that they do not uniquely determine a graph for which they are the cycle matroid; i.e., several nonisomorphic graphs may have isomorphic cycle matroids. So, given two embedded graphs, G and H, the question arises: Is $M(G)$ isomorphic to $M(H)$? It turns out that the geometric duals, provided that they exist, can often shed some light on this question. However, when dealing with large graphs, the duals are just as difficult to deal with as the original graphs. In this section, we discuss a way to simplify the structure of the dual graph without losing any information we may obtain about the cycle matroid. First, however, we need some definitions.

Let G be a graph, with v a vertex in G. Assume the degree of v is n. The *fracture of* G *at* v is formed by deleting the vertex v, and replacing it with n new vertices $y_1, \ldots, y_2$, and adding the following edges. Consider the set $\{\{x_i, v\} \in E(G) : x_i \in V(G)\}$. Since vertex v has degree n, the subscript i varies from 1 to n. (Note that the x_i's need not be distinct.) Further, each of these edges is removed when v is deleted. Now to the graph, add the edges $\{x_1, y_1\}, \ldots, \{x_n, y_n\}$. this process results in the fracture.

The *fractured dual*, G^o, of an embedded plane block G is obtained by taking the geometric dual of G and fracturing the vertex corresponding to the infinite face of G. For a fixed embedding of a graph, the fractured dual is unique, and is thus well-defined, but, as with the regular geometric dual, since there is nothing special about the infinite face, the fractured dual is not unique for a general. graph. However, if the fractured dual is a tree, there are some things which we can say about the" fractured dual. Some examples of fractured duals are given below in Figure 1.

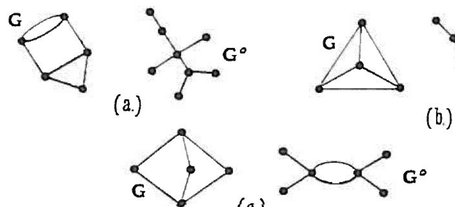

Figure 1.

We now wish to prove a theorem which classifies those graphs which have a tree as a possible fractured dual. First, however, we need to state a theorem about outerplanarity. From Harary **[1]**, we know that a graph is outerplanar if and only if it contains no **subgraph** homeomorphic to K_4 or $K_{2,3}$, **with** the exception of $K_4 - x$, **i.e.**, K_4 with an edge deleted, which is homeomorphic to $K_{2,3}$, but has an outerplanar embedding.

Theorem **1**. A graph has an outerplanar embedding if and only if it has an embedding for which its fractured dual is a tree.

Proof: Assume that G is not in an outerplanar embedding. Then there is a vertex v which does not lie on the infinite face.

The faces containing v will form a cycle in the dual, and since v is not on the infinite face, this cycle will not be broken by the fracture of the vertex corresponding to the infinite face. Thus the fractured dual of this embedding will contain a cycle, and thus not be a tree.

Conversely, if G is in an outerplanar embedding, every vertex will lie on the infinite face. Then, in the dual, every cycle in the dual will pass through the vertex in the infinite face. So, in the fractured dual, when the vertex in the infinite face is fractured, all of the cycles will be broken. Thus, the fractured dual will contain no cycles, and thus will be a tree. ∎

Matroid Isomorphism

We now wish to prove two theorems about the relationship between cycle matroids and fractured duals. With these theorems, we wish to address two questions. One, how does one determine when the cycle matroids of two given graphs are isomorphic? Second, how many graphic matroids are there? (A graphic matroid is a matroid which is the cycle matroid of some graph.)

Theorem **2**. If G and H are plane blocks and G^o is isomorphic to H^0, then $M(G)$ is isomorphic to $M(H)$.

The proof of this fact is a straightforward but technical argument, which uses more matroid theory than has been introduced here. Thus, the proof is omitted.

What does **this** tell us? If G and H are two blocks with outerplanar embeddings, and their fractured duals are isomorphic trees, then their cycle matroids are isomorphic. However, this only tells us what happens when the fractured duals are ***isomorphic;*** it does not tell us what happens when the fractured duals are not isomorphic.

The reason for this is that Theorem 2 is an implication and not an equivalence. The converse of Theorem 2 is not true in general. However, we now wish to prove the converse for a special case, that of G^o and H^0 being nonisomorphic trees.

Before proving this theorem, we need one more definition. A ***twisting*** of a graph is defined as follows: Consider a minimal **cutset** of a graph of connectivity 2, $\{u,v\}$. The removal of these two vertices disconnects the graph into several components. Using these components, we wish to form two subgraphs, Gl' and $G2'$. We do this by dividing the remaining components up between the two subgraphs such that each component appears in exactly one of Gl' and $G2'$. Now let G1 be Gl' joined with $\{u,v\}$ and any edges between $\{u,v\}$ and the vertices of $G1'$, and define G2 in an analogous way. The twisting of G is the graph formed by attaching G2 to G1 such that u in G1 is identified with v in $G2$ and v in G1 is identified with u in G2.

Theorem 3. Let G and H be blocks with outerplanar embeddings. Then $M(G) \approx M(H)$ if and only if G° as H° for the outerplanar embeddings.

Proof: The reverse direction is equivalent to Theorem 2. If $M(G) \approx M(H)$, we have a theorem from Welsh **[2]**, due to Whitney, which says that G can be obtained by a series of twistings from H. An example of this is shown below in Figure 2.

Now, in a series of twistings, we do not change the structure of the fractured dual, **since we** are only changing the spatial arrangement of the graph, and are not changing any adjacencies. Therefore, the fractured dual is unchanged under a series of twistings. Thus $G^o \approx H^0$. ∎

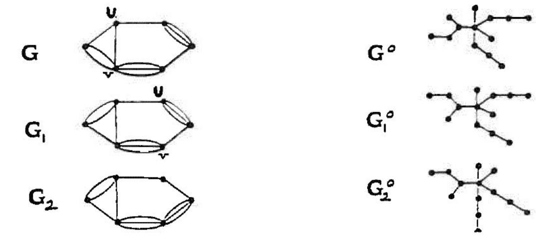

Figure 2: Twistings on a graph and its fractured dual.

Now this theorem gives us the second part of what we sought earlier. If we have two blocks with outerplanar embeddings, and their fractured duals ***for those particular embeddings*** are **noniso**morphic, then their cycle matroids are nonisomorphic.

Corollary. All outerplanar embeddings of a given outerplanar block have the same fractured dual.

Proof: From Theorem 3, we showed that if G and H are outerplanar blocks and $M(G) \approx M(H)$, then G^o a H^o, regardless of embedding. Now, just take H $= G$, and we have the **result**.∎

Due to this corollary, when we talk about the fractured dual of an outerplanar embedding, this is the fractured dual of ***every*** outerplanar embedding of G. Note that if G is not in **an** outerplanar embedding, its fractured dual need not be a tree, **so** we cannot talk about a unique fractured dual.

Now, we can make the followingstatement: If G and H are graphs in an outerplanar embedding, then every cycle matroid isomorphism is equivalent to fractured dual isomorphism.

These technical results allow us to address our second question. Since fractured dual isomorphism is equivalent to cycle matroid isomorphism, we can take the set of all trees with n edges and invert the fractured dual process. (This can be done by identifying all vertices of degree one in the tree and dualizing the resulting graph.) All members of this set of graphs have nonisomorphic cycle matroids, by construction. So, the number of matroids on n elements (**i.e.**, the size of the set E is n) which are the cycle matroids of outerplanar, 2-connected graphs is given by the number **of** trees on n edges. In fact, this is a very poor estimate for the actual number of matroids, as this count ignores many large classes of matroids. However, this technique combined with other counting techniques, does give a fair lower bound on the number of matroids which are both graphic and cographic.

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UNIFORM EMBEDDINGS OF GRAPHS

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1. Introduction In the first book written on graph theory (published in 1936), Dénes Konig [3] described a procedure where, given any graph G with maximum degree $\Delta(G) = d$, a d-regular graph H can be constructed so that G is an induced subgraph of H. (A graph G is an induced subgraph of H if H is obtained by adding edges and vertices (possibly none) to G such that no new edges join two vertices of G.) It will be advantageous for us to describe this technique.

Let G be a graph with $\Delta(G) = d$. If G is a regular graph, then take H = G. Otherwise, we take a new copy of G, which we denote by G', and join corresponding vertices of G and G' whose degrees are less that d. We refer to the resulting graph as G_1. If G_1 is d-regular, then take H = G_1. If not, we continue this process until a d-regular graph G_n is obtained. Figure 1 illustrates this process.

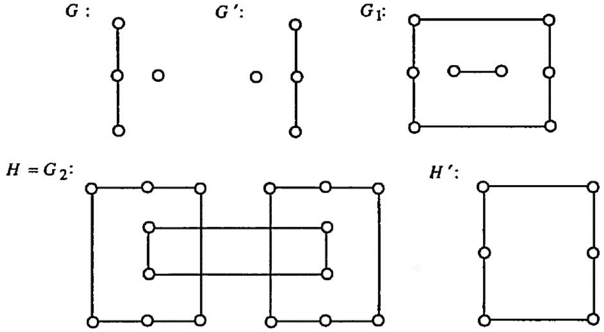

Figure 1. An illustration of Konig's method

Konig's technique, therefore, has the following consequence.

Theorem. (Konig) For every graph G and every integer $r \geq \Delta(G)$, there exists an r-regular graph H containing G as an induced subgraph.

From Konig's technique, we can also see that for every vertex v of H, there exists an induced subgraph of H that is isomorphic to G and contains v. This leads us into the main topic of this article. A graph G is said to be uniformly embedded in a graph H if, for every vertex v of H, there exists an induced subgraph of H isomorphic to G that contains v. It follows from the proof of the theorem that every graph G is uniformly embedded in some r-regular graph for each $r \geq \Delta(G)$. This technique, however, does not guarantee that the graph H produced has minimum order. In fact, for the graph G of Figure 1, the graph H of Figure 1 has order 16, while the minimum order of a 2-regular graph containing G as an induced subgraph is only 6. The graph H' of Figure 1 has this property.

In 1963, Erdös and Kelly [2] developed a formula for determining the minimum order of a d-regular graph H containing a given graph G (with $\Delta(G) = d$) as an induced subgraph. We describe this formula. Let G be a graph with maximum degree d whose vertex set is $V(G) = \{v_1, v_2, \ldots, v_n\}$. Let d_i denote the degree of v_i and let $e_i = d - d_i$ $(1 \leq i \leq n)$ denote the deficiency of v_i. Further, let $e = \max\{e_i\}$ represent the maximum deficiency and $s = \sum_{i=1}^{n} e_i$ the total deficiency. We can now state the theorem of Erdos and Kelly.

Theorem. (Erdös and Kelly) Let G be a graph of order n and let r be an integer such that $r \geq \Delta(G)$. A necessary and sufficient condition that m + n be the least order of an r-regular graph H containing G as an induced subgraph is that m be the least integer satisfying the following four conditions:

(1) $mr \geq s$,

(2) $m^2 - (r+1)m + s \geq 0$,

(3) $m \geq e$,

(4) $(m + n)r$ is even.

Figure 2 shows examples of graphs G_i and $H_i (1 \leq i \leq 4)$ such that H_i is $\Delta(G_i)$-regular, has minimum order, and contains G_i as an induced subgraph. The solid vertices in each graph H_i indicate the vertices added to G_i, while the edges incident to each solid vertex are the added edges.

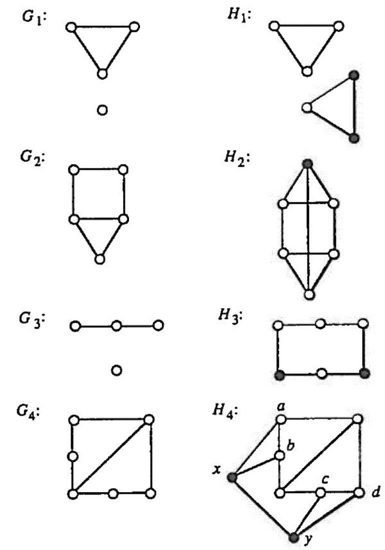

Figure 2. Smallest regular graphs containing a given graph as an induced subgraph

In Figure 2, G_i is uniformly embedded in H_i, for $i = 1,2,3$. However, G_4 is not uniformly embedded in H_4 since there does not exist an induced subgraph of H_4 that contains x and is isomorphic to G_4. We verify this fact next. Suppose, to the contrary, that G_4 is uniformly embedded in H_4. The graph G_4 contains no 3-cycles, so either a or b must be removed, as well as one of c, d, and y. Since G_4 contains no vertex of degree 1, the vertex y cannot be removed. However, if a or b is removed, then the resulting graph has three consecutive vertices of degree 2, and this graph is not isomorphic to G_4. This produces a contradiction.

2. The Uniformity Number of a Graph Let G be a graph and r an integer with $r \geq d = \Delta(G)$. Then we define the r-uniformity number $u_r(G)$ of G as the minimum number of vertices needed to be added to G to produce an r-regular graph H in which G is uniformly embedded. We write $u(G)$ for $u_d(G)$ and call it simply the uniformity number of G. For a given graph G of order p, an r-regular graph H of order p + $u_r(G)$ in which G is uniformly embedded is called an r-uniformity graph of G while a d-uniformity graph is called more simply a uniformity graph. The set of r-uniformity graphs will be denoted by $\mathbf{U}_r(G)$ and the set of uniformity graphs by $\mathbf{U}(G)$.

We now illustrate the above concepts. For a positive integer n, let P_n denote the path with n vertices and for $n \geq 3$, let C_n denote the cycle **with** n vertices. Then $\Delta(P_n) = 2$ if $n \geq 3$. Thus, for $n \geq 3$, the uniformity number $u(P_n) = 1$ since we need only add one vertex to P_n to produce a 2-regular graph in which P_n can be uniformly embedded. Since C_{n+1} is the only graph with this property (see Figure *3)*, it follows that $\mathbf{U}(P_n) = \{C_{n+1}\}$.

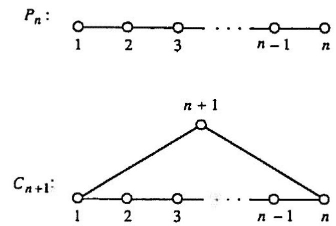

Figure *3*. The path $P_n(n \geq 3)$ and its uniformity graph C_{n+1}

It is now useful to describe some classes of graphs which we will encounter soon. The complete graph K_p is that graph of order p in which every two vertices are adjacent. A graph G is a bipartite graph if its vertex set $V(G)$ can be partitioned into two subsets V_1 and V_2 such that every edge of G joins a vertex of V_1 and a vertex of V_2. If, in addition, $|V_1| = m, |V_2| = n$, and every vertex of V_1 is adjacent to every vertex of V_2, then G is referred to as the complete bipartite graph $K_{m,n}$. The graph $K_{1,n}$ is called a star, with the vertex of degree n referred to as the center of the star. For positive integers m and n, the double star $S_{m,n}$ consists of adjacent vertices u and v, where u and v are adjacent to $m-1$ and $n-1$ additional vertices of degree *1*, respectively. The vertices u and v are the centers of the double star $S_{m,n}$. The double star $S_{1,n}$ is therefore the star $K_{1,n}$. These concepts are illustrated in Figure *4*.

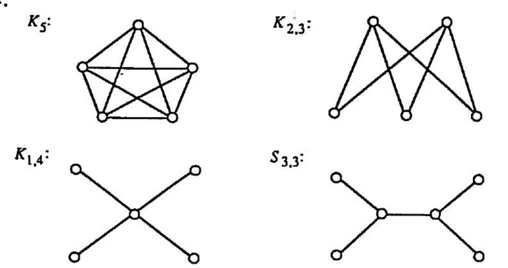

Figure *4*. Some special types of graphs

Suppose G is the complete graph K_p. Thus G is $(p-1)$-regular. If r is an integer with $r \geq p-1$, then any r-uniformity graph of G contains at least $r+1$ vertices. On the other hand, G is uniformly embedded in K_{r+1}. These remarks provide the basis for **the** following result.

Theorem 1. Let p be a fixed positive integer. If r is an integer with $r \geq p-1$, then $u_r(K_p) = r+1-p$ **and** $\mathbf{U}_r(K_p) = \{K_{r+1}\}$.

By Theorem *1*, the complete graph K_{r+1} is the r-uniformity graph for all the graphs $K_1, K_2, \ldots, K_{r+1}$. This observation gives us the following result.

Corollary 2. For every positive integer r, there exists a graph that is the r-uniformity graph of at least r distinct graphs.

We next determine the uniformity number of a star.

Theorem 3. The star $K_{1,n}$ has uniformity number $n-1$ and $\mathbf{U}(K_{1,n}) = \{K_{n,n}\}$.

Proof: Let **H** be an n-uniformity graph of $K_{1,n}$, and suppose that M is the set of vertices added to $K_{1,n}$ to produce H, where $|M| = m$. Thus, $u(K_{1,n}) = m$. Let v_0 be the center of the star $K_{1,n}$, and let $v_1, v_2, \ldots, v_n$ denote the remaining vertices of $K_{1,n}$. (See Figure *5.)* The deficiency d_i of v_i $(0 \leq i \leq n)$ is then given by $d_0 = 0$ and $d_1 = d_2 = \cdots = d_n = n-1$. Since the maximum deficiency e of $K_{1,n}$ is $n-1$, it follows that $m \geq n-1$. On the **other** hand, if **we let** $M = \{u_1, u_2, \ldots, u_{n-1}\}$, define $V_1 = \{v_1, v_2, \ldots, v_n\}$ and $V_2 = \{v_0\} \cup M$, and join every vertex v_i $(1 \leq i \leq n)$ to every vertex u_j $(1 \leq j \leq n-1)$, we produce the graph $K_{n,n}$. Since $K_{1,n}$ is uniformly embedded in $K_{n,n}$, we have $u(K_{1,n}) = n-1$ and, further, since $K_{n,n}$ is the unique n-regular graph of order $2n$ with this property, $\mathbf{U}(K_{1,n}) = \{K_{n,n}\}$. ■

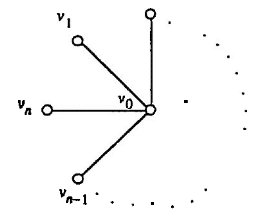

Figure 5. The star $K_{1,n}$.

We now turn to a more complicated problem, namely, the investigation of the uniformity numbers of double stars of the type $S_{n,n}$.

Theorem 4. Let n be a positive integer.

(1) If $n = 1$, then $u(S_{n,n}) = 0$ and $\mathbf{U}(S_{n,n}) = \{P_2\}$.

(2) If $n = 2$, then $u(S_{n,n}) = 1$ and $\mathbf{U}(S_{n,n}) = \{C_5\}$.

(3) If $n \geq 3$ and n is odd, then $u(S_{n,n}) = 2(n-1)$.

(4) If $n \geq 4$ and n is even, then $2n-3 \leq u(S_{n,n}) \leq 2(n-1)$.

Proof: If $n = 1$, then $S_{n,n} = P_2$, while if $n = 2$, then $S_{n,n} = P_4$. Since P_2 is regular, $\mathbf{U}(P_2) = \{Pa\}$ and $u(P_2) = 0$. On the other hand, we have already seen that $u(P_4) = 1$ and $\mathbf{U}(P_4) = \{C_5\}$.

Suppose now that $n \geq 3$. Let u and v be the centers of the double star $G = S_{n,n}$. Let $U_1 = \{u_1, u_2, \ldots u_{n-1}\}$ be the set of vertices adjacent to u and $V_1 = \{v_1, v_2, \ldots, v_{n-1}\}$ the set of vertices adjacent to v. (See Figure *6.)*

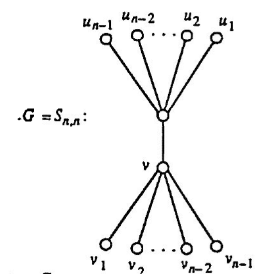

Figure 6. The double star $S_{n,n}$.

The maximum degree of $S_{n,n}$ is n, which is the degree of the centers. Since all the other vertices have degree 1, the total deficiency s of the double star is

$$(n-1) \cdot 2(n-1) = 2(n-1)^2$$

Let H be a uniformity graph of $S_{n,n}$, where m vertices are added to $S_{n,n}$ to produce H. Then

$$m \cdot n \geq 2(n-1)^2, \quad \text{or} \quad m \geq 2n - 4 + \frac{2}{n}.$$

Since $n \geq 3$,

$$m \geq \lceil 2n - 4 + \frac{2}{n} \rceil = 2n - 3.$$

Also, $\sum_{v \in V(H)} \deg v = (2n+m)n$, which is twice the number of edges of H. Hence, mn must be even. Consequently, if n is odd, we have that $m \geq 2n-2$.

We prove by construction that if $n \geq 3$ and n is odd, then $u(S_{n,n}) = 2(n-1)$, while if $n \geq 4$ and n is even, then $2n-3 \leq u(S_{n,n}) \leq 2(n-1)$.

Consider the graph G with centers u and v as shown in Figure 6. Join v_1 to all vertices in $M_1 = \{x_1, x_2, \dots x_{n-1}\}$, and join u_1 to all vertices in $M_2 = \{x_n, x_{n+1}, \dots, x_{2n-2}\}$. (See Figure 7.) Now, join all the vertices in M_2 to every vertex in $\{v_2, v_3, \dots, v_{n-1}\}$ and join all vertices in M_1 to every vertex in $\{u_2, u_3, \dots, u_{n-1}\}$. As there are $n-1$ vertices in each of M_1 and M_2 and $n-2$ vertices in each of $\{u_2, u_3, \dots, u_{n-1}\}$ and $\{v_2, v_3, \dots, v_{n-1}\}$, the degrees of the vertices in $\{u_2, u_3, \dots, u_{n-1}\}$ and $\{v_2, v_3, \dots, v_{n-1}\}$ are now n. But, each vertex in $M_1 \cup M_2$ has degree $n-1$. To increase the degrees of the vertices in $M_1 \cup M_2$ to n, we join x_i to x_{i+n-1} $(1 \leq i \leq n-1)$. Let H' be the graph just constructed. We claim that this G is uniformly embedded in H'. To prove our claim, we must show that for each vertex x_i $(1 \leq i \leq 2n-2)$ of H', there is an induced subgraph of H' isomorphic to $S_{n,n}$ that contains x_i. Any vertex $x_i \in M_1$ belongs to the subgraph induced by $M_1 \cup V_1 \cup \{u,v\}$, while each vertex $x_i \in M_2$ belongs to the subgraph induced by $M_2 \cup U_1 \cup \{u,v\}$. In both cases, the subgraph is isomorphic to $S_{n,n}$. Thus, $S_{n,n}$ is uniformly embedded in H'. ∎

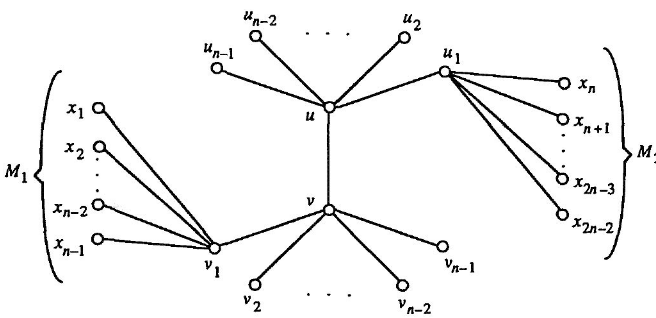

Figure 7. Constructing a uniformity graph for $S_{n,n}$.

The reader might well find it interesting to investigate the problems of finding $u(S_{m,n})$ and $U(S_{m,n})$ for $m \neq n$.

From the proof of Konig's theorem described at the beginning of this article, it follows that the number of vertices added to a graph G of order p to make it r-regular is $(2^n - 1)p$, where $n = r - \Delta(G)$. So $(2^n - 1)p$ is an upper bound for $u_r(G)$. Also, since the resulting graph from Erdös and Kelly's theorem is r-regular, the number m of vertices added to obtain its graph is a lower bound for $u_r(G)$. Therefore,

$$m \leq u_r(G) \leq (2^n - 1)p \tag{1}$$

The number $u_r(G)$ may lie strictly between these two bounds, as we next show.

Consider the graph G of Figure 8. Observe that G has order 13, namely, ten vertices of degree 3 and three vertices of degree 2. Further, G contains one 3-cycle, all three vertices of which have degree 3. Also, G has three 5-cycles, no two of which share more than two edges.

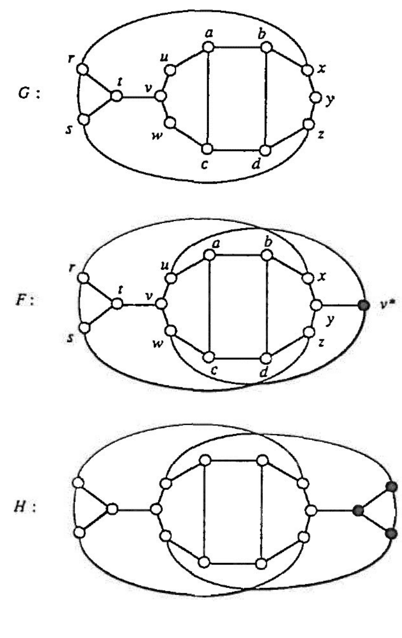

Figure 8. A graph whose uniformity number is obtained neither by Konig's nor by Erdos and Kelly's formula.

It is clear that we need add only one vertex to G and join it to the three vertices of degree 2 in G to produce a 3-regular graph containing G as an induced subgraph. Indeed, the graph F so produced is unique and is shown in Figure 8. Thus the number m in (1) has the value 1 for this graph G while $(2^n - 1)p = 13$. We show, however, that G is not uniformly embedded in F. Observe that F has one 3-cycle and four 5-cycles (the 5-cycle v^*, u, a, c, w, v^* is added). Also, two 5-cycles of F have three common edges.

If G were uniformly embedded in F, then the deletion of some vertex of F different from v^* must produce a graph isomorphic to G. Because G contains a 3-cycle, none of r, s, or t can be deleted. Because every vertex of the 3-cycle in G has degree 3, none of v, x, or z can be deleted. Because G has three 5-cycles, none of u, a, c, or w can be deleted. Because G does not contain two 5-cycles sharing exactly three edges, none of b, d, or y can be deleted. Therefore, G is not uniformly embedded in F and so $u(G) \geq 3$.

On the other hand, G is clearly uniformly embedded in the graph H, so that $u(G) = 3$.

3. Uniformity Sequences of Graphs

We have seen that for a given graph G and an integer $r \geq \Delta(G)$, the r-uniformity number $u_r(G)$ always exists. This then suggests a sequence associated with G. Let G be a graph with $\Delta(G) = d$. Then the ***uniformity sequence*** $s(G)$ of G is the sequence $s_1, s_2, s_3, \dots$, where $s_k = u_{k+d-1}(G)$ for $k = 1, 2, 3, \dots$. We write $s(G) = \{s_1, s_2, s_3, \dots\}$. It follows from Theorem 1 that for a fixed positive integer p, the uniformity sequence $s(K_p) = \{0, 1, 2, \dots\}$. Indeed, we can say more.

Theorem 5. The sequence $0, 1, 2, \dots$ is the uniformity sequence of a graph G if and only if $G = K_p$

for some positive integer p.

Proof: We have already seen from Theorem 1 that $s(K_p) : 0, 1, 2, \ldots$. Next assume G is agraph with uniformity sequence $0, 1, 2, \ldots$. Suppose that G has order p and $\Delta(G) = d$. Then $s_1 = u_d(G) = 0$, which implies that G is d- regular. Since $s_2 = u_{d+1}(G) = 1$, it is possible to add one new vertex v to G and $d+1$ new edges (all incident with v), so, that the resulting graph is (d+ 1)-regular. This, however, implies that p = d + 1 and that G $\cong K_p$.■

The proof of the preceding theorem actually provides a somewhat stronger result.

Corollary 6. A sequence s is the uniformity sequence of a complete graph if and only if the second term is 1.

We now consider uniformity sequences of other specific graphs. The following concept and theorem will be useful to us. The complement G of a graph G is that graph with $V(\overline{G}) = V(G)$ such that uv is an edge of $\overline{G}$ if and only if uv is not an edge of G.

Theorem 7. If a graph G is uniformly embedded in a graph H, then $\overline{G}$ is uniformly embedded in $\overline{H}$.

Proof: Suppose that G is uniformly embedded in H, and let $v \in V(H)$. Then thereexists $U \subseteq V(H)$ with $v \in U$ such that the subgraph induced by U in H is isomorphic to G. However, the subgraph induced by U in $\overline{H}$ is isomorphic to $\overline{G}$, so there exists an induced subgraph of $\overline{H}$ containing v that is isomorphic to $\overline{G}$.■

From Theorem 7, we have an immediate corollary.

Corollary 8. Let G be a graph with $\Delta(G)$ = d and $\Delta(\overline{G}) = \overline{d}$. Then, for every nonnegative integer k,

$$\mathbf{U}_{d+k}(G) = \mathbf{U}_{\overline{d}+k}(\overline{G}),$$

so $s(G) = s(\overline{G})$.

From Theorem 7, we know that $\overline{K}_{1,3}$ is uniformly embedded in every graph belonging to $\mathbf{U}(\overline{K}_{1,3})$ for all $r \geq 3$. We can see that a graph belonging to $\mathbf{U}_r(\overline{K}_{1,3})$ must have a 3-cycle in addition to a vertex not adjacent to any vertex of the 3-cycle. We can also see that a 2-regular graph G belonging to $\mathbf{U}_r(\overline{K}_{1,3})$ will be a graph of smallest order for which $\overline{G}$ has the largest possible degree such that $K_{1,3}$ is uniformly embedded in $\overline{G}$. Therefore, $\overline{G}$ belongs to $\mathbf{U}(K_{1,3})$. The graphs H_1, H_2, and H_3 whose complements are given in Figure 9 correspond to the first three terms of $s(K_{1,3})$. By following the general pattern set in Figure 9, we have that $s(K_{1,3}) = \{2, 3, 4, \ldots\}$.

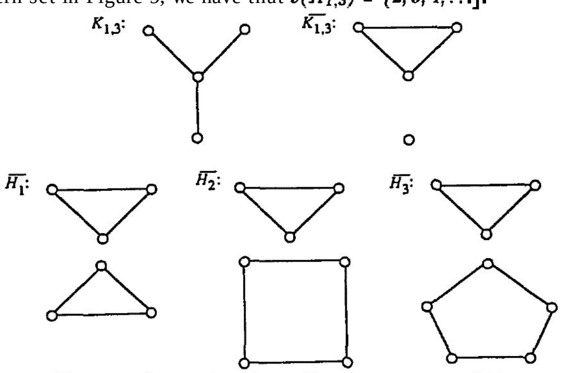

Figure 9. Computing the uniformity sequence of $K_{1,3}$.

We now determine the uniformity sequence of all stars. It is useful to state the following result.

(See [1], for example.)

Theorem A. Let r and n be integers with $0 \leq r < p$. There exists an r-regular graph of order p if and only if rp is even.

Theorem 9. Let r and n he integers with $r \geq n \geq 2$. Then

$$u_r(K_{1,n}) = \begin{cases} r & \text{if n is even or } r \text{ is odd} \\ r-1 & \text{otherwise.} \end{cases}$$

Proof: Since the star $K_{1,n}$ contains vertices of degree 1, at least $r-1$ vertices must be added to produce an r-regular graph containing $K_{1,n}$ as an induced subgraph. Thus, $u_r(K_{1,n}) \geq r - 1$. If $u_r(K_{1,n}) = r - 1$, then there exists an r-regular graph of order (n + 1) + (r - 1) = n + r containing $K_{1,n}$ as an induced subgraph. If n is even and r is odd, then no such graph exists (by Theorem A), in which case $u_r(K_{1,n}) \geq r$.

Suppose that it is not the case that n is even and r is odd. We show that there exists an r-regular graph of order n + r containing $K_{1,n}$ as an induced subgraph. Let V_1 and V_2 be two disjoint sets of vertices, where $|V_1| = r$ and $|V_2|$ = n. On the set V_1, we construct an (r - n)-regular graph (of order r). By Theorem A, such a graph exists. We then join every vertex of V_1 to each vertex of V_2, producing a graph G. The graph G is r-regular of order $r + m$. If $v \in V_1$, then the subgraph induced by $\{v\} \cup V_2$ is $K_{1,n}$. Thus, $K_{1,n}$ is uniformly embedded in G.

Suppose n is even and r is odd so that $r \geq$ n + 1. Then we let V_1 and V_2 be disjoint sets of vertices with $|V_1| = r$ and $|V_2| = n+1$. We construct an (r - n - 1)-regular graph on V_1, which can be done by Theorem A. Let H be the graph produced by joining every vertex of V_1 to each vertex of V_2. Then H is an r-regular graph of order r + n + 1. If $v \in V_1$ and x $\in V_2$, then each subgraph of H induced by {v} $\cup$ (V_2 - {x}) is $K_{1,n}$. Therefore, $K_{1,n}$ is uniformly embedded in H. ■

Corollary 10. If n $\geq$ 3 is an odd integer, then $s(K_{1,n}) = \{n-1, n-2, n-3, \ldots\}$; while if $n \geq 2$ is an even integer, then $s(K_{1,n}) = \{n-1, n+1, n+1, n+3, n+3, \ldots\}$.

We next investigate the uniformity sequence of C_4. For the purpose of doing this, we present a formula for $u_r(C_4)$.

Theorem 11. For $r \geq 2$,

$$u_r(C_4) = \begin{cases} r-1 & \text{if } r \text{ is odd} \\ r-2 & \text{if } r \text{ is even.} \end{cases}$$

Proof: We consider two cases.

Case 1. Assume $r \geq 3$ is odd. Every graph in $\mathbf{U}(C_4)$ is r-regular and $\overline{C_4}$ is 1-regular; so the order of every graph in $\mathbf{U}(C_4)$ is at least r + 2. Therefore, $u_r(C_4) \geq r - 2$. However, for r odd, every r-regular graph has even order, so $u_r(C_4) \geq r-1$. If $u_r(C_4) = r-1$, then every graph in $\mathbf{U}_r(\overline{C_4})$ is 2-regular. Now $\overline{C_4}$ is uniformly embedded in C_n for n $\geq$ 6. Thus, C_4 is uniformly embedded in $\overline{C}_{r+3}$, which implies that $u_r(C_4) = r-1$.

Case 2. Assume $r \geq 2$ is even. As in Case 1, we know $u_r(C_4) \geq r - 2$. However, $\overline{C_4}$ is uniformly embedded in all 1-regular graphs of order at least 4. So, $\overline{C_4}$ is uniformly embedded in $\overline{C}_{r+2}$, and $u_r(C_4) = r - 2$. n

The following corollary is now immediate.

Corollary 12. $s(C_4) = \{0, 2, 2, 4, 4, 6, 6, \ldots\}$.

With the aid of the preceding two results, we can now establish the following:

Theorem 13. For every positive integer n, there exists a graph G and an integer r for which there" are at least n r-uniformity graphs.

Proof: The result is certainly true if n = 1, so we may assume that $n \geq 2$. Let $G \cong C_4$ and let $r = 2n+1$. By Theorem 11, $u_r(C_4) = r-1$. Observe that $\overline{C_4}$ is uniformly embedded in every graph in the set

$$S = \{C_p \cup C_q | p+q = r+3, \quad 3 \leq p \leq (r+3)/2\},$$

which is a subset of $\mathbf{U}_r(\overline{C_4})$. Therefore,

$$|\mathbf{U}_r(C_4)| = |\mathbf{U}_r(\overline{C_4})| \geq |S| = (r-1)2 = n,$$

completing the proof.''

From the definition of uniformity sequences, it may seem that such a sequence is nondecreasing. However, that is not always the case.

For example, it can be shown that the first ten terms of the uniformity sequence of C_5 are **0,3,5,5,5,7,9,9,11,10**.

We have seen that two nonisomorphic graphs may have the same uniformity sequence. For example, for every integer $n \geq 2$, $s(K_n) = \{0,1,2,\ldots\}$. However, of course, for $n \neq m$, the complete graphs of K_n and K_m have distinct orders. Even if two nonisomorphic graphs have the same order, though, this does not imply that their uniformity sequence must be different. Of course, by Corollary 8, complementary graphs have the same uniformity sequence. However, the graphs G and H of Figure 10 are nonisomorphic, noncomplementary, and have order 4, yet $s(G) = s(H) = \{2,3,4,5,\ldots\}$.

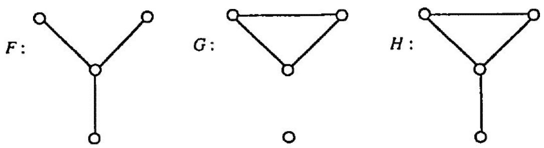

Figure 10. Graphs with the same uniformity sequence.

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The authors prepared this paper while they were undergraduate students at Western Michigan University.

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INTRINSIC REACTION COORDINATE METHODOLOGIES: COMPARATIVE ANALYSES

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INTRODUCTION

The more progress physical sciences make, the more they tend to enter the domain of mathematics, which is a kind of centre to which they all converge. We may even judge the degree of perfection to which a science has arrived by the facility with which it may be submitted to calculation.' Adolphe Quetelet, 1796-1874

Chemical calculations that predict structures, energetics, and other properties of experimentally known or unknown molecules provide a fundamental resource for chemical research today. The basis of these calculations lies in an area of theoretical chemistry called molecular quantum mechanics. This is a science that relates molecular properties to the motion and interaction of electrons and nuclei. Since the chemical properties of atoms and molecules are determined by their electronic structure, it is necessary to understand the nature of the motions and energies of the electrons and nuclei.

This, in turn, requires understanding the highly mathematical formulations that predict molecular structure and properties, and thermodynamic and reaction processes. This paper discusses the numerical techniques used to calculate reaction paths; paths which lead from reactant to product species in reaction processes.

BACKGROUND

Soon after the formulation of molecular quantum mechanics in **1925**,[2] it was determined that solving the Schrodinger differential equation lead to direct quantitative predictions of chemical phenomena from first principles. This *ab initio* method provided a theoretical approach to chemistry independent of laboratory experimentation.

A key computational problem in solving the molecular Schrodinger equation is the solution of the real symmetric eigensystem, known as the Hartree-Fock equations:

$$F\psi_i = \lambda_i \psi_i \tag{1}$$

Here, F is a given n x n real symmetric matrix, and (λ_i, ψ_i) is one of n **eigenvalue/eigenvector** pairs to be determined. In a given molecule, the energy of the system is dependent on both the geometry of the molecule and the placement of the electrons in orbitals around the atoms in the molecule. These energy contributions are expressed in the kinetic and potential energy terms of the n x n F matrix, the Fock matrix. The eigenvalues, λ_i, represent energy levels of the molecular orbitals, ψ_i. These molecular orbitals are represented as linear combinations of basis functions or, in chemical terms, atomic orbitals (i.e., *s* orbitals, *p* orbitals, etc.).

The matrix dimension **n** (i.e., the number of basis functions in the computation) varies with the number of electrons in the molecule and the desired accuracy of the molecular orbital function representation. Values of n on the order of a few hundred are easily reached for even moderately sized molecular systems. The individual matrix elements, which represent electron-electron interactions, involve the evaluation of $O(n^4)$ integrals, which tends to dominate the $O(n^3)$ floating point operations required for solution of the eigensystem.

An iterative technique is used to solve the self-consistent field (SCF) computation (1). This is because the Fock operator depends on its own eigenfunctions, and the Fock matrix is usually constructed from orbitals computed on the previous iteration. Thus, a sequence of eigensystems must be solved until convergence (or self-consistency) is attained. Moreover, the SCF computation often is the inner iteration in a geometry optimization in which the nuclear coordinates are optimized with respect to energy. This means that a single geometry optimization for a molecule with even a few heavy atoms may require the solution of hundreds of real symmetric eigensystems.

These calculations begin with a Cartesianal representation of the molecular system. In a many-atom molecule, three coordinates define the location of each atom in space. Of these $3N$ total coordinates, 3 translational and 2 (linear molecules) or 3 (non-linear) rotational degrees of freedom can be ignored because energy is invariant to these motions in the overall molecule. The remaining $3N-6$ (or $3N-5$) coordinates define the vibrations of the molecule, **i.e.,** bond stretches and angle distortions.

In a chemical reaction, the key structures are the reactants (molecules present at the onset of a reaction), the products (molecules resulting from some chemical reaction between the reactant species), and the transition state (a high energy complex through which the reactants must traverse for the reaction to occur). Mathematically (Figure **1**), the reactants and products are at the bottom of a well on the potential energy surface (PES), having a zero gradient and positive curvature. The transition state is located at a saddle point on the PES. This point has a zero gradient, but in contrast to the stationary points (reactants and products), has one imaginary frequency (obtained from the diagonalization of the second derivative matrix of energy with respect to coordinates). This frequency corresponds to the one and only one downward curvature. Following this frequency, from the transition state to either the reactants or the products, provides a preferred path along the bottom of the valleys connecting these structures, called the minimum energy path (MEP).

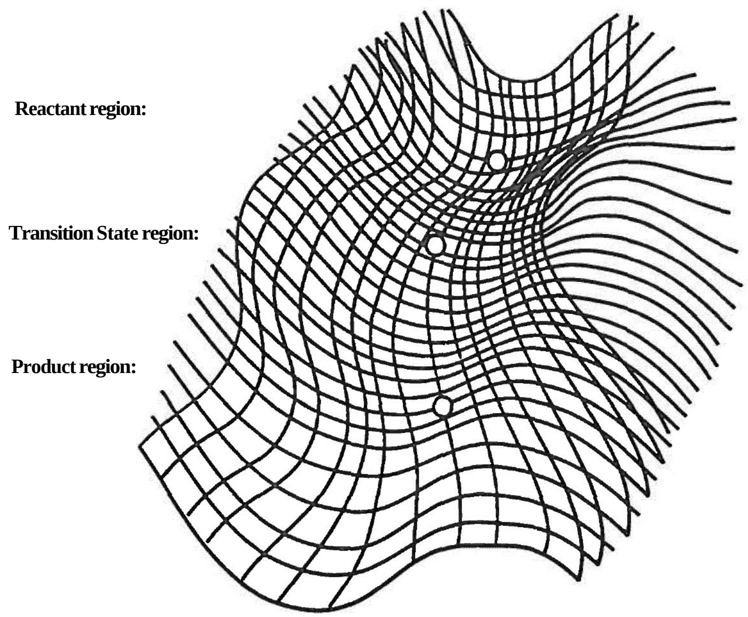

Figure I.

The mathematical determination of the MEP requires solving a set of simultaneous differential equations. The reaction path is defined in terms of the intrinsic reaction coordinate (IRC)[3], s, which is followed in moving along the MEP from reactants to products. This reaction coordinate represents a structural and energetic progression as the system proceeds from reactants to products. The following set of IRC equations gives the desired path:

$$\begin{aligned}\frac{dx}{ds} &= f(s,x)\\ x &= [(m_A/\mu)^{1/2}x_A,(m_B/\mu)^{1/2}x_B,\ldots]\end{aligned} \tag{2}$$

where $x_A, x_B, \ldots$ are the coordinates of atoms $A, B, \ldots$; μ is reduced mass of reactants; $m_A, m_B, \ldots$ are the masses of the atoms; $f(s,x) = -\nabla V/|\nabla V|$ is defined to be a unit vector in the negative direction of the normalized gradient of the potential.

The complexity of the reaction path problem is due to these multidimensional equations. When these equations are integrated, the following equation is obtained:

$$x(s_{n+1}) = x(s_n) + \int_{s_n}^{s_{n+1}} f(s,x(s))ds \tag{3}$$

Because $x(s)$ is unknown, an interpolating polynomial is used for $f(s,z)$. Various solution methods can be obtained by inserting a different interpolating polynomial. Although the resulting methods are not new to mathematics, their particular application to quantum chemistry has yet to be fully understood and developed. One of the first discussions of reaction path following was presented by Fukui[4] in 1970. Today, researchers are interested in finding the most efficient methods for following the MEP.

Calculations of chemical reaction structures can help in understanding the kinetics of a reaction. An upper bound on the kinetics for the reaction can be calculated from solving the transition states, reactant, and product structures. Knowledge of even more points along this path allows one to include such effects as reaction path curvature and tunneling effects, both of which improve approximation of the predicted reaction kinetics.

Ab *initio* prediction of accurate rate constants is limited by the cost of calculating sufficient information on a PES. Advancements made in gradient calculations and higher-order derivatives has been an important factor in reducing the computational effort[5]. Another concern is the accuracy with which the MEP for the reaction must be calculated to obtain a converged thermal rate constant.

The methods presented here include both basic one-step and complex methods. All methods considered require only single-point energy and first derivative calculations. The complex methods are required for chemical reaction paths governed by a stiff set of differential equations because the time constants of the variables differ greatly (stiff terms). Applying standard numerical techniques to differential equations governing the dynamic behavior of very stiff systems is often difficult.[6] To maintain stability, the step size must be extremely small in these systems since the small time constants decay rapidly. For example, the IRC equations for a reaction in which the frequency of one internal coordinate is diminishing rapidly while another is increasing very slowly as a function of reaction time s. Most standard numerical techniques have poor round-off characteristics when applied to stiff systems because round-off errors tend to cover up the decay of the solution.

METHODS

Because the gradient at the saddle point is zero, the IRC is initiated by a small displacement in the direction of the imaginary normal mode. Within the harmonic approximation, the energy lowering, ΔE, for a given step, h, is approximately

$$\Delta E = \frac{kh^2}{2} \tag{4}$$

where ΔE represents the desired energy change to take the first step away from the transition state structure; k is the negative force constant from the magnitude of the imaginary frequency; and h is the resultant step size.

Along the MEP, it is assured that the slope will not become infinite because the occurrence of an infinite derivative at a particular point implies multiple energies for that geometric structure, which is not possible. In fact, any drastic changes in energy for infinitesimal changes in the geometry are not seen, as such energy changes could result in surface hopping from the given potential energy surface to another higher energy excited surface. It is generally the case that the chemical surfaces are smooth, as long as the level of theory is high enough. The supposition that the Lipschitz condition is satisfied for the steps taken along the PES (which is a necessary condition for the application of these methods) appears valid.[7]

ONE-STEP METHODS

In one-step methods[8] (Table I, opposite), approximating a new IRC point, x_{i+1}, involves information from only one of the previous points, x_i. Although these methods use function evaluation information at points between x_i and x_{i+1}, they do not retain that information for approximating new points along the IRC.

All the information these methods use is obtained within the interval over which the solution is approximated. Euler methods are the most simplistic methods to solve Initial Value Problems (IVPs). Euler methods involve computing a discrete set of solutions knowing only the derivative at the previous point.

While it has been demonstrated[9] that Euler methods are qualitatively accurate in predicting the MEP for simple reactions process, in applications which require quantitatively accurate MEP's (e.g., prediction of reaction dynamics), results from Euler methods may not be sufficient. It is possible that more sophisticated, higher-order techniques will permit a larger step size to balance the greater amount of computer time required for complex methods. The Runge-Kutta methods have been applied to such cases.

The Runge-Kutta methods were developed to avoid the computation of higher-order derivatives required by methods such as Taylor methods.[8] Instead of the higher-order derivatives, extra function values are used to duplicate the accuracy of the Taylor methods. The major computational effort in Runge-Kutta methods is the evaluation of the function $dx/ds = y' = g(s,x)$. In the second (fourth) order methods, the local truncation error is the square (quartic) of the step size, h, i.e., $O(h^2)$ ($O(h^4)$), while the cost is two (four) function evaluations per step. Thus, there is a tradeoff between number of function values and step size in using higher order methods over lower order methods. In addition, it has been pointed out[10a,10b] that Runge-Kutta methods of order higher that 4 may not be valid for systems of equations, although this fact has shown to be controversial based on the literature. For this particular application, we have tested these methods and have shown that the Runge-Kutta methods of order 2 give unconverged reaction path properties, as do methods of order greater than 4.[10c]

The following paragraphs identify each one-step method that was considered. Details of any of these one-step methods, including derivations, can be found in standard numerical analysis texts.[8]

Traditional Euler Single Step:

$$x_{n+1} = x_n + \delta s\, V(x_n) \tag{5}$$

$V(x_n)$ is the unit vector in a direction opposite to the gradient of the potential at the point x_n, and δs is the size of the step taken along this vector. This method is a standard to compare with all other techniques because it converges to a unique solution in the limit of small step size. This method works for systems with stable gradients. However, for a fixed-step size (as far as computational time costs), the path generated will not be completely accurate.

TABLE I

Method	Error Term*	CPU Time**	N***
Traditional Euler	O(h**2)	0.3	61
ES2	O(h**2)	1	84
BEM	O(h**2)	3.7	271
TRAP	O(h**3)	1.5	61
RK2	O(h**3)	0.7	61
RK4	O(h**5)	3.5	240
QFAP	O(h**3)		
FAB2	O(h**3)		
AMPC3	O(h**4)	2.7	230
AMPC4	O(h**5)	3.1	219

One Step Methods:

Traditional Euler Method (FAPO)
Euler Method With Reaction Path Stabilization (ES2)
Backward Euler Method (BEM)
Trapezoidal Method (TRAP)
Runge-Kutta of Order 2 (RK2)
Runge-Kutta of Order 4 (RK4)
General Equation:

$$y_{(n+1)} = y_{(n)} + w_i k_i$$

$$k_i = h^* f\,(x_n + c_i h,\ y_0 + a_{ij} k_j) \qquad i = 1, v\ ;\ c_i = 0$$

Multistep Methods:

Quadratic Fixed Step Adams Predictor (QFAP)
Fixed-Stride Adams-Bashforth Method of Order 2 (FAB2)
Adapted-Stride Adams-Moulton Predictor-Corrector
Method of Order 3 (AMPC3)
Method of Order 4 (AMPC4)
General Equation:

$$y_{(n+1)} = a_1 y_n + a_2 y_{n-1} + \ldots + a_k y_{n+1-k} + h\,[\,b_0 y'_{n+1} + b_1 y'_n + \ldots + b_k y'_{n+1-k}\,]$$

explicit: use the previous k known points and gradients
implicit: use the previous k known points and gradients plus the predicted $y_{(n+1)}$ point.

* A method is conventionally called nth order if its error term is $O(h^{n+1})$
** Ratio of CPU lime of the given method to that of ES2 with step size $0.5a_0$
*** Number of steps required to walk from 0 to $-3.02a_0$ along the MEP for the reaction: $CH_3 + H_2 \rightarrow CH_4 + H$.

Euler Method With Reaction Path Stabilization **(ES2):**

To correct for the implicit reaction path deviation in the Euler method, the minimum energy point along a perpendicular bisector of successive gradients is sought. This is a better estimate of the new point along the path. The first step x^0_{n+1} is obtained in the usual fashion with the Euler method. The bisector vector is then defined as follows:

$$d_{n+1} = (V_n - V^0_{n+1})/|V_n - V^0_{n+1}|$$

The corrector step along the bisector can either be obtained iteratively, based on a small fixed step, or can be determined by a parabolic fit, based on the potential along the bisector and a finite difference approximation of the derivative.[11]

$$x_{n+1} = x^0_{n+1} + \xi d_{n+1} \tag{6}$$

If, however, the point generated is already on the minimum energy path, this correction step can introduce large errors. Careful analysis has shown stabilization should only be implemented when the angle between the two gradient vectors is less than 176°. This particular method allows larger steps than does the traditional Euler method, thus, significantly reduces the computation time by requiring fewer gradient calculations.

Runge-Kutta Order 2 **(RK2):**

There are many Runge-Kutta methods of order 2, including the midpoint method, Heun's method, and the modified Euler's method. Each of these methods was tested in our application, although only the midpoint method will he discussed here. The local error in these methods do not exceed the order of the Taylor method of order two.

$$\begin{aligned} k_1 &= \delta s V(x_n) \\ x_{n+1} &= x_n + \delta s V(x_n + 0.5k_1) \end{aligned} \tag{7}$$

The k_j's in the Runge-Kutta Methods represent intermediate points between the last known point and the one being predicted. These intermediate points are not saved after the desired point has been obtained.

Runge-Kutta Order 4 **(RK4):**

The Runge-Kutta method of order 4 can he derived using Simpson's rule for numerical integration and Euler approximations of gradients. As was true for the order 2 method, this method avoids calculation of higher derivatives.

$$\begin{aligned} k_1 &= \delta s V(x_n) \\ k_2 &= \delta s V(x_n + 0.5k_1) \\ k_3 &= \delta s V(x_n + 0.5k_2) \\ k_4 &= \delta s V(x_n + k_3) \\ x_{n+1} &= x_n + (k_1 + 2k_2 + 2k_3 + k_4)/6 \end{aligned} \tag{8}$$

Both fixed step size and variable step size RK2 and RK4 methods can be used. The variable step method predicts the next step size hased on percentage of geometrical change.[12]

MULTISTEP METHODS

Multistep methods require approximating more than one previous point along the IRC to determine the new point approximation. The Adams-Bashforth techniques are explicit methods; that is, they determine the next point explicitly in terms of previous points along the IRC. The Adams-Moulton techniques are implicit methods because the new point is determined using the value of the new point and previous points. The two techniques typically are used together and collectively in predictor-corrector methods. These involve the Bashforth formula to predict the first or the next point, followed by the Moulton formula for improvements. Although these methods are complex, they estimate error from successive approximations to each x_k value. The procedure uses an (n - 1)-step implicit Adams-Moulton method to improve an approximation from an nth step Adams-Bashforth method. Both of these methods have local truncation error of order O(h^n).

Adams-Moulton methods have been observed[8] to give considerably better results than the Adams-Bashforth method of the same order. This is partially explained by comparing an m-step Adams-Bashforth explicit method with an (m − 1)-step Adams-Moulton implicit method. Both require m evaluations of the function per step, and both have local truncation errors proportional to h^m. In general, the coefficients of the terms involving the function and the local truncation error are smaller for the Adams-Moulton methods. This leads to greater stability for the implicit methods and smaller rounding error. However, the implicit methods have the inherent weakness of having to first convert to an explicit representation for x_{n+1}, which can be difficult algebraically.

Thus, the best compromise is the explicit methods for predicting a new point, followed by refining of this prediction by using the Adams-Moulton method.

The Adams-Moulton methods involve stride adaptation. Two approximations (one from the predictor and one from the corrector) are available for each MEP point, and comparison of these allows one to estimate the uncertainty in the step. The difference in these two approximations (the error approximation) is then used to adapt the stride. This stride adaptation controls the local truncation error, and as a consequence, the global error, as one proceeds along the path. It allows one to specify a larger nominal stride and still retain accuracy in regions that are difficult to integrate.

Both advantages and disadvantages of multistep methods are pronounced as the order is increased. A higher order gives a smaller error term and a more efficient algorithm, while it requires more storage and special provisions for starting the integrator. Order four multistep methods appear to be the most useful compromise.

The general equation for any predictor or corrector equation is:

$$x_{n+1} = a_1 x_n + a_2 x_{n+1} + \ldots + a_k x_{n+k-1} + h(b_0 x'_{n+1} + b_1 x'_n + \ldots + b_k x'_{n+1-k})$$

When $b_0 = 0$, the method is called an explicit or open method and this equation gives x_{n+1} explicitly in terms of previously determined values. When b_0 does not equal zero, the method is called an implicit or closed method, since x_{n+1} occurs on both sides of the equation and is determined only in an implicit manner. The following paragraphs briefly define the multistep methods considered in this work. As with the one-step methods, details of any of these multistep methods, including derivations, can be found in standard numerical analysis texts.[8]

Adapted-Stride **Adams-Moulton** Predictor-Corrector Method of Order 3 **(AMPC3):**

Including a non-zero b_0 term in the linear k-step difference equation results in a recursive method that successively approximates a given point with the inclusion of the slope at that point. Corrections made on the step size give a tighter control on the truncation errors, and, therefore, also on the accumulated errors. The following are the predictor and corrector equations:
Predictor:

$$\begin{aligned} x^{(0)}_{n+1} &= x_n + (\delta s^{(0)}/12)[23\,V(x_n) - 16\,V(x_{n-1}) + 5\,V(x_{n-2})] \\ \delta s^{(1)} &= \delta s^{(0)}(14\epsilon/3\,|x^{(1)}_{n+1} - x^{(0)}_{n+1}|)^{1/(p+1)} \\ \epsilon &= 5 \times 10^{-6} a_0 \text{ and } p = 3 \end{aligned}$$

Corrector:

$$x^{(i)}_{n+1} = x^{(0)}_{n+1} + (\delta s^{(i-1)}/12)[5\,V(x^{(0)}_{n+1} + 8\,V(x_n) - V(x_{n-1})] \qquad i = 1, 2$$

The initiation of this method is based on one-step methods in the following way:

x_0 = transition state point (known).

x_1 = first step taken away from the saddle point (based on ΔE).

x_2 = second step (use Euler method on points x_0 and xi).

x_{2+i} = subsequent points obtained using **AMPC3** acting on the most recent three points known along the MEP.

Other methods to generate the starting values for this method can be employed; however, this method appears the most efficient. It should be noted also that, whenever the step size is changed, the method must be restarted using the one-step procedure.

Adapted-Stride **Adams-Moulton** Predictor-Corrector Method of Order 4 **(AMPC4):**

Similarly, one can arrive at the fourth order method with these equations:
Predictor:

$$x_{n+1}^{(i)} = x_n + (\delta s^{(i)}/24)\,[55\,V(x_n) - 59\,V(x_{n+1}) + 37\,V(x_{n-2}) - 9\,V(x_{n-3})]$$

Corrector:

$$x_{n+1} = x_{n+1}^{(0)} + (\delta s^{(i)}/24)\,[9\,V(x_{n+1}) + 19\,V(x_n) - 5\,V(x_{n-1}) + V(x_{n-2})]$$

The same stride correction is used as with the **AMPC3** method, p = 4.

DISCUSSION

Comparison of the methods is based on the number of function evaluations for a particular method; the CPU time required to calculate the particular IRC; and the accuracy of the method. From these criteria, it is clear that no one class of methods is better than all the others. The choice depends on the particular reaction system to which the methods are applied. One can, however, assess the behavior of methods within each class and extract guidelines on which method to use for a particular problem.

In general, one should first try the simple one-step ES2 **method** for applications that involve qualitative information about the reaction path (for example, verification that a particular transition state leads to an indicated set of reactants and products). The Euler method and its **ES2** extension are the most commonly used methods for solving the MEP equations. Provided the step size is small enough and the chemical reaction not stiff, sufficient accuracy can be **obtained.**[13] However, very small step size will in turn demand considerable CPU time investment.

The ES2 method should also be used for other more quantitative applications, ones in which the reaction system is known to be stable. This is true especially if the reaction involves many atoms. If one knows in advance that a particular reaction involves internal coordinates with widely varying time constants, then a stiff method should be used. If results require a particularly small error tolerance, one should try the multistep methods.

Comparing RK2 and RK4, the higher order method is recommended because of more accurate prediction of reaction path properties. Results obtained in this work clearly show instability in the RK2 methods. In particular, discontinuous functions of structural and energetic properties are obtained.

The Ad ——Moulton predictor-corrector methods are multistep formulas for **computation.**[8] Results indicate these methods are reliable in predicting a converged reaction path with reasonable computer time. In general, Adams-Moulton type methods give results comparable to both the traditional Euler method, with very small step size, as well as the stiff methods, **RK4.** However, not every type of multistep method is appropriate for any given type of problem. The intervals of integration may be so short that the multistep methods have little chance to demonstrate their advantage over the Runge-Kutta methods.

Overall, for complex reactions, it is best to choose the variable step RK4 method, even though this method uses more computer time, because RK4 competes with both the one-step ES2 method and the Ad ——Moulton methods in convergence properties. Mathematically, various second derivative approximation techniques include the effects of curvature with the expectation of **better convergence** properties. However, the calculation of the gradient takes up to three times as long as the calculation of a single point energy. Calculation of the Hessian (second derivative matrix) takes up to ten times that of the single point energy. Perhaps a compromise, such as the calculation of a second derivative at every n points, and at that point taking a more sophisticated step would be feasible. These modifications are currently being investigated.

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12. For the Runge-Kutta method of order 4, a value for the next step is determined based on percentage of geometrical change,

    $$q = \left(\frac{\epsilon h}{2\,|y_{i+1} - y_{L+1}|}\right)^{1/4}$$

    is used. In this formula, h = ***previous stepsize*** and $|y_{i+1} - y_{L+1}|$ = norm of the change in geometry of previous point and newly predicted point.
13. K. K. Baldridge, M. S. Gordon, and D. G. **Truhlar,** unpublished results.

The authors prepared this paper while Lisa Pederson was a senior and Kim Baldridge on staff at the San Diego Supercomputer Center. This work provided the basis for Lisa's presentation at the 1990 Pi Mu Epsilon National Meeting in Columbus, Ohio. Lisa is currently a graduate student in chemistry at Johns Hopkins University.

A NOTE CONCERNING FARE FUNCTIONS

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INTRODUCTION. Recently [1], we presented a bit of mathematics in a story-like setting. We defined a function f on a one-dimensional, ordered array of points, $x(0), x(1), x(2), \ldots, x(n)$, by means of the recursion relation

$$f(x(1)) = \frac{f(x(i-1)) + f(x(i+1))}{2} + 1$$

with $f(x(0)) = f(x(n)) = 0$ and $i = 1, 2, 3, \ldots, n-1$. For reasons growing out of the scenario, we called the functions "fare functions." Even more recently [2], we generalized the fare functions to higher dimensional arrays, giving the extended functions the somewhat imposing name "complete fare functions."

Then, not being content to give our functions their well deserved rest, we wondered what would happen if, rather than adding a constant, we subtracted it from the average of f on $x(i-1)$ and $x(i+1)$ as given in the recursion relation. With the new defining equation

$$f(x(i)) = \frac{f(x(i-1) + f(x(i+1))}{2} - C \qquad (1)$$

we proceeded to investigate the functions obtained by subtracting the positive constant C, To our pleasure, we found them to have at least one natural application.

THE FUNCTIONS. To fix the geometry in out minds, let us imagine $n+1$ points $x(i)$ on a coordinate axis so that the coordinate of $x(i)$ is simply i for $i = 0, 1, 2, \ldots, n$.

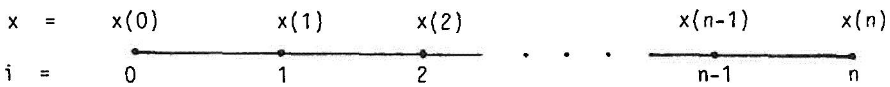

Figure 1. The points on a Coordinate Line.

Following the same sort of reasoning as given in our first reference, we made the guess that

$$f(x(i)) = Ci(i-n) + k, \qquad (2)$$

where k is any real constant, would satisfy Equation *(1)* for $i = 1, 2, 3, \ldots, n-1$. The guess turned out to be correct, as the reader can verify if so desired. Letting $i = 0$ or n yields the boundary condition that $f(x(0)) = f(x(n)) = k$. If we suppose that we have a second function $g(x(i))$ also satisfying *(1)* with $g(x(0)) = g(x(n)) = k$ for the same constant k, then the difference function $f - g$ is harmonic with value zero on the boundary of its domain. Therefore, $f(x(i)) - g(x(i)) = 0$ for every i in its domain, and we may conclude that f is the unique function satisfying Equation *(1)* for any given k.

The functions are symmetric with respect to $i = n/2$, whether or not n is even. An interesting choice of k is $Cn^2/4$ so that f, with i extended to the odd half integers if necessary, is zero at the median of $x(0), x(1), x(2), \ldots, x(n)$. If the $x(i)$'s are equally spaced on their coordinate line, then $x(n/2)$ is the midpoint of the segment with endpoints $x(0)$ and $x(n)$.

AN APPLICATION. Our application is probably misnamed as such. It is simply an observation that something happens in the "real world" that mirrors (or is mirrored by) the behavior of our functions. As others have done who think of themselves as mathematicians first rather than physicists or engineers, we went to the "real world" to seek a problem to which our functions would give a solution.

We tried to associate with our discrete linear array some process through which we could imagine a cost at each point such that the cost would increase with the distance of the point from its nearest boundary point. One idea was that of stockpiling supplies at $n+1$ points along a highway. We assumed that supplies could be assembled with equal ease at either terminus of the highway. Then it would certainly be more costly to carry one unit of supplies to a point farther from an endpoint than nearer. Our function would describe the quantity of supplies to be deposited at the i-th point. However, the "story line" soon became too artificial to hold anyone's interest for long.

Next we asked, "Why not build a bridge to be supported at $n+1$ equally spaced locations?" Would not the supporting structures farther from the banks of the stream be more difficult to put in place than those which were closer? At first we were thinking of constructing equally spaced buttresses across a stream with the most massive at the banks on either side. Our function might describe the weight of the i-th buttress. Again, we seemed to be developing a scenario too far removed from reality.

Then, rather suddenly, we realized that we were actually describing the central span of a suspension bridge. After writing down a simple differential equation with tensions and tangents, it is quite straightforward to show that a light, taut cable bearing a very heavy, horizontally uniform load (such as a roadway) takes the form of a parabolic arc.[3] The cables of a real suspension bridge in which the roadway hangs from many, equally spaced strands closely approximate arcs of parabolas.

Thus, we arrived at our application. The functions describe the lengths of the $n+1$ strands linking the roadway to a supporting cable in a suspension bridge. The scenario becomes "natural" if one feels that the first try at functions to model spatially distributed physical quantities ought to be "as close to" harmonic as possible. If C were taken to be zero, Equation *(1)* would become the defining relation for a discrete harmonic function.[4]

AN EXAMPLE. Assume that n is a very large even integer. Find C so that the lengths of the first and last of the $n+1$ strands are *500* units longer than the middle strand.

SOLUTION. Let $k = Cn^2/4 = 500$. Then $C = 2000/n^2$.

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ON QUOTIENT STRUCTURES OF Z^n

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Cyclic groups, direct products, quotient (factor) groups, group generators, and isomorphisms are fundamental concepts in an undergraduate abstract algebra course. Moreover, the group of lattice points in Euclidean n-space arises in many contexts. This note was prompted by a question that involves all of these notions; namely, what is the structure of the group $Z \oplus Z/\langle(a,b)\rangle$? This question naturally leads to related ones **such** as the structures of the group $Z^n/\langle(a_1,a_2,\ldots,a_n)\rangle$, the ring $Z^n/\langle(a_1,a_2,\ldots,a_n)\rangle$, the group $Q \oplus Q/\langle(a,b)\rangle$, the group $R \oplus R/\langle(a,b)\rangle$ and the vector space $R \oplus R/\langle(a,b)\rangle$ over R. To make the matter even more encompassing, the analysis invokes a bit of linear algebra **as** well.

It is our opinion that specific instances of these questions are useful as classroom examples, exercises, and exam problems as they challenge students to synthesize many important concepts (see [1, pp. *154,1581* and [2,p. 154]).

We proceed with the answer to our first question.

Theorem 1. $Z \oplus Z/\langle(a,b)\rangle \approx Z \oplus Z_{gcd(a,b)}$

Proof. To simplify the notation, we let $G = Z \oplus Z$, $H = \langle(a,b)\rangle$ and $d = gcd(a,b)$. First we observe that G/H is infinite. For if $a \neq b$, then $(1,1) + H$ has infinite order, while if $a = b$, $(1,0) + H$ has infinite order.

Next, write $a = a'd$, $b = b'd$, and $1 = a's + b't$. We claim that G/H is generated by $(t,-s) + H$ and $(a',b') + H$. To verify the claim, let $(m,n) + H$ be an arbitrary element of G/H and observe that because $1 = a's + b't$ is the determinant of the linear system

$$\begin{aligned} tx + a'y &= m \\ (-s)x + b'y &= n \end{aligned}$$

there are *integers* x and y so that

$$\begin{aligned} (m,n) + H &= (tx + a'y, -sx + b'y) + H \\ &= x(t,-s) + H + y(a',b') + H \\ &= x((t,-s) + H) + y((a',b') + H). \end{aligned}$$

This establishes the claim.

Next, note that $d((a',b') + H) = (a,b) + H = H$ so that $((a',b') + H)$ is isomorphic to Z_d. Moreover, since G/H is infinite, it follows that $((t,-s) + H)$ must have infinite order and therefore is isomorphic to Z. We complete the proof by noting that $((t,-s) + H) \cap ((a',b') + H)$ is the identity (since every element of the subgroup on the right has finite order while every nonidentity element in the subgroup on the left has infinite order. ∎

Since Z_1 is the trivial group, we have the following corollary.

Corollary 1. $Z \oplus Z/\langle(a,b)\rangle$ is cyclic if and only if $gcd(a,b) = 1$.

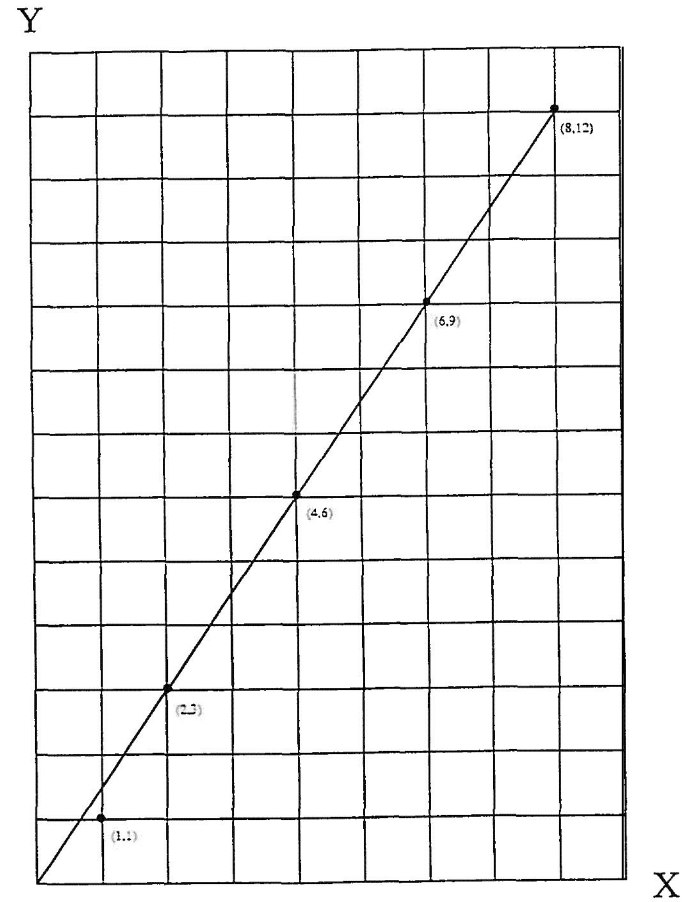
In an **Abelian** group the subgroup comprised of the elements of finite order is called the *torsion* subgraup. As another corollary of Theorem 1, we have the structure of the torsion subgroup of $Z \oplus Z/\langle(a,b)\rangle$.

Corollary 2. The torsion subgroup of $Z \oplus Z/\langle(a,b)\rangle$ is isomorphic to $Z_{gcd(a,b)}$.

The structure of $Z \oplus Z/\langle(a,b)\rangle$ as well as generators of **the** finite and infinite direct factors can be readily determined geometrically **as** follows. In the real plane, let $L(a,b)$ be the line segment from $(0,0)$ to (a,b) with $(0,0)$ deleted. Then $Z \oplus Z/\langle(a,b)\rangle$ is cyclic if and only if (a,b) is the only lattice point on $L(a,b)$; the order of the finite direct factor is the number of lattice points on $L(a,b)$; a coset representative of a generator of the finite direct factor is the lattice point on $L(a,b)$ nearest to $(0,0)$; a **coset** representative of a generator of the infinite direct factor is the lattice point colsest to $L(a,b)$ and nearest to $(0,0)$.

To illustrate, we consider $Z \oplus Z/\langle(8,12)\rangle$. From the figure below, we see that the group is not cyclic; the order of the finite direct factor is 4; a generator of the finite direct factor is $(2,3) + \langle(8,12)\rangle$; and a generator of the infinite direct factor is $(1,1) + \langle(8,12)\rangle$.

Continuing with the notation introduced in the proof of Theorem *1*, letting T denote the torsion subgroup of G/H (i.e., the subgroup isomorphic to $Z_{gcd(a,b)}$), and L the line in the real plane joining $(0,0)$ and (a,b), we can also give a description of the cosets of T in G/H. For $k > 0$, the elements of $k(t,-s) + T$ are the lattice points in the plane that are above L and a distance of $k/\sqrt{a'^2 + b'^2}$ from L; for $k < 0$, the elements of $k(t,-s) + T$ are the lattice points in the plane that are below L and a distance of $|k|/\sqrt{a'^2 + b'^2}$ form L.



Theorem 1 and its corollaries have natural extensions to higher dimensions. The proof of the general case is analogous to the $n = 2$ case and entails a (non-routine) induction argument to prove the existence of the generators.

Theorem 2. $Z^n/\langle(a_1, a_2, \dots, a_n)\rangle \approx Z^{n-1} \oplus Z_{gcd(a_1,a_2,\dots,a_n)}$.

Corollary 1. $Z^n/\langle(a_1, a_2, \dots, a_n)\rangle$ is torsion-free if and only if $\gcd(a_1, a_2, \dots, a_n) = 1$.

Corollary 2. The torsion subgroup of $Z^n/\langle(a_1, a_2, \dots, a_n)\rangle$ is isomorphic to $Z_{gcd(a_1,a_2,\dots,a_n)}$.

For $Z \oplus Z \oplus Z/\langle(a,b,c)\rangle$, we may obtain explicit generators by putting

$$d = \gcd(a,b,c),\quad a = a'd,\quad b = b'd,\quad c = c'd;$$
$$d' = \gcd(b', c'),\quad b' = b_1 d',\quad c' = c_1 d';$$
$$a'x_1 + d'y_1 = 1$$
$$b_1 x_2 + c_1 y_2 = 1.$$

The three coset representatives of the generators are: (a', b', c'), $(-y_1, b_1x_1, c_1x_1)$, and $(0, -y_2, x_2)$. Verification is left to the reader.

In sharp contrast to the simple description of the structure of the factor group $Z \oplus Z/\langle(a, b)\rangle$, a determination of the structure of the corresponding group with Z replaced by the additive group of the rational numbers, real numbers, or complex numbers is a bit beyond the scope of an undergraduate abstract algebra text. It turns out that in all of these cases the factor group is isomorphic to the direct product of groups that are isomorphic to the group of rational numbers and the group Q/Z. (See Section 5.2 of [3] for details.)

To round out our discussion we answer three related questions that might naturally occur to students.

1. What is the structure of $Z^n/\langle(a_1, 0, \dots, 0)\rangle \times \langle(0, a_2, 0, \dots, 0)\rangle \times \cdots \times \langle(0, 0, \dots, a_n)\rangle$?
2. Viewing Z^n as a ring and $\langle(a_1, a_2, \dots, a_n)\rangle$ as the principal ideal generated by $(a_1, a_2, \dots, a_n)$, what is the structure of the ring $Z^n/\langle(a_1, a_2, \dots, a_n)\rangle$?
3. Viewing R^n as a vector space over R and $\langle(a_1, a_2, \dots, a_n)\rangle$ as the subspace spanned by $(a_1, a_2, \dots, a_n)$, what is the structure of the vector space $R^n/\langle(a_1, a_2, \dots, a_n)\rangle$?

It is straightforward (in fact, good exercises for students) to prove that (yes, you guessed it) the answer to Question 1 is the group $Z_{a_1} \oplus Z_{a_2} \oplus \cdots \oplus Z_{a_n}$; the answer to Question 2 is the ring $Z_{a_1} \oplus Z_{a_2} \oplus \cdots \oplus Z_{a_n}$; and the answer to Question 3 is the vector space R^{n-1}.

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PRODUCTS OF TRIANGLE TRISECTORS

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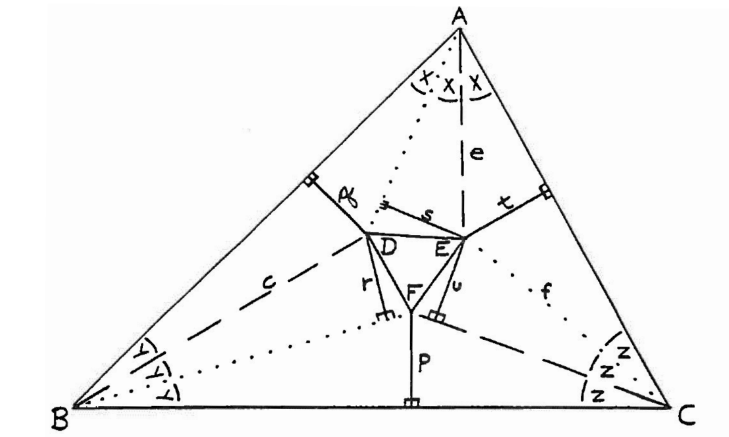
Theorem. In the accompanying arbitrary triangle, the product of the dotted line segments is equal to the product of the broken line segments.



Proof:

1) $s = t = u$

2) $q = r$

3) $u = \dfrac{f \cdot p}{FC}$

4) $s = \dfrac{e \cdot q}{AD}$

5) $r = \dfrac{c \cdot p}{BF}$

6) $\dfrac{f \cdot p}{FC} = \dfrac{e \cdot q}{AD}$

7) $e \cdot q \cdot FC = f \cdot p \cdot AD$

8) $e \cdot r \cdot FC = f \cdot p \cdot AD$

9) $\dfrac{e \cdot c \cdot p \cdot FC}{BF} = f \cdot p \cdot AD$

10) $e \cdot c \cdot FC = f \cdot AD \cdot BF$

11) $AE \cdot BD \cdot FC = CE \cdot AD \cdot BF$ ■

BEAUTIFUL THEOREMS

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A concise theorem or formula which relates ALL of the "basic" elements in some context seems somewhat of a rarity, mathematically speaking. Well known in such an extraordinary category is the Euler formula

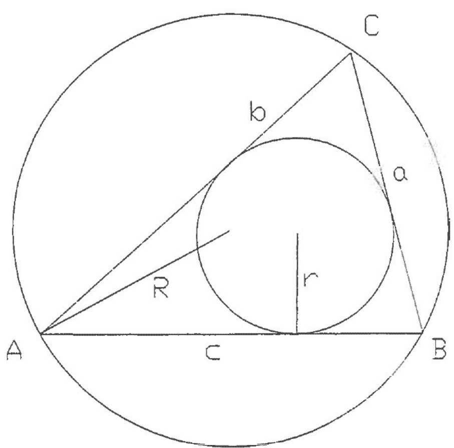
$$e^{i} + 1 = 0.$$

It is noteworthy that this formula contains five of the most crucial constants from the study of numbers, namely, 0, 1, e, π, and i. Also included is the most basic of operation symbols (+) as well as the most fundamental of relation symbols (=). Such a less-than-obvious theorem will be considered BEAUTIFUL as a consequence, not only of its importance, but also of its concise and highly inclusive nature. Other beautiful theorems appear on the mathematical landscape. Some are fairly well known. A remarkable result, one of the geometer's favorites, seems quite fitting in this overall category. Though not so well known, it involves various lengths and concerns triangles in particular. Somehow its basic components, as now follows, all come together in one impressive statement.

Consider the triangle ABC in which a, b, and c are side measures, s is the semi-perimeter, and r and R denote, respectively, the **inradius** and circumradius. Is it possible to relate all of these notables in a single, concise, and easy-to-remember equation? The answer is YES. Use of all six of the fundamental symbols is accomplished by the formula

$$4rRs = abc.$$

It provides still another look at a beautiful theorem. This result, likely known in varying forms in ancient times, is also associated with the works of Euler. In some accounts, it bears his name.



$$4rRs = abc$$

Such a beautiful theorem can be established by lengthy and complicated methods. However, the novel one below is quite instructive and definitely within the solving range of the secondary school or college trigonometry student. Begin by letting the area K equal $\frac{1}{2}ab\sin C$, or equivalently, $4Kc = 2abc\sin C$. By writing this result as a proportion, it follows that

$$\frac{c}{2\sin C} = \frac{abc}{4K}.$$

The circumradius is given by $R = \frac{c}{2\sin C}$, meaning that

$$R = \frac{abc}{4K}.$$

But $K = rs$, in which case

$$R = \frac{abc}{4rs} \quad \text{or} \quad 4rRs = abc.$$

Since $4s$ is twice the perimeter, the formula may be expressed in the alternate but highly impressive form

$$\frac{abc}{a+b+c} = 2rR.$$

Other beautiful theorems come to mind. Some are more advanced, such as the Law of Quadratic Reciprocity (which was conjectured but not proved by Euler). Others are fairly close at hand. Note, for example, the concise relationship

$$\mathrm{Tan}^{-1}1 + \mathrm{Tan}^{-1}2 + \mathrm{Tan}^{-1}3 = \pi,$$

or the formula for triangles given by

$$\tan A + \tan B + \tan C = (\tan A)(\tan B)(\tan C),$$

or the famous Eulerian formula relating the number of faces, vertices, and edges of a polyhedron, namely,

$$F + V = E + 2.$$

The beautiful theorem need not take the form of an equation as happens above. It may express a relationship among notable elements in a way which does not suggest the equation but, instead, something just as impressive. The Euler Line Theorem, for example, fits this mold nicely. It reveals that the centroid, the orthocenter, and the circumcenter of ANY triangle will always lie on a line. And then, by the various standards, there are the "pretty" theorems of Desargues, **Pappus,** Pascal, Brianchon, and others.

The word "beautiful" of course refers to the subjective. It likely casts mathematics in the light of an ART as opposed to a SCIENCE. Within the framework of diverse criteria, quite a range of theorems, some practical and some not so practical, easily become the object of aesthetic interest. In the judgment of many, Euler's concise results prove outstanding, insightful, and – beautiful. Do you have a prime candidate for a theorem or result in such a category?

A NOTE ON A DIE'FERENCE EQUATION

Russell *Euler*
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Let a and b be nonzero parameters. If $a \neq b$, then two linearly independent solutions of the difference equation

$$y_{n+2} - (a+b)y_{n+1} + aby_n = 0 \tag{1}$$

are a^n and b^n, and the general solution of (1) is

$$y_n = c_1 a^n + c_2 b^n, \tag{2}$$

where c_1 and c_2 are arbitrary constants.

When $a = b$, the two fundamental solutions of (1) are $y_n^{(1)} = a^n$ and $y_n^{(2)} = na^n$. This is easy to check but not so easy to motivate, especially $y_n^{(2)}$. The motivation of the form of the general solution in the case of equal roots of the characteristic equation can be accomplished by rearranging the terms in equation (2) for the case when $a \neq b$, renaming the constants, and considering the limit as the parameter b approaches a. To achieve this, by adding and subtracting the term c_2a^n, equation (2) can be written as

$$y_n = (c_1 + c_2)a^n + c_2(b^n - a^n).$$

For $a \neq b$, multiplying and dividing the second term of this equation by $b-a$ will change the solution into a form that will lend itself to using L'Hospital's rule when the limit is taken. So, for $a \neq b$, the general solution of (1) becomes

$$y_n = (c_1 + c_2)a^n + c_2(b-a)\frac{b^n - a^n}{b-a},$$

or

$$y_n = c_3a^n + c_4\frac{b^n - a^n}{b-a}, \tag{3}$$

where $c_3 = c_1 + c_2$ and $c_4 = c_2(b-a)$.

Notice that when $a = b$, the second term in equation (3) is of the indeterminate form $\frac{0}{0}$. So, employing L'Hospital's rule to compute the limit as b approaches a in (3) yields

$$\begin{aligned} y_n &= c_3a^n + c_4na^{n-1}, \\ &= c_3a^n + c_5na^n, \end{aligned} \tag{4}$$

where $c_5 = c_4/a$. Equation (4) is the general solution of (1) when $a = b$ and the technique utilized clearly shows how $y_n^{(2)} = na^n$ arises.

As an alternative to using L'Hospital's rule on the second term in (3), the following method can be used. Since

$$b^n - a^n = (b-a)(b^{n-1} + b^{n-2}a + b^{n-3}a^2 + \cdots + ba^{n-2} + a^{n-1}),$$

we have

$$\begin{aligned} \lim_{b\to a}\frac{b^n - a^n}{b-a} &= \lim_{b\to a}(b^{n-1} + b^{n-2}a + b^{n-3}a^2 + \cdots ba^{n-2} + a^{n-1}), \\ &= a^{n-1} + a^{n-1} + \cdots a^{n-1} \\ &= na^{n-1}. \end{aligned}$$

Hence, (3) becomes $y_n = c_3a^n + c_4na^{n-1}$ as before.

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EDITOR'S NOTE

Several careful readers have pointed out some miscalculations in the paper **"Fractorials!"** by Nataniel **Greene**, which appeared in the **Fall**, 1992, issue of the Journal. On p. 431, $9!_3$ should equal **162**; on p. 433, $18!_3$ should equal $[(6)(3)]!_3$; in the example following the proof of Corollary 4 on p. 433, x should equal $1/[2 \cdot 189^{1/4}]$; on p. 435 in the statement of Theorem 8, $a!_b$ should equal $a!_{bh}(a-b)_{bh}(a-2b)_{bh}\cdots[a-(h-1)b]!_{bh}$. Finally, in Example 3 on p. 436, $(10!_3)/(2^4) = 17.5$ and not 35 as indicated. Thus we use the inequality $10!_3/2^4 = 17.5 < 20 = (2x)!/(2^4) < 12!_3/(2^4)$. Solving as in the example we find that $x \approx 5.0463$

The Editor apologizes for any confusion that may have been caused.

WHAT'S YOUR SINE?

It was Robert of Chester's translation from the Arabic that resulted in our word "sine." The Hindus had given the name jiva to the half chord in trigonometry, and the Arabs had taken this over as jiba. In the Arabic language there is also a word jaib meaning "bay" or "inlet." When Robert of Chester came to translate the technical word *jiba*, he seems to have confused this with the word jaib (perhaps because vowels were omitted); hence he used the word sinus, the Latin word for "bay" or "inlet."

Carl B. Boyer, A History of Mathematics, John Wiley & Sons, 1968, p. 278.

Editor's note: This same translation story is also attributed to Gerard of Cremona. (See Howard Eves, An *Introduction* to the History of *Mathematics*, Fourth Edition, Holt, Rinehart and Winston, 1976, p. 194.) Both references indicate that the time of the translation into Latin was in the year 1150 A.D.

CHANGES OF ADDRESS

Subscribers to the Journal should keep the Editor informed of changes in mailing address. Journals are mailed at bulk rate and are not forwarded by the postal system. The cost of sending replacement copies by first class mail is prohibitive.

A PARTIAL FRACTIONS APPROACH TO A FAMILIAR IDENTITY

M. A. Khan, RDSO
Lucknow, India

The identity

$$\sum_{k=1}^{n}(-1)^{n-k}\binom{n}{k}k^n = n!$$

can be established either by using the combinatorial argument of **distributing** n balls in n boxes, or the operator technique. (See [1].) Here is an alternative approach based on partial fractions wherein we show that the LHS is the expansion of the RHS rather than proving that the RHS is the closed form of the **LHS.**

To this end, we start with

$$n! = 1\cdot 2\cdot 3\cdots n$$
$$= \frac{1}{1}\cdot\frac{1}{(1-1/2)}\cdot\frac{1}{(1-2/3)}\cdots\frac{1}{[1-(k-1)/k]}\cdots\frac{1}{[1-(n-1)/n]}$$

Now **consider** the following continued product:

$$\frac{1}{x(x-1/2)\cdots[x-(k-1)/k)]\cdots[x-(n-1)/n]} \tag{1}$$

We resolve *(1)* into partial fractions of the form *(2):*

$$\frac{a(1)}{x}+\frac{a(2)}{x-1/2}+\cdots+\frac{a(k)}{x-(k-1)/k}+\cdots\frac{a(n)}{x-(n-1)/n} \tag{2}$$

We need only determine the general coefficient $a(k)$ in expression *(2)*. To accomplish this, we set expression (1) *identically* equal to (2), multiply both sides by $[x-(k-1)/k]$, and take the limit as x tends to $(k-1)/k$. **This** yields:

$$a(k) = \frac{k}{k-1}\frac{2k}{k-2}\cdots\frac{(k-1)k}{1}\frac{(k+1)k}{-1}\frac{(k+2)k}{-2}\cdots\frac{nk}{-(n-k)}$$
$$= (-1)^{(n-k)}\frac{n!}{k!(n-k)!}k^{(n-1)} \quad \text{(on multiplying numerator and denominator by k)}$$
$$= (-1)^{(n-k)}\cdot\binom{n}{k}\cdot k^{(n-1)}$$

Since the partial fraction representation of *(1)* is valid for all values of x except those for which $x = (k-1)/k$, for $k = 1,\ldots,n$, we may put $x = 1$ on both sides of *(1)* and *(2)* to obtain:

$$\sum_{k=1}^{n}\frac{a(k)}{1-(k-1)/k} = n!$$

which, on restoring the value of $a(k)$, implies that

$$\sum_{k=1}^{n}(-1)^{(n-k)}\cdot\binom{n}{k}\cdot k^n = n!$$

This technique can be applied to more general problems of this type. For instance, it can readily be shown by resolving the RHS into partial **fractions** that:

$$\sum_{k=0}^{n}\frac{(-1)^k}{k+x}\binom{n}{k} = \frac{n!}{x(x+1)(x+2)\cdots(x+n)}$$

and

$$\sum_{k=0}^{n}(-1)^k\frac{P_m(k)}{x+k}\binom{n}{k} = \frac{P_m(-x)\cdot n!}{x(x+1)(x+2)\cdots(x+n)},$$

where $P_m(k)$ is a rational polynomial in k of degree m and m $\leq n$.

This article is dedicated to Sir Syed Ahmad Khan, founder of Muslim University, Aligarb.

References

1. M. R. Spiegel, *Calculus of Finite Differences and Difference Equations*
2. M. R. Spiegel, *Theory and Problems of Laplace Transforms*

MESSAGE FROM THE SECRETARY-TREASURER

Copies of the new, revised Constitution and Bylaws are now available. The prices are: *$1.50* for each of the first four copies and **$1** for each copy thereafter. **I.e., $(1.50 n)** for $n < 4$ and $\$(n+2)$ for $n \geq 4$.

The videotape of Professor **Joseph** A. **Gallian's** AMS-MAA-PME Invited Address, "The Mathematics of Identification Numbers," given as part of **PME's** 75th Anniversary Celebration at Boulder, **CO,** in August, *1989,* **is** also still available. The tape may be borrowed free of charge by PME chapters, and by others upon an advance payment of **$10.** Please contact my office if you desire to borrow the tape, telling me the date on which you would like to use it. I prefer to mail the tape directly to faculty advisors, and expect them to take responsibility for returning it to my office. Please submit your request in writing and include a phone number and a time that I might reach you if there are problems. Robert **M.** Woodside, Secretary-Treasurer, Department of Mathematics, East Carolina University, Greenville, NC *27858.*

Seen on the back of a Math Club T-shirt:

Top N reasons for being a mathematician:

1. When people don't understand you, they think it's their own fault.
2. see # 1.
3. See # 1.

N. See #*1*. •

⋮

• If you don't understand this, see # *1.*

A THEOREM ON CIRCUMSCRIBED CIRCLES

Jun Ozone
Tochigi Minami Senior High School, Japan

Within the angle formed by intersecting rays, inscribe a chain of circles $C_1, C_2, C_3, \ldots$ such that C_1 is closest to the vertex of the angle and each circle C_n, $n > 1$, is tangent to the two rays and tangent externally to the two circles C_{n-1} and C_{n+1}. Then it is easy to show that the radii of the circles form a geometric sequence. The circles $A_{-2}, A_{-1}, C_0, A_1, A_2$ of Figure *1* form such a sequence of circles, which we shall call a *vee sequence* (of *circles*).

Vee sequences are a suitable project topic for high school students, and related questions are sometimes given in entrance examinations for Japanese universities. Furthermore, we find this type of question in *Wasan*, the old **mathematics** of 17th to 19th century Japan. In this article we shall point out some properties of vee sequences of circles, especially in the light of Casey's and Monge's theorems.

Theorem **1**. Suppose $\{C_0, A_i(i = \pm1, \pm2, \ldots)\}$ and $\{C_0, B_i(i = \pm1, \pm2, \ldots)\}$ are two vee sequences sharing the common central circle C_0 and whose intersecting rays have vertices A and B, respectively, as shown in Figure *1*. Let $r_0, a_i, b_i(i = \pm1, \pm2, \ldots)$ denote the radii of the circles C_0, A_i, and B_i, respectively. Then, for each i, we have that $a_{-i}a_i = b_{-i}b_i$.

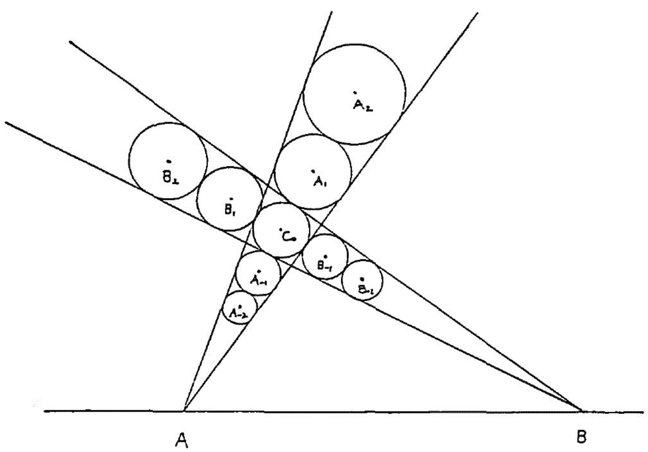

Figure 1.

The proof follows easily from the fact that the radii ate in geometric progression, so for any **fixed** i, r_0 is the geometric mean of a_i and a_{-i}, whence $a_{-i}a_i = r_0^2$. Similarly, $b_{-i}b_i = r_0^2$, so $a_{-i}a_i = b_{-i}b_i$. •

Monge's Theorem, which is fundamental to projective geometry, states that if three circles are given, of three **different** radii and no two of which are concentric, then the line connecting the centers of similitude of two pairs of the circles will **pass** through the center of similitude of the third pair [3, Thm. *25.9*, p. **109**]. The next theorem is an immediate consequence of Monge's Theorem.

Theorem 2. In the vee sequence of Theorem **1**, the intersection point of the external common tangents to arbitrary circles A_i and B_j lies on the line AB. If the radii of the two circles are equal, then their common tangents are parallel to line AB.

Casey's Theorem, stated next, is a delightful extension of Ptolemy's cyclic quadrilateral theorem. [Ptolemy's theorem states that if a convex quadrilateral $ABCD$ is *cyclic* (can be inscribed in a circle), then the product of its diagonals is equal to the sum of the products of its opposite sides; **that** is, $AC \cdot BD = AB \cdot CD + AD \cdot BC$. The converse of this theorem is *also* true: if the equation $AC \cdot BD = AB \cdot CD + AD \cdot BC$ holds, then quadrilateral $ABCD$ is cyclic.] In fact, Ptolemy's theorem is the special case where the four circles of Casey's Theorem **all** have radius zero.

Casey's Theorem. Let C be a given circle and let C_1, C_2, C_3, and C_4 be four circles with distinct centers that form a convex quadrilateral $C_1C_2C_3C_4$ having diagonals C_1C_3 and C_2C_4. If **circle** C is tangent to each of the four circles C_1, C_2, C_3, and C_4, then

$$t_{12}t_{34} + t_{14}t_{32} = t_{13}t_{24},$$

where t_{ij} is the length of the common external tangent to circles C_i and C_j if the two circles C_i and C_j lie both outside or neither one outside of circle C, and t_{ij} is the length of the common internal tangent to circles C_i and C_j if these two circles lie one outside and the other not outside of circle C.

Casey's proof of this theorem [1, Prop. *10*, p. **103**] cleverly uses inversion. We shall present a proof that is readily accessible to any high school student possessing a reasonable knowledge of trigonometry and geometry.

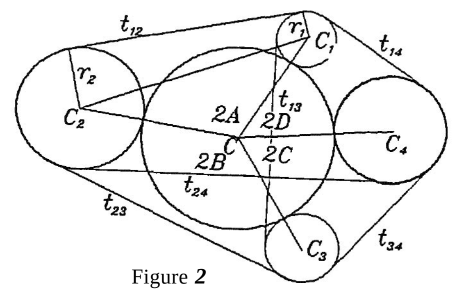

Figure *2*

Proof. Let there he an inscribed circle of center C and radius r for the four given circles C_1, C_2, C_3, and C_4, with distinct centers, and let each circle with center C_i have radius r_i. Let the angles $C_1CC_2, C_2CC_3, C_3CC_4$, and C_4CC_1 be denoted by $2A, 2B, 2C$, and $2D$, respectively. (See **Figure** *2*.) Then $A + B + C + D = \pi$. Now by the law of cosines, we have that

$$C_1C_2^2 = (r + r_1)^2 + (r + r_2)^2 - 2(r + r_1)(r + r_2)\cos 2A,$$

and by the Pythagorean theorem,

$$t_{12}^2 = C_1C_2^2 - (r_1 - r_2)^2.$$

After some **simplification**, we have that

$$t_{12}^2 = 2(r + r_1)(r + r_2)(1 - \cos 2A)$$

and finally,

$$t_{12}^2 = 4(r + r_1)(r + r_2)\sin^2 A.$$

Similar expressions hold for the other tangent lengths *tjj*. For convenience we let

$$q = 4\sqrt{(r + r_1)(r + r_2)(r + r_3)(r + r_4)}.$$

Then

$$t_{12}t_{34} = q\sin A\sin C, \quad t_{23}t_{41} = q\sin B\sin D, \quad \text{and} \quad t_{12}t_{34} = q\sin(A+B)\sin(B+C).$$

Now we have that

$$\begin{aligned}
\sin B\sin D &= \sin B\sin(\pi-(A+B+C)) \\
&= \sin B\sin(A+B+C) \\
&= \sin B\sin(A+B)\cos C+\sin B\cos(A+B)\sin C \\
&= \sin(A+B)\sin B\cos C+\sin B\cos A\cos B\sin C-\sin A\sin^2 B\sin C \\
&= \sin(A+B)\sin B\cos C+\sin B\cos A\cos B\sin C+\sin A\cos^2 B\sin C-\sin A\sin C \\
&= \sin(A+B)\sin B\cos C+(\sin B\cos A+\sin A\cos B)\cos B\sin C-\sin A\sin C \\
&= \sin(A+B)(\sin B\cos C+\cos B\sin C)-\sin A\sin C \\
&= \sin(A+B)\sin(B+C)+\sin A\sin C,
\end{aligned}$$

which establishes that

$$\sin(A+B)\sin(B+C) = \sin A\sin C+\sin B\sin D,$$

proving that the equation of the theorem holds.

If any circle C_i does not lie outside the circle C that touches the four given circles, that is, if the interior of circle C and the interior of circle C_i have a nonempty intersection, then the above argument holds if each occurrence of r_i is replaced by $-r_i$ and any resulting negative $(r-r_i)$ or $(\pm r_i \pm r_j)$ is replaced by its absolute value $|r-r_i|$ or $|\pm r_i \pm r_j|$. ■

Theorem 3. Let A_{-1}, C_0, A_1 be a vee chain of circles with vertex A, and let C be any circle that circumscribes circles A_{-1} and A_1. Then point A lies on the radical axis of circles C and C_0.

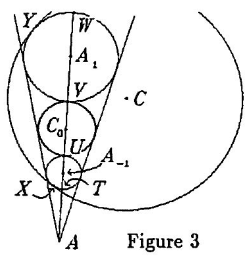

Figure 3

Proof: Let T, U, V, and W be the intersections of the ray AC_0 with the three circles of the vee chain emanating from A as shown in Figure 3. Invert the figure in a circle centered at A and orthogonal to circle C_0. Then circle C_0 is self- inverse and the two rays emanating from A are self-inverse. Circle A_{-1} inverts to a circle tangent to the two rays and to circle C_0, namely, circle A_1. So, the point X of tangency of circles A_{-1} and C inverts to a point Y on circle A_1.

Now any circle through X and Y is self-inverse, but there is only one circle passing through both X and Y and tangent to circle A_{-1} at X. Since inversion preserves angles between curves, that self-inverse circle is tangent to circle A_1 at Y; that is, it is circle C, the unique circle tangent to both circles A_1 and A_{-1} and passing through X. Now circles C and C_0 both are self-inverse with respect to the stated inversion in center A, so they have equal powers from point A. Thus A is on their radical axis. ■

Theorem 4. Let A and B be any two circles tangent to both rays of an angle with vertex P, and let C be any circle externally tangent to both circles A and B. Then circles A and B are images of one another in the inversion in the circle centered at P and orthogonal to circle C.

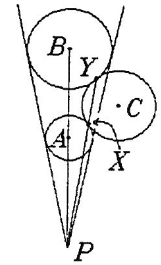

Figure 4

Proof: Let X be the point of tangency of circles A and C, and let Y be the image of X under the stated inversion. (See Figure 4.) Then Y lies on circle C and on the image circle of circle A, which image circle must also be tangent to the two rays. The only circles externally tangent to circle C and tangent to both rays emanating from P are circles A and B. Since circle A is not self-inverse, then the image circle is circle B. ■

Theorem 5. Let A_{-1}, C_0, A_1 be a vee chain of circles with vertex A, and B_{-1}, C_0, B_1 a vee chain with vertex B, each vertex lying external to the angle of the other vee chain. Then there is a circle C that circumscribes circles A_{-1}, A_1, B_{-1}, and B_1.

Proof: The radical axis in the proof of Theorem 3 is the line through A that is perpendicular to the line of centers of circles C and C_0. By adjusting C_0, that radical axis can be made to be any line through A that lies external to the angle containing the vee chain with vertex A. For example, if the center of circle C_0 lies on the line of centers of the given vee chain, then the radical axis is the line through A perpendicular to that line of centers. By increasing the radius of circle C_0 and letting its points of tangency with circles A_{-1} and A_1 slide along the left side of those circles, one can see that the radical axis revolves about A and approaches the left bounding ray of the vee chain.

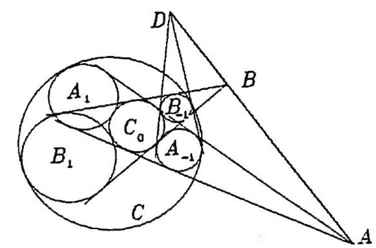

Figure 5.

Thus draw the circle C_0 that circumscribes A–, and A_1 and whose radical axis with circle C is the line AB. By Theorem 2, the common external tangents to circles A_{-1} and B_{-1} meet at a point D on line AB. (See Figure 5.) Since D is then on the radical axis of circles C and C_0, Theorem 4 applies. That is, in an inversion in center D and circle orthogonal to circles C and C_0, circle A_{-1} maps to circle B_{-1}. Since circle A_{-1} is tangent to the self-inverse circle C, then so also is circle B_{-1} tangent to circle C.

Now invert the vee chain with vertex B in a circle centered at B and orthogonal to circle C_0. Then circle B_{-1} maps to circle B_1, and circle C is fixed. Since circle B_{-1} is tangent to circle C, then its image B_1 is also tangent to circle C.■

Corollary 1. For each i, there is a circle that circumscribes the four circles A_i, A_{-i}, B_i, and B_{-i}. (See Figure 6.)

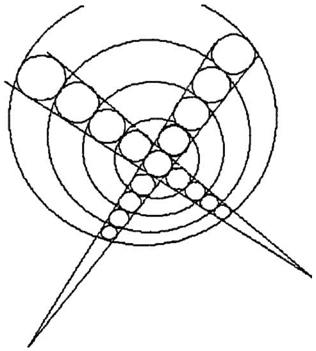

Figure 6.

Proof. The inversion arguments above all hold when the subscripts 1 and -1 are replaced by i and $-i$, respectively. ∎

Corollary 2. Theorem 5 can be applied repeatedly to vee chains emanating from points A and B in Figure 5 and tangent to circle C to produce the chains of Figure 7.

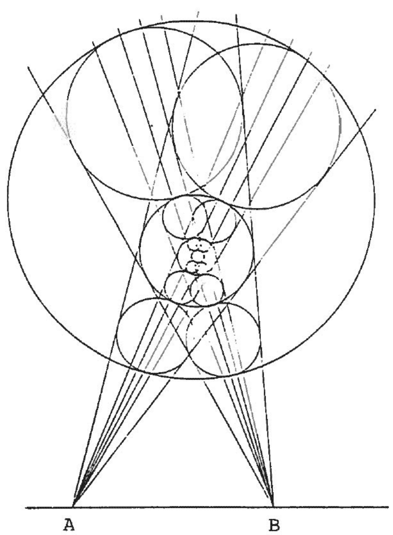

Figure 7.

The main theorem of this paper is an immediate corollary of Theorem 5.

Theorem 6. From any point P on line AB, draw tangent rays to circle C_0. Then the vee sequence of circles C_{-1}, C_0, C_1 thus determined is circumscribed by circle C. (See Figure 8.)

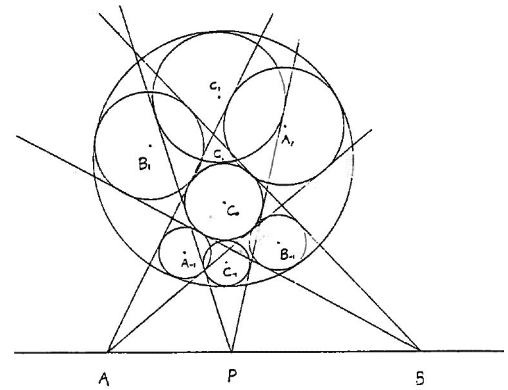

Figure 8.

One final result completes our study of vee sequences of circles.

Theorem 7. Let d denote the distance between the centers C and C_0 of Figure 3 and let r and r_0 be the radii of circles C and C_0. Then $d^2 = r^2 - 2rr_0 - 3r_0^2$.

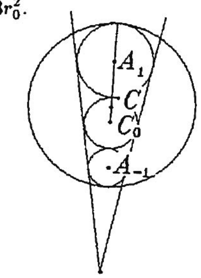

Figure 9.

Proof. Take that vee chain of circles A_{-1}, C_0, A_1 whose vertex A is the intersection of the radical axis and the line of centers of the two given circles C and C_0. (See Figure 9.) Then we have $r = r_1 + r_0 + r_{-1}$ and $r_0^2 = r_1 r_{-1}$ because these radii are in geometric progression. Multiply the former equation by r_1 and then replace its last term using the latter equation to get

$$rr_1 = r_1^2 + r_0 r_1 + r_0^2 \qquad \text{and} \qquad -r_0^2 = r_1^2 + r_0 r_1 - rr_1.$$

Now, $d = 2r_1 + r_0 - r$, so

$$d^2 = 4r_1^2 + 4r_0 r_1 - 4rr_1 + r_0^2 + r^2 - 2rr_0$$

and finally,

$$d^2 = -4r_0^2 + r_0^2 + r^2 - 2rr_0 = r^2 - 2rr_0 - 3r_0^2. \blacksquare$$

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THE LEAST SQUARES LINE WITHOUT CALCULUS

Norman Schaumberger
Hofstra University

Let

$$y = b_0 + b_1 x \tag{1}$$

be the straight line that best fits the n points $(x_1, y_1), (x_2, y_2), \ldots (x_n, y_n)$ by the method of least squares. In this note we use basic algebra to verify that the familiar equations

$$\begin{aligned} nb_0 + b_1 \sum x_i &= \sum y_i \\ b_0 \sum x_i + b_1 x_i^2 &= \sum x_i y_i \end{aligned} \tag{2}$$

can be used to get the b_0 and b_1 in (1).

The standard method for deriving (2) uses partial derivatives to minimize the function

$$f(b_0, b_1) = \sum_1^n (y_i - b_0 - b_1 x_i)^2.$$

Thus the method of least squares will more readily fit into a precalculus, survey, or statistics course which does not require calculus.

If $y = a_0 + a_1 x$ is any straight line in the plane, it follows that

$$\sum (y_i - a_0 - a_1 x_i)^2 - \sum (y_i - b_0 - b_1 x_i)^2 =$$
$$\sum [(a_0 + a_1 x_i)^2 - (b_0 + b_1 x_i)^2 + (2b_0 - 2a_0) \sum y_i + (2b_1 - 2a_1) \sum x_i y_i.$$

Using (2), this becomes

$$\begin{aligned} &\sum [(a_0 + a_1 x_i)^2 - (b_0 + b_1 x_i)^2 + (2b_0 - 2a_0)(b_0 + b_1 x_i) + (2b_1 - 2a_1)(b_0 x_i + b_1 x_i^2)] = \\ &\sum [(a_0 + a_1 x_i)^2 - (b_0 + b_1 x_i)^2 + 2(b_0 + b_1 x_i)^2 - 2(a_0 + a_1 x_i)(b_0 + b_1 x_i)] = \\ &\sum [(a_0 + a_1 x_i)^2 + (b_0 + b_1 x_i)^2 - 2(a_0 + a_1 x_i)(b_0 + b_1 x_i)] \geq 0. \end{aligned}$$

MATCHING PRIZE FUND

If your chapter presents award for Outstanding Mathematical Papers or for Student Achievement on Mathematics, you may apply to the National Office for an amount equal to that spent by your Chapter, up to a maximum of fifty dollars. Contact Professor Robert Woodside, Secretary-Treasurer.

PUZZLE SECTION

SOLUTION TO MATHACROSTIC NO. 35 (FALL, 1992)

WORDS:

A.	kohlrabi	O.	asthenosphere
B.	assurgent	P.	Neujmin
C.	nervy	Q.	drift
D	decussate	R.	lang syne
E.	incarnadine	S.	Invisible
F.	Nude Descending	T.	neon sign
G.	salmagundi	U.	esthesis
H.	kookaburra	V.	twiddle
I.	yawp	W.	overstrew
J.	pollan	X.	plash
K.	oxeye	Y.	lamelliform
L.	invective	Z.	asyndetic
M.	nugatory	a.	nisse
N.	twinge	b.	Edelweiss

AUTHOR AND TITLE: KANDINSKY - POINT AND LINE TO PLANE

QUOTATION: Just as an explorer penetrates deeply into new and unknown lands, one makes discoveries in everyday life and erstwhile mute surroundings begin to speak alanguage which becomes increasingly clear. In this way lifeless signs turn into living symbols and the dead is revived.

SOLVERS: THOMAS F. BANCHOFF, Brown University; JEANETTE BICKLEY, St. Louis Community College at Meramec, MO; CHARLES R. DIMINNIE, St. Bonaventure University, NY; ROBERT FORSBERG, Lexington, MA; JENNIFER HAKE, Newton High School, Newton, IL; META HARRSEN, Durham, NC; TED KAUFMAN, Brooklyn, NY; BETH KAYROS, Trenton State College, NJ; STEPHANIE SLOYAN, Georgian Court College, NJ.

MATHACROSTIC NO. 36

Proposed by Charlotte Maines, Rochester, NY.

The 304 letters to be entered in the numbered spaces in the grid will be identical to those in the 29 keyed words at the matching numbers. The key numbers have been entered in the diagram to assist in constructing the solution. When completed, the initial letters on the Words will give the name of an author and the title of a book; the completed grid will be a quotation from that book.

Solutions to Mathacrostic No. 36 should be sent to: Richard Poss, Pi Mu *Epsilon Journal*, St. Norbert College, 100 Grant Street, De Pere, WI 54115. Solutions must be received by September 15.

DEFINITIONS	WORDS
A. Any procedure involving statistical sampling techniques in obtaining a probabilistic approximation to the solution of a mathematical or physical problem (3)	146 260 90 175 84 267 61 205 192 3 70 298 185 44 252 180
B. Formal mathematical system consisting of **undefined** objects and **axioms** of a geometric nature (2)	299 115 150 259 47 187 66 121 176 54 266 79 134
C. Device for regulating strength of an electric current	99 238 126 225 82 156 279 39
D. Pigment made from carbonate of lead (2)	10 179 210 43 191 249 114 287 83 167 277 152 274
E. Showing lack of desire	208 16 51 223 177 101 26 145 257 110
F. Biblical prophet who rebuked David for the death of Uriah	136 206 166 18 173 304
G. System of eliminating a variable from two algebraic equations (3)	57 118 234 161 296 122 28 281 153 89 63 148 213 138 265 97 254 142 4 195 41 190 106 288
H. Sharpens	273 112 214 198 80
I. Name given to the set of points which satisfy the equation $x^2 + y^2 = -r^2$ (2)	78 108 81 251 32 182 88 158 56 219 230 135 168 285 14
J. Illegitimate sons of medieval prelates	170 45 291 29 244 92 184
K. Quantity of anything made in one operation	124 256 17 247 55
L. Prayer for the repose of the dead	228 91 282 23 165 204 8 243 246 77
M. Ultimate goals	94 216 125 9 59 248 245
N. The **cubic** curve defined by the equation $xy = ax^3 + bx^2 + cx + d$ $(a \neq 0)$ (3)	207 242 302 258 131 62 5 157 212 31 276 104 22 227 271
O. The paths of moving particles or celestial bodies	35 224 103 217 11 200 128 289 144 269 239 60
P. American anthropologist (1887-1954)	73 292 241 235 116 33
Q. Surface that lies between **two** parts of matter and forms their **common** boundary	188 280 237 37 264 69 284 300 154
R. **German-American** algebraist (1882-1935)	232 169 162 295 98 7 297
S. **Exogamous groups reckoning descent only through the male lines**	189 218 21 93 100 163
T. **Amplifying** device that effects a certain relation between input and output signals	155 15 141 262 130 64 178 201 6 183 113 294 171 75
U. Plane cubic curve consisting of a single loop, a node, and two branches **asymptotic** to the same line (3)	240 270 86 221 65 159 215 50 117 34 105 19 233 2 133 143 278
V. Arrangement of flowers on the axis of inflorescence	46 149 301 67 160 229 71 132 25
W. Meager; cheerless	203 36 174 196 74 250
X. Forgetfulness	52 263 72 197 202
Y. To **comment** upon	147 95 194 120 48 137 255 20
Z. Place at which two branches of a curve have a **common** tangent and lie on opposite sides of it (3)	226 68 193 253 172 27 107 49 293 127 283 85 151 220 58 209 12
a. Furnace formerly used in alchemy to maintain a uniform and constant heat	76 268 40 286 119 140 24
b. Method of calculating an unknown by making an estimate and working from it and properties of the unknown to secure the value of the latter (4)	186 123 53 261 211 139 1 199 102 222 30 290 236 96 181 272 111 231 38
c. In machinery, having **double** cranks forged upon it, usually situated near and at right angles to each other (hyph.)	13 164 303 129 42 275 109 87

							1 b	2 U	3 A	4 G		5 N	6 T	7 R	
8 L	9 M	10 D	11 O	12 Z	13 c	14 I	15 T	16 E	17 K	18 F		19 U	20 Y	21 S	22 N
23 L	24 a	25 V		26 E	27 Z		28 G	29 J	30 b		31 N	32 I	33 P	34 U	35 O
36 W	37 Q	38 b	39 C	40 a		41 G	42 c	43 D		44 A	45 J	46 V	47 B	48 Y	
49 Z	50 U		51 E	52 X	53 b		54 B	55 K	56 I	57 G	58 Z	59 M	60 O		61 A
62 N	63 G		64 T	65 U	66 B	67 V		68 Z	69 Q		70 A	71 V	72 X	73 P	74 W
75 T	76 a	77 L	78 I	79 B	80 H		81 I	82 C		83 D	84 A	85 Z	86 U		87 c
88 I	89 G		90 A	91 L	92 J	93 S	94 M	95 Y	96 b		97 G	98 R	99 C	100 S	101 E
	102 b	103 O	104 N	105 U		106 G	107 Z		108 I	109 c	110 E	111 b	112 H	113 T	
114 D		115 B	116 P	117 U	118 G		119 a	120 Y	121 B		122 G	123 b	124 K	125 M	126 C
127 Z	128 O		129 c	130 T		131 N	132 V	133 U	134 B	135 I	136 F	137 Y	138 G		139 b
140 a	141 T	142 G	143 U		144 O	145 E	146 A	147 Y	148 G	149 V	150 B		151 Z	152 D	
153 G	154 Q	155 T	156 C		157 N	158 I		159 U	160 V	161 G	162 R	163 S		164 c	165 L
166 F	167 D		168 I	169 R	170 J	171 T	172 Z	173 F	174 W	175 A		176 B	177 E	178 T	179 D
180 A		181 b	182 I		183 T		184 J	185 A	186 b	187 B	188 Q	189 S	190 G	191 D	
192 A	193 Z	194 Y	195 G		196 W	197 X	198 H		199 b	200 O	201 T	202 X	203 W	204 L	205 A
206 F	207 N	208 E	209 Z	210 D		211 b	212 N		213 G	214 H		215 U	216 M	217 O	218 S
219 I	220 Z		221 U	222 b		223 E	224 O	225 C	226 Z	227 N	228 L	229 V	230 I	231 b	232 R
233 U	234 G		235 P	236 b		237 Q	238 C	239 O		240 U	241 P	242 N	243 L	244 J	245 M
	246 L	247 K	248 M	249 D	250 W	251 I		252 A	253 Z		254 G	255 Y		256 K	257 E
258 N		259 B	260 A		261 b	262 T	263 X	264 Q	265 G		266 B	267 A	268 a	269 O	270 U
271 N		272 b	273 H	274 D	275 c	276 N		277 D	278 U		279 C	280 Q		281 G	282 L
283 Z	284 Q	285 I		286 a	287 D	288 G		289 O	290 b	291 J	292 P	293 Z	294 T	295 R	296 G
	297 R	298 A	299 B	300 Q	301 V	302 N	303 c	304 F							

PROBLEM DEPARTMENT

Edited by Clayton W. Dodge
University of Maine

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

All communications should be addressed to C. W. Dodge, 5752 Neville/Math, University of Maine, Orono, ME 04469-5752. Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly identified with name and address. Solutions to problems in this issue should be mailed by December 15, 1993.

Problems for Solution

797. *Proposed by Alan Wayne, Holiday, Florida.*

Restore the enciphered digits of the addends in the following base four addition:

$$A + RAP + AT + A + RAT = 1230.$$

By what means was the RAP caused?

*798. *Proposed by Dmitry P. Mavlo, Moscow, Russia.*

Since 1993 is a prime year, it seems reasonable to ask which is larger,

$$\frac{10^{1992}-1}{10^{1993}-1} \text{ or } \frac{10^{1993}-1}{10^{1994}-1}?$$

799. *Proposed by Stan Wagon, Macalester College, St. Paul, Minnesota.*

a) Find all years that are palindromes in both the standard and the Hebrew calendars. (To get the Hebrew year, add 3761 if it is after the Jewish New Year in September, add 3760 otherwise. *A palindrome* is a number, such as 17271, that reads the same backwards and forwards.)

b) Find all positive integers x such that there are infinitely many positive integers n for which n and $n + x$ are palindromes.

800. *Proposed by Michael D. Williams, Wake Forest University, Winston-Salem, North Carolina.*

Prove that for positive integral n,

$$(2^n)! = \prod_{i=1}^{n} \left(\frac{2^i}{2^{i-1}}\right)^{2^{n-i}}.$$

801. *Proposed by Norman Schaumberger, Bronx Community College, Bronx, New York.*

If a, b, and c are real numbers, then prove that

$$e^a(a-b) + e^b(b-c) + e^c(c-a) \ge 0 \ge e^a(c-a) + e^b(a-b) + e^c(b-c).$$

802. *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.*

Let a and b be positive real numbers. Determine the maximum value of

$$f(x) = (a - x)(x + \sqrt{x^2 - b^2})$$

over all real x with $x^2 \ge b^2$. A non-calculus solution is requested.

803. *Proposed by R. S. Luthar, University of Wisconsin Center, Janesville, Wisconsin.*

In any triangle ABC prove that

$$\sum \sqrt{\tan \frac{A}{2}} \le \sqrt{3} \sum \sqrt{\csc A}.$$

(In a triangle ABC, $\Sigma f(A)$ means $f(A) + f(B) + f(C)$.)

804. *Proposed by Robert C. Gebhardt, Hopatcong, New Jersey.*

Show that

$$4 \arctan \frac{1-x}{1+x} = \pi - 4 \arctan x.$$

Student solutions are especially invited.

805. *Proposed by David Ivy, Baltimore, Maryland.*

a) For all integers k ≥ -2 evaluate the integral

$$I_k = \int_0^1 \left(\frac{y-1}{\ln y}\right)^k dy.$$

*b) Can you evaluate the integral for other values of k?

806. *Proposed by Robert C. Gebhardt, Hopatcong, New Jersey.*

The integral

$$I = \int \frac{dx}{(x^{1/3} - x^{4/3})^{3/2}}$$

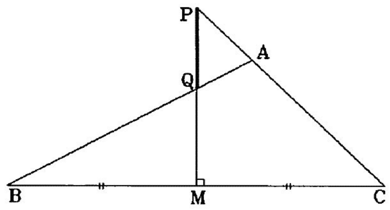
was evaluated by one student as follows:

$$I = \int \frac{dx}{x^{1/2} - x^2} = \int \frac{dx}{x^{1/2}} - \int \frac{dx}{x^2} = 2x^{1/2} + \frac{1}{x} + C.$$

Provide a correct evaluation. Student solutions are especially invited.

807. *Proposed by Florentin Smarandache, Phoenix, Arizona.*

In terms of the lengths a, b, and c of the sides of a given triangle ABC, find the length of the segment PQ of the normal to side BC at its midpoint M cut off by the other two sides. See the accompanying figure.



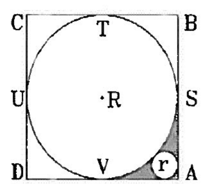
Problem 807

808. Proposed **by** *Scott H. Brown, Stuart Middle School, Stuart, Florida,*

Student solutions **are** especially solicited. *A* circle *(R)* is inscribed in the unit square *ABCD* and touches the sides of the square at *S*, T, ***U***, and *V*, as shown in the accompanying figure. Another circle (r) is inscribed in the region ***ASV*** outside circle *(R)* and inside the square at vertex *A*.

a) Find the **area** of the shaded region inside region *ASV* and outside circle (*r*). Give the answer in radical, not just decimal, form.

*b) If the sequence of smaller circles is continued indefinitely, each successive circle inscribed between the preceding circle and the comer *A* of the square, find the limit of the shaded region. That is, find **the area** of region *ASV* less the **sum** of the areas of the circles in **the** resulting infinite chain.



Problem *808*

809. Proposed **by** *David Ivy, Baltimore, Maryland,*

In triangle *ABC* let AD and *BE* be any two **cevians** intersecting at a point *F. (A* ***cevian*** *AD* for triangle *ABC* is a line through the vertex *A* of the triangle and intersecting the opposite side *BC*, perhaps extended, at a point ***D***, different from both *B* and *C.)* Find the ratios *BDIDC* and ***AF/FD*** in terms of **the** ratios ***AE/EC*** and ***BF/FE***.

Solutions

76*1.* [Fall **1991,** Spring **1992]** *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta,* ***Canada.***

Determine all **functions** ***f(x)*** such that

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad \frac{1}{f(x)} = \sum_{n=0}^{\infty} (-1)^n a_n x^n.$$

I. Solution by Richard I. ***Hess,*** *Rancho Palos Verdes, California.*

The given equations require that $f(x) \cdot f(-x) = 1$. Suppose $f(x) = a_0 e^{q(x)}$ where $q(0) = 0$. Since $f(0) \cdot f(-0) = 1$, then $a_0 = \pm 1$. Furthermore, since

$$e^{q(x)} e^{q(-x)} = 1, \text{ then } q(x) = -q(-x).$$

Thus $f(x) = \pm e^{q(x)}$ for any odd function $q(x)$.

II. *Solution by the Proposer.*

Let $f(x) = E(x) + O(x)$, where E and O are even and odd functions, respectively. Then we have

$$\frac{1}{f(x)} = E(x) - O(x), \text{ so that } 1 = E^2(x) - O^2(x).$$

Hence

$$E(x) = \pm\sqrt{1 + O^2(x)} \text{ and } f(x) = O(x) \pm \sqrt{1 + O^2(x)}$$

where $O(x)$ is an arbitrary odd function analytic at the origin. Two simple examples **are**

$$O(x) = \sinh x, \text{ whence } f(x) = e^{\pm x},$$

and

$$O(x) = Wax, \text{ whence } f(x) = \tan x \pm \sec x.$$

in. *Comment by the Editor.*

By setting $O(x) = \sinh q(x)$, we see that Solutions I and II **are** equivalent. Then $E(x) = \cosh q(x)$ and $f(x) = E(x) + O(x) = \cosh q(x) + \sinh q(x) = e^{q(x)}$.

The conditions of the problem were misstated originally as

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad \frac{1}{f(x)} = \sum_{n=0}^{\infty} (-1)^{n+1} a_n x^n,$$

which implies that $f(x) \cdot f(-x) = -1$. The following two solutions are based on this misstatement.

IV. Solution by ***Jayanthi Ganapathy,*** *University of Wisconsin-Oshkosh,* ***Oshkosh,*** *Wisconsin,*

Since $f^2(0) = -1$, no real-valued function has the properties mentioned in the problem.

V. Solution ***by Seung-Jin*** *Bang, Seoul, Republic of Korea.*

Since $f(x)$ and its reciprocal are holomorphic, there exists a holomorphic function $g(x)$ such that $f(x) = e^{g(x)}$. See **[1].** Since

$$e^{g(x)+g(-x)} = -1, \text{ then } g(x) + g(-x) = (2m(x) + 1)\pi i,$$

where $m(x)$ is an integer-valued function. Since $g(x) + g(-x)$ is holomorphic, then $m(x)$ is a constant m. Hence $g(0) = (m + 112)ni$. From $f(x) \cdot f(-x) = -1$ it follows that $h(x) = g(x) - (m + 112)ni$ is an odd function. Thus we conclude that

$$f(x) = \pm i e^{h(x)}$$

for some odd holomorphic function $h(x)$.

Reference

1. W. Rudiin, *Real* ***and*** *Complex Analysis, 2nd ed.,* **McGraw-Hill,** New York, p. *292,* **Theorem** *13.11.*

Also solved by SEUNG-JIN BANG, *Seoul, Korea,* PAUL S. BRUCKMAN (*2 solutions*), *Edmonds, WA,* MARK EVANS, *Louisville. KY,* STEPHEN I. GENDLER, *Clarion University of Pennsylvania,* RICHARD I. HESS, *Rancho Palos Verdes, CA,* DAVID IVY, *Baltimore, MD,* and REX H. WU, *Brooklyn, NY.*

771. [Spring 1992] *Proposed by Alan Wayne, Holiday, Florida.*

In the base six addition

$$EVE + EVE + EVE + AND = 1310$$

the digits of the addends have been unambiguously replaced by letters. Restore the digits. Where was *EVE*?

Solution by Laurel Benn, Brooklyn, New York.

From the 6^2 column, since there must be a carry from the 6 column, we have that $3E + A < 9$, so $E = 1$ or 2. Hence, from the units column, $D = 3$ or 0, respectively.

If $D = 0$, then $E = 2$ and, since 3 divides N, $N = 3$. Now $V = 1$ or 5. If $V = 1$, then $A = 2$, a contradiction since $E = 2$. If $V = 5$, then $A = 0$, which is not permitted.

Therefore, $D = 3$ and $E = 1$. Now $N = 0$ and $V = 2$ or 4. If $V = 4$, then $A = 4$, a contradiction. So $V = 2$ and $A = 5$. Hence $EVE = 121$, $AND = 503$, and EVE was in 1310 = EDEN.

Also solved by MATT AMOROSO, *St. Bonaventure University, NY,* JOHN T. ANNULIS, *University of Arkansas-Monticello,* CHARLES ASHBACHER, *Cedar Rapids, IA,* STEVE ASCHER, *McNeil Pharmaceutical, Spring House, PA,* FRANK P. BATTLES, *Massachusetts Maritime Academy, Buzzards Bay,* SCOTT H. BROWN, *Stuart Middle School, FL,* PAUL S. BRUCKMAN, *Edmonds, WA,* MARK EVANS, *Louisville, KY,* VICTOR G. FESER, *University of Mary, Bismarck, ND,* ROBERT C. GEBHARDT, *Hopatcong, NJ,* STEPHEN I. GENDLER, *Clarion University of Pennsylvania,* RICHARD I. HESS, *Rancho Palos Verdes, CA,* YOSHINOBU MURAYOSHI, *Eugene, OR,* ANDY PULKSTENIS, *Messiah College, Grantham, PA,* WILLIAM STENZLER, *Gorton High School, Yonkers, NY,* KENNETH M. WILKE, *Topeka, KS, and the* PROPOSER.

772. [Spring 1992] *Proposed by Robert C. Gebhardt, Hopatcong, New Jersey.*

Let $xx44$ be a fourdigit number and yy be a two-digit number in base $b > 4$. Find x and y in terms of b so that $(yy)^2 = xx44$ in every such base $b > 4$ (such as $88^2 = 7744$ in base ten).

I. *Solution by William H. Peirce, Rangeley, Maine, and Delray Beach, Florida.*

We have that

$$(yb + y)^2 = xb^3 + xb^2 + 4b + 4,$$

which reduces to

$$y^2(b + 1) = xb^2 + 4 = x(b^2 - 1) + (x + 4).$$

Hence $b + 1$ must divide $x + 4$. Since x is a nonzero digit in base b, it follows that $x = b - 3$ and $b \geq 4$. Now substitute $x = b - 3$ into either displayed equation to get that $y = b - 2$.

This problem is readily generalized to $(yy)^2 = xxzz$ in base b. In addition to the solution set above, other solutions do exist, and their existence for a given base b is related to the prime factors of $b - 1$. This question is not considered further, other than to list some additional solutions for selected values of b:

Base b	YY	$(yy)^2 = xxzz$
9	5,5	3,3,7,7
13	5,5	2,2,12,12
13	7,7	4,4,10,10
16	11,11	8,8,9,9
25	7,7	2,2,24,24
25	11,11	5,5,21,21
25	13,13	7,7,19,19
25	17,17	12,12,14,14
25	19,19	15,15,11,11

II. *Solution by Scott H. Brown, Stuart Middle School, Stuart, Florida.*

Let N have the j-digit ($j = 2, 3, 4, \ldots$) representation in base b, $b > j + 3$, each digit equal to $b - 2$. Then N^2 has $2j$ digits, the first (from the left) $j - 1$ in ascending order beginning with $b - 3$, the jth digit being $b + j - 5$, the next $j - 1$ in descending order beginning with $j + 2$, and the last digit is 4. Thus, for $j = 2, 3,$ and 4, we have

$$[(b - 2)(b + 1)]^2 = (b - 3)b^3 + (b - 2)b^2 + 4b + 4,$$

$$[(b - 2)(b^2 + b + 1)]^2 = (b - 3)b^5 + (b - 2)b^4 + (b - 2)b^3 + 5b^2 + 4b + 4,$$

and

$$[(b - 2)(b^3 + b^2 + b + 1)]^2 =$$

$$(b - 3)b^7 + (b - 2)b^6 + (b - 1)^5 + (b - 1)b^4 + 6b^3 + 5b^2 + 4b + 4.$$

Reference

Problem 4272, *School Science and Mathematics,* vol. 91 (3), March 1991.

Also solved by JOHN T. ANNULIS, *University of Arkansas-Monticello,* CHARLES ASHBACHER, *Cedar Rapids, IA,* FRANK P. BATTLES, *Massachusetts Maritime Academy, Buzzards Bay,* PAUL S. BRUCKMAN, *Edmonds, WA,* KENNETH B. DAVENPORT, *Pittsburgh, PA,* MARK EVANS, *Louisville, KY,* VICTOR G. FESER, *University of Mary, Bismarck, ND,* RICHARD A. GOOD, *University of Maryland, College Park,* STAN HARTZLER, *Messiah College, Grantham, PA,* RICHARD I. HESS, *Rancho Palos Verdes, CA,* RANDY HO, *University of Arizona, Tucson,* DAVID E. MANES, *SUNY at Oneonta,* YOSHINOBU MURAYOSHI, *Eugene, OR.* LAWRENCE SOMER, *Catholic University of America, Washington, D.C.,* WILLIAM STENZLER, *Gorton High School, Yonkers, NY,* KENNETH M. WILKE, *Topeka, KS, and the* PROPOSER.

773. [Spring 1992] *Proposed by Leon Bankoff, Los Angeles, California.*

In a given circle (*O*) a chord *CD* is drawn to intersect diameter *AOB* at point *E*. Three circles are inscribed, the first two in the sectors *BEC* and *BED*, and the third in the opposite segment *CED*.

Let the circle in sector BEC touch CE at J and let the circle in sector BED touch DE at N. See the figure. If the three inscribed circles have equal radii,

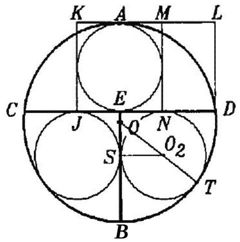
a) show that CD is perpendicular to AB, b) **find** the ratio AE/EB,

c) **find** the ratio AD/AB,

d) **find** the ratio $CD/$AB,

e) show that the rectangle $JKMN$ on JN as base and with opposite side KM passing **through** A circumscribes the third inscribed circle, and

f) show that the rectangles $JKLD$ and $NMLD$ are golden rectangles.



Problem 773

Solution *by* Richard I. *Hess*, Rancho *Palos Verdes, California.*

a) S i c e the three inscribed circles have equal radii, the **figure** $CEDB$ is symmetric in diameter BOEA, whence CD is perpendicular to AB.

b) Let the radii of the large and small circles be R and r, respectively. Draw the **line** OO_2T **through** the center O_2 of the small lower right circle to its point of tangency T with the large circle. Draw radius O_2S of circle (O_2) perpendicular to AB, as shown in the **figure.** Then AS $= 3r$ and from the Pythagorean theorem applied to right triangle SOO_2 we have

$$(OO_2)^2 = (R - r)^2 = r^2 + (3r - R)^2,$$

from which it follows that

$$4R = 9r \text{ and } r = \frac{4}{9}R.$$

Now we get that

$$\frac{AE}{EB} = \frac{2r}{2R - 2r} = \frac{4r}{9r - 4r} = \frac{4}{5}.$$

c) By the Pythagorean theorem applied to triangle ODE, since OE $=$ R - $2r$, we get that

$$ED = \sqrt{OD^2 - OE^2} = \sqrt{R^2 - (R - 2r)^2} = \sqrt{4rR - 4r^2} = r\sqrt{5}$$

since 4R $= 9r$. By applying the Pythagorean theorem to triangle AED, we **find** that

$$AD = \sqrt{4r^2 + 5r^2} = 3r.$$

Finally, $AD/$AB $= 3r/2R =$ 213 since 4R $= 9r$.

d) **Since** CD = W E $= 2r\sqrt{5}$, then $CD/AB = 2r\sqrt{5}/2R = (4/9)\sqrt{5}$. e) **Since** the three circles have equal radii r, then J N $= 2r$. By the **symmetry** of the entire figure about **line** AB, the third small circle is the circle on AE as diameter, so rectangle $JKMN$ circumscribes that circle and therefore is a square.

f) Now J D = J E $+ ED = r + r\sqrt{5}$, whence $JD/JK = (1 + \sqrt{5})/2$, the golden ratio. **Since** $JKMN$ is a square cut **from** a golden rectangle, then the remaining rectangle $NMLD$ is another golden rectangle.

Also solved by PAUL S. BRUCKMAN, *Edmonds*, WA, YOSHINOBU MURAYOSHI, Eugene, OR, and the PROPOSER.

Editorial comment. *Fifty* lashes to the editor for *faulty* terminology. A sector is *the figure bounded by* two radii of a circle and a subtended arc. A segment is the figure bounded *by* a chord of a circle and a *subtended* arc. So in *the figure for* this problem CEOB and DEOB are not *sectors since* EC and ED are not radii, but should properly have been *called* semi-segments. It is *true* that CEDA is a segment.

774. **[Spring 1992]** *Proposed by* Robert *C. Gebhardt, Hopatcong,* New Jersey.

The first player in a game who acquires 250 points is the winner. Because player A is a better player **than** player B, he **gives player** B a SO-point handicap. Similarly player B **gives player** C a SO-point handicap and player C gives player D a SO-point handicap. What handicap should player A give player D?

I. Solution *by* Richard I. *Hess*, Rancho *Palos* Verdtx, *California.*

S i c e A makes 250 points while B makes 200, and B makes 250 points while C makes 200, then B **makes** 200 while C makes 160. So A should give C a 90 point handicap. Since C makes 250 points while D makes **200, then** C makes **160 while D** makes **128. Hence A** should **give D** a **122 point** handicap.

This d l sounds very **logical,** but consider the simpler **4-point** game where **A** gives **B** a 2-point handicap, and B gives C a 2-point handicap. Here, **by** the same logic, **A** should give C a 3-point handicap.

Consider, however, the following model: Points are accumulated one at time. When A plays B, she has a probability p of winning any point and B has probability q $=$ 1 - p of winning the point. The handicap is set so as to give **each** player a probability of 112 of winning the 4-point game.

Define A's chance of winning the game when she has m points and B has n points to be $P(m, n)$. Then $P(m,n) =$ p $\cdot P(m + 1,n) +$ q $\cdot P(m,n +$ I), where $P(4,x) = 1$ and $P(x,4) = 0$ for any $x = 0, 1, 2, 3$. With some **algebra** we get $P(0,3) =$ p^4 and $P(0,2) = p^4(1 + 4q)$. Using the logic of the solution above, we would have p $=$ 213 and q $=$ 113. We **would** expect that $P(0,2) =$ 112, but **actually** $P(0,2) =$ **112/243.** To obtain $P(0,2) =$ 112, we must take p $\approx$ **.6862. This earlier** approach would **give** p $=$ 415 when A plays C, but this gives $P(0,3) =$ 2561625 $\neq$ 112. To get $P(0,3) =$ 112, we must take p $\approx$ **.8409** and $(.8409)^2 \approx .7071 \neq .6962$. **In** this (more accurate?) model there is no basis for **determining** A's **probability** of winning a point from C when the probabilities are **known** when A plays B and when B plays C. Thus the question of handicapping has no exact answer.

II. Solution *by David Ivy,* Baltimore, *Maryland.*

We first develop a model for handicaps and player **skills.** Let r_{AB} **represent** the average number of points player A expects to score against player B on any given turn. When $r_{AB} \geq r_{BA}$, then A is considered better than B and A gives a **handicap** h_{BA} to B based on

$$(1) \qquad \frac{250}{r_{AB}} = \frac{250 - h_{BA}}{r_{BA}}, \text{ that is, } h_{BA} = 250\left(1 - \frac{r_{BA}}{r_{AB}}\right).$$

Note that this handicap does not equalize the chances of either player winning. Rather, when $r_{AB} <<$ 250, we *can* **regard** $250/r_{AB}$ **as** the approximate number of turns A **needs** to amass 250 points, and this value is set **equal** to the approximate number of turns B needs to amass $250 - h_{BA}$ points. Furthermore, $250/r_{AB}$ is not the expected number of turns that A needs to amass 250 points, as on some of the turns he will do better than on others, and we need to consider the distribution of his scores. None-the-less, our tournament directors have decided to use the Formula (1) to save the expense of buying a supercomputer to use with alternative, more complicated formulations..

We consider a game with three basic skills that are easily measured and can be used with Formula (1). The game rules call for players to alternate turns trying to score unless a player succeeds in scoring. If a player does score, he is given a chance to create another scoring opportunity. That is, if he succeeds in forming this opportunity, he goes again. In order to score, a player must perform a successful offensive maneuver, and then he scores if his opponent fails to perform a successful parry. Thus there are three skills: carrying out an offensive maneuver, performing a defensive maneuver, and generating a scoring opportunity. Let (p_A, q_A, t_A) be a triplet that defines player A's ability in the three skill areas, respectively. Thus the probability that A will score against B on any given scoring opportunity is $x_{AB} = p_A(1 - q_B)$. On any given turn, then, the number of points player A is expected to score is given by

$$r_{AB} = x_{AB} + (x_{AB})^2 t_A + (x_{AB})^3 t_A^2 + \cdots,$$

that is,

$$(2) \qquad r_{AB} = \frac{x_{AB}}{1 - t_A x_{AB}}.$$

Now we can calculate the handicaps. For the given problem we have $h_{BA} = h_{CB} = h_{DC} = 50$. Then (1) yields $r_{DC} = (415)\, r_{CD}$, $r_{CB} = (415)\, r_{BC}$, and $r_{BA} = (415)\, r_{AB}$. We now consider three cases.

Case 1: $t_A = t_B = t_C = t_D$ and $q_A = q_B = q_C = q_D$. In this single skill degeneracy case, the opponent doesn't affect the player's ability to score. Thus $r_{AB} = r_{AC} = r_{AD} = r_A$ and $r_D = (415)\, r_C = (415)^2 r_B = (415)^2 r_A$, so

$$h_{DA} = 250\left[1 - \left(\frac{4}{5}\right)^3\right] = 122.$$

Case 2. $t_A = t_B = t_C = t_D = 1$, two skill degeneracy. Given any choice of p_A, q_A, q_B, q_C, and q_D, we can use (1) and (2) to determine p_B, p_C, and p_D. The solution is rejected, however, if it does not satisfy $0 \le p_B, p_C, p_D \le 1$. For example, if $p_A = 3/4$, $q_A = 0$, $q_B = 114$, $q_C = 112$, and $q_D = 3/4$, then $x_{AB} = 9116$, $r_{AB} = 917$, $r_{BA} = 36135$, $x_{BA} = 36/71$, and $p_B = 36171$. Similar calculations show that $p_C = 961337$ and $p_D = 19211661$. Finally, $h_{DA} = 15925011469 \approx 108.4$.

Case 3. We now set $h_{BA} = h_{CB} = h_{DC} = 50$. One way to do this is with the values in the following table:

Player	p	q	t
A	0.6	0	1
B	0.9677	0	0.2
C	0.9600	0	0
D	0.00007681	0.9999	0

Now we calculate that $r_{DA} > r_{AD}$, so that player D needs to give A a handicap $h_{AD} = 54.67582$. This example is non-transitive; if A is a better player than B and B is better than C, then it is not necessarily true that A is better than C.

Also solved by JOHN *T.* ANNULIS, *University of Arkansas-Monticello,* CHARLES ASHBACHER, *Cedar Rapids, IA,* PAUL S. BRUCKMAN, *Edmonds, WA,* MARK EVANS, *Louisville, KY,* STEPHEN I. GENDLER, *Clarion University of Pennsylvania,* LEE LIAN KIM, *Messiah College, Grantham, PA,* CARL LIBIS, *Granada Hills, CA, and the* PROPOSER.

775. [Spring **1992**] ***Proposed by Norman Schaumberger, Bronx Community College, Bronx, New York.***

If H is the harmonic mean of the positive numbers $a_1, a_2, \ldots, a_n$, prove that

$$H^{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}} \ge a_1^{\frac{1}{a_1}} a_2^{\frac{1}{a_2}} \cdots a_n^{\frac{1}{a_n}}$$

Comment by David Ivy, Baltimore, Maryland.

I guess a hundred people must have pointed out that Problem 775 is worked out on pages 384-385 [of the Spring 1992 issue] by the proposer!

Editorial reply. No, only six!

Also solved by SEUNG-JIN BANG, *Seoul, Korea,* SCOTT *H.* BROWN, *Stuart Middle School, FL,* PAUL S. BRUCKMAN, *Edmonds, WA,* RICHARD I. HESS, *Rancho Palos Verdes, CA,* DAVID IVY *(2 solutions), Baltimore, MD,* DAVID *E.* MANES, *SUNY at Oneonta,* YOSHINOBU MURAYOSHI, *Eugene, OR, and the* PROPOSER.

776. [Spring **1992**] ***Proposed by Russell Euler, Northwest Missouri State University, Maryville, Missouri.***

Let n be a fixed positive integer and let

$$P_k = 1^k + 2^k + \cdots + n^k.$$

Write as a polynomial in P_1 the expression

$$15^4(P_1^{\,4} + P_2^{\,4} + P_3^{\,4} + P_4^{\,4}).$$

Solution by Kenneth M. Wilke, Topeka, Kansas.

We have

$$P_1 = \frac{n(n+1)}{2},$$

from which it follows that

$$\left(\frac{2n+1}{3}\right)^2 = \frac{8P_1 + 1}{9} \text{ and } \frac{3n^2 + 3n - 1}{5} = \frac{6P_1 - 1}{5}.$$

Then

$$P_2 = \frac{n(n+1)(2n+1)}{6} = \frac{2n+1}{3}P_1 = P_1\sqrt{\frac{8P_1 + 1}{9}},$$

$$P_3 = \left(\frac{n(n+1)}{2}\right)^2 = P_1^2,$$

and

$$P_4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} = P_1 P_2 \frac{6p_1 - 1}{5}.$$

By straightforward but tedious algebra we find that

$$15^4(P_1^4 + P_2^4 + P_3^4 + P_4^4) =$$

$$P_1^4(82944P_1^6 - 34560P_1^5 + 51921P_1^4 + 1056P_1^3 + 39896P_1^2 + 9992P_1 + 51251).$$

Also solved by SEUNG-JIN BANG, *Seoul, Korea,* FRANK P. BATTLES, *Massachusetts Maritime Academy, Buzzards Bay,* SCOTT H. BROWN, *Stuart Middle School, FL,* PAUL S. BRUCKMAN, *Edmonds, WA,* J. S. FRAME, *Michigan State University, Lansing,* RICHARD I. HESS, *Rancho Palos Verdes, CA,* DAVID IVY, *Baltimore,* MD, DAVID E. MANES, *SUNY at Oneonta,* YOSHINOBU MURAYOSHI, *Eugene, OR,* WILLIAM H. PEIRCE, *Rangeley, ME,* KEVIN ROBINSON, *Messiah College, Grantham, PA, and the* PROPOSER. *Ivy gave several interesting formulas regarding the* P_k, *including that* P_k *is expressible as a polynomial in* P_1 *whenever k is an odd positive integer, and that* P_k^2 *is expressible as a polynomial in* P_1 *whenever k is any positive integer.*

778. [Spring 1992] *Proposed by Laura L. Kelleher and Frank P. Battles, Massachusetts Maritime Academy, Buzzards Bay, Massachusetts.*

It is readily established that the arc length along the curve $y = \cosh x$ on any interval $[a,b]$ and the area under the graph of this same function on this same interval are numerically equal. For what other functions, if any, is this curious fact tme?

I. *Solution by Paul S. Bruckman, Edmonds, Washington.*

We assume that any function $y = f(x)$ with the stated property is continuous and has a continuous first derivative on $[a,b]$. Our equation that the length of arc L equals the area A on that interval takes the form

$$A - L = \int_a^b \left[y - \sqrt{1 + (y')^2}\right] dx = 0.$$

Since this equation is to be tme for all intervals $[a,b]$, we must have, for all x,

$$y^2 = 1 + (y')^2, \text{ whence } \frac{dx}{dy} = \frac{1}{\sqrt{y^2 - 1}},$$

whose solution is $x = \cosh^{-1} y - C$ for any real constant C. Therefore,

$$y = f(x) = \cosh(x + C).$$

II. *Solution by David E. Manes, State University of New York, Oneonta, New York*

Besides the obvious solution $f(x) = 1$, any function of the form

$$f(x) = Ae^x + Be^{-x}, \text{ with } AB = 114$$

for any constants A and B, satisfies the above property; i.e.,

$$\int_a^b \sqrt{1 + (f'(x))^2}\, dx = \pm\int_a^b f(x)\, dx.$$

To this end, the integral equation requires that

$$(1) \qquad \sqrt{1 + (f'(x))^2} = f(x).$$

Since the left side is always at least 1, then $f(x)$ cannot be zero for any x. So we square and then differentiate this equation and simplify the result to get

$$f'(x)[f(x) - f''(x)] = 0.$$

If $f'(x) = 0$, then f is a constant function and we have $f(x) = 1$. Otherwise, we have the homogeneous differential equation $f(x) = f''(x)$, which has the family of solutions $f(x) = Ae^x + Be^u$. Then Equation (1) yields $4AB = 1$, as required.

Also solved by SEUNG-JIN BANG, *Seoul, Korea,* RUSSELL EULER, *Northwest Missouri State University, Maryville,* ROBERT C. GEBHARDT, *Hopatcong, NJ,* DAVID IVY, *Baltimore,* MD, YOSHINOBU MURAYOSHI, *Eugene, OR, and the* PROPOSER. BOB PRIELIPP, *University of Wisconsin-Oshkosh, located this same problem as* **Problem E1549, proposed by** *C. R. MacCluer and solved by D. A. Moran, in the American Mathematical Monthly 70 (1963). p 893. In addition to the two solutions above, Moran gives*

$$y = \begin{cases} \cosh(x - a), & 0 \le x \le a \\ 1, & a < x < b \\ \cosh(x - b), & x \ge b. \end{cases}$$

AMM solvers located this problem as Ex. 9, p. 45, Ordinary Differential Equations, by R. E. Langer, as Ex. 8, p. 25, Elementary Differential Equations, by C. E. F. Sherwood and A. E. Taylor, and on pp. 149-50 of Through the Mathescope, by C. S. Ogilvy.

779. [Spring 1992] *Proposed by W. Moser, McGill University, Montreal, Canada.*

If $0 C a \le x \le y \le 1/a$, then prove that

$$x + \frac{1}{x} \le a + \frac{1}{a}, \quad \frac{x}{y} + \frac{y}{x} \le \frac{y}{a} + \frac{a}{y},$$

$$\frac{x}{y} + \frac{y}{x} \le ax + \frac{1}{ax}, \quad \text{and} \quad (x + y)\left(\frac{1}{x} + \frac{1}{y}\right) \le \left(a + \frac{1}{a}\right)^2$$

Solution by Jonathan Hartzel, Messiah College, Grantham, Pennsylvania.

Let $f(x) = x + 1/x$, so that $f(x) = f(1/x)$. By elementary calculus, $f(x)$ is decreasing on $(0,1]$ and increasing on $[1,\infty)$. Hence we have the following lemma.

Lemma. For any t in the interval $(0,1)$, $f(x)$ achieves absolute maximum on $[t,1/t]$ at either endpoint.

Since we are given $a \le x \le 1/a$, then by our lemma

$$f(x) \le f(a), \text{ that is, } x + \frac{1}{x} \le a + \frac{1}{a}.$$

Since $a \le x \le y$, then $a/y \le x/y$ 5 $1 \le y/a$. By our lemma,

$$f\left(\frac{x}{y}\right) \le f\left(\frac{y}{a}\right), \text{ that is, } \frac{x}{y} + \frac{y}{x} \le \frac{y}{a} + \frac{a}{y}.$$

Now ax *S* x/y *5 1* $\le 1/ax$ *because y 5 lla, x 5 y, and* $x \le$ *lla.* **Our lemma** *yields*

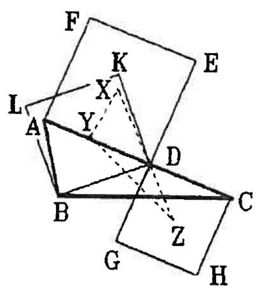
$$f\left(\frac{x}{y}\right) \le f(ax), \text{ whence } \frac{x}{y} + \frac{y}{x} < ax + \frac{1}{ax}.$$

Since $a \le x$ **and** a *5* $1/y$, **then** $a^2 \le x/y$. **Since** $x \le y$ **and** $a \le 1$, **then** x/y *5 1 5 lla*^{2}*. Now we have* a^2 *5* $x/y \le$ *lla*^{2}*. By our lemma,* $x/y + ylx \le a^2 + lla^2$*. Add 2 to each side of this inequality and then* **factor** *the sides to get*

$$(x + y)\left(\frac{1}{x} + \frac{1}{y}\right) \le \left(a + \frac{1}{a}\right)^2.$$

Also *solved by CHARLES* **ASHBACHER,** *Cedar* ***Rapids,*** *IA,* **SEUNG-JIN** *BANG, Seoul, Korea,* **PAUL S.** *BRUCKMAN,* ***Edmonds, WA,*** *STEPHEN I. GENDLER, Clarion* ***University*** *of* ***Pennsylvania,*** **RICHARD** *I. HESS, Rancho* ***Palos Verdes, CA,*** *DAVID E. MANES, SUNY at* ***Oneonta,*** **YOSHINOBU MURAYOSHI,** *Eugene, OR, ANDREW F. PINGITORE, Fredonia State* ***University College, NY,*** **LONG** *PHI* **VO,** *Arlington, TX, and the PROPOSER.*

781. **[Spring** *19921 Proposed by the late Jack* ***Garfunkel,*** *Flushing, New York.*
Erect squares ***ADEF***, *BDKL,* **and** *CDGH as shown in the* **figure,** *on the segments AD, DC, and BD, where D is any point on side CA of given triangle ABC. Let X,* ***Y,*** *and Z be the centers of the erected squares.* **Prove** *that triangles ABC* **and** ***XYZ*** *are similar and the ratio of similarity is* $\sqrt{2}$.



Problem 781

Solution by A. T. E. ***Levin,*** *Closed Bar Company, Dntown, Georgia.*
Since *X is the center of the square DEFA, then a* **45°** *counterclockwise rotation and a* **homothety** *or stretch of ratio* $\sqrt{2}$, *both about point D, will* **carry** *X to A. Similarly, that* **same rotation-***homothety* **carries** *Y to B and Z to C.* **The theorem** *follows.*

Also solved by PAUL S. BRUCKMAN, ***Edmonds,*** *WA,* **DAVID** *IVY, Baltimore, MD,* **YOSHINOBU MURAYOSHI** *(two solutions), Eugene, OR, and the PROPOSER.*

782. **[Spring 1992]** *Proposed by Murray S.* ***Klamkin, University*** *of Alberta,* ***Edmonton,*** *Alberta,* ***Canada.***

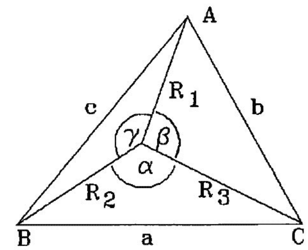
In *O. Bottema et al, Geometric Inequalities,* **Wolters-Noordhoff, Gronigen, 1969, item 12.55, p.** *118, it is stated that for a triangle* ***ABC*** *with no angle* $\ge 2\pi/3$ *that*

$$2(R_1 + R_2 + R_3)^2 \ge (a^2 + b^2 + c^2) + 4F\sqrt{3},$$

where R_1, R_2, *and* R_3 **are** *the respective distances from an* **arbitrary** *point P inside the* **triangle** *to its vertices, a, b, and c are the triangle's side lengths, and F is its* **area.** *Item* **12.55 further** *states* **that for** *a* **triangle** *in which*
LA $\ge 2\pi/3$,

$$(R_1 + R_2 + R_3)^2 \ge (b + c)^2.$$

Show that the first inequality is **true** *for all triangles.*



Problem **782**

Solution by David ***Ivy,*** *Baltimore, Maryland.*
Label the **central** *angles* α, β, *and* γ *as shown in the* **figure.** *By the law of cosines we have*

$$b^2 = R_1^2 + R_3^2 - 2R_1R_3 \cos \beta, \; c^2 = R_2^2 + R_1^2 - 2R_2R_1 \cos \gamma,$$

and

$$a^2 = R_3^2 + R_2^2 - 2R_3R_2 \cos \alpha.$$

Also

$$2F = R_1R_2 \sin \gamma + R_2R_3 \sin \alpha + R_3R_1 \sin \beta,$$

whence the stated inequality is equivalent to

$$\sum(4 + 2\cos\alpha - 2\sqrt{3}\sin\alpha)R_2R_3 \ge 0.$$

Since $4 + 2\cos\theta - 2\sqrt{3}\sin\theta = 4[1 - \sin(\theta - \pi/6)] \ge 0$, *the stated inequality trivially follows with equality if and* **only** *if* $\alpha = \beta = \gamma = 2\pi/3$.

Also solved by PAUL S. BRUCKMAN, ***Edmonds, WA,*** *and the PROPOSER.*

783. [Spring **1992]** *Proposed by the late Jack* ***Garfunkel,*** *Flushing, New York.*
If, A, B, and C **are** *the angles of a triangle ABC, then prove that*

$$\frac{\sum \sin^2 A}{\sum \cos^2\left(\frac{A}{2}\right)} \geq \frac{\prod \sin A}{\prod \cos\left(\frac{A}{2}\right)}.$$

Solutwn by J. S. Frame, Michigan State University, East Lansing, Michigan.

For triangle ABC with sides a, b, c, inradius r, semiperimeter s, and area F, we establish the inequality and show that equality holds only if $a = b = c$, by proving that

$$\frac{\sum \sin^2 A}{\prod \sin A} - \frac{\sum \cos^2 \frac{A}{2}}{\prod \cos \frac{A}{2}} = \sum \frac{(c - b)^2}{2rs}$$

Clearly, the right side of this equation is nonnegative and is zero if and only if the triangle is equilateral. Recall that $\Sigma(s - a) = s$, that $r/(s - a) = \tan(A/2)$, and that $r^2s^2 = F^2 = s(s - a)(s - b)(s - c)$, so $r/(s - a) = (s - b)(s - c)/(rs)$. Then

$$\frac{\sum \sin^2 A}{\prod \sin A} = \sum \frac{\sin(B + C)}{\sin B \sin C} = 2\sum \cot A = \sum (\cot\frac{A}{2} - \tan\frac{A}{2})$$

and

$$\frac{\sum \cos^2 \frac{A}{2}}{\prod \cos \frac{A}{2}} = \sum \frac{\sin \frac{B + C}{2}}{\cos \frac{B}{2} \cos \frac{C}{2}} = 2\sum \tan \frac{A}{2}.$$

The difference between these two sums is seen to be

$$\sum (\cot\frac{A}{2} - 3\tan\frac{A}{2}) = \sum\left(\frac{s - a}{r} - 3\frac{r}{s - a}\right) = \frac{s}{r} - \sum \frac{3(s - b)(s - c)}{rs}$$
$$= \frac{(\sum (s - a))^2 - 3\sum (s - b)(s - c)}{rs} = \frac{\sum [(s - b) - (s - c)]^2}{2rs} = \frac{\sum (c - b)^2}{2rs}.$$

Also solved by PAUL S. BRUCKMAN, *Edmonds, WA,* DAVID IVY, *Baltimore, MD,* YOSHINOBU MURAYOSHI, *Eugene, OR,* BOB PRIELIPP, *University of Wisconsin-Oshkosh,* PAUL YIU, *Florida Atlantic University, Boca Raton, and the* PROPOSER.

Late solutions were received from KENNETH B. DAVENPORT, *Pittsburgh, PA, to problem 750, 753, 754, 764, 769 and 770.*

INQUIRIES

Inquiries about certificates, pins, posters, matching prize funds, support for regional meetings, and travel support for national meetings should be directed to the Secretary-Treasurer, Robert M. Woodside, Department of Mathematics, East Carolina University, Greenville, NC 27858, 919-757-6414.

PI MU EPSILON 1993 NATIONAL MEETING

The 1993 National Meeting of the Pi Mu Epsilon National Honorary Mathematics Society will be held in Vancouver, British Columbia, in Canada, from August 16-19. The meeting will be held in conjunction with the AMS-MAA meetings, which run from August 15-19. Pi Mu Epsilon will again co-host this national meeting with the MAA student chapters.

The Pi Mu Epsilon meeting will begin with a reception on the evening of Monday, August 16. On Tuesday, August 17, the Pi Mu Epsilon Council will have its annual meeting. Also on that day, the student presentations will begin. The presentations will continue on Wednesday, August 18. The Pi Mu Epsilon banquet will take place that evening, followed by the J. Sutherland Frame lecture. This year's Frame lecture will be given by George E. Andrews, of Pennsylvania State University. The meetings will conclude on Thursday, August 19, with the final student presentations.

TRAVEL SUPPORT FOR STUDENT SPEAKERS

Pi Mu Epsilon will provide travel support for student speakers at the national meeting. If a chapter is not represented by a student speaker, Pi Mu Epsilon will provide one-half support for a student delegate. Full support is defined to be full round-trip air fare (including ground transportation) from the student's school or home to Vancouver, BC, Canada, up to $600. (Delegates will receive up to $300.) A student who chooses to drive will receive 25 cents per mile for the round trip from school or home to Vancouver, up to $600. (Delegates will receive $12\frac{1}{2}$ cents per mile, up to $300.) If several students from the same chapter wish to attend, they may share the travel support, if they choose to do so.

The National Council of Pi Mu Epsilon haa approved, on a temporary basis, a more generous travel allowance for student speakers at this year's meeting. The first speaker from a given chapter will be eligible for the same travel allowance as before, but if there is more than one speaker from a given chapter, the additional speakers (up to four) will be eligible for an allowance of 20% of what the first speaker receives. For example, if the distance traveled (by car or van) is over 2400 miles (round trip distance), a single student speaker would receive $600, two student speakers would receive $720 (to share in any way they wish), three speakers would share $840, four speakers would share $960, and five or more speakers from this single chapter would share $1080.

The purpose of this more generous travel allowance is to encourage as many students as possible to speak at the Vancouver meeting. If you are a student member of Pi Mu Epsilon, and won't have received a master's degree before May of this year, you are eligible to submit a paper to present at the meeting.

For further information about the meeting and the travel support:

SEE YOUR PI MU EPSILON ADVISOR

GLEANINGS FROM THE CHAPTER REPORTS

CONNECTICUT GAMMA (Fairfield University) During the fall semester, the chapter sponsored a "Research in Undergraduate Mathematics Night." Members Laura Davey and Charles **Ragozzine** spoke about their NSF sponsored summer research at Mills College/ **UC-Berkeley** and Worcester Polytechnic Institute, respectively. In the spring, members of Pi Mu Epsilon assisted the Mathematics Department in coordinating the activities for Math Counts, which is a mathematics contest for junior high school students. At the annual spring initiation ceremony, twenty new members were inducted and Henry **O. Pollak** (former president of the MAA and researcher at Bell Labs and currently on faculty at Columbia) delivered the Pi Mu Epsilon Lecture entitled "Some Mathematics of Baseball." The third annual Math Bowl Contest was also held in the spring. Six teams of four students competed in a "GE College Bowl" type of competition, in which all of the questions were mathematical. During the annual Arts and Sciences Awards Ceremony, three members, James **Klosowski,** Charles Ragozzine, and Margaret **Sweeney** received recognition for their outstanding performance in mathematics. Each was given a hook in an area of mathematics, and a one-year membership in the MAA.

FLORIDA KAPPA (The University of West Florida) At the induction meeting in December, Dr. Donald **Byrkit** spoke on the history of number systems. A total of 17 new members were inducted during the year. The chapter worked with the MAA Student Chapter to raise money for social events and to sponsor a trip to the Florida Section Meeting of the MAA. Professor James **R.** Weaver (Faculty Correspondent) and PME chapter president Tracey Polsgrove took two vehicles filled with students to the meeting. Shannon Pugh, Greg **Scible,** and Jeff Wallace gave student talks entitled: "Subdivy, Exploration into a Winning Strategy," "Remarks on the Generalized Riemann Integral," and 'The Wondering Mathematician," respectively. The chapter, along with the UWF Mathematics Association (Student Chapter of the MAA), helped the Florida Association of Professional Engineers with their annual Northwest Florida Math Counts program in February. The joint efforts of the PME chapter and the MAA Student Chapter resulted in solving the "Vacillating Mathematician" problem in the *College* Mathematics *Journal.*

KANSAS GAMMA (The Wichita State University) The chapter sponsored several speakers during the year. The speakers, and the titles of their talks, were: Dr. J. **Chaudhuri,** "Materials Science and Engineering"; Ms. Lynette **Bikos,** "Careers with a Math Degree"; Dr. W. D. Wallis, "Hadamard Matrices"; and **Apurvna** Sheth, "Vedic Mathematics." There were two group presentations during the year. One was "Mathematics in Other Countries" was discussed by **Zaheer Aziz** (Pakistan), **Satoshi Kume** (Japan), **Naruatheap Puangpathumanond** (Thailand), Kent **Rowe** (USA, and Wee Meng Tan (Malaysia). (This presentation was repeated at the annual joint meetings of the MAA and the Kansas Association of Teachers of Mathematics.) The other group presentation was on "Vedic Mathematics," by **Tamim Arif,** Supriya **Madan,** and **Apurva** Sheth. David C. Ogden gave a talk on "A Combinatorial Queuing Model Related to the Ballot Problem" at the joint **MAA/KATM** meeting. In October, the chapter sponsored the showing of the movie "Stand and Deliver." During the year, the chapter also provided free help sessions for students in courses through Calculus **III.**

MICHIGAN EPSILON (Western Michigan University) Chapter member Mark Kust presented his paper "Singular Value Decay in the Numerical Inversion of the Weierstrass 'Kansform" at the national meeting of Pi Mu Epsilon in Orono, Maine. There were several talks on campus during the year. Professor George **Piranian,** University of Michigan, presented the talk "Geometric Meditations on Function Theory." At **PME's** annual Initiation Banquet, where a total of 32 new members were initiated, Dr. Tom **Vidmar,** The **Upjohn** Company, presented the after dinner talk entitled "Statistics: Helping to Improve Productivity Through Laboratory Automation." WMU graduate student Heather **Jordon Gavlas** presented the talk "Framed." Mark Kust spoke on "Approximation Methods in Tomography." University of Michigan undergraduate student Cheryl P. Grood presented the talk "Dihedral Rewriteability." Professor Robert Devaney, Boston University, presented two talks: "Chaos, Fractals, and Dynamics," and "The Mathematics Behind the Mandelbrot Set." Professor Timothy **Pennings,** Hope College, spoke on "Further Insights into Dynamical Systems and Chaos." Finally, Michigan State University graduate student Lisa **Hansen** presented a talk entitled "Least Common Divisors and Least Common Multiples of Graphs." At its Annual Book Sale, Pi Mu Epsilon raised $370 to help support chapter activities for the coming year.

MONTANA ALPHA (The University of Montana) Professors Rudy A. Gideon and Mary Jean Brod are the new faculty advisors of-the chapter. The chapter had three meetings during the year. Although the meetings were mostly organizational, several students discussed some mathematical topics.

NEW YORK OMEGA (St. Bonaventure University) The chapter continued its cooperation with the SBU Student Chapter of the MAA in sponsoring the Mathematics Forum. This year's Forum lectures were: "Some Irrational numbers by an Irrational Person", by Albert White, SBU; "On Maximizing the Product of Partitions," by Jeffe Boats, SBU student; "Stochastic Calculus and the Valuation of Option Contracts," by **Larry** Lardy, Syracuse University; "The Higher Derivative Test for Extreme Values," by Chuck **Diminnie,** SBU; "Arrow's Paradox: Why Democracy Does Not Exist," by Doug Cashing, SBU; "The Actuarial Profession," by Kerry Fitzpatrick, Senior Actuarial Associate, Aetna Insurance Co.; "Some Mathematics of Computer Graphics," by Dalton **Hunkins,** SBU Department of Computer Science; and "Differential Equations, I Can't Solve Them," by Harry Sedinger, SBU. Our third annual celebration of Mathematics Awareness Week included the talk by Sedinger, the Pi Mu Epsilon induction ceremony, and a showing of Joe **Gallian's** videotape lecture on "The Mathematics of Identification Numbers."

WISCONSIN DELTA (St. Norbert College) Seven students attended the Pi Mu Epsilon National Meeting at Orono, Maine: Sandra Gestl, Amy Krebsbach, Mike Lang, Roxann **Leisemann,** Linda Mueller, Shawn **Volk,** and Dave Ward. Gestl, Krebsbach, and **Lang** presented papers at the conference. St. Norbert College had the honor of hosting Jaime Escalante in February. Mr. Escalante addressed the community and also conducted a unique class in order to demonstrate some of his teaching techniques to prospective high school teachers. In November, the chapter hosted its Sixth Annual Pi Mu Epsilon Regional Undergraduate Mathematics Conference. The featured speaker was J. Douglas **Faires** (Youngstown State University), who spoke on "Some Puzzles I have Known," and "How Much Company Will You Have When You Retire?" There were 18 student presentations at the conference, including those by SNC students Laura **Donzelli,** Mark **Fahey,** Amy **Gerrits,** Mark **Geske,** Sandra Gestl, Amy Krebsbach, Mike **Lang,** and Linda Mueller. Another significant event was the tenth annual SNC High School Math Meet, held in conjunction with **SNC's** math club, Sigma Nu Delta. Also in cooperation with SNA, the chapter held the annual Brenda Roebke Volleyball Tournament. The proceeds from the tournament were divided between the American Cancer Society and a scholarship fund for SNC students majoring in math. In October and February, members of the chapter helped recruit donors for the on-campus blood drive.

ATTENTION FACULTY ADVISORS

To have your chapter's report published, send copies to Robert M. Woodside, **Secretary-**Treasurer. Department of Mathematics, East Carolina University, Greenville. NC 27858 and to Richard L. **Poss,** Editor, St. Norbert College, De Pere, WI 54115.

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