PI MU EPSILON

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PI MU EPSILON JOURNAL

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Editorial correspondence, including manuscripts, chapter reports, and news items should be sent to Underwood Dudley, Mathematics Department, DePauw University, Greencastle, Indiana 46135 (e-mail, dudley@depauw.edu). Students submitting manuscripts should give their school and date of graduation. Others should provide their affiliation, academic or otherwise.

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ON MUTUAL AND PAIRWISE INDEPENDENCE: SOME COUNTEREXAMPLES

Anwer Khurshid and Haredo Sahai
University of Exeter and University of Puerto Rico

In elementary probability theory the concepts of pairwise and mutual or stochastic independence play a useful role. Three events A, B, and C, defined on the same sample space, are said to be ***pairwise independent*** if

$$P(A\cap B) = P(A)P(B),$$
$$P(A\cap C) = P(A)P(C),$$
$$P(B\cap C) = P(B)P(C).$$

The events are said to be ***completely independent*** if

$$P(A\cap B\cap C) = P(A)P(B)P(C).$$

The events are said to be ***mutually independent*** if both conditions hold. One might think that the first relations imply the second, i. e., that pairwise independence implies complete independence. Generally, it is almost always true but there are instances where the events are pairwise independent and yet the second condition does not hold. However, such occurrences are not very common and it takes some effort to construct a nontrivial natural example. In fact, Feller (1957, p. 117) has remarked that "... practical examples of pairwise independent events that are not mutually independent apparently do not exist." It was the famous Russian mathematician S. N. Bernstein who first gave an artificial example to illustrate such a possibility. Similarly, the second condition does not imply the first, i. e., events which are completely independent are not necessarily pairwise independent.

The fact that pairwise independence is a strictly weaker condition than mutual independence is noted with some surprise by most students in a probability course. The purpose of this note is to assemble some examples when the events are pairwise independent but not completely independent and ***vice versa.*** The examples are readily constructed and can be easily presented in an elementary probability course.

Pairwise independence does not imply complete independence.

Example 1. Suppose that a regular tetrahedron has one red face, one green face, one blue face, and its fourth face colored with red, green, and blue stripes. Toss the tetrahedron and observe the face that appears (on the bottom). Let A be the event that red appears, B the event that green appears, and C the event that blue appears. We see that A, B, and C each have probability **1/2.** Furthermore, each of $A\cap B$, $A\cap C$, and $B\cap C$ has probability **1/4,** since the only way that two colors can appear is for the striped face to appear. So, A, B, and C are **pairwise** independent. But $A \cap B \cap C$ also has probability **1/4, so** that A, B, and C are not completely independent.

This example appears in **[Gnedenko,** 1963, p. **62],** where credit is given to S. N. Bernstein. It also appears in may other texts. For example, [Freund, 1973, p. **151]** gives a Venn diagram with those probabilities. But most of the time, "Bernstein's tetrahedron" wears some kind of disguise. Here are some other examples. We leave it to the reader to figure out the disguises.

1a: **[Cramér,** 1946, p. 162; Lindgren, 1976, p. **46].** Let the sample space be $S = \{(1,0,0), (0,1,0), (0,0,1)\}$ with all points equally likely. Let A (respectively, B, C) be the event that the first (respectively, second, third) coordinate is a 1.

1b: [Goldberg, 1960, p. **111].** In order to maintain quality control in a manufacturing process, each item undergoes three inspections. Of four units in a sample, unit 1 passed only the first inspection, unit 2 passed only the second inspection, unit 3 passed only the third, and unit 4 passed all three. Select one of the four units at random. Let A (respectively, B, C) be the event that the unit passed the first (respectively, second third) inspection.

1c: [Eisen, 1969, p. **49].** The following four combinations of symbols for apples, pears, and lemons appear on the face of a slot machine: (a, a, a), (p, p, p), (l, l, l), and (a, p, l), each with probability **1/4.** Let A (respectively, B, C) be the event that an apple (respectively, a pear, a lemon) shows up.

1d: [Parzen, 1960, p. 90; Hogg and **Tanis,** 1988, p. **42].** An urn contains balls numbered 1 to 4. Draw a ball at random. Let A be the event that ball number 1 or 2 is selected, B the event that ball number 1 or 3 is selected, and C the event that ball number 1 or 4 is selected.

1e: [Ash, 1972, p. 204; **Subrahmaniam,** 1979, p. 109; **Berman,** 1968, p. 69; Larsen and Marx, 1986, p. 61; Mendenhall, Schaeffer, and Wackerly, 1986, p. 1091. Suppose that a fair coin is tossed twice. Let A be the event that the first toss is a head, B the event that the second toss is a head, and C the event that both tosses yield the same outcome.

Example 2.

Roll two fair dice. Let A be the event of an odd number showing up on the first die, B the event of an odd number showing up on the second die, and C the event of an odd total from the two faces. Each of A, B and C has probability 1/2 and each of $A \cap B$, $A \cap C$, and $B \cap C$ has probability **1/4,** so A, B, and C are **pairwise** independent. But $A \cap B \cap C$ has probability 0, since two odd numbers have an even sum, so the events are not completely independent.

This example appears in [Mood, Graybill, and Boes, 1974, p. 42; Feller, 1968, p. 143; Larsen and **Marx,** 1986, p. **63].** Similar examples based on rolling **two** dice appear in several textbooks. We leave to the reader the calculation of the probabilities.

2a: [Goldberg, 1960, p. **111].** Let A be the event that 6 shows up on the first die, B the event that 6 shows up on the second die, and C the event-Of, an odd total.

2b: [Blake, 1979, p. **122].** Let A be the event that 1 shows up on the first die, B the event that 1 shows up on the second die, and C the event that the same number shows up on both dice.

2c: [Ash, 1970, p. 28]. Let A be the event that the first die shows a 1, 2, or 3, B the event that the second die shows a 4, 5, or 6, and C the event that the sum is 7.

Example 3 [Feller, 1968, p. **127].** Consider the sample space consisting of nine triplets: the six permutations of a, b, and c together with (a, a, a), (b, b, b), and (c, c, c), each with probability 1/9. Let A (respectively, B, C) be the event that the **first** coordinate ia a (respectively, the second coordinate is b, the third coordinate is c). Each of A, B, and C has probability 1/3 and each of $A \cap B$, $A \cap C$, and $B \cap C$ has probability **1/9,** so A, B, and C are **pairwise** independent. But $A \cap B \cap C$ has probability **1/9** also, so the events are not completely independent.

Example 4 [Larsen and Marx, 1986, p. **62].** A roulette wheel has 36 numbered slots colored red or black as follows:

red: (1, 2, 3, 4, 5, 10, 11, 12, 13, 24, 25, 26, 27, 32, 33, 34, 35, 36)
black: (6, 7, 8, 9, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 28, 29, 30, 31).

Spin the wheel once and observe the **number** and color of the slot in which the ball lands. Let A be the event that the slot is red, B the event that the slot has an even number, and C the event that its number is less than or equal to 18. Then each of A, B, and C has probability 1/2, and each of $A \cap B$, $A \cap C$, and $B \cap C$ has probability **1/4.** But $A \cap B \cap C$ has probability **1/9.**

Example 5 [Geisser and Mantel, 1962, p. **290].** Consider a sphere and select, randomly and independently, three great circle segments on it. Let A be the event that segments 1 and 2 intersect, B the event that **segments** 1 and 3 intersect, and C the event that segments 2 and 3 intersect. The specific probabilities involved will depend on the lengths of the segments, but whatever they are, **pairwise** independence is inherent in the experiment. If any two of the three events occur, then one of the segments intersects both of the other two; intuitively, this increases the chance that the other two segments will intersect each other. So, complete independence fails.

Example 6 [Geisser and Mantel, 1962, p. 290]. Suppose that three halls are distributed independently and at random into two or more urns. Let A be the event that balls 1 and 2 are placed in the same urn, B the event that balls 1 and 3 are placed in the same urn, and C the event that balls 2 and 3 are placed in the same urn. If there are exactly two urns, then we have yet another disguised example of the Bernstein tetrahedron. If there are exactly three urns, then we have the same probabilities as in Example 3. If there are more than three urns, then the probabilities are new.

Geisser and Mantel actually gave a generalized Example 6, where the number of balls is $n \geq 3$. Then we get $n(n - 1)/2$ events A,., where A_{ij} occurs if balls i and j are placed in the same urn. Similarly, Bernstein's example has been generalized to any number of dimensions in [Lancaster, 19651.

Other examples, not significantly different from those above can be found in [Neuts, 1973, p. 76; Kreysig, 1970, p. 57; **Giri,** 1974, p. 33; **DeGroot,** 1975, p. **42].**

Complete independence does not imply pairwise independence

Example 1 [Ash, 1970, p. **27].**
Suppose that two dice are tossed and let S be the sample space comprising all ordered pairs $\{(i, j)\}$, $i, j = 1, 2, 3, 4, 5, 6$, with a probability of 1/36 assigned to each point. **Let** A be the event that the face of the first die is **1, 2,** or 3; B the event that the face of the **first** die is 3, 4, or 5; and C the event that the sum of the two faces is 9. It is easily verified that $P(A) = P(B) = 1/2, P(C) = 1/9$, $P(A \cap B) = 1/6$, $P(A \cap C) = 1/36$, $P(B \cap C) = 1/12$, and $P(A \cap B \cap C) = 1/36$. Thus, it follows that A, B, and C are completely independent but not **pairwise** independent.

Example 2 [Larsen and Marx, pp. 61-62].
In Example 1, let A be the event that the face of the first die is 1 or 2; B the event that the face of the second die is **3, 4,** or 5; and C the event that the sum of the two faces is 4, 11, or 12. It is readily shown that A, B, and C are wmpletely independent but not **pairwise** independent.

Example 3 [Crow, 1957, pp. 716-7171.
An urn contains one red, one blue, one white, two yellow, and three black balls. A ball is drawn randomly from the urn. Let A be the event that a red, yellow, or white ball is drawn; B the event that a blue, yellow, or white ball is drawn; and C the event that a black or white ball is drawn. It is easily shown that $P(A) = P(B) = P(C) = 1/2$, $P(A \cap B) = 3/8$, $P(A \cap C) = P(B \cap C) = 1/8$, and $P(A \cap B \cap C) = 1/8$. Thus, it follows that A, B, and C are completely independent but not **pairwise** independent. Essentially the same example is given in [Lindgren (1976, p. **48)].**

Example 4 [Subrahmaniam, 1979, p. **110].**
In a certain town there are three editions of a daily newspaper: morning (M), evening (E), and weekend (W). Suppose the probability of a randomly chosen household subscribing to any one of the editions is illustrated by the Venn diagram in Figure 1. From the figure, $P(M) = .50$, $P(E) = .80$, $P(W) = .60$, $P(M \cap E) = .42$, $P(M \cap W) = .24$, $P(E \cap W) = .42$ and $P(M \cap E \cap W) = .24$. Thus, M, E, and Ware wmpletely independent but not **pairwise** independent.

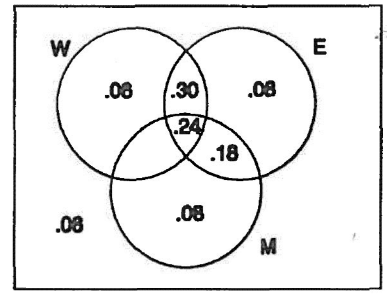

Figure 1

Example 5 [Freund, **1962,** p. **50].**
Suppose the events A, B and C and their respective probabilities are illustrated by the Venn diagram in Figure 2, where the sample space contains 100 equally likely points and the numbers on the diagram indicate the number of distinct outcomes contained in the respective events. From the figure, $P(A) = 1/2$, $P(B) = 1/4$, $P(C) = 8/100$, $P(A \cap B) = 1/4$, $P(A \cap C) = P(B \cap C) = 11100$, and $P(A \cap B \cap C) = 1/100$. **Thus,** it **follows that** A, B, and C are wmpletely independent but not **pairwise** independent.

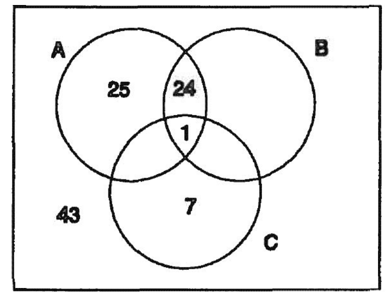

Figure 2

Example 6 [Goldberg, 1960, p. **112].**
A card is selected at random from a standard deck of 52 cards. Let A be me event that the selected card is a spade or club, B the event that it is a spade, and C the event that it is the ace of spades or the ace, king, **... ,** 8 of diamonds. It is readily verified that A, B, and C arc completely independent but not **pairwise** independent.

Example 7 [Mood, **Graybill,** and Boes, 1974, p. **43].**
Consider two events A and B that are not independent and another event C of probability zero. Then it immediately follows that A, B, and C are wmpletely independent but not **pairwise** independent

As suggested by the examples given in this paper, some **authors** give only one type of counterexample while others include both types. Furthermore, most examples can be criticized as **"frivolous" in** nature, or at least artificial. We conclude with two examples which definitely are not frivolous but have important practical applications.

Example 1 [Geisser and Mantel, 1962, p. 2901.

Let r_{12}, r_{13}, and r_{23} be the **pairwise** sample correlation coefficients based on a random sample of n observations from a trivariate nonsingular normal distribution having a diagonal variance-covariance matrix. Using the methods of mathematical statistics, it can be proven that the joint density of the sample correlation coefficients, r_{12}, r_{13}, and r_{23} is given by

$$f(r_{12}, r_{13}, r_{23}) = C(n)(1 - r_{12}^2 - r_{13}^2 - r_{23}^2 + 2r_{12}r_{13}r_{23})^{(n-5)/2}$$

when $1 - r_{12}^2 - r_{13}^2 - r_{23}^2 + 2r_{12}r_{13}r_{23} > 0$ and zero elsewhere. Now, using the considerations of continuity and positive definiteness of the correlation matrix, it can be shown that the three random variables are not mutually independent. However, it can be shown directly that the variables r_{12}, r_{13}, and r_{23} are **pairwise** independent.

The above result can be extended to the general case of $p(p - 1)/2$ (withp > 3) jointly distributed correlation coefficients when a random **sample** of n observations is drawn from a p-variate nonsingular normal distribution having a diagonal variance-covariance matrix. The result has an important statistical application to the effect that it simplifies the evaluation of the variance, in the null case (when the corresponding population correlations are zero), **i.** e.,

$$\mathrm{Var}\left(\sum_{i=1}^{p}\sum_{j=i+1}^{p} a_{ij}r_{ij}\right) = \sum_{i=1}^{p}\sum_{j=i+1}^{p} a_{ij}^2\,\mathrm{Var}(r_{ij}).$$

Example 2 [Driscoll, 1978, p. **432].**

Consider two independent random variables X and Y each having the rectangular distribution on the unit interval. Further, define a random variable $Z = (X + Y)$ (mod **1**), i. e.

$$Z = \begin{cases} X + Y & \text{if } 0 \le X + Y \le 1 \\ X + Y - 1 & \text{if } 1 < X + Y \le 2. \end{cases}$$

Then, using the methods of calculus and analytical probability, it can be shown that X, Y, and Z are identically distributed and **pairwise** independent but not mutually independent.

The above result has an important application in the characterization of the rectangular distribution. For example, using the methods of advanced probability, it can be proven that among the absolutely continuous distributions having the closed unit interval for their support, the rectangular distribution is the only one satisfying the above properties [Driscoll, 19781. The result also extends to intervals other than **[0, 1]**, as well as to discrete rectangular distributions, and thus provides characterizations for all rectangular distributions.

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Chapter Reports

Professors Rudy Gideon and Mary Jean **Brod,** faculty advisors of the MONTANA ALPHA chapter (University of Montana), report that there are thirty new members and plans for biweekly meetings. Four students presented talks to the chapter last year:

Dean **Risinger—Finiteness** and rings of continuous functions
Doug Holstein—Exact sequences of topological spaces
Patricia **Olsen—Optimal** jury selection
Scott McRae—Analysis of a heuristic algorithm for optimally scheduling assignments with constraints.

The NEW **YORK** OMEGA chapter (St. Bonaventure University) held five meetings and, in cooperation with the MAA student chapter, sponsored the Mathematics Forum, a series of ten lectures, including one by this year's Frame Lecturer, George Andrews.

ON THE SOLUTIONS OF $a^a = b^b$

Jeffrey D. Bomberger
University of *Nebraska—Lincoln*

The equation $x^y = y^x$ was solved by **Euler** and has been considered many times since. However, the equation $x^x = y^y$ is not as well known. In this paper we will find all pairs (a, b) which satisfy $a^a = b^b$.

Let us consider the function $f(x) = x^x$, $x > 0$. Calculation shows that f attains its minimum at $x = 1/e$ and that $\lim_{x\to 0^+} x^x = 1$. Thus the graph of f is as shown in Figure 1. From the graph we can see that if $e^{-1/e} < y < 1$, then there is a unique pair of real numbers (a, b) that satisfies $a^a = b^b = y$. Therefore, by allowing y to vary in the interval $(e^{-1/e}, 1)$, it follows that there are infinitely many pairs (a, b) that satisfy $a^a = b^b$.

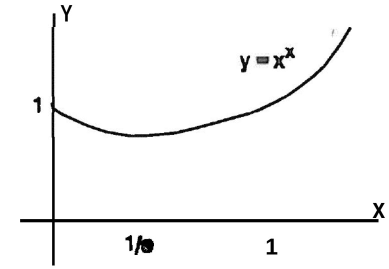

Figure 1

Let S denote the set of all pairs (a, b) with a < b which satisfy $a^a = b^b$. Then it is clear that if $(a, b) \in S$ then $0 < a < 1/e < b < 1$. Also, since f is one-to-one on each of (0, lie) and $(1/e, 1)$, if (a, b) and (a, c) are in S, then b = c, and if (a, b) and (c, b) are in S, then a = c.

It is not hard to see that $(1/4, 1/2) \in S$. Thus, there is y so that $(1/2y, 1/y) \in S$. Is there y so that $(1/3y, 1/y) \in S$? If so, then $(1/3y)^{1/3y} = (1/y)^{1/y}$, $(1/3y) = (1/y)^3$, and $y^3 - 3y = 0$. Solving, we get $y = 3^{1/2}$. Replacing 3 with 4 gives $y^4 - 4y = 0$ and $y = 4^{1/3}$.

In general, if $(1/xy, 1/y) \in S$, we have $y^x - xy = 0$, from which we get

LEMMA 1: For any $x > 1$, $(x^{-x/(x-1)}, x^{-1/(x-1)}) \in S$.

Proof-. Let $a = x^{-x/(x-1)}$ and $b = x^{-1/(x-1)}$. Then $0 < a < b$ and since $xa = b$,

$$a^a = (x^{-x/(x-1)})^a = (x^{-1/(x-1)})^{xa} = b^b.$$

LEMMA 2: For $x > 1$, let $A(x) = x^{-x/(x-1)}$ and $B(x) = x^{-1/(x-1)}$. Then

(i) A and B are continuous on $(1, \infty)$.

(ii) A is strictly decreasing and B is strictly increasing on $(1, \infty)$.

(iii) $\lim_{x\to 1^+} A(x) = e^{-1}$, and $\lim_{x\to\infty} A(x) = 0$.
(iv) $\lim_{x\to 1^+} B(x) = e^{-1}$, and $\lim_{x\to\infty} B(x) = 1$.

Proof: (i): Since $A(x) = \exp(-x\ln x/(x-1))$ and $x\ln x/(x-1)$ is continuous on $(1,\infty)$, then so is A. Similarly, B is continuous on $(1,\infty)$. In fact, A and B are infinitely differentiable on $(1,\infty)$. (ii): Let $h(x) = x - 1 - \ln x$. Then $h'(x) = 1 - 1/x > 0$ for $x > 1$. So, h is strictly increasing on $(1,\infty)$, and $x - 1 - \ln x = h(x) > h(1) = 0$ for all $x > 1$. Now, since $A'(x) = -h(x)A(x)/(x-1)^2$ and $A(x) > 0$, it follows that $A'(x) < 0$ for $x \in (1,\infty)$. The proof that $B'(x) > 0$ for $x \in (1,\infty)$ is similar. Here, we need only show that $x\ln x - (x-1) > 0$ for all $x > 1$, which is not difficult. Finally, (iii) and (iv) follow from L'Hôpital's rule.

THEOREM: $S = \{(x^{-x/(x-1)}, x^{-1/(x-1)}) \mid 1 < x < \infty\}$.
Proof: By Lemma 1, we need only show that
$$S \subset \{(x^{-x/(x-1)}, x^{-1/(x-1)}) \mid 1 < x < \infty\}.$$
Let $(a, b) \in S$ and $A(x)$, $B(x)$ as in Lemma 2. Then, from Lemma 2, $A(x)$ and $B(x)$ are one-to-one wntinuous functions on $(1, \infty)$, $A(1,\infty) = (0, e^{-1})$, and $B(1,\infty) = (e^{-1}, 1)$. Now, since $a \in (0, e^{-1})$ and $b \in (e^{-1}, 1)$ then by the intermediate value theorem, there exist unique numbers x and $z \in (1,\infty)$ such that $A(x) = a$ and $B(z) = b$. So, $(A(x), B(z)) \in S$, but $(A(x), B(x)) \in S$ by Lemma 1. So, $B(z) = B(x)$. Since $B(x)$ is one-to-one on $(1,\infty)$, then $z = x$. Hence, $(a, b) = (A(x), B(x))$ for a unique $x \in (1,\infty)$, completing the proof.

By letting $x = n/(n-1)$ for integer $n > 1$, we get an infinite sequence of rational solutions (a, b): $\{((n-1)/n)^n, (n-1)/n)^{n-1}\}$, $n = 2, 3, 4, \ldots$. The first four terms of this sequence are (1/4, 1/2), (8/27, 4/9), (811256, 27/64), (1024/3125, 256/625).

Jeffrey Bomberger is an actuarial science major at the University of *Nebraska—Lincoln*. This paper was written when he was a first-year student in calculus, under the direction of Professor Mohummad *Rammaha*.

Chapter Report

Professor Paul Eloe reports from the OHIO ZETA chapter (University of Dayton) that Kristine Fromm and Kristen Toft participated in summer research programs in 1992, that nine students participated in the Putnam Examination, and that a Dayton team was entered in the annual **mathematical** modeling contest.

ELLIPSES AS PROJECTIONS OF CIRCLES

Ali R. *Amir-Moéz*
Texas Tech University

One can study some properties of an ellipse through the orthogonal projection of circles. Two properties which are carried through the projection are quite interesting.

(i) Tangency is transformed into tangency.

(ii) Areas are all multiplied by the cosine of the angle between the plane of the circle and the plane of its projection.

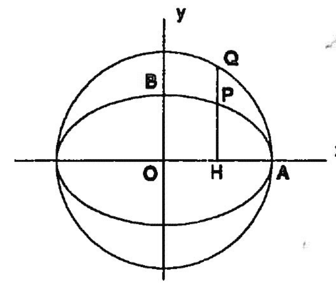

Figure 1

1. The Projection of a Circle: Each ellipse can be considered as the orthogonal projection of its principle circle. Let Q be a point on the principle circle of an ellipse (Fig. 1). Then the perpendicular to OA through Q intersects the ellipse at P. Let the foot of the perpendicular be H. We may say that HP is the projection of HQ. In order to make this idea clear, we rotate the principle circle about the x-axis through an angle 9 such that $\cos 9 = b/a$; then the projection of this circle on the xy-plane is the ellipse. A cross section, with OA being the edge of the plane of the circle, is shown in Figure 2.

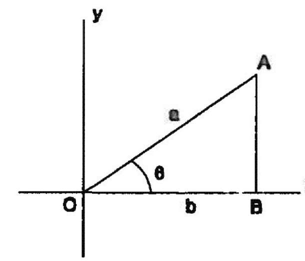

Figure 2

Let the equation of the principle circle be

(1) $$X^2 + Y^2 = a^2,$$

Then Q has coordinates (X, Y) which satisfy (1). Since
$$\frac{PH}{QH} = \cos\theta = \frac{b}{a},$$
if the coordinates of P are (x, y) then we observe that
$$x = X \text{ and } y = \frac{b}{a}Y.$$
Substituting in (1) for X and Y, we obtain

$$x^2 + \frac{a^2}{b^2}y^2 = a^2$$

or

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

2. Tangency: We observe that a tangent line to an ellipse at a point P and the tangent line to the corresponding point Q of the principle circle intersect at a point K which is on the x-axis (Fig. 3). The proof is quite simple. Now suppose we would like to draw a tangent line to the ellipse from any point L outside the ellipse (Fig. 3). We draw the perpendicular line to the x-axis through L. Then we obtain the point M on this line such that

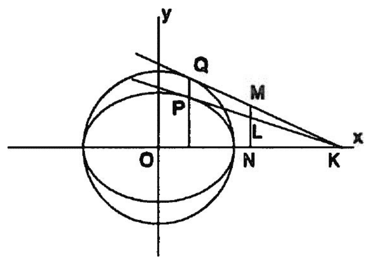

Figure 3

$$\frac{LN}{MN} = \cos\theta = \frac{b}{a}.$$

Then the tangent line through M to the circle intersects the x-axis at K. The line KL is tangent to the ellipse.

3. Areas: It is well-known that a projection of an area (as in Figure *2)* is equal to the area of the original surface multiplied by the cosine of the angle between the two planes. So the area of the ellipse with semi-major axis a and semi-minor axis b will be

$$A = na^2\cos\theta = \pi a^2\left(\frac{b}{a}\right) = nab.$$

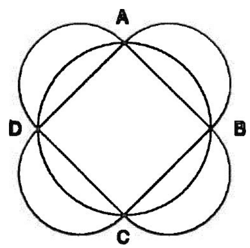

Figure 4

4. Hippocrates' Theorem: Let ***ABCD*** be a square inscribed in a circle. Draw four half circles with diameters ***AB, BD, DC,*** and ***CA.*** We obtain four crescent-shaped configurations over the arcs (Fig. 4). The sum of the areas of these crescent shapes is the same as the area of the square ***ABCD.*** The proof is quite simple and can be found in [1].

Now consider the orthogonal projection of this configuration on a plane through the line ***CB.*** We shall get an ellipse and four crescent-shaped areas that

we would like to describe. Each of the areas is situated between two ellipses; one of them is the projection of the circumscribed circle of the square. We shall look at the outer ellipse in the first quadrant. The center M of the circle with diameter ***AB*** projects into M', the midpoint of $A'B$ (Fig. *5).* The half circle of diameter ***AB*** projects into half of an ellipse. Let the projection of the circumscribed circle be

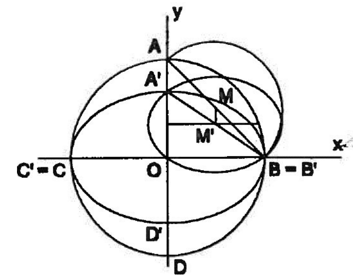

Figure 5

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Then half of the major axis of the other ellipse will be $a\sqrt{2}/2$ and half of its minor axis will be

$$\frac{a\sqrt{2}}{2}\cos\theta = \frac{a\sqrt{2}}{2}\cdot\frac{b}{a} = \frac{b\sqrt{2}}{2}.$$

The set of coordinates of M is $(a/2, a/2)$. So the set of coordinates of M' will be

$$\left(\frac{a}{2}, \frac{a}{2}\cos\theta\right) = \left(\frac{a}{2}, \frac{b}{2}\right).$$

Consequently, the equation of this ellipse is

$$\frac{(x - a/2)^2}{a^2/2} + \frac{(y - b/2)^2}{b^2/2} = 1.$$

This way the four crescent-shaped areas become four areas the sum of which is the same as the area of the rectangle, that is, the projection of the square ***ABCD.***

Reference

1. Amir-Moéz, A. R. and J. D. Hamilton, Hippocrates, ***Journal of Recreational Mathematics 7*** (1974) #2, 105-107.

Ali R. Amir-Moéz, Professor Emeritus since 1988, is the author of, among many other things, Elements of Multilinear Algebra and Three Persian Tales.

$PSL(2, 7)$ IS SIMPLE, BY COUNTING

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Proving that $PSL(2, 7)$ is a simple group can be done by the method of conjugation. This paper will use the more efficient method of counting. The paper will cover some definitions that arc referred to in the proof and then address the proof. This method of proof also applies to A_5. I will begin by defining the group.

$SL(2, 7)$ is the group of 2-by-2 matrices with determinant one with entries from F_7, the integers modulo 7. To find its order, we look at the following elements, with a, b, c, d nonzero entries. There are 6 of the form $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$, 6 of the form $\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}$ and 36 of each of the four forms $\begin{bmatrix} 0 & b \\ c & d \end{bmatrix}$, $\begin{bmatrix} a & 0 \\ c & d \end{bmatrix}$, $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$, and $\begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$. The entries a and b in $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ can be chosen in 36 ways and for each choice there are 5 ways of choosing c and d (since both are nonzero). So, the order of $SL(2, 7)$ is $2\cdot 6 + 4\cdot 36 + 5\cdot 36 = 336$.

The center of a group is

$$Z(G) = \{ s \in G \mid sg = gs \text{ for all } g \in G \}.$$

$PSL(2, 7)$ is the group $SL(2, 7)/Z(SL(2, 7))$. To find the order of $PSL(2, 7)$, we find $|Z(SL(2, 7))|$. We arc looking for the matrices that commute with all the elements of $SL(2, 7)$. Since such a matrix commutes with all the elements, in particular it commutes with matrices of the form $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, with a and b not equal to 0. Thus

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} c & d \\ e & f \end{bmatrix} = \begin{bmatrix} c & d \\ e & f \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

or

$$\begin{bmatrix} ac & ad \\ be & bf \end{bmatrix} = \begin{bmatrix} ca & db \\ ea & fb \end{bmatrix}.$$

So we know that $be = ea$ for all b, a in F_7 so $(b - a)e = 0$ from which it follows that $e = 0$. We also know that $ad = db$ for all a, b, so we have that $d = 0$. Since $d = e = 0$, any matrix in the center has the form $\begin{bmatrix} c & 0 \\ 0 & f \end{bmatrix}$. Now, $\begin{bmatrix} c & 0 \\ 0 & f \end{bmatrix}$ also commutes with $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ so

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} c & 0 \\ 0 & f \end{bmatrix} = \begin{bmatrix} c & 0 \\ 0 & f \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

or

$$\begin{bmatrix} 0 & f \\ c & 0 \end{bmatrix} = \begin{bmatrix} 0 & c \\ f & 0 \end{bmatrix},$$

from which it follows that $f = c$. Since $\begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$ must have determinant 1, $c^2 = 1$ and so $c = 1$ or $c = -1 = 6$ (in F_7). Also, it is easy to see that $\begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$ does indeed commute with all the elements of $SL(2, 7)$. Hence the order of $PSL(2, 7)$ is $336/2 = 168$.

Now we state and prove a theorem.

THEOREM: Let H be a normal subgroup of a finite group G and let x be in G. If $\gcd(|x|, |G/H|) = 1$, then x is in H. ([1], p. 156)

Proof: Let $|x| = r$, $|G/H| = s$. Then $x^r = e$, so $|xH| \mid r$ because $(xH)^r = (x^r)H = eH$. Now, by Lagrange's Theorem, $|xH| \mid |G/H|$, and so $|xH| \mid s$. Since $|xH| \mid r$ and $|xH| \mid s$, we have that $|xH| \mid \gcd(r, s)$. Since $\gcd(r, s) = 1$, we have $|xH| = 1$, and x is in H.

A straightforward but long calculation shows that the elements of $PSL(2, 7)$ have the following orders:

Order	1	2	3	4	7
Number of elements	1	21	56	42	48

Now we can prove that $PSL(2, 7)$ is simple. If $PSL(2, 7)$ has a nontrivial proper subgroup H, then

$$|H| = 2, 3, 4, 6, 7, 8, 12, 14, 21, 24, 28, 42, 56, \text{ or } 84.$$

We consider different cases.

i.) If $|H| = 7, 14, 21, 28$, or 42, then $|PSL(2,7)/H|$ is relatively prime to 7, and so by the theorem, H would have to contain all 48 elements of order 7. Since $|H| \leq 42$, that is impossible.

ii.) If $|H| = 3, 6, 12$, or 24, then $|PSL(2,7)/H|$ is relatively prime to 3, and so by the theorem, H would have to contain all 56 elements of order 3, but $|H| \le 24$.

iii.) If $|H| = 8$, then $|PSL(2,7)/H|$ is relatively prime to 4, and so by the theorem H would have to contain all 42 elements of order 4, but $|H| = 8$.

iv.) If $|H| = 56$, then $|PSL(2,7)/H|$ is relatively prime to both 4 and 7, and so by the theorem, H would have to contain all 90 elements of orders 4 and 7, but $|H| = 56$,.

v.) If $|H| = 84$, then $|PSL(2,7)/H|$ is relatively prime to both 3 and 7, and so by the theorem, H would have to contain all 104 elements of orders 3 and 7, but $|H| = 84$,.

vi.) If $|H| = 2$ or 4, then $|PSL(2,7)/H| = 84$ and 42. We know from the Sylow theorems that any group of order 84 and any group of order 42 has only one Sylow-7 subgroup, which is therefore normal. If H is normal in $PSL(2, 7)$, then $PSL(2, 7)/H$ has a normal subgroup N_H of order 7. So now we consider the projection homomorphism

$$\psi : PSL(2, 7) \to PSL(2, 7)/H.$$

$\psi^{-1}(N_H)$ is a normal subgroup of G and is of order $2 \cdot 7$ or $4 \cdot 7$. However, we have already shown that there does not exist a normal subgroup of order 14 or 28, so this case also is impossible.

We have examined all possible orders of nontrivial proper normal subgroups H of $PSL(2, 7)$ and shown that all lead to contradictions. Hence $PSL(2, 7)$ has no nontrivial proper normal subgroups and is a simple group of order 168.

Reference

1. **Gallian,** Joseph A., Contemporary Abstract Algebra, second edition, D. C. Heath & ***Co.,*** Boston, 1990.

Sandra ***Lepsi,*** now a graduate of Illinois Benedictine, wrote this ***paper*** under the direction of Dr. Lisa ***Townsley Kulich. She plans*** to continue studying mathematics.

3507! – 1 is a big prime. How many digits?

EQUIVALENCE CLASSES IN THE REAL WORLD

James Ramaley
Ziff-Davis Publishing Company

1. Introduction — what does a mathematician do in business?

Perhaps the last two words in the title of this section are **superfluous—many** people **simply** wonder what **mathematicians** do in general without restricting the question to the "business world". But while I was teaching I was often asked precisely that question by my students and, at the **time,** I really couldn't answer them with any kind of authority. I knew that there were many jobs in which a knowledge of elementary statistics was useful. And I even suspected that one might be able to use some linear algebra or calculus, but aside from jobs that are viewed as "technical", I didn't really have much of a clue as to the real value of a mathematical education to the business community.

Over the last twenty **years** in business I have come to realize that the value of **mathematical** studies is not specifically in the mastery of certain tools. Rather, it is that the study of niathematics leads one into a "mathematical" approach to problem solving that places great emphasis upon precise definitions. The **fact** of the matter is that niost people are not skilled in, nor do they appreciate the importance of, precise definition-making. Yet a mathematician, almost by instinct, will turn a question of problem-solving into one of **problem**-identification.

The matching problem outlined below gives a good example of a problem that is of great strategic and tactical interest to the publishing industry and which has been "solved" before many times. However, as you will see, a mathematician's approach will give a new twist to this old problem.

2. The matching problem.

Recognizing whether or not two things are the same is a **theme** common to many parts of mathematics, as well as of human existence. Even the statement that $1 + 1 = 2$ is a simple illustration of this theme, but it has great consequences in the development of the **Peano** Postulates for Arithmetic.

The matching I am interested in is that of **names** and addresses. In the magazine industry the inadvertent entry onto the subscription files of two orders for the same person is usually a minor annoyance to the subscriber, but it is a costly mistake for the company. Not only will the company waste copies by sending unwanted copies to the subscriber, but the failure to **detect** such duplicates has the undesirable side effect of reducing the **total** "paid **circulation"** of the **magazine.** Paid circulation is a technically defined **term** that is used by advertisers to indicate the **number** of people who have paid to **receive** copies of the magazine. It is the basis for advertising rates in the magazine and is there-;

fore of great interest to the advertiser and publisher alike.

Correctly matching names and addresses has many other raniifications. For example, credit reporting bureaus link together records of transactions such as credit card payments or loan payments and prepare credit reports requested by companies looking to extend credit to borrowers. An example that is very important to direct mail advertisers (magazines themselves are often sold through direct mail promotions) has to do with name suppression. If a list of prospects is rented from an outside source, before the list is mailed one would want to identify all the prospects on the list who are already subscribers to the magazine and "suppress" their names from the mailing.

Making an error in the suppression of names is not nearly as serious as erroneously reporting credit information. In the former case it simply means than an existing subscriber gets a wasted promotion; in the latter case an erroneous credit report could result in a costly lawsuit.

But consider the cost of mailing promotions to already subscribing prospects. The cost of a promotion may be as great as *50* cents per name. It is not unusual to mail out as many as 1 million pieces in a single campaign. If as few as 5% of these are already on file, a not uncommon duplication percentage, some *$25,000* will be wasted trying to promote people who already subscribe!

3. The traditional solution.

The common approach to this problem is quite straightforward. For each name and address record a "matchcode" is defined by extracting portions of the record in a specific way. Two records having the same matchcode are, by definition, declared to be duplicates.

For example, a matchcode might be defincd by concatenating the zip code with the subscriber's last name. Such a simple matchcode would have a couple of obvious failings. In the first place, any two people having the same last name actually residing in the same zip code area would be declared duplicates. A glance in any phone book would indicate that it is, in fact, relatively common for families to have members with the same surname living near each other. Another problem conics from the fact that many lists are constructed in such a way that the last name is not easily extracted. Sometimes the last name might be first, sometimes it might be last. There might he a suffix (Ph.D. or Jr.) or a prefix (Mr. or Ms.) that one has to avoid. Several techniques can help minimize these shortcomings. For example, instead of using the entire last name, a common trick is to drop all non-initial vowels from the last name instead. This partially compensates for two of the most common errors in transcribing names: transposition of characters and the interchange of vowels within a name. Another idea is to append the first initial of the first name—this helps distinguish different family members. Sometimes a portion of the address

is added too. Clearly any matchcoding scheme can fall prey to two errors—two different records might give rise to the same matchcode (a type II error, in statistical terms) or really duplicate records give rise to different matchcodes (a type I error). It is well-known that it is impossible to minimize both types of errors simultaneously. So, the question usually boils down to which type of error is the more serious in the context. The publisher can decide to use a "tight" matchcode (one that extracts many characters from each record and requires matching on all characters) or a "loose" matchcode. A tight matchcode minimizes making an error in saying that two records are duplicates (when they are not) but will overlook duplicate records that are not nearly perfect duplicates; a loose matchcode minimizes the chance of overlooking two duplicate records by asserting some are equal even when they are not. Aside from the observations above, matchcodes have an extremely vulnerable shortcoming—they are hierarchical. That is, in comparing two matchcodes, even a very simple error early in the hierarchy will cause two match codes to be significantly different. Two records could be nearly identical, but a mistake in the first letter of the last names would be fatal.

4. A High Tech Approach.

I like to think of the process of matching to a "search and rescue" operation. I think of each record as broadcasting a signal that indicates its presence while a matching procedure picks up these signals and determines (by a scoring mechanism) whether two records' signals are sufficiently strong to determine them as duplicates. The signals broadcast are called "tokens" and are extracts of the subscriber record chosen in a way that they "represent" portions of the record. For example, a name field might be represented by a first name token, a middle name token, a last name token, and a set of initials. Thomas J. Watson might become four tokens: THMS, J, WTSN, TJW. The same concept could apply to companies: International Business Machines would become INTRN, BSNSS, MCHNS, IBM. In a high tech approach each record is first broken into tokens and the tokens are put into an indexed file. Then every record is again read sequentially and, for each token, every record having a matching token is selected and put into a pool of matching "candidates". A scoring mechanism can be set up that is used to determine whether a potential match has a sufficiently high score to be declared an actual match. This method has the advantage of minimizing the chance of missing records which are, in fact, duplicates (providing the scoring mechanism is good). But there is a tremendous cost. Clearly there will be many, many instances where one, two, or even more tokens might match but the records are simple not matches. Several techniques could be used to reduce the number of potential matching records, but still the number of data fetches could overwhelm even a very powerful computer.

5. A mathematician's approach.

We have seen two approaches to the matching problem. The **matchcode** technique is fast but is very sensitive to variations in the name and address, A token approach decreases the chance of missing duplicate records but requires tremendous computer capacity since many records are read hundreds of times. What would a mathematician do? A mathematician is trained to look for ways in which a problem can be broken down into **smaller** problems — hopefully ones which are simpler and perhaps have even already been solved. Also **reasoning** by analogy is a common approach and so one tries to find analogies to **prior** experiences. First note that it is possible to think of matching records as a **metric** problem; or more generally, as a problem in defining an **equivalence** relation whereby two records are "related" if they match. Secondly, it is important to realize that matching has two important **components—(1)** the search for candidates for matching and (2) a scoring mechanism to declare actual matches. Considering the two aspects of matching — searching and **scoring** — gives a key idea. We want a search procedure that leverages prior search activities. That is, suppose we had five candidates for matching a given record. Because we want matching to be an equivalence relation, each of these records should be a candidate for matching each other. Therefore, when we create a candidate pool for the "first" of these records we want to use this pool for **all** these records simultaneously. This leads to the idea of a "window" of candidates. We sort all the records by some criterion that maximizes the chances that candidates are "near" each other and then open a sliding window that considers all of the records visible in the window as potential candidates for matching. The advantage of this view is that records are read only once and **are** candidates for matching with all other records in the window. From a **processing** standpoint, since each record is read only once the process is linear with **the** **number** of records.

Now that we have a way to locate candidates we still need a **scoring** mechanism to declare matches. The scoring mechanism should be **symmetric** since the order of comparing two records should be irrelevant. This suggests some kind of additivity, the **simplest** being just to add points for matching tokens from both records. We will declare that two records match if the score exceeds some predetermined threshold. The easiest way of insuring transitivity is simply to take the transitive closure of this matching relation–. if a record matches any member of a family it is defined to **be** in that family.

This idea also has an extremely valuable consequence that considerably widens its usefulness. **A** "helper file" is a file which contains some certified linking field that exists between records within the file. For **example,** it may be

possible to obtain a file that contains "certified" variations of a company name all linked by a single company number. Sometimes these variations are non-trivial, as in the case of the Scripps Oceanographic Institute — a part of the University of California at San **Diego!** If the helper file links these two records and is merged with the file to be matched, the transitivity of the matching relation insures that otherwise unmatchable records are, in fact, matched.

Without going into further detail here I just want to recall the point that an equivalence relation gives rise to equivalence classes so that two records are "related" (or **matched)** if they are in the same class. This point is fundamental to a person who has been trained in mathematics but it would **be completely** overlooked by a non-mathematician.

6. Epilogue.

There are many further applications of mathematical thought in the matching problem. Just to name a couple, consider the probleni of measuring the accuracy of a matching run. To assert that a given **match** is either correct or incorrect requires that you have some underlying scoring **method** (other than the niatching program itself). This would be the case if, for example, one has a certified linkage between records (perhaps you secretly have social security numbers). The problem, then, is how to define a **metric** that reflects the accuracy of the matching run.

Another **problem** would be how to best define a match "threshold", a score that minimizes the changes of erroneously declaring matches and which **simultaneous** maximizes the likelihood of correctly declaring matches (it is mathematically impossible to actually do both, but still one needs to define a threshold for use).

So, how can one recognize uses of mathematics in the "real world"? The key is to be able to recognize general structures and the procedures used to build and analyze such structures. And while the iniportance of precise definitions is well understood by a **mathematician,** it is rarely understood just how important precision is in understanding **the** exact nature of a problem **before** trying to solve it.

James Ramaley is Vice President for Circulation Systems for Ziff-Davis Publishing, where he has been employed Jor almost twenty years. His Ph. D. work at the University of New Mexico was in category theory. His career ran he explained by the genetic influences of his grandfather (professor at the University of Colorado from 1898 to 1943) and great-grandfather (printer and publisher in Minneapolis from AS70 to 1917).

A Theorem on Invertible Matrices

Russell Euler
Northwest Missouri State University

By the Cayley-Hamilton Theorem, every square matrix satisfies its characteristic equation. Let A be a 2×2 invertible matrix with real entries. When will A^{-1} satisfy the characteristic equation of A? This note answers the question.

Let $p(\lambda)$ denote the characteristic polynomial of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $p(\lambda) = |A - \lambda I| = \lambda^2 - (\mathrm{tr}\,A)\lambda + |A|$, where $|A| = ad - bc$ and $\mathrm{tr}\,A = a + d$ (which is called the trace of A). Since A is invertible, $|A| \neq 0$.

THEOREM. $p(A^{-1}) = O$ if and only if $|A| = 1$ or $(\mathrm{tr}\,A)A = (1 + |A|)I$.

Proof. Since $p(A) = 0$,

$$\begin{aligned} A^2 p(A^{-1}) &= A^2 p(A^{-1}) - |A|\, p(A) \\ &= A^2[(A^{-1})^2 - (\mathrm{tr}\,A)A^{-1} + |A|I] - |A|[A^2 - (\mathrm{tr}\,A)A + |A|I]. \end{aligned}$$

This simplifies to give

$$\begin{aligned} A^2 p(A^{-1}) &= I - (\mathrm{tr}\,A)A + (\mathrm{tr}\,A)|A|A - |A|^2 I, \\ &= (1 - |A|^2)I + (|A| - 1)(\mathrm{tr}\,A)A, \\ &= (1 - |A|)\big((1 + |A|)I - (\mathrm{tr}\,A)A\big). \end{aligned}$$

If $p(A^{-1}) = O$, then either $|A| = 1$ or $(\mathrm{tr}\,A)A = (1 + |A|)I$.

Conversely, if $|A| = 1$ or $(\mathrm{tr}\,A)A = (1 + |A|)I$, then $A^2 p(A^{-1}) = O$ and so $p(A^{-1}) = O$.

COROLLARY. If $|A| = -1$ and $\mathrm{tr}\,A = 0$, then $p(A^{-1}) = O$.

Proof. Since $|A| = -1$ and $\mathrm{tr}\,A = 0$, $(\mathrm{tr}\,A)A = (1 + |A|)I = O$ and so $p(A^{-1}) = O$ from the theorem.

The restriction $\mathrm{tr}\,A = 0$ in the corollary is necessary: if $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$, then $|A| = -1$, $\mathrm{tr}\,A = 2$, and $p(\lambda) = \lambda^2 - 2\lambda - 1$. Also, $A^{-1} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$ and

$$p(A^{-1}) = \begin{bmatrix} 4 & -8 \\ -4 & 4 \end{bmatrix} \neq O.$$

The question of when a 3×3 matrix A has an inverse that satisfies the characteristic equation of A remains open.

Russell Euler, the author of many analytical and pedagogical papers, received his Ph. D. degree from the University of *Missouri—Kansas* City. He is a professor in the department of mathematics and statistics at Northwest Missouri State University.

Are Proofs Hard?

Here is an amazing theorem, giving a necessary and sufficient condition for two numbers to be equal, connecting the operations of addition and multiplication:

THEOREM: $x = y$ if and only if $xy = (x + y)^2/4$.

Prove half of the theorem, your choice of which half.

There, was that hard? Now, if you feel like it, prove the other half. Then see how the theorem would be changed if the last equation had a 2 instead of a 4 in the denominator. Then, if you still feel like it, see what a 2n instead of a 4 would do, thus proving infinitely many theorems all at once. Then let $n \to \infty$. Then what? I don't know—mathematics is endless.

Chapter Report

The Historian of the OHIO XI chapter (Youngstown State University), Lori Kaminski produced an eight-page chapter Newsletter, not all of which can be reproduced here. The winner of the annual calculus competition, and \$50—who says that mathematics does not pay?—was a major in mechanical engineering, but second place was taken by Patrick **DiRusso,** a mathematics major. The Pi Mu Epsilon T-shirt Sale resulted in twenty-seven purchases. Among the advantages of preparing a student paper are

> A student paper presentation at a national meeting will often draw a positive reaction from interviewers when discussing a resume.

The chapter held a mathematics careers panel discussion at which six alumni served as panel members. Their occupations are: Vice President of a software company, Senior Actuarial Analyst at a pension consulting firm, Computer Analyst for the National Security Agency, Business Relations Specialist for Electronic Data Systems, Mathematics Instructor at a high school, and graduate, student in mathematics.

How to Find Sines Without Knowing Any

Andrew Cusumano
Great Neck, New York

This note describes an easily programmable procedure for approximating the sine of a given angle, using the double-angle formula for the sine.

The approach is to divide the angle in half enough times until we obtain an angle small enough to approximate its sine as the angle itself. We can then work backwards, repeatedly using the double-angle formula, until we have the sine of the original angle.

Since $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$, the error in approximating $\sin x$ by x is $< x^3/6$. If we want 10-place accuracy, then we can use the approximation it $x^3/6 < 5\cdot 10^{-11}$, or $x < \sqrt[3]{3\cdot 10^{-10}} \approx .0006694$. For an angle between $0°$ and $90°$, we will need no more than k divisions by 2, where $\frac{\pi/2}{2^k} < .0006694$, so k = 12 will do. Then, since $\sin 2x = 2\sin x \cos x = 2\sin x\sqrt{1-\sin^2 x}$, we apply

$$(1) \qquad \sin a_{n+1} = 2\sin a_n\sqrt{1-\sin^2 a_n}$$

k times, with $\sin a_0 = a_0$, where a_0 is the original angle bisected k times.

The procedure can easily be carried out on **non-programmable** calculators without any data entry other than the original angle as long as one memory location is available to store the current value of sin a . It takes only a few minutes to go from $\frac{\pi/6}{2^{10}} = .0005113$ back to $.5000000$ by applying (1) 10 times.

Andrew Cusumano was a mathematics major at C. W. Post College, graduating in 1976. He is now a software engineer and, besides that, is interested in sequences, series, and geometry.

Computer Arithmetic on an Abacus

Don Bloomquist, Jr.
Albertson College

That ancient calculating device, the abacus, can be used to do calculations in base 16, the base in which the contents of computer files are usually displayed.

The Chinese version of the abacus, the *suan* pan, is divided into two sections, as indicated in Figure 1. In the lower section there are five beads, the one-point beads, on each reed. Each one-point bead on the rightmost reed—the unit's reed—represents one unit. In the upper section there are two beads, the five-point beads, on each reed. When the suan pan is used to represent decimal numbers, each one-point bead represents ten beads on the reed to its right, and each five-point bead represents five one-point beads on the same reed. Figure 2 shows the decimal number 1993 represented on the suan pan.

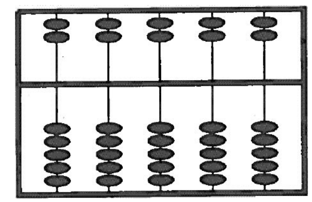

Figure 1: Suan pan, displaying 0

The suan pan, with its five one-point beads and two five-point beads, has more beads than it needs for calculating in decimal. (In contrast, the modern Japanese version of the abacus, the *soroban*, has only four one-point and one **five-point** bead on each reed. and thus has no beads to spare.) Some numbers can be displayed in more than one way. For example, 10 can be shown in three different ways: one one-point bead on the second reed from the right, both five-point beads on the rightmost reed, or one five-point bead and the five one-point beads on the rightmost reed. If they were used with maximum efficiency, the beads on the units reed could be used to represent all of the integers from 0 to 15 ($2\cdot 5 + 5$).

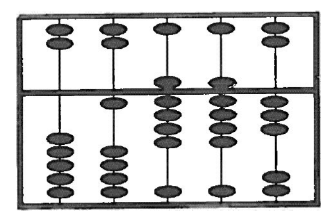

Figure 2: 1993 (decimal)

Thus, one reed of a suan pan can represent all of the digits (0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F) used in hexadecimal (base-16) arithmetic. In order to represent all numbers in hexadecimal on a suan pan, each one-point bead on the second reed from the right (the sixteen's reed) must represent sixteen one-point beads on the unit's reed. Each one-point bead on the third reed from the right (the 256's reed) represents sixteen one-point beads on the sixteen's reed. Similarly, each one-point bead on the fourth reed from the right (the 4096's reed) represents sixteen one-point beads on the **256's** reed, and so on. Starting with the rightmost reed, the value that each one-point bead represents increases by powers of sixteen.

Zero is displayed on the suan pan by pushing all beads away from the middle bar. To display other numbers, push the appropriate beads toward the middle bar. For example, to display 1000 (3E8 in hexadecimal), push three one-point beads and one five-point bead to the middle bar on the unit's reed. Then push four one-point and both **five-point beads** to the middle bar on the sixteen's reed to display the E; finally, push three one-point beads on the **256's** reed to the middle bar. See Figure 3 for this representation, and Figure 4 for the display of D7FC (854012 in decimal).

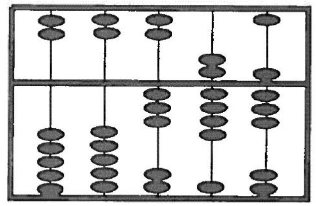

Figure 3: 3E8 (hexadecimal)

At times there are two ways of representing hexadecimal numbers on the suan pan. The digit 5 can be represented by pushing one five-point bead or five one-point beads to the middle bar. The digit A can be represented by pushing the two five-point beads to the center bar or by moving one five-point and all the one-point beads to the middle bar.

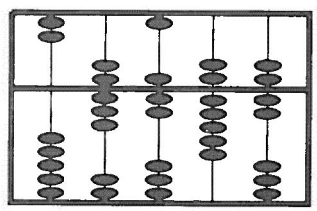

Figure 4: D7FC (hexadecimal)

To add two numbers, for example 7CE + 217, first display 7CE on the suan pan. Display E by pushing two five-point beads and four one-point beads to the middle bar on the unit's reed. Next, display C on the sixteen's reed by pushing two five-point and two one-point beads to the **middle** bar. Lastly, display the 7 on the 256's reed. To add 217, first add 7 to the unit's reed. This manipulation cannot be done immediately because there are not enough one- and five-point beads to move to the middle bar. Carrying is necessary: since 7 = 16 - 9, add 16 by moving a one-point bead on the sixteen's reed to the middle bar. Then subtract 9 from the unit's reed by moving one five-point and four one-point beads away from the **middle** bar. Another way to subtract 9 is to slide both five-point beads on the unit's reed away from the middle bar (subtracting 10) and then to move one unit bead to the middle bar.

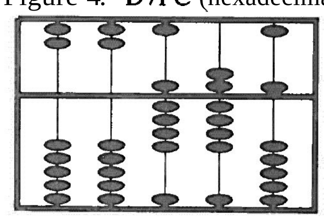

Figure 5: 9E5 (hexadecimal)

No carrying is needed to add 1 on the sixteen's reed and 2 on the 256's reed. The correct sum, **9E5,** is now displayed. See Figures 5 and 6 for the two ways the result could appear, depending on which of the two methods described was used to subtract 9.

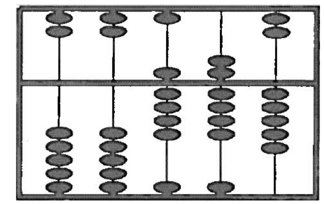

Figure 6: Also 9E5 (hexadecimal)

Sometimes, as in adding 57 to **4EF,** it will be necessary to carry more than one hexadecimal place to the left, as it is necessary to carry more than one decimal place in the addition of 57 to 489 in base 10.

Subtraction may be done as easily as addition. With some practice, a person with a suan pan will be able to do hexadecimal arithmetic much more quickly than almost anyone who uses pencil and paper; however, computers will still be faster.

Don Bloomquist, Jr., a mathematics major, is in his junior year at the Albertson College of Idaho. He wrote this paper during his freshman year under the direction of Dr. L. R. Tanner.

Was Dirichlet Smart?

Even the great mathematicians can make mistakes. As J. W. **Dauben** tells us on page 7 of ***Georg Cantor*** (Princeton University Press, **1979),** Cauchy once made an assertion equivalent to saying that if $\Sigma\, a$ converges and if $\lim_{n\to\infty} a_n/b_n = 1$, then $\Sigma\, b_n$ converges also. Dirichlet, who read critically, found an example showing that Cauchy was wrong. Can you do as well? We have all been taught, as Cauchy and Dirichlet were not, that Cauchy's assertion is true if the series have positive terms, so we know to look first at alternating series. Even with that hint, it is not all that likely that an average, or even above-average, student of mathematics would be likely to duplicate **Dirichlet's** accomplishment, which is why **Dirichlet's** example is on page 598.

SQUARE-FREE LUCAS PSEUDOPRIMES

Paul S. *Bruckman*
Everett, Washington

The purpose of this paper is to give a condition necessary and sufficient for all Lucas pseudoprimes to be square-free.

We will begin with some preliminaries. The Fibonacci numbers are defined by

$$F_{n+2} = F_{n+1} + F_n,\ n = 0, 1, 2, \ldots;\ F_0 = 0,\ F_1 = 1.$$

The sequence $\{F_n\}_0^\infty$ is called the Fibonacci sequence, after the 13th-century mathematician Leonardo of Pisa, also known as Fibonacci. The sequence has non-negative terms and, for $n \ge 2$, is strictly increasing.

The Lucas numbers are defined similarly, but with different initial values:

$$L_{n+2} = L_{n+1} + L_n,\ n = 0, 1, 2, \ldots;\ L_0 = 2,\ L_1 = 1.$$

The Lucas sequence $\{L_n\}_0^\infty$ is named after the 19th-century French mathematician Edouard Lucas, whose seminal work [5] generated much of the subsequent research into the sequences and their generalizations. The sequence has positive terms and, for $n \ge 1$, is strictly increasing.

If $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$, then

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},\quad L_n = \alpha^n + \beta^n,\quad n = 1, 2, \ldots.$$

Broadly speaking, there are two categories of properties of (and relationships between) the Fibonacci and Lucas numbers: additive and divisibility. The first includes representations of integers as sums (or differences) of Fibonacci (or Lucas) numbers. The second, as its name implies, includes representations of integers as products (or ratios) of Fibonacci (or Lucas) numbers. In this paper, we shall be most concerned with the divisibility properties.

We list below (without derivation) some of the divisibility properties that we will make frequent use of in this paper. For derivations, see, for example, [1, 6].

(1) $F_{2n} = F_n L_n$.

(2) $(F_m, F_n) = F_{(m,n)}$, where (a, b) denotes the greatest common

divisor of a and b.

(3) $(F_n, F_{n+1}) = (L_n, L_{n+1}) = 1$.

(4) $F_m | F_n$ if and only if $m | n$.

(5) $$L_{2n+1} - 1 = \begin{cases} 5F_nF_{n+1}, & n \text{ even} \\ L_nL_{n+1}, & n \text{ odd.} \end{cases}$$

We next define the Fibonacci entry-point. It may be shown that any integer m is a divisor of some element of the Fibonacci sequence. This is by no means a foregone conclusion. Not all integers m divide some element of the Lucas sequence since 5 divides no Lucas number. (The reason for this is that the Lucas numbers, modulo 5, are 2, 1, 3, 4, 2, 1, 3, 4, 2, ...) The smallest positive index n such that $m | F_n$ (where $m > 1$) is called the Fibonacci ***entry-point*** of m in the Fibonacci sequence, and is denoted by $Z(m)$. Other authors have referred to $Z(m)$ as the "rank of apparition" of m, an odious appellation brought about by a mistranslation of the French word apparition, which means "appearance", not "apparition", in English. Another more acceptable and frequently used alternative is "rank of appearance". To illustrate, since $F_1 = F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, $F_6 = 8$, $F_7 = 13$, we find $Z(2) = 3$, $Z(3) = 4$, $Z(5) = 5$, $Z(8) = 6$, $Z(13) = 7$. We may also verify that $Z(4) = 6$, $Z(6) = 12$, $Z(7) = 8$, $Z(9) = 12$, $Z(10) = 15$, $Z(11) = 10$, $Z(12) = 12$, and so on.

For all $m > 1$ we have the following properties:

(6) $Z(m) = n$ if and only if $m | F_n$ and $m \nmid F_r$ for all r with $1 \le r \le n-1$ (this is actually the definition of $Z(m)$);

(7) $Z(m) | Z(n)$ if and only if $m | n$ (if and only if $F_m | F_n$, by (4));

(8) $m | F_n$ if and only if $Z(m) | n$;

(9) if $m = \prod_{i=1}^{n} p_i^{e_i}$ is the prime-power decomposition of m, then

(10) $Z(m) = \mathrm{LCM}\{Z(p_i^{e_i})\}$;

(11) if p is any prime and $e \ge 1$ any integer, then $Z(p^e) = p^f Z(p)$, for some integer f with $0 \le f < e$;

(12) if p is any prime with $p^e \ne 2$ for some integer $e \ge 1$, and if $Z(p^{e+1}) \ne Z(p^e)$, then $Z(p^r) = p^{r-e} Z(p^e)$ for all $r \ge e+1$.

See, for example, [2, 6].

We need one more property which involves (x/p), the ***Legendre*** symbol, defined for odd primes p and integers x relatively prime to p as follows. If an integer y exists such that $y^2 \equiv x \pmod p$, we call x a quadratic residue (mod p) and write $(x/p) = 1$; otherwise we write $(x/p) = -1$. The **Legendre symbol** is also known as the quadratic character of x (mod p). The final property is

(13) if $p \neq 2, 5$ is prime, then $Z(p) \mid (p - (p/5))$.

Since for odd primes $\neq 5$

$$(p/5) = \begin{cases} 1 & \text{if } p \equiv \pm 1 \pmod{10} \\ -1 & \text{if } p \equiv \pm 3 \pmod{10}. \end{cases}$$

we may restate (13) as

$$\text{if } p \neq 2, 5 \text{ is prime, then } \begin{cases} Z(p) \mid (p - 1) & \text{if } p \equiv \pm 1 \pmod{10}, \\ Z(p) \mid (p + 1) & \text{if } p \equiv \pm 3 \pmod{10}. \end{cases}$$

In a 1982 paper [8], H. C. **Williams** reported that he had verified that, for $p \neq 2, 5$,

(14) $$p^2 \nmid F_{p-(5/p)}$$

for all primes $p < 10^9$. Although he did not assert that (14) holds for ***all*** primes $\neq 2, 5$, we will nevertheless call (14) the "Williams conjecture". In light of (8), we may restate it as

(15) $Z(p^2) \nmid (p - (5/p))$ for all primes $p \neq 2, 5$.

However, (11) implies that $Z(p^2) = Z(p)$ or $pZ(p)$. From (12), we see that (15) implies that

(16) $Z(p^2) = pZ(p)$, for all primes $p \neq 2, 5$.

On the other hand, if we assume that (16) holds and if it were true that $p^2 \mid F_{p-(5/p)}$, then, using (8), $Z(p^2) = pZ(p) \mid (p - (5/p))$. This is impossible since $p - (5/p) = p \pm 1$, and cannot be divisible by p. Therefore (16) implies (14) and we have shown that (14) and (16) are equivalent. Moreover, $Z(2^2) = Z(4) = 6 = 2 \cdot 3 = 2Z(2)$. We can also verify that $Z(25) = Z(5^2) = 25 = 5 \cdot 5 = 5Z(5)$. Therefore we **may** restate the Williams conjecture in the slightly stronger form

$$Z(p^2) = pZ(p) \text{ for all primes p.}$$

In a 1984 paper [3], J. J. Heed, evidently unaware of Williams' prior work, verified the conjecture for all $p < 10^6$.

The Williams conjecture is related to the study of Wieferich primes and their generalizations. A Wieferich prime is a prime p that satisfies $2^{p-1} \equiv 1 \pmod{p^2}$. There are only two Wieferich primes $< 6 \cdot 10^9$, namely 1093 and **3511**. More generally, prime solutions of $a^{p-1} \equiv 1 \pmod{p^2}$, where $a > 1$ is not a multiple of p, are exceedingly rare. Accordingly, we should expect solutions to the **"counter-conjecture"** $Z(p^2) = Z(p)$ to be also very rare. Indeed, the Williams conjecture states that such solutions are non-existent.

Note that (13) implies $p \mid F_{p-(5/p)}$, or, equivalently,

$$F_{p-(5/p)} \equiv 0 \pmod p, \text{ for all primes } p \neq 2, 5.$$

We can ask if the congruence

(17) $F_{n-(5/n)} \equiv 0 \pmod n$, with $(n, 10) = 1$,

might hold for composite values of n, with $(5/n)$ being the Jacobi symbol, the generalization of the **Legendre** symbol to composite n defined by $(x/n) = \prod_{p \mid n} (x/p)$. The answer is affirmative. Any **composite** integer which satisfies (17) is called a Fibonacci *pseudoprime*, or FPP. The first two **FPP's** are $323 = 17 \cdot 19$ and $377 = 13 \cdot 29$ and it is known that there are infinitely many [4].

It is known that $L_p \equiv 1 \pmod p$, holds for all primes p [7]. We can also ask if there are composite n such that $L_n \equiv 1 \pmod n$. Again, the answer is affirmative, and such n are called Lucas *pseudoprimes* (or LPP's). The first three LPP's are $705 = 3 \cdot 5 \cdot 47$, $2465 = 5 \cdot 17 \cdot 29$, and $2737 = 7 \cdot 17 \cdot 23$, and it is known that there are infinitely many [7]. It is also known that all LPP's are odd [7].

All known **FPP's** and LPP's are products of distinct primes, and so are square-free. It is not known if this is true in general, in spite of efforts made to prove it. Though it will not **be** proved here either, we will establish that a slightly weaker version **of the** Williams conjecture is equivalent to the conjecture that all LPP's are square-free. (A comparable version may **be** shown to be equivalent to the conjecture that all **FPP's** are square-free, but we will not **prove** that here.) We will attempt to show that

(*) $Z(p^2) = pZ(p)$ for all primes p

and

(**) All LPP's are square-free

are equivalent.

Suppose (*) is true and that n is a LPP. Also, assume that $p^2 \mid n$ for some prime p. Since all LPP's are odd, p is odd. Since, by definition, $n \mid (L_n - 1)$, we have $p^2 \mid (L_n - 1)$. Let $m = (n - 1)/2$. We consider three **cases.**

CASE 1: $n \equiv 1 \pmod 4$, $5 \nmid n$.

By (5), $L_n - 1 = 5F_mF_{m+1}$. Since $p \neq 5$, $p^2 \mid F_mF_{m+1}$. By (3), F_m and F_{m+1} are relatively prime, so either $p^2 \mid F_m$ or $p^2 \mid F_{m+1}$. That is, $p^2 \mid F_{m+\theta}$, where $\theta = 0$ or 1. By (1), $F_{2m+2\theta} = F_{m+\theta}L_{m+\theta}$. Thus $p^2 \mid F_{2m+2\theta}$, i. e., $p^2 \mid F_{n\pm 1}$. By (8), $Z(p^2) \mid (n \pm 1)$, which implies $p \mid (n \pm 1)$. However, since $p^2 \mid n$, $p \mid n$ also. This is impossible.

CASE 2: $n \equiv 3 \pmod 4$.
By (5), $L_n - 1 = L_m L_{m+1}$, which implies $p^2 \mid L_m L_{m+1}$. By (3), $(L_m, L_{m+1}) = 1$, which implies either $p^2 \mid L_m$ or $p^2 \mid L_{m+1}$. As in Case 1, $p^2 \mid L_{m+\theta}$, which again implies $p^2 \mid F_{2m+2\theta}$. The remaining steps are identical to those in Case 1.

CASE 3: $n \equiv 5 \pmod{20}$.
If $p \neq 5$, we proceed as in Case 1. If $p = 5$, then $5^2 \mid 5F_m F_{m+1}$, so $5 \mid F_m F_{m+1}$. Proceeding as above, we conclude that $5 \mid (n \pm 1)$. However, $5 \mid n$ also, since $5^2 \mid n$. Once again, we are led to a contradiction.

Since the three cases are exhaustive, we conclude that (*) implies (**).

Conversely, suppose that (**) is true. Let n be any LPP and suppose that $p \mid n$, where p is an odd prime. By hypothesis, $p^2 \mid n$. By definition, $n \mid (L_n - 1)$. Following the steps used to show that (*) implies (**), we find that $p \mid F_{n\pm1}$ and $p^2 \nmid F_{n\pm1}$. Then $Z(p) \mid (n \pm 1)$, but $Z(p^2) \nmid F_{n\pm1}$. Hence, $Z(p^2) \neq Z(p)$ for all primes p which divide some LPP. Since $Z(2^2) = 2Z(2)$, we see that all that is needed to complete the proof that (**) implies (*) is the assertion that all odd primes p divide some LPP. We will forego the proof of this assertion here, since it involves concepts that arc somewhat more complicated than intended in the scope of this paper.

So, all that we have proven is the weaker equivalence

(#) $\quad Z(p^2) = pZ(p)$ for all prime p dividing some LPP. $\iff$ All LPP's are square-free.

However, as stated above, (#) may be replaced by the stronger statement (*).

Based on the empirical evidence, it appears highly likely that (*) and (**) are true, but the proof of either conjecture is equally likely to meet with considerable difficulties.

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Paul Bruckman received an M. S. degree in mathematics from the University of Illinois in *1971*. He is a frequent contributor to ***The Fibonacci Quarterly,*** has been a member of The Fibonacci Society for more than twenty years, and subscribed to the Pi Mu ***Epsilon Journal*** in ***1992.***

Chapter Report

The FLORIDA EPSILON chapter (University of South Florida) held thirteen meetings during the 1992-93 academic year, devoted to talks by students, faculty members, and visitors. A selection of titles is

Off into space
An explanation of Strang's strange figures
Applied mathematics in engineering
The Euler summation formula
Computer image techniques in medical imaging
Perfect and perfectly useless numbers
Dynamics for the college student
Careers for mathematics majors
Paradoxes in mathematics
Laser sensing of the atmosphere.

In addition, there was a mathematical game party before the Christmas vacation and the chapter, in collaboration with the student chapter of the MAA, sponsored the twice-yearly Hillsborough County Math Bowl competitions, with more then 200 participants in each. Professor Fredric Zerla also noted that Suzanne Josephs, the chapter's Outstanding Scholar, completed her university career with a perfect 4.0 grade-point average.

COUNTING ICE-CREAM FLAVORS TO PROVE AN IDENTITY

James Chew
North Carolina A & T State University

The identity in question is

$$1 + 2 + \dots + n = \frac{n(n+1)}{2}$$

How many kinds of double-dipped cones can a customer order at an ice-cream parlor that sells n flavors of ice cream? We will allow two scoops of the same flavor but will consider $\frac{\text{chocolate}}{\text{vanilla}}$ the same as $\frac{\text{vanilla}}{\text{chocolate}}$. Let N be the number of different cones.

One line of reasoning is to think of N as being the sum of the number of double-flavored cones and the number of single-flavored cones, so

$$N = \binom{n}{2} + n = \frac{n(n-1)}{2} + n = \frac{n(n+1)}{2}.$$

The same result comes from thinking that N = total number of possible pairs − number of duplicated pairs, so

$$N = n^2 - \binom{n}{2} = n^2 - \frac{n(n-1)}{2} = \frac{n(n+1)}{2}.$$

Let us take a different approach, with n = 4 for definiteness. Let A, B, C, and D be the flavors. There are 4 single-flavored cones: AA, BB, CC, and DD. The rest are double-flavored. There are 3 cones in which A is picked first: AB, AC, and AD. Next come the 2 cones in which B is picked first: BC and BD. Finally, we have the 1 cone in which C is picked first: CD. Hence $N = 4 + 3 + 2 + 1$.

The argument generalizes to n flavors, so that

$$N = n + (n-1) + \dots + 2 + 1.$$

James Chew has lived in Indonesia, Australia, and Ethiopia, though his Ph. D. degree is from the Virginia Polytechnic Institute and he has been at *NC A&T* State for the past fifteen years.

A REGULAR POLYGON EQUATION

See Chin Woon
Imperial College, London

In analytic geometry, geometrical objects are defined algebraically by equations. In this note, we give a polar coordinate equation for a regular polygon with any number of sides.

Let R be the distance from the center of the polygon to a vertex. Place the polygon so that its center is at the origin and a vertex is at $(R, 0)$.

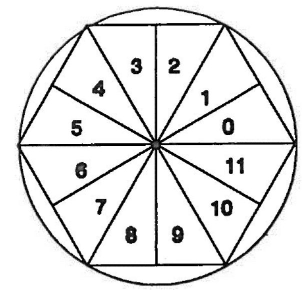

Figure 1

For $9 \in [0, 2\pi)$, let

$$9 \equiv \theta' \pmod{\pi/n},$$

so that

$$9 = \theta' + q\pi/n, \text{ with } 0 \le \theta' < \pi/n \text{ and } 0 \le q < 2n.$$

The polygon can be divided into triangles with different values of q, as illustrated in Fig. 1 for a hexagon.

Let

$$\phi(q) = \begin{cases} 0 \text{ if } q \text{ is odd} \\ 1 \text{ if } q \text{ is even.} \end{cases}$$

The function may also be defined in a single equation as $\phi(q) = 2[q/2] - q + 1$ or as $\phi(q) = ((-1)^q + 1)/2$.

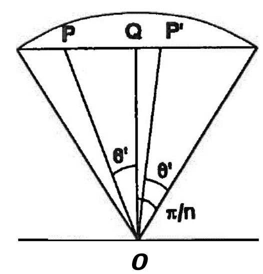

Figure 2

In Fig. 2, if q is odd, then

$$\frac{OQ}{OP} = \cos 9 \quad \text{so}$$

$$OP = \frac{R \cos \pi/n}{\cos 9'}$$

If q is even, $\frac{OP'}{OQ} = \cos(\pi/n - \theta')$, so

$$OP' = \frac{R \cos \pi/n}{\cos(\pi/n - 6')}$$

The two equations for OP and OP' may be combined into

$$r = \frac{R \cos \pi/n}{\cos(\phi(q)\pi/n + (-1)^{\phi(q)}\theta')}$$

This gives the following single polar coordinate equation for the regular polygon:

$$r = \frac{R\cos\pi/n}{\cos(\phi(q)\pi/n + (-1)^{\phi(q)}(\theta - q\pi/n))}.$$

As $n \to \infty$ this equation becomes $r = R$, as it should.

See Chin Woon a member of the Class of *1994* at Imperial College. He says that he has sometimes felt that some of the concepts in mathematics seem really ahead of their time.

Yes, He Was

Dirichlet said, let

$$a_n = \frac{(-1)^n}{\sqrt{n}} \quad \text{and} \quad b_n = \frac{(-1)^n}{\sqrt{n}}\left(1 + \frac{(-1)^n}{\sqrt{n}}\right).$$

Then it is clear that a_n/b_n approaches 1 as $n \to \infty$, but $a - b_n = -1 - 1/2 - 1/3 - \dots$, which diverges. Thus, not both $2a_n$ and $2b_n$ can converge.

Chapter Reports

Professor Joan Wyzkoski Weiss reports that **CONNECTICUT** GAMMA (Fairfield University) had twenty-three new initiates in the spring. Members of the chapter assisted in coordinating the activities for Math Counts, a mathematics contest for junior high school students.

MASSACHUSETTS GAMMA (Bridgewater State College) sponsored a colloquium, "Aspects of real-time object-oriented systems" by John **McNulty** of the MITRE Corporation. Vice-president Keith **Desrosiers** also reports that the chapter's advisor, Professor Thomas E. Moore, was awarded the **MAA's** Northeastern Section award for distinguished teaching.

SOME PARTITIONS OF THE INTEGERS

Joseph M. Moser and Genoveno Lopez
San Diego State University

Let m be an arbitrary positive integer. The congruence **relation,** $\equiv$ modulo m, on the set $\boldsymbol{Z}$ of all integers is defined by

$$x \equiv y \pmod m \quad \text{if and only if} \quad x - y = fen \text{ for some integer m.}$$

The congruence relation is an equivalence relation with equivalence classes $\{x + km \mid k \in \boldsymbol{Z}\}$, $x = 0, 1, \dots m - 1$. It is likely that other equivalence relations on the set $\boldsymbol{Z}$ are not well known. We would like to present a few more equivalence classes on $\boldsymbol{Z}$ which may be useful as exercises or examples.

First, for a fixed integer k, define R_k by

$$x R_k y \quad \text{if and only if} \quad x^2 + kx = y^2 + ky.$$

It is easy to show that R_k is an equivalence relation which partitions the integers into an infinite number of classes, $\{x, -(x + k)\}$.

Next, let $i = \sqrt{-1}$, and define R_i by

$$x R_i y \quad \text{if and only if} \quad i^x = i^y.$$

It again is easy to show that R_i is an equivalence relation which partitions the integers into four equivalence classes. Also, $R_i = R_m$ when m = 4.

Let us define R_e by

$$x R_e y \quad \text{if and only if} \quad e^{2\pi i x/m} = e^{2\pi i y/m}.$$

It is not difficult to see that $R_e = R_m$, where R_m is as defined above.

Now, let us define $R_{\sin}$ by

$$x R_{\sin} y \quad \text{if and only if} \quad \left|\sin\frac{\pi y}{m}\right| = \left|\sin\frac{\pi x}{m}\right|.$$

It is easy to show that $R_{\sin}$ is an equivalence relation which partitions the **integers** into k equivalence classes, where m = 2k or m = 2k − 1. For example, when m = 5, the equivalence classes are

$$\{0, \pm 5, \pm 10, \pm 15, \dots\}, \quad \{\pm 1, \pm 4, 26, \pm 9, \dots\}, \quad \{\pm 2, \pm 3, \pm 7, \pm 8, \dots\}.$$

Finally, let us define R^* by $x R^* y$ if and only if

$$(-1)^{[y/m]}\left(\frac{y}{m} - \left[\frac{y}{m}\right]\right) + \frac{1 + (-1)^{[y/m+1]}}{2}$$

$$= (-1)^{[x/m]}\left(\frac{x}{m} - \left[\frac{x}{m}\right]\right) + \frac{1 + (-1)^{[x/m+1]}}{2}$$

where [z] is the greatest integer in z. It is easy to show that $R^{\bullet}$ is an equivalence relation. The partition of the integers is

$\{x = mk \mid k \in Z\}$,
$\{x = 1 + 2mk \mid k \in Z\} \cup \{x = (2m - 1) + 2mk \mid k \in Z\}$,
$\{x = 2 + 2mk \mid k \in Z\} \cup \{x = (2m - 2) + 2mk \mid k \in Z\}$, ...
$\{x = m - 1 + 2mk \mid k \in Z\} \cup \{x = (2m - m + 1) + 2mk \mid k \in Z\}$.

Professors Moser and Lopez earned Ph. D. degrees *from*, respectively, St. Louis *U.* and *UCLA* before joining *SDSU.*

Chapter Reports

At the OHIO NU chapter (University of Akron), nineteen new members were inducted in April and thirty-five awards were made, including seventeen memberships in various mathematical organizations and thirteen scholarships.

One of the new initiates of the MARYLAND DELTA chapter (Hood College) has an interdisciplinary major in the political economy of the third world. The annual Pi Mu Epsilon Lecture was given by Dr. Lida K. Barrett, on "Emmy Noether and Grace Chisholm Young: two women mathematicians of this century".

A Steeply Puzzling Question

Does anyone, anywhere, know why *m* is always and invariably used to denote the slope of a line? If so, many readers of the Journal would like to know as well,

NON-EXISTENCE OF CERTAIN UNITARY PERFECT NUMBERS

Jennifer DeBoer
Michigan Technological University

Let N be a positive integer. The **unitary divisors** of N are all the integers d such that $d \mid N$ and $(d, N/d) = 1$. A positive integer N is unitary perfect when the sum of its unitary divisors is $2N$.

We use $\sigma^*(N)$ to denote the sum of the unitary divisors of N. It can be easily shown that if $N = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$, where $p_1, \dots, p_k$ are distinct primes, then

$$\sigma^*(N) = (p_1^{a_1} + 1)(p_2^{a_2} + 1)\dots(p_k^{a_k} + 1).$$

Therefore, N is unitary perfect if and only if

$$\frac{\sigma^*(N)}{N} = \frac{p_1^{a_1} + 1}{p_1^{a_1}} \cdots \frac{p_k^{a_k} + 1}{p_k^{a_k}} = 2.$$

As Guy [2] pointed out, any unitary perfect number must be even. Subbarao and Warren [4] proved that the first four unitary perfect numbers are

(A) $6 = 2 \cdot 3$, $60 = 2^2 \cdot 3 \cdot 5$, $90 = 2 \cdot 3^2 \cdot 5$, and $87360 = 2^6 \cdot 3 \cdot 5 \cdot 7 \cdot 13$

Wall [5] has shown that

$$2^{18} \cdot 3 \cdot 5^4 \cdot 7 \cdot 11 \cdot 13 \cdot 19 \cdot 37 \cdot 79 \cdot 109 \cdot 157 \cdot 313$$

is the fifth unitary perfect number. Furthermore, Graham [1] has shown that the only three unitary perfect numbers of the form $2^m s$, where s is odd and squarefree, are the first, second, and fourth numbers in (A).

In this paper, we will look at the third unitary perfect number and show that it is the only one of the form $2^m 3^2 s$.

For a given m, there is a simple procedure to find out if there is a unitary perfect number of the form $2^m 3^2 s$.. For example, suppose $m = 1$. Then we want

$$\frac{2^1 + 1}{2^1} \frac{3^2 + 1}{3^2} \frac{\sigma^*(s)}{s} = 2.$$

or

$$\frac{5}{3} \frac{\sigma^*(s)}{s} = 2.$$

This cannot equal 2 unless there is some factor $p_i = 5$ in the denominator to cancel the 5 in the numerator. Therefore, $5 \mid s$. On writing $s = 5 \cdot s_1$, we get

$$\frac{5}{3}\frac{6}{5}\frac{\sigma^*(s_1)}{s_1} = 2.$$

This is true when $s_1 = 1$, so we see that $2 \cdot 3^2 \cdot 5$ is unitary perfect.

However, this does not work for every m. For example, m = 2 does not yield an unitary perfect number of the form $2^m \cdot 3^2 \cdot s$, for in the product

$$\frac{5}{2^2}\frac{2 \cdot 5}{3^2}\frac{2 \cdot 3}{5} \cdots$$

there is no way to cancel out both factors of 5 in the numerator with a squarefree s.

As we see, a number N cannot be unitary perfect if $5 \mid (2^m + 1)$ or if there exists $p_i \mid s$ such that $5 \mid (p_i + 1)$. In addition, since s is **squarefree,** s must not be divisible by the squares of any odd primes.

For another example,

$$\frac{3^2}{2^3}\frac{2 \cdot 5}{3^2}\frac{2 \cdot 3}{5} \cdots$$

cannot equal 2 because there are three factors of 3 in the numerator, and the highest power of 3 in the denominator is 2. Therefore, if $3^2 \mid (2^m + 1)$, then N is not unitary perfect. Finally, suppose m = 5. Then

$$\frac{3 \cdot 11}{2^5}\frac{2 \cdot 5}{3^2}\frac{2 \cdot 3}{5}\frac{2^3 \cdot 3}{11} \cdots$$

cannot equal 2, again because there are too many factors of 3 in the numerator.

So, if one of the following cases holds, then $2^m \cdot 3^2 \cdot s$ cannot be unitary perfect:

(1) $3^2 \mid (2^m + 1)$.

(2) $3 \mid (2^m + 1)$ and there exists $p_i \mid s$ such that $3 \mid (p_i + 1)$.

(3) $5 \mid (2^m + 1)$ or there exists $p_i \mid s$ such that $5 \mid (p_i + 1)$.

(4) there exists $p_i \mid s$ such that $p_i + 1$ is divisible by the square of an odd prime.

(5) **there exists $p_i \mid s$ and $p_j \mid s$ such that $3 \mid (p_i + 1)$ and $3 \mid (p_j + 1)$.**

Now we always have $5 \nmid s$ since $5 \mid (3^2 + 1)$. For the remainder of this paper, we will write $N = 2^m \cdot 3^2 \cdot 5 \cdot s$, where s is such that $(s, 3) = (s, 5) = 1$, and we will let $s = p_1 p_2 \cdots p_k$, where $p_1, \ldots, p_k$ are distinct primes. If N is unitary perfect, then

$$\frac{2^m + 1}{2^m}\frac{4}{3}\frac{p_1 + 1}{p_1} \cdots \frac{p_k + 1}{p_k} = 2.$$

Furthermore, if any of criteria (1) through (5) **hold,** then N cannot be unitary perfect.

We will use criteria (1) through (5) to establish the result **in all cases** except $m \equiv 0 \pmod 8$.

LEMMA 1. If $m \equiv 3 \pmod 6$, then $2^m \cdot 3^2 \cdot 5 \cdot s$ cannot be unitary perfect.

Proof. If $m \equiv 3 \pmod 6$, then $9 \mid (2^m + 1)$. Thus we may apply case (1).

LEMMA 2. If $m \equiv 5 \pmod 6$, then $2^m \cdot 3^2 \cdot 5 \cdot s$ cannot be unitary *perfect.*

Proof. Suppose $m \equiv 5 \pmod 6$, Then $2^m \equiv 5 \pmod 9$, so $2^m + 1 \equiv 6 \pmod 9$, and $3 \mid (2^m + 1)$. **Also,** $a = (2^m + 1)/3 \equiv 2 \equiv -1 \pmod 3$. Now a has at least one prime divisor $p \equiv -1 \pmod 3$, for if all prime divisors of a are $\equiv 1 \pmod 3$, then $a \equiv 1 \pmod 3$. Thus, $3 \mid (p + 1)$. This meets case (2).

LEMMA 3. If $m \equiv 1 \pmod 6$ and $m > 1$, then $2^m \cdot 3^2 \cdot 5 \cdot s$ *cannot* be unitary perfect.

Proof. Let $m \equiv 1 \pmod 6$. Then $2^m \equiv 2 \pmod 9$, so $3 \mid (2^m + 1)$, and $(2^m + 1)/3 \equiv 1 \pmod 3$. Furthermore, $2^m + 1 \equiv 1 \pmod 8$, and $(2^m + 1)/3 \equiv 3 \pmod 8$.

We may assume that every prime divisor p of $(2^m + 1)/3$ has $p \equiv 1 \pmod 3$; otherwise, criterion (2) is satisfied. If $p \mid (2^m + 1)/3$ then $2^m \equiv -1 \pmod p$; consequently, $2^{6a+2} \equiv -2 \pmod p$, where $m = 6a + 1$.

We now see that $x = 2^{3a+1}$ is a solution of $x^2 \equiv -2 \pmod p$. A result from quadratic residue theory tells us that this **congruence** is solvable if and **only** if $p \equiv 1$ or $3 \pmod 8$. **Thus** every prime divisor of $(2^m + 1)/3$ satisfies $p \equiv 1$ or $3 \pmod 8$.

Now $(2^m + 1)/3 = p_1 \cdots p_r \equiv 3 \pmod 8$, $p_i \equiv 1$ or $3 \pmod 8$, and $p_i \equiv 1 \pmod 3$ for every i. There must be at least one i such that $p_i \equiv 3 \pmod 8$. Then $p_i + 1 \equiv 4 \pmod 8$, and $(p_i + 1)/4 \equiv 2 \pmod 3$. Therefore, there exists j such that $p_j \mid (p_i + 1)$ and $p_j \equiv 2 \equiv -1 \pmod 3$. But $3 \mid (p_j + 1)$ and we noted previously that $3 \mid (2^m + 1)$. This again meets case **(2),** so $2^m \cdot 3^2 \cdot 5 \cdot s$ cannot be unitary perfect.

LEMMA 4. If $m \equiv 2 \pmod 4$, then $2^m \cdot 3^2 \cdot 5 \cdot s$ cannot be **unitary perfect.**

Proof. Suppose $m \equiv 2 \pmod 4$. Then $2^m \equiv -1 \pmod 5$ and

$5 \mid (2^m + 1)$. This is case (3).

LEMMA 5. If $m \equiv 4 \pmod 8$, then $2^m \cdot 3^2 \cdot 5 \cdot s$ cannot be unitary perfect.

Proof. Let $m \equiv 4 \pmod 8$. Then $2^m \equiv -1 \pmod{17}$, $17 \mid (2^m + 1)$, and so $17 \mid s$. However, $17 + 1 = 18$, which is case (4).

From these lemmas, we see that if $2^m \cdot 3^2 \cdot 5 \cdot s$ is unitary perfect, then $m \equiv 0 \pmod 8$. The proof for this case requires a different technique. To begin with, we will show that there exists at least one Mersenne prime p_i such that $p_i \mid s$. Note that $2^m + 1 \equiv 2 \pmod 3$, and there exists a unique $p_1 \mid (2^m + 1)$ such that $p_1 \equiv -1 \pmod 3$. At least one such term is needed to cancel out one of the factors of 3, but two such numbers would meet 5 and violate our conditions. The other primes $p_2, \ldots, p_k$ are $\equiv 1 \pmod 3$. Let p_i be the smallest prime in $p_2, \ldots, p_k$. Then $p_i + 1$ is not divisible by any p_j in $\{p_2, \ldots, p_k\}$, so $p_i + 1 = 2^n$ for some integer n. Therefore p_i is a Mersenne prime.

Now, we will look at "chains" of prime divisors of s. To explain this concept, we examine the unitary perfect number $2^6 \cdot 3 \cdot 5 \cdot 7 \cdot 13$. (This is not germane to our main theorem, but it is a helpful example.) The Mersenne prime divisors of this number are 3 and 7. First, we look for a prime $p \mid s$ such that $3 \mid (p + 1)$. By inspection, we see that this number is 5. **Furthermore,** $5 \mid (2^6 + 1)$, **completing** the chain. If 5 did not divide $2^6 + 1$, then we would look for another prime $p_2 \mid s$ such that $5 \mid (p_2 + 1)$. The "chain" would continue in this way until we reached a prime that did divide $2^6 + 1$. Thus, the elements in this chain are 3 and 5. Next, we consider the chain beginning with the same prime 7. Because $7 \mid (13 + 1)$ and $13 \mid (2^6 + 1)$, the elements in this chain are 7 and 13. (Note also that if either of our Mersenne primes had **divided** $2^6 + 1$, that chain would have contained only the single prime.)

More **formally,** there is a set of Mersenne primes $q_1, \ldots, q_l$ that divide s. **Each** q_i begins a chain of primes $R_1(q_i), \ldots, R_j(q_i)$, **where** $q_i \mid (R_1(q_i) + 1)$, $R_a(q_i) \mid (R_{a+1}(q_i) + 1)$ for every a such that $1 \le a \le j - 1$, and, finally, $R_j(q_i) \mid (2^m + 1)$. The primes found in these chains, in addition to p_1, account for every prime that divides s. Some factors may be repeated; for example, if $433 \mid s$, the Mersenne primes 7 and 31 both lead up to 433, since $7 \mid (433 + 1)$ and $31 \mid (433 + 1)$. From this point, we will let $Q_i = q_i R_1(q_i) R_2(q_i) \ldots R_j(q_i)$, or the product of all the primes in the chain beginning with q_i. If N is unitary perfect, then

(B) $$\frac{2^m + 1}{2^m} \frac{4}{3} \frac{p_1 + 1}{p_1} \frac{\sigma^*(Q_1)}{Q_1} \cdots \frac{\sigma^*(Q_l)}{Q_l} \ge 2 = \frac{\sigma^*(N)}{N}.$$

To complete the proof, we **will show that** (B) does not happen when $m \equiv 0 \pmod 8$.

First, we may assume that $m \ge 8$, so

Second, we will look at p_1, which is $\equiv 2 \pmod 3$. We may write $m = 8b$. Then, as $p_1 \mid (2^m + 1)$, $2^{8b} \equiv -1 \pmod{p_1}$. Thus $2^{16b} \equiv 1 \pmod{p_1}$. This

$$\frac{2^m + 1}{2^m} \le \frac{257}{256}.$$

shows that the **order** of $2^b \pmod{p_1}$ is 16. **Therefore,** $16 \mid (p_1 - 1)$. (Note **that** $p - 1$ is the order of Z_p and that the order of 2^b must divide $p - 1$.) Now, with a little arithmetic, one can see that the smallest prime p such that $p \equiv 2 \pmod 3$, $p \equiv 1 \pmod{16}$, and $p + 1$ has odd squarefree part is 113. Therefore,

$$\frac{p_1 + 1}{p_1} \le \frac{114}{113}.$$

Finally, we will look at the terms contributed by each Q^i. This is a product of the form

$$\frac{q + 1}{q} \frac{p + 1}{p} \cdots,$$

where each **numerator** is divisible by the previous **denominator.** The product is largest if $p_{i+1} + 1 = 2p_i$ for every i. Then the product is at most

(C) $$\frac{q + 1}{q} \frac{2q}{2q - 1} \frac{4q - 2}{4q - 3} \cdots.$$

Now an easy induction shows that the nth partial product of (C) is

$$\frac{2^n q + 2^n}{2^n q - (2^n - 1)}$$

Thus

$$\frac{\sigma^*(Q)}{Q} \le \lim_{n \to \infty} \frac{2^n q + 2^n}{2^n q - 2^n + 1} = \frac{q + 1}{q - 1},$$

where Q is any Q_i.

Remember that $q \ne 3$. The next two Mersenne primes are 7 and 31, so

$$\frac{\sigma^*(Q_1)}{Q_1} \frac{\sigma^*(Q_2)}{Q_2} \le \frac{8}{6} \frac{32}{30} = \frac{4}{3} \frac{16}{15}.$$

To find the upper bound for the **contribution** of the Mersenne primes > 31, we will suppose that every odd power of 2 yields a Mersenne prime. Then the next Mersenne prime is 127, and

(D) $$\frac{\sigma^*(Q_3)}{Q_3} \frac{\sigma^*(Q_4)}{Q_4} \cdots \le \frac{64}{63} \frac{156}{255} \frac{1024}{1023} \cdots.$$

This is bounded by the product obtained by replacing each numerator with 4 times the previous denominator, so is

(E) $$\le \frac{64}{63} \frac{252}{251} \frac{1004}{1003} \cdots.$$

It is easy to show that the nth partial product of (E) is

$$\frac{4^{n-1}\cdot 64}{4^{n-1}\cdot 64-(4^{n-2}+4^{n-2}+\dots+1)}=\frac{64}{64-(1+4^{-1}+\dots+4^{-n+1})}$$

On letting n tend to infinity, we get

$$\frac{64}{64-4/3}=\frac{48}{47}.$$

Now we are ready to **finish** the proof by noting that

$$\frac{\sigma^*(N)}{N}\le\frac{257}{256}\,\frac{4}{3}\,\frac{114}{113}\,\frac{4}{3}\,\frac{16}{15}\,\frac{48}{84}=1.96\ldots<2.$$

This contradicts (B). Therefore, there are no unitary perfect numbers of the form $2^m\cdot 3^2\cdot 5\cdot s$, when $m>1$.

Acknowledgements. This work was supported by a grant through the Michigan Technological University Undergraduate Research Project in Mathematics. I also wish to thank **S.** W. Graham for his guidance throughout this project.

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SURPRISE! AN IMPROPER INTEGRAL

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Computer mathematics packages, useful as they are, must be treated in the same way as pocket calculators: whenever something appears on the display screen, the user should ask, "**Is** the answer reasonable?" (For that **matter,** the same question could profitably be asked after many paper-and-pencil calculations as well.)

Some packages **can,** for some functions f, find in closed form the antiderivative F. When asked for the exact value of $\int_a^b f(x)\,dx$, they will return $F(b)-F(a)$. They can also approximate the value using **formulas,** such as **Simpson's** Rule, that do not depend on the antiderivative.

As an example, Derive **(version 1.62),** asked for the exact value of $f(x)dx$ with $f(x) = 1/(3\cos^2 x+1)$, gave 0. This is obviously wrong since $f(x)>0$ on $[0,\pi]$. When asked for an approximation, the program gave a value of 1.57079.

When asked for the antiderivative **of** f, the answer was

$$F(x) = (112)\arctan((1/2)\tan x).$$

Sure **enough,** $F(\pi)-F(0)=0$, and a few **minutes'** work with pencil and paper shows that $F'(x)=f(x)$.

So what went **wrong?** Let us look at one way of stating the Fundamental Theorem of Calculus:

If $f(x)$ is continuous on [a, b] then f has an antiderivative G defined by

$$G(x)=\int_a^x f(t)\,dt$$

and if H is any antiderivative of f on [a, b] then G − H is a constant on [a, b] and

$$\int_a^b f(x)\,dx = H(b)-H(a).$$

For G and H to be antiderivatives, they must have antiderivatives on [a, b] and are thus continuous there. In the example, f is continuous **on** $[0,\pi]$ but F is not. Thus F is not an antiderivative on the entire **interval** [a, b] and cannot be used as the function H to evaluate $\int_0^\pi f(x)\,dx$.

If we begin with $F(x)=(1/2)\arctan((1/2)\tan x)$, since F is undefined at $\pi/2$ we have that

$$F'(x)=f(x)=\frac{1}{3\cos^2 x+1}$$

is not defined at $\pi/2$ and thus is discontinuous. (The discontinuity, of course, is removable.)

Surprise! We have an improper integral. To evaluate it, we have

$$\begin{aligned}\int_0^{\pi} f(x)\,dx &= \lim_{t\to\pi/2^-}\int_0^{t} f(x)\,dx + \lim_{t\to\pi/2^+}\int_t^{\pi} f(x)\,dx \\ &= \lim_{t\to\pi/2^-} F(x)\Big|_0^{t} + \lim_{t\to\pi/2^+} F(x)\Big|_t^{\pi} \\ &= F(\pi) - F(0) - \left(\lim_{t\to\pi/2^+}F(t) - \lim_{t\to\pi/2^-}F(t)\right) \\ &= F(\pi) - F(0) - (-\pi/4 - \pi/4) = \pi/2.\end{aligned}$$

Note that $\lim_{t\to\pi/2^+}F(t) - \lim_{t\to\pi/2^-}F(t)$ is the jump of $F(x)$ at $x = \pi/2$.

If we ask the computer to draw the graphs of $F(x)$ and the continuous antiderivative $G(x) = \int_0^x f(t)\,dt$ we get Figures **1** and 2. On any closed interval where both are continuous their values differ at most by a constant and they have the same shape. If we define a new function F^* by

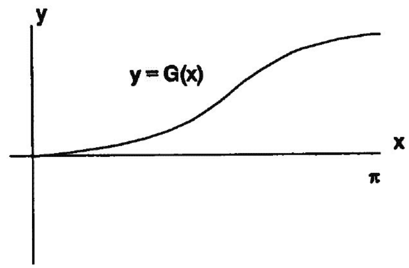

Figure 1

$$F^*(x) = \begin{cases} F(x) & \text{if } 0 \le x < \pi/2 \\ \lim_{x\to\pi/2^-}F(x) & \text{if } x = \pi/2 \\ F(x) - J & \text{if } \pi/2 < x \le \pi \end{cases}$$

where J is the jump in F at $\pi/2$, then F^* is a continuous antiderivative of f and in the example is equal to G. It **follows** that $\int_0^{\pi} f(x)\,dx = F^*(\pi) - F^*(0) = F(\pi) - J - F(0) = \pi/2$. *An* updated version of Derive (2.06) gives the **anti**-derivative of f as

$$\frac{1}{2}\arctan\left(\frac{\tan x}{2}\right) + \frac{x}{2} - \frac{1}{2}\arctan(\tan x),$$

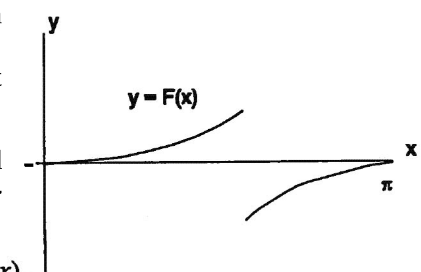

Figure 2

whose **only** discontinuities are **removable**. The discontinuities of $(1/2)\arctan((1/2)\tan x)$ are smoothed out by $x/2 - (1/2)\arctan(\tan x)$, which has a jump at odd multiples of $\pi/2$.

Other programs give different antiderivatives. A version of Maple gives

$$\frac{1}{2}\arctan\left(2\tan\frac{x}{2} - \sqrt{3}\right) + \frac{1}{2}\arctan\left(2\tan\frac{x}{2} + \sqrt{3}\right),$$

discontinuous at $(2k+1)\pi$; a version of Mathematica

$$-\frac{1}{2}\arctan(2\cot x),$$

discontinuous at $k\pi$; and another **version** of Mathematica

$$\frac{i}{4}\ln(1 + 3(\cos 2x - i\sin 2x)) - \frac{i}{4}\ln(3 + \cos 2x - i\sin 2x)),$$

discontinuous at $(2k+1)\pi/2$.

The proper rephrasing of the Fundamental Theorem of Calculus for a function f with a piecewise antiderivative, continuous except for jumps, is left for the reader.

So, when using the computer to find a **definite** integral, we should ask for both the exact answer and the approximate answer. The program uses different methods to find them, so if they agree we should feel **confident** with the result. If they do not agree, then the antiderivative should be examined closely.

Howard Lambert is a graduate of Texas Tech University. In 1992, he received a Teaching Excellence Award *from* the East Texas State University chapter of the Texas *Association* of College Teachers.

Pin Prices Going Up

The cost of a Pi Mu Epsilon Pin will increase from \$8 to \$12 at **midnight,** June 30,1994. Those of you who have lost yours and need a replacement or who are buying quantities of pins for their investment value are thus advised to act soon. Pins are available **from** the Secretary-Treasurer, Professor Robert M. Woodside, Department of Mathematics*East Carolina University, **Greenville,** North Carolina 27858.

USING POWER SERIES TO COMPLETE THE BASIC INTEGRATION FORMULA

Margaret Webb
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Recently [1], Schaumberger used the mean-value theorem to show that

$$\lim_{n \to -1} \int_a^b x^n \, dx = \int_a^b \frac{dx}{x}.$$

In this note we will use power series to obtain the same result. We have

$$\lim_{n \to -1} \int_a^b x^n \, dx = \lim_{n \to -1} \frac{b^{n+1} - a^{n+1}}{n+1}$$

$$= \lim_{n \to -1} \frac{e^{(n+1)\ln b} - e^{(n+1)\ln a}}{n+1}$$

$$= \lim_{n \to -1} \frac{1}{n+1} \sum_{k=0}^{\infty} \left[\frac{((n+1)\ln b)^k}{k!} - \frac{((n+1)\ln a)^k}{k!} \right]$$

$$= \lim_{n \to -1} \left[\ln b - \ln a + \sum_{k=2}^{\infty} \frac{(n+1)^{k+1}}{k!} \cdot ((\ln b)^k - (\ln a)^k) \right]$$

$$= \ln b - \ln a = \int_a^b \frac{dx}{x}.$$

The interchange of limit and summation is justified by the uniform convergence of the series.

Reference

1. N. Schaumberger, Using the mean value theorem to complete the basic integration formula, this Journal 9 (1991), 226-227

The author prepared *this* paper under the supervision of Professor Javier Gomez-Calderon while she was a freshman at Penn State *University—New* Kensington campus.

EVALUATING A DEFINITE INTEGRAL USING PROBABILITY

M. A. Khan
Research Design and Standards Organisation
Lucknow, India

We shall show that

$$\int_0^1 x^p (1-x)^q \, dx = \frac{p!\,q!}{(p+q+1)!}.$$

The integral is known as **Euler's** Integral (see, for example [1], [2]). It is interesting that it can be evaluated by using a probabilistic argument.

Let us select r + q random numbers from the interval [0, 1] and call them $a_1, a_2, \ldots a_r$, $b_1, b_2, \ldots b_q$. What is the probability that $a_i < b_j$ for all i and j? We will answer the question in two ways.

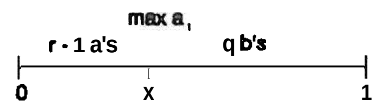


Figure 1

First, consider the $(r+q)!$ ways of arranging the $r+q$ numbers. If we are to have $a_i < b_j$, then the as must come first. Thus, the first number may be chosen in r ways, the second in r − 1, and so on until the last of the as is selected. Then the first of the bs may be chosen in q ways, the second in q - 1 ways, and so on. There are then r! q! ways to have the numbers appear in the desired fashion, and so the probability $a_i < b_j$ for all i and j is $\dfrac{r!\,q!}{(r+q)!}$

Second, consider the location of the largest of the as on [0, 1]. Suppose it is between x and dx, which has probability dx. (See Figure 1.) Then, if we are to have $a_i < b_j$, all of the bs must be greater than x, and the probability of this is $(1-x)^q$. Also, the other r − 1 as must be less than x, and the probability of this is x^{r-1}. Finally, the largest of the as may be selected in r ways, so the probability of a suitable arrangement with the largest of the as between x and dx is $x^{r-1} r \, dx (1-x)^q$. The total probability is obtained by integrating over all possible values of x, and thus is

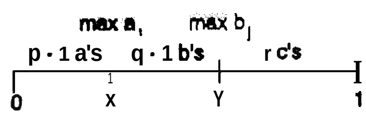


Figure 2

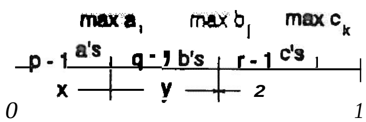
$$\int_0^1 r x^{r-1} (1-x)^q \, dx.$$

Since the probability is the same no matter which method we use to find it, we have

$$\int_0^1 r x^{r-1} (1-x)^q \, dx = \frac{r!\, q!}{(r+q)!}.$$

Putting $r = p + 1$, we have the result.

The argument may be generalized to any number of sets. For example, with three sets of numbers, $\{a_1, \ldots, a_p, b_1, \ldots, b_q, c_1, \ldots, c_r\}$, the probability that $a_i < b_j < c_k$ for all i, j, k is, by considering arrangements, $\dfrac{p!\,q!\,r!}{(p+q+r)!}$ On the other hand (see Figure 2) it is



$$\iint_{0 \le x \le y \le 1} p x^{p-1} q (y-x)^{q-1} (1-y)^r \, dx\, dy.$$

Figure 3

The integral may be put in various other forms by change of variables. One of these forms, known as Dirichlet's Integral, is

$$\iiint_{\substack{x,y,z \ge 0 \\ x+y+z \le 1}} x^{p-1} y^{q-1} z^{r-1} \, dx\, dy\, dz = \frac{(p-1)!\,(q-1)!\,(r-1)!}{(p+q+r)!}$$

This result can be obtained on similar lines by a slight modification of the number line (see Figure 3). The details are left to the reader.

References

1. Spiegel, Murray R., *Theory and Problems of Advanced Calculus*, McGraw-Hill, Singapore, 1981.
2. Srivastava, R. S. L., *Engineering Mathematics*, vol. 1, Tata McGraw-Hill, New Delhi, 1980.

M. A. Khan has a degree in electrical engineering and is presently Deputy Director of RDSO, Lucknow. Besides mathematics, his interests include contract bridge.

LETTER TO THE EDITOR

Dear Professor Poss,

I am sending here a brief followup to my article "Fractorial!" which appeared in the Fall 1992 issue of the *Journal*. Even when I had finished it I was troubled by what is a very cumbersome development to get from Definition 3, the definition of fractorial, to Theorems 1 and 2. I have simplified the whole thing greatly by slightly reformulating the definition of fractorial so that now the results of Theorems 1 and 2 follow in a very straightforward manner.

DEFINITION 3. $a!_b = a(a-b)(a-2b)\ldots(a-kb)$, where a and b are positive real numbers, $a \ge b$, and k is a natural number such that $k < a/b \le k+1$, so that $0 < a - kb \le b$.

THEOREM. For all natural numbers a, b, c, and d, the number, n, of factors in the fractorial expression $(a/c)!_{(b/d)}$ is given by $n = \left\lceil \dfrac{ad-1}{bc} \right\rceil + 1$.

Proof. From $k < a/b \le k+1$ we get $0 < a - kb \le b$, or $l \le a - kb < b + 1$. Replace a with ad and b with bc to get $1 \le ad - kbc < bc + 1$. Dividing by bc gives $1/bc \le ad/bc - k < + 1/bc$, so

$$k \le ad/bc - 1/bc < k + 1.$$

Since $n = k + 1$, the result follows.

Sincerely yours,

Nataniel Greene
Yeshiva University

New Giant Twin Primes

$$4650828 \cdot 1001 \cdot 10^{3429} \pm 1.$$

Questions for the reader: 1. Do they add to our store of mathematical knowledge? 2. Can we trust the computer that calculated them? If your answers are, respectively, "Not much" and "Not necessarily" then 3. Why were they included here and why did you read about them?

Mathacrostics

Solution to Mathacrostic 36, by Charlotte Maines, (Spring, 1993).

Words:

A. Monte **Carlo** method
B. abstract space
C. rheostat
D. venetian white
E. inappetent
F. Nathan
G. Sylvester's dialytic method
H. hones
I. imaginary circle
J. nephews
K. batch
L. **requiescat**
M. objects
N. trident of Newton
O. trajectories
P. **Hooton**
Q. interface
R. **Noether**
S. gentes
T. servomechanism
U. folium of Descartes
V. anthotaxy
W. **lenten**
X. lethe
Y. annotate
Z. point of osculation
a. **athanor**
b. rule of false position
c. two-throw

Author and title: Marvin **Shinbrot,** Things Fall Apart,

Quotation: From the seventeenth century to the nineteenth, the heart of all physics and much of mathematics as well was Newton's three laws of motion: a body not subject to external force remains at rest or moves with constant speed in a straight line, the acceleration of an object os proportional to the forces acting on it, and to every action there is an equal and opposite reaction.

Solvers: THOMAS BANCHOFF, Brown University, JEANETTE BICKLEY, St. Louis Community College—Meramec, CHARLES R. **DIMINNIE,** St. Bonaventure University, VICTOR G. **FESER,** University of Mary, ROBERT FORSBERG, Lexington, Massachusetts, META HARRSEN, Durham, North Carolina, THEODOR KAUFMAN, Brooklyn, New York, **HENRY** S. LIEBERMAN, Waban, Massachusetts, DON PFAFF, University of Nevada, Reno, STEPHANIE SLOYAN, Georgian Court College

Mathacrostic 37, proposed by PATTI **VAHEDI,** follows. To be listed as a solver, send your solution to Underwood Dudley, Pi Mu Epsilon Journal, Math. Dept., **DePauw** Univ., Greencastle, Indiana 46135.

Clue	Letters
A. Vector cross-product convention. [Hyph + wd]	5 127 158 111 119 183 21 239 36 216 58 170
B. He secured election of first German Pope. [2 wds]	234 114 39 79 184 136 65 207
C. Enlightened one. [Sanskrit]	139 47 199 226 95 71 2 147 218 178 162
D. A solution to the **Sturm-Liouville** problem.	194 241 186 152 118 230 37 92 15 75
E. FDR's director of the U. S. Mint. [Surname]	121 150 84 38
F. Belgian art group of ***la Belle Epoque***. [2 wds]	25 64 100 8 52 224 163 81 122
G. It salivated when the bell rang. [2 wds]	78 68 99 55 10 173 211 204 229 138
H. Its limit, by definition. = integral of function. [2 wds]	180 4 134 16 157 23 109 69 197 228
I. German gelato or sorbet.	203 238 96
J. Fictive beast, clarifies uncertainty principle. [2 wds]	149 20 53 42 72 94 106 80 176 117 131 164
K. Simple problem in astronomy. [Hyph]	130 145 220 233 24 103 181 206 217 246
L. Incidental remark; if from a judge, it has no bearing on the case and is not binding. [2 wds]	88 26 9 155 179 62 237 171 124 104 212 74
M. It brought attention to fundamental law in classical mechanics. [2 wds]	57 129 50 76 210 11 107 143 86 153 198 102
N. Descartes' parabola = __ __ Newton. [2 wds]	191 19 227 208 35 22 49 142 73
O. Narrow-minded; prejudiced.	222 146 240 28 168 51 120 3 91
P. He estimated earth radius with trig, circa 230 BC	161 101 12 221 46 245 209 126 90 188 172 33
Q. $0.4\cos(\pi/4)$.	1 159 133 41 223 113
R. Indian mathematician b. 476 BC.	215 202 140 7 60 156 77 110 132
S. Polynomial expression of a function. [2 wds]	67 98 125 201 193 34 177 6 219 169 242
T Fool's dullness = "Whetstone _____ ." [3 wds, from As You Like *It*.]	196 48 144 166 32 225 174 44 89
U. Realm of $a + bi$. [2 wds]	97 167 83 66 13 231 39 87 135 190 185 56
V Designer, intensity interferometer [Init, surname]	232 214 141 31 54 165 189
W Chinese moon guitar [Hyph]	154 85 18 40 213 115 187 175

X. Its examples include 5, 23, 14009. [2 wds]

30 160 45 59 112 137 244 151 123 27 236

Y. __ Atlas; __ telescope; __ time; __ expansion.

105 192 200 17 128 61

Z. 19th century pioneer in theory and analysis of Word U. [2 wds]

43 82 243 205 63 182 108 14 116 235 70 195 148 93

			1 Q		2 C	3 O		4 H	5 A	6 S	7 R	8 F	9 L	10 G
11 M	12 P	13 U		14 Z	15 D	16 H	17 Y	18 W	19 N		20 J	21 A	22 N	23 H
24 K	25 F		26 L	27 X		28 D	29 U	30 X	31 V	32 T	33 P	34 S	35 N	36 A
	37 D	38 E		39 B	40 W	41 Q		42 J	43 Z	44 T	45 X	46 P		47 C
48 T		49 N	50 M	51 O		52 F	53 J	54 V	55 G	56 U		57 M	58 A	59 X
60 R	61 Y	62 L	63 Z		64 F	65 B	66 U	67 S	68 G	69 H	70 Z	71 C		72 J
73 N		74 L	75 D	76 M	77 R	78 G	79 B	80 J	81 F	82 Z	83 U		84 E	85 W
86 M	87 U	88 L	89 T	90 P	91 O	92 D	93 Z		94 J	95 C	96 I	97 U	98 S	99 G
100 F	101 P	102 M	103 K		104 L	105 Y	106 J	107 M		108 Z	109 H		110 R	111 A
112 X		113 Q	114 B	115 W		116 Z	117 J	118 D	119 A	120 O	121 E	122 F		123 X
124 L		125 S	126 P	127 A	128 Y	129 M		130 K	131 J	132 R	133 Q	134 H	135 U	136 B
137 X	138 G		139 C	140 R		141 V	142 N	143 M	144 T		145 K	146 O	147 C	148 Z
	149 J	150 E	151 X	152 D		153 M	154 W	155 L	156 R	157 H	158 A	159 Q	160 X	161 P
162 C	163 F	164 J		165 V	166 T	167 U		168 O	169 S	170 A	171 L	172 P	173 G	174 T
175 W	176 J		177 S	178 C	179 L	180 H	181 K	182 Z	183 A	184 B	185 U	186 D		187 W
188 P		189 V	190 U	191 N	192 Y	193 S	194 D		195 Z	196 T	197 H	198 M	199 C	
200 Y	201 S		202 R	203 I	204 G	205 Z	206 J	207 A	208 N		209 P	210 M		211 G
212 L	213 W	214 V		215 R		216 A	217 J	218 C	219 S	220 K		221 P	222 O	223 Q
224 F	225 T		226 C	227 N	228 H		229 G	230 D	231 U	232 V	233 K	234 B	235 Z	236 X
237 L		238 I	239 A		240 O	241 D	242 S	243 Z	244 X	245 P	246 J			

PV Box 6414 Albany CA 94706-0414

PROBLEM DEPARTMENT

Edited by Clayton W. Dodge
University of Maine

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

All communications should be addressed to C. W. Dodge, 5752 Neville/Math, University of Maine, Orono, ME 04469-5752. Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly identified with name and address. Solutions to problems in this issue should be mailed by July 1, 1994.

This department seeks to present a wide variety of problems in each issue, preferably not more than two problems from any one category. Hence appropriate proposals are sought for all categories, but especially for those that are empty or nearly so. To aid (and entice) you in your submissions, we list each category along with the number of problem proposals in its file: algebra 10, alphametics 1, analysis 10, geometry 3, logic and combinatorics 0, number theory 3, probability and statistics 0, trigonometry 0, and miscellaneous 0.

Corrections

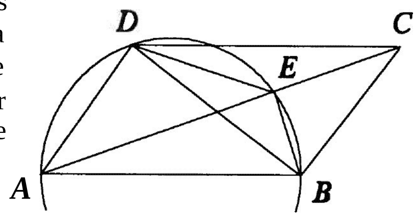
Several times in the Spring 1993 Problem Department the name of proposer and solver David Iny was inadvertently printed as David Ivy. Our sincere apologies.

Thanks to William Peirce for pointing out that in the solution to problem 765 on page 475 of the Fall 1992 issue, the last equation should read $410^2 = (12122_4)^2 = 0221002210_4$.

780. [Spring 1992, Fall 1992] *Corrected again. Proposed by R. S. Luthar, University of Wisconsin Center, Janesville, Wisconsin.*

Let $ABCD$ **be a parallelogram with** $\angle A = 60°$. **Let the circle through** A, B, **and** D **intersect** AC at E. **See the figure. Prove** that $BD^2 + AB \cdot AD = AE \cdot AC$.

Editor's comment: The statement of the problem was correct. The *figure* was perhaps misleading: ABCD is a parallelogram, not necessarily a rhombus. The accompanying figure shows this situation more clearly. Our apologies for any inconvenience the original figure may have caused.



Problems for Solution

810. Proposed by Alan Wayne, Holiday, Florida.

In the following base eight **multiplication,** the digits of the two multipliers have been replaced in a one to one **manner** by letters:

$$(I)(CLUED) = 437152.$$

Restore the digits. Similarly replace 437152 to find out who might have said "I clued."

811. Proposed by Tom Moore, Bridgewater State College, Bridgewater, ***Massachusetts.***

If $a < b < c$ are positive integers with $\gcd(a, b) = 1$ and $a^2 + b^2 = c^2$, then (a, b, c) is called a primitive Pythagorean triple (PPT). If both a and c are primes, then we shall call it a prime PPT (P^3T).

a) If (a, b, c) is a P^3T, deduce that $b = c - 1$.

b) Find all P^3Ts in which a and c are

i) twin **primes.**

ii) both Mersenne primes.

iii) both Fermat **primes.**

iv) one a Mersenne, the other a Fermat prime.

812. Proposed by George P. ***Evanovich,*** Saint Peter's College, Jersey City, New Jersey.

If $n \geq 2$ is a positive integer, prove that

$$\sum_{j=1}^{n} \cos\left(\frac{2j\pi}{n}\right) = \sum_{j=1}^{n} \sin\left(\frac{2j\pi}{n}\right) = 0.$$

813. Proposed by the late Jack ***Garfunkel,*** Flushing, New York.

Given a triangle ABC with sides a, b, c and a triangle $A'B'C'$ with sides $(b + c)/2$, $(c + a)/2$, $(a + b)/2$. Prove that $r' \geq r$, where r and r' are the **inradii** of triangles ABC and $A'B'C'$ respectively.

814. Proposed by Nathan ***Jaspen,*** Stevens Institute of Technology, ***Hoboken,*** New Jersey.

For any decimal integer n, prove that n^5 and n end in the same **digit,** that n^6 and n^2 end in the same digit, that n^7 and n^3 end in the same digit, and so forth.

815. Proposed by Bill ***Correll,*** Jr., Cincinnati, Ohio.

Let $[x]$ denote the greatest integer not exceeding x. Solve for x:

$$\left[\frac{x+1}{2}\right]\left[\frac{x+2}{3}\right]\left[\frac{x+3}{4}\right] = 819.$$

816. Proposed by Robert ***C. Gebhardt,*** Hopatcong, New Jersey.

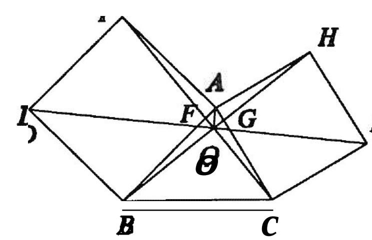
a) From the integers 1, 2, 3, ..., n, a state lottery selects at random k numbers ($k < n$). A person who had previously chosen at random m of those k numbers ($m \leq k$) is a winner. Find the probability of being a winner.

*b) The Tri-State Megabucks (Maine, New Hampshire, and Vermont) tickets cost $1 each. A participant selects m = 6 numbers out of n = 40 and is a winner if all six numbers match the k = 6 numbers the game selects. The winnings are paid in 20 equal annual installments. How large does the pot have to be before a ticket is worth $1?

817. Proposed by Andrew ***Cusumano,*** Great Neck, New York.

In the accompanying figure squares ***CEHA*** and AIDE are erected externally on sides CA and AB of triangle ABC. Let BH meet IC at **O** and AC at G, and let CI meet AB at F.

a) Prove that points D, O, and E are collinear.



b) Prove that angles HOE, EOC, AOH, and AOI are each **45°.**

c) If ACB is a right angle, then prove that E, F, and G are collinear.

Find an "elegant" proof for parts (a) and (b), both of which are known to be true whether the squares are erected both externally or both internally (see The American Mathematical Monthly, problem **E831,** vol. 56, 1949, 406-407). Part (c) is a delightful result that also should be known, but appears to be more difficult to prove.

***818.** Proposed by *Dmitry* P. Mavlo, Moscow, Russia.
From the SYMP-86 Entrance Examination, solve the inequality

$$\frac{1}{x^3 - x} \le \frac{1}{|x|}.$$

819. Proposed by Morris *Katz,* Macwahoc, Maine.
Evaluate the integral

$$\int \ln x \sin^{-1} x \, dx.$$

820. Proposed by William *Moser, McGill* University, Montreal, Quebec, Canada.

Let $a_{n,k}$ $(0 \le k < n)$ denote the number of n-bit strings (sequences of **0's** and **1's** of length n) with exactly k occurrences of of two consecutive **0's.** Show that

$$a_{n,k} = \sum_{r=0}^{n-1} \binom{r-k}{k} \binom{n-r-1}{r-k},$$

where $\binom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$ if $0 \le k \le n$ and $\binom{n}{k} = 0$ **otherwise.**

821. Proposed by Zeev *Barel, Hendrix* College, *Conway,* Arkansas.

Problem B-2 at the fifty-second annual William Lowell Putnam Mathematical Competition (1991) stated: Suppose f and g are non-constant, differentiable, real-valued functions defined on $(-\infty,\infty)$. Furthermore, suppose that for each pair of real numbers x and y,

$$f(x+y) = f(x)f(y) - g(x)g(y) \quad \text{and} \quad g(x+y) = f(x)g(y) + g(x)f(y).$$

If $f'(0) = 0$, prove that $(f(x))^2 + (g(x))^2 = 1$ for all x.

In fact, one can do a little more under the same hypothesis. Prove that there exists a real number k such that $f(x) = \cos kx$ and $g(x) = \sin kx$ for all x.

822. Proposed by Stanley *Rabinowitz, MathPro* Press, *Westford,* Massachusetts.

If a is a root of the equation $x^5 + x - 1 = 0$, then find an equation that has $a^4 + 1$ as a root.

Solutions

777. [Spring 1992, Fall **1992]** Corrected. Proposed by *Seung-Jin* Bang, Seoul, Korea.

It is well known that, for $n \ge 2$, $\ln(n+1) < S_n < 1 + \ln n$, where

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}.$$

It is also known (Crux Mathematicorum 11 **(1985),** 109) that, for $n \ge 2$,

$$n(n+1)^{1/n} - n < S_n \le n - (n-1)n^{-1/(n-1)}.$$

Prove that

$$\ln(n+1) < n(n+1)^{1/n} - n \quad \text{and} \quad n - (n-1)^{-1/(n-1)} < 1 + \ln n$$

for all $n \ge 2$.

Solution by Alma College Problem Solving Group, Alma College, Alma, Michigan.

Recall that $e^u > 1 + u$ for all $u \ne 0$, so that $\ln x < x - 1$ for all $x > 0$, $x \ne 1$, by setting $x = u + 1$.

Because $(n+1)^{1/n} > 0$ (and never equal to 1) when $n \ge 1$, then

$$\ln (n+1)^{1/n} < (n+1)^{1/n} - 1$$

for all $n \ge 1$. Finally, multiply both sides by n to get that

$$\ln (n+1) = n \ln (n+1)^{1/n} < n(n+1)^{1/n} - n.$$

Again using $x > 1 + \ln x$, we have, for $n \ge 2$,

$$n^{-1/(n-1)} > 1 + \ln n^{-1/(n-1)}, \quad (n-1)n^{-1/(n-1)} > (n-1) - \ln n,$$

and

$$n - (n-1)n^{-1/(n-1)} < 1 + \ln n.$$

Also solved by PAUL S. BRUCKMAN, Everett, *WA*, RICHARD I. HESS, *Rancho Palos* Verdes, *CA,* DAVID INY, *Westinghouse* Electric Corporation, Baltimore, *MD,* MURRAY S. KLAMKIN, University of *Alberta,* Canada, DAVID E. MANES, *SUNY* at *Oneonta,* JOHN D. MOORES, *Westbrook,* ME, YOSHINOBU MURAYOSHI, *Eugene,* OR, PAUL D.

SHOCKLEE, Memphis, TN, JORGE-NUNO SILVA, Albany, CA, REX H. WU, Brooklyn, *NY,* and the PROPOSER.

784. [Fall **1992]** Proposed by Alan Wayne, Holiday, Florida.
Restore the enciphered digits in the decimal **computation:**

$$(TWO)(TWO + TWO) = EIGHT.$$

Solution by Kenneth M. Wilke, *Topeka,* Kansas.

Since $98765 \geq EIGHT = 2(TWO)^2$, then $TWO < 223$, T is even so T = 2, and O = 1, 4, 6, or 9. Therefore TWO = 201, 204, 206, 209, 214, 216, or 219. Testing these possibilities, we find that only TWO = 209 yields a solution, and then EIGHT = 87362.

Also solved by ALMA COLLEGE PROBLEM SOLVING GROUP, MI, CHARLES ASHBACHER, Cedar Rapids, IA, SEUNG-JIN BANG, Seoul, Korea, FRANK P. BATTLES, Massachusetts Maritime Academy, Buzzards Bay, SCOTT H. BROWN, Auburn University, *AL,* PAUL S. BRUCKMAN, Everett, WA, MARK EVANS, Louisville, *KY,* VICTOR G. **FESER,** University of Mary, Bismarck, ND, STEPHEN I. GENDLER, Clarion University of Pennsylvania, RICHARD I. HESS, Rancho *Palos* Verdes, CA, PETE JOHNSON, Hebron, *CT,* JOHN D. **MOORES,** Westbrook, ME, YOSHINOBU MURAYOSHI, Eugene, OR, PAUL D. SHOCKLEE, Memphis, TN, LAURA SILVA, Albany, CA, SONNY VU, University of Illinois at *Urbana-Champaign,* REX H. WU, Brooklyn, *NY,* and the PROPOSER.

785. [Fall 19921 Proposed by Charles *Ashbacher,* Cedar Rapids, Iowa, and dedicated to the memory of Joseph *Konhauser.* Student solutions are especially solicited.

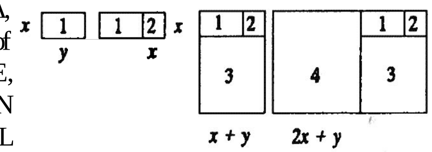
A tiling of the plane by non-overlapping, non-congruent rectangles P_1, P_2, ... is defined in the following way: P_1 is an arbitrary x by y rectangle; P_2, P_3, . are all squares such that the side of each square P_{k+2} is equal to the sum of the sides of the two previous squares P_k and P_{k+1} for all $k > 1$. Show this tiling.

Solution by Matthew Amoroso, St. *Bonaventure* University, St. *Bonaventure,* New York.

To begin, let rectangle 1 be an x by y rectangle. See the figure on the next page. Let rectangle 2 be a square of side x and place it so that it is resting on a side of rectangle 1 with length x. Again see the figure. Next, rectangle 3 is a square of side x + y which we place to rest against both rectangles 1 and 2. Now rectangle 4 is a square of side $2x$ + y that rests against the two immediately preceding rectangles 2 and 3. Continue in this manner to tile the plane in the

prescribed way. Since each square must have its side equal to the sum of the sides of the two previous squares, the coefficients of x and of y in these side lengths are successive Fibonacci numbers: $f_0 = 0, f_1 = 1$ and $f_{n+2} = f_n + f_{n+1}$ for n ≥ 1. That is, the side of square n is equal to $f_{n-1}x + f_{n-2}y$ for n ≥ 2.

Also solved by PAUL S. BRUCKMAN, Everett, WA, RICHARD I. HESS, Rancho *Palos* Verdes, CA, MURRAY S. KLAMKIN, University of Alberta, Canada, TOM MOORE, Bridgewater State College, MA, JOHN D. **MOORES,** Westbrook, ME, PAUL D. SHOCKLEE, Memphis, TN, REX H. WU, Brooklyn, NY, and the PROPOSER.



786. [Fall **1992]** Proposed by *Dmitry* P. Mavlo, Moscow, Russia.

From two towns A and B, 48 km apart, two groups of hikers march toward each other starting at the same time. The group leaving A marches at 4 km/hr by marches of not more than 6 **hr** at one time. The group from B hikes at 6 **km/hr** for not more than 2 **hr** at a time. After marching t **hr,** the first group must rest for at least t hr. The second group has to rest not less than $2t$ **hr** after t **hr** of hiking. Find the least time until the two groups meet and describe the hiking patterns necessary for that solution.

Solution by William H. Peirce, Delray *Beach,* Florida,

The time until the two groups meet will be least if each group hikes as long as possible without resting. Therefore, we let **them** meet as A is completing a 6-hr hike at 4 **km/hr** and B is completing a 2-hr hike at 6 **km/hr,** assuming the time needed to meet is at least 6 hours. These final hikes span 24 + 12 = **36** km.

If the total time until they meet is T > 6 hours, then group A must hike and rest equal times for the first T − 6 hours, averaging **2 km/hr** during that time. Group B has T − **2** hours during which they average 2 **km/hr.** Hence

$$48 = 36 + 2(T - 6) + 2(T - 2)$$

and T = 7 hours. Group A must hike 30 minutes and rest 30 minutes during the first hour, and then hike the last 6 hours. Group B hikes 1 hour 40 minutes and rests 3 hours 20 minutes during the first 5 hours, and then hikes the last 2 hours. Except for the fact that the initial hike-and-rest period can be broken down into smaller segments, this solution is unique.

Also solved by ALMA COLLEGE PROBLEM SOLVING GROUP, MI, CHARLES ASHBACHER, Cedar Rapids, IA, PAUL S. BRUCKMAN, Everett, WA, MARK EVANS, Louisville, KY, STEPHEN I. GENDLER, Clarion University of Pennsylvania, RICHARD I. HESS, Rancho Palos Verdes, CA, JOHN D. MOORES, Westbrook, ME, WILLIAM H. PEIRCE (second solution), Delray Beach, FL, PAUL D. SHOCKLEE, Memphis, TN, MANUEL SILVA, Albany, CA, and REX H. WU, Brooklyn, NY. One incorrect solution was received.

787. [Fall 1992] Proposed by R. S. Luthar, University of Wisconsin Center, *Janesville*, Wisconsin.

If a, b, c, d are the roots of

$$x^4 + px^3 + qx^2 + rx + s = 0,$$

then evaluate the expression

$$(a + b + c - 2d)(b + c + d - 2a)(c + d + a - 2b)(d + a + b - 2c)$$

in terms of p, q, r, and s.

Solution by Harry *Sedinger*, St. Bonaventure University, St. Bonaventure, New York.

It is well known (and easily seen) that if

$$x^4 + px^3 + qx^2 + rx + s = (x - a)(x - b)(x - c)(x - d),$$

then

$$a + b + c + d = -p, \quad ab + ac + ad + bc + bd + cd = q,$$

$$abc + acd + abd + bcd = -r, \quad \text{and} \quad abcd = s.$$

Then the given expression is equal to

$$\begin{aligned}(-p - 3d)&(-p - 3a)(-p - 3b)(-p - 3c)\\ &= p^4 + 3(a + b + c + d)p^3 + 9(ab + ac + ad + bc + bd + cd)p^2\\ &\quad + 27(abc + acd + abd + bcd)p + 81abcd\\ &= p^4 - 3p^4 + 9qp^2 - 27rp + 81s\\ &= -2p^4 + 9qp^2 - 27rp + 81s.\end{aligned}$$

Also solved by SEUNG-JIN BANG, Seoul, Korea, SCOTT H. BROWN, Auburn University, AL, PAUL S. BRUCKMAN, Everett, WA, JILL CARNAHAN, Eastern Kentucky University, Richmond, BILL CORRELL, JR., Cincinnati, OH, DAVID DELSESTO, North Scituate, RI, RUSSELL EULER, Northwest Missouri State University, Maryville, GEORGE P. EVANOVICH, Saint Peter's College, Jersey City, NJ, MARK EVANS, Louisville, KY, STEPHEN I. GENDLER, Clarion University of Pennsylvania, RICHARD I. HESS, Rancho Palos Verdes, CA, MURRAY S. KLAMKIN, University of Alberta, Canada, DAYONG LI, Eastern Kentucky University, Richmond, HENRY S. LIEBERMAN, Waban, MA, DAVID E. MANES, SUNY at Oneonta, JOHN D. MOORES, Westbrook, ME, YOSHINOBU MURAYOSHI, Eugene, OR, WILLIAM H. PEIRCE, Delray Beach, FL, BOB PRIELIPP, University of Wisconsin-Oshkosh, PAUL D. SHOCKLEE, Memphis, TN, LAWRENCE SKAGGS, Eastern Kentucky University, Richmond, KENNETH M. WILKE, Topeka, KS, J. ERNEST WILKINS, JR., Clark Atlanta University, GA, REX H. WU, Brooklyn, NY, and the PROPOSER.

788. [Fall 1992] Proposed by the late Jack *Garfunkel*, Flushing, New York.

Given positive numbers x, y, z such that $x + y + z = 1$, prove that

$$xy + yz + zx \ge x^2y^2 + y^2z^2 + z^2x^2 + 8xyz.$$

I. Solution by Sammy *Yu* and Jimmy *Yu*, students, Vermillion Middle School, Vermillion, South Dakota.

Since $x + y + z = 1$, the desired inequality is equivalent to

$$(x + y + z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) \ge \frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y} + 8,$$

and hence to

$$\text{(1)} \qquad \frac{y + z}{x} + \frac{z + x}{y} + \frac{x + y}{z} \ge \frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y} + 5.$$

Now consider that

$$\frac{(y + z)^2}{x} + \frac{(z + x)^2}{y} + \frac{(x + y)^2}{z} - \frac{yz}{x} - \frac{zx}{y} - \frac{xy}{z}$$

$$= \frac{y^2 + yz + z^2}{x} + \frac{z^2 + zx + x^2}{y} + \frac{x^2 + xy + y^2}{z}$$
$$= \left(\frac{y^2}{x} + \frac{zx}{y} + \frac{y^2}{z}\right) + \left(\frac{yz}{x} + \frac{x^2}{y} + \frac{x^2}{z}\right) + \left(\frac{z^2}{x} + \frac{z^2}{y} + \frac{xy}{z}\right)$$
$$\geq 3y + 3x + 3z = 3$$

by applying the arithmetic mean-geometric mean inequality to each parentheses separately. Equality holds if and only if $x = y = z$. Thus

$$(2) \qquad \frac{(y+z)^2}{x} + \frac{(z+x)^2}{y} + \frac{(x+y)^2}{z} \geq \frac{yz}{x} + \frac{zx}{y} + \frac{xy}{z} + 3.$$

The left side of (2) can be rewritten as

$$(y+z)\left(\frac{x+y+z}{x} - 1\right) + (z+x)\left(\frac{x+y+z}{y} - 1\right) + (x+y)\left(\frac{x+y+z}{z} - 1\right)$$
$$= (x+y+z)\left(\frac{y+z}{x} + \frac{z+x}{y} + \frac{x+y}{z}\right) - 2.$$

Hence equation (2) implies (1). The desired result follows with equality if and only if $x = y = z$.

II. Solution by Murray *S. Klamkin*, University of Alberta, Edmonton, Alberta, Canada.

We rewrite the inequality in the following homogeneous form:

$$(yz + zx + xy)(x + y + z)^2 \geq y^2z^2 + z^2x^2 + x^2y^2 + 8xyz(x + y + z)$$

or

$$(1) \qquad T_1^2T_2 \geq T_2^2 - 2T_1T_3 + 8T_1T_3 = T_2^2 + 6T_1T_3$$

where $T_1 = x + y + z$, $T_2 = yz + zx + xy$, and $T_3 = xyz$.

We give a stronger result by proving that the "best inequality" of the form

$$(2) \qquad T_1^2T_2 \geq aT_2^2 + bT_1T_3$$

is

$$(3) \qquad T_1^2T_2 \geq 4T_2^2 - 3T_1T_3$$

or equivalently,

$$(3') \qquad yz(y-z)^2 + zx(z-x)^2 + xy(x-y)^2 \geq 0,$$

so (3) is clearly true. For (2) to be valid we have, on setting $x = y = z = 1/3$, that $9 \geq 3a + b$; on setting $x = y = 1/2$ and $z = 0$, that $4 \geq a$. That (3) is "best" will now follow if

$$(4) \qquad 4T_2^2 - 3T_1T_3 \geq aT_2^2 + bT_1T_3$$

for all a and b satisfying the latter two inequality conditions. Since $T_2^2 \geq 3T_1T_3$ is a known equality equivalent to $\Sigma x^2(y - z)^2 \geq 0$, then (4) follows if

$$\frac{b-3}{4-a} \leq 3,$$

which is equivalent to the condition $9 \geq 3a + b$.

Working backwards from (3), we find that the "best inequality" for the original problem is

$$xy + yz + zx \geq 4(x^2y^2 + y^2z^2 + z^2x^2) + 5xyz.$$

Also solved by SEUNG-JIN BANG, Seoul, Korea, PAUL S. BRUCKMAN, Everett, WA, DAVID E. MANES, SUNY at Oneonta, YOSHINOBU MURAYOSHI, Eugene, OR, WILLIAM H. PEIRCE, Delray Beach, FL, BOB PRIELIPP, University of Wisconsin-Oshkosh, J. ERNEST WILKINS, JR., Clark Atlanta University, GA, and the PROPOSER.

789. [Fall 1992] Proposed by David *Iny*, Baltimore, Maryland.

Evaluate the integral

$$\int_0^1 \frac{y-1}{\sqrt{y}\ln y}\,dy.$$

I. Solution composed from those submitted by Paul S. Bruckman, Everett, Washington, and George P. *Evanovich*, St. Peter's College, Jersey City, New Jersey.

Let the integrand be denoted by $f(y)$. Using L'Hôpital's rule, we see that

$$\lim_{y\to 1} \frac{y-1}{\ln y} = \lim_{y\to 1} \frac{1}{1/y} = 1,$$

so $\lim_{y\to 1} f(y) = 1$ and f is continuous at $y = 1$. Accordingly, we make the definition

$$(1) \qquad I_n = \int_{e^{-2n}}^1 f(y)\,dy, \quad n = 1, 2, 3, \ldots.$$

Now if $\lim_{n\to\infty} I_n$ exists, then this limit must be equal to the given integral, which we denote as I.

We make the substitution $y = e^{-2x}$ in *(1)*. Then $dy = -2e^{-2x}\,dx$, and we obtain

$$I_n = \int_0^n \frac{e^{-x} - e^{-3x}}{x}\,dx, \quad \text{so} \quad I = \int_0^\infty \frac{e^{-x} - e^{-3x}}{x}\,dx,$$

if it exists.

Now, more generally, we let

$$F(a,b) = \lim_{n\to\infty} \int_0^n \frac{e^{-bx} - e^{-ax}}{x}\,dx.$$

Then

$$\frac{\partial F(a,b)}{\partial b} = \lim_{n\to\infty} \int_0^n (-e^{-bx})\,dx = \lim_{n\to\infty} \frac{1}{b} e^{-bx}\Big|_0^n = -\frac{1}{b}.$$

Therefore

$$F(a,b) = -\int \frac{1}{b}\,db = -\ln b + g(a) + C$$

for some function g and constant C. Since $F(b, a) = -F(a, b)$, it follows that $g(a) = \ln a$. Also $F(a, a) = 0 = \ln a - \ln a + C$, so $C = 0$. That is,

$$F(a,b) = \ln a - \ln b = \ln\frac{a}{b}, \quad \text{whence } I = \ln 3.$$

II. *Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.*

More generally, we evaluate

$$I(m,n) = \int_0^1 \frac{y^m - 1}{y^n \ln y}\,dy.$$

Let $y = e^{-x}$, so that

$$I(m,n) = \int_0^w \frac{e^{-(1-n)x} - e^{-(m+1-n)x}}{x}\,dx.$$

This is a Frellani integral and it is known that

$$\int_0^\infty \frac{F(ax) - F(bx)}{x}\,dx = F(0)\cdot\ln\frac{b}{a},$$

provided the integral exists and that

$$\lim_{n\to\infty} \int_{n/b}^{n/a} \frac{F(bx)}{x}\,dx = 0.$$

Hence $I(m, n) = \ln\,[(m+1-n)/(1-n)]$. For the given problem, $I(1, 1/2) = \ln 3$.

For the case when $\lim_{x\to\infty} F(x) = F(\infty)$, then Elliot has shown that

$$\int_0^\infty \frac{F(ax) - F(bx)}{x}\,dx = [F(\infty) - F(0)]\ln\frac{a}{b}.$$

As an example,

$$\int_0^\infty \frac{\tan^{-1} ax - \tan^{-1} bx}{x}\,dx = \frac{\pi}{2}\ln\frac{a}{b}.$$

Also solved by BORIS BAEUMER, Louisiana State University, Baton Rouge, SEUNG-JIN BANG, Seoul, Korea, FRANK P. BATTLES, Massachusetts Maritime Academy, Buzzards Bay, SCOTT H. BROWN, Auburn University, AL, DAVID E. MANES, *SUNY at Oneonta, JOHN D. MOORES, Westbrook, ME, HARRY SEDINGER, St. Bonaventure University, NY, STAN WAGON, Macalester College, St. Paul, MN, J. ERNEST WILKINS, JR., Clark Atlanta University, GA, and the PROPOSER.*

Several solvers found references to the integrals $F(a, b)$ of Solution I and $I(m, n)$ of Solution II: Sokolnikoff, Advanced Calculus, p. 364, Gradshteyn & Ryzhik, Table of Integrals, Series, and Products, 4th ed, Academic Press, 1965, Formula 8, p. 543, Borden, A Course in Advanced Calculus, North-Holland, 1983, Problem 37, p. 364. A common method of solution for integral $F(a, b)$ was to rewrite it as a double integral and then apply Fubini's theorem to reverse the order of integration.

790. [Fall **1992]** *Proposed by **Florentin** Smarandache, Phoenix, Arizona.*

In base 6 how many digits does the nth prime contain?

*Solution by Paul T. **Bateman**, University of Illinois, Urbana, Illinois.*

More generally, the number of digits of the number N to the base b is the integral part of $1 + (\ln N)/(\ln b)$. The known formula

$$n \ln n < p_n < n \ln n\,(1 + \delta(n)),$$

where $\delta(n)$ is a positive quantity that approches zero when n gets large, makes it possible to approximate the number of digits in the nth prime p_n within one unit for large n. That is, the number of digits in p_n to base b is given by the integral part of

$$1 + \frac{\ln(n \ln n)}{\ln b}$$

for large n.

Also solved by PAUL S. *BRUCKMAN, Everett, **WA**, and* DAVID *E. MANES, **SUNY** at Oneonta.*

791. [Fall **1992**] Proposed by Seung-Jin Bang, Seoul, Republic of Korea.

Prove that $2^n + 1$, where n is a nonnegative integer, is never a multiple of 143.

I. Solution by Kenneth M. Wilke, Topeka, Kansas.

Suppose that $2^n + 1 \equiv 0 \pmod{143}$ for some nonnegative integer n. Then n simultaneously satisfies both congruences $2^n + 1 \equiv 0 \pmod{11}$ and $2^n + 1 \equiv 0 \pmod{13}$. By trial one finds that $2^5 + 1 = 33 \equiv 0 \pmod{11}$. That is,

$$2^5 \equiv -1 \pmod{11}, \quad \text{so} \quad 2^{10} \equiv (-1)^2 \equiv 1 \pmod{11}.$$

Thus the congruence $2^n + 1 \equiv 0 \pmod{11}$ is satisfied when $n = 5 + 10r$ for any nonnegative integer r. By similar trial one finds that $2^n + 1 \equiv 0 \pmod{13}$ is satisfied when $n = 6 + 12s$ for any nonnegative integers. Hence we must have that $5 + 10r = 6 + 12s$ for integral r and s. Since the left side of the equation is always odd while the right side is always even, there is no solution.

II. Solution by David E. Manes, State University of New York College, Oneonta, New York.

We disregard 0 since $2^0 + 1 = 2$ is not a multiple of 143. The order of 2 modulo 143 is 60; i.e., 60 is the smallest positive integer t such that $2^t \equiv 1 \pmod{143}$. Also $2^m \equiv 1 \pmod{143}$ if and only if m is a multiple of 60. Now assume there is a positive integer n such that $2^n + 1 \equiv 0 \pmod{143}$. Then $2^n \equiv -1 \pmod{143}$, which implies $2^{2n} \equiv 1 \pmod{143}$. Then $2n = 60k$ for some positive integer k, or $n = 30k$. We have, however, that $2^{30} \equiv -12 \pmod{143}$. Accordingly,

$$2^n = 2^{30k} = \begin{cases} 1 \pmod{143}, & \text{if } k \text{ is even} \\ -12 \pmod{143}, & \text{if } k \text{ is odd;} \end{cases}$$

that is, $2^n \not\equiv -1 \pmod{143}$ for any positive integer n.

Also solved by CHARLES ASHBACHER, Cedar Rapids, IA, who calculated that $2^{60} = 1152921504606846976$, SCOTT H. BROWN, Auburn University, AL, PAUL S. BRUCKMAN, Everett, WA, BILL CORRELL, JR., Cincinnati, OH, CHARLES R. DIMINNIE, St. Bonaventure University, NY, MARK EVANS, Louisville, KY, STEPHEN I. GENDLER, Clarion University of Pennsylvania, RICHARD I. HESS, Rancho Palos Verdes, CA, MURRAY S. KLAMKIN, University of Alberta, Canada, HENRY S. LIEBERMAN, Waban, MA, JOHN D. MOORES, Westbrook, ME, KANDASAMY MUTHUVEL, University of Wisconsin-Oshkosh, WILLIAM H. PEIRCE, Delray Beach, FL, BOB PRIELIPP, University of Wisconsin-Oshkosh, HARRY SEDINGER, St. Bonaventure University, NY, REX H. WU, Brooklyn, NY, and the PROPOSER.

792. [Fall **1992**] Proposed by Seung-Jin Bang, Seoul, Republic of Korea.

Given any thirteen distinct real numbers, prove that there exists at least one subset {x, y, z} of three of them such that

$$0 < \frac{(x - y)(y - z)(x - z)}{(1 + xy)(1 + yz)(1 + xz)} < \frac{1}{3\sqrt{3}}.$$

Solution by Oxford Running Club, University of Mississippi, University, Mississippi.

The arctangents of the thirteen numbers are thirteen distinct numbers in the interval $(-\pi/2, \pi/2)$. Then some three of these must lie in one of the subintervals $(-\pi/2, -\pi/3)$, $[-\pi/3, -\pi/6)$, $[-\pi/6, 0)$, $[0, \pi/6)$, $[\pi/6, \pi/3)$, $[\pi/3, \pi/2)$. Say that $x > y > z$ are three of the original thirteen and that $\arctan x > \arctan y > \arctan z$ are in the same subinterval. Then each of the differences

$$\arctan x - \arctan y, \quad \arctan y - \arctan z, \quad \arctan x - \arctan z$$

is positive and strictly less than $\pi/6$. Hence, by the increasing nature of the tangent function, each difference has a tangent between 0 and $1/\sqrt{3}$, so the product of their three tangents is less than $1/(3\sqrt{3})$. Since

$$\tan(\arctan x - \arctan y) = \frac{x - y}{1 + xy}$$

and two similar relationships, the result follows.

Also solved by PAUL S. BRUCKMAN, Everett, WA, BILL CORRELL, JR., Cincinnati, OH, and the PROPOSER.

793. [Fall **1992**] Proposed by Dieter *Bennewitz, Koblenz*, Germany.

Given any trapezoid, its diagonals divide its interior area into four triangular areas: A and B adjacent to the parallel bases, and C and D adjacent to the nonparallel sides, as shown in the figure.

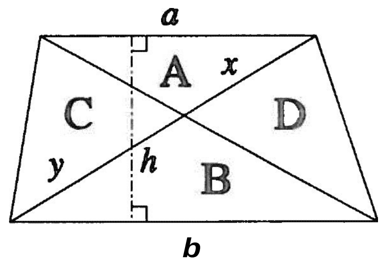
a) Prove that the areas C and D are equal and that $A \cdot B = C \cdot D$.

b) Find area C in terms of the lengths of the altitude and the bases of the trapezoid.

Solution by Richard I. *Hess, Rancho Palos* Verdes, California.

Let the upper and lower bases of the trapezoid have lengths a and b and let its altitude be h. Let the segment of the diagonal common to triangles A and D have length x and that common to B and C have length y. See the figure.

a) Since the triangle formed by B + C and that formed by B + D have the same base b and altitude h,



$$B + C = B + D = bh/2$$

and hence C = D. Similarly, A + C = A + D = $ah/2$. Thinking of the diagonal x + y as base, then triangles B and D have a wmmon altitude, so $D/B = x/y$. Similarly $A/C = x/y$, whence AB = CD.

b) Since AB = CD = C^2, then B = C^2/A and we have

$$B + C = \frac{C^2}{A} + C = \frac{bh}{2} \quad \text{and} \quad A + C = \frac{ah}{2}.$$

We solve these equations simultaneously to get that

$$C = D = \frac{abh}{2(a+b)}, \quad A = \frac{a^2h}{2(a+b)}, \quad \text{and} \quad B = \frac{b^2h}{2(a+b)}.$$

Also solved by ALMA COLLEGE PROBLEM SOLVING GROUP, MI, SEUNG-JIN BANG, Seoul, Korea, PAUL S. BRUCKMAN, Everett, WA, BILL CORRELL, JR., Cincinnati, OH, MARK EVANS, Louisville, *KY,* STEPHEN I. GENDLER, Clarion University of Pennsylvania, HENRY S. LIEBERMAN, Waban, MA, BARBARA J. LEHMAN, Brigantine, NJ, DAVID E. MANES, SUNY at Oneonta, JOHN D. MOORES, Westbrook, ME, YOSHINOBU MURAYOSHI, Eugene, OR, HARRY SEDINGER, St. Bonaventure University, *NY,* REX H. WU, Brooklyn, *NY,* SAMMY YU and JIMMY YU (jointly), *Vermillion,* SD, and the PROPOSER.

794. [Fall **1992**] Proposed by Peter A. *Lindstrom,* North Lake College, Irving, Texas.

For $-3 \le x \le 6$, show that 2π is equal to the sum of the zeros of

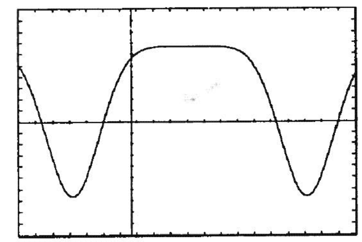
$$f(x) = \sin(x + \cos x).$$

Solution by George P. Evanovich, St. Peter's College, Jersey City, New Jersey.

Because ws (–1) ≈ .54 and cos 0 = 1 and the cosine function is continuous, there exists a number a such that $0 < a < 1$ and $\cos(-a) = a$. (Actually, $a \approx 0.739085$.) Therefore $-a$ is a zero of $\sin(x + \cos x)$. Also note that

$$\cos(a + \pi) = -a, \ \cos(-a + 2\pi) = a, \text{ and } \cos(a - \pi) = -a$$

Thus $-a$, $a + \pi$, $-a + 2\pi$, and $a - \pi$ are the zeros of sin $(x$ + ws $x)$ in the interval (–3, 6). Their sum is 2π. See the figure for the graph of y = sin(x + ws x) over this interval.



$$y = \sin(x + \cos x).$$

Also solved by SEUNG-JIN BANG, Seoul, Korea, PAUL S. BRUCKMAN, Everett, WA, MARK EVANS, Louisville, *KY,* MURRAY S. KLAMKIN, University of Alberta, Canada, HENRY S. LIEBERMAN, Waban, MA, DAVID E. MANES, SUNY at Oneonta, JOHN D. MOORES, *Westbrook,* ME, OXFORD RUNNING CLUB, *University of Mississippi, University,* BOB PRIELIPP (who supplied the figure), University of *Wisconsin-Oshkosh,* REX H. WU, Brooklyn, *NY,* and the PROPOSER.

795. [Fall **1992**] Proposed by Russell *Euler,* Northwest Missouri State University, *Maryville,* Missouri.

Find all solutions on the interval $[0, 2\pi]$ to

$$2\text{ws}^3x - 2\cos x + 1 = 0.$$

I. Solution by Henry S. Lieberman, Waban, Massachusetts.

The equation has no solutions. We have that

$$1 = 2\cos x - 2\cos^3 x = 2\cos x\,(1 - \cos^2 x) = 2\sin^2 x \cos x = \sin x \sin 2x.$$

Then $\sin x = \sin 2x = \pm 1$. Now $\sin x = \pm 1$ for $x = k\pi/2$ where k is an odd integer. Then sin $2x$ = sin hi = 0. Hence, there are no solutions to $\sin x \sin 2x = 1$, and hence to the original equation also.

II. Solution by Rex H. Wu, Brooklyn, New *York.*

If $t = \cos x$, then we must solve

$$(1) \qquad 2t^3 - 2t + 1 = 0, \text{ where } -1 \le t \le 1.$$

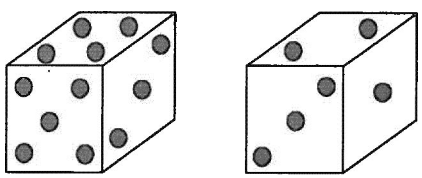
That is, ? = $t - 112$. However, $t^3 > t - 112$ in the interval $(-1,1)$. Hence the original equation has no real solution. In fact, the only real solution to equation (1) is

$$t = \left(-\frac{1}{4} + \sqrt{\frac{11}{432}}\right)^{1/3} + \left(-\frac{1}{4} - \sqrt{\frac{11}{432}}\right)^{1/3} \approx -1.1915 < -1.$$

Also solved by CHARLES **ASHBACHER,** ***Cedar Rapids, IA,*** SEUNG-JIN BANG, ***Seoul, Korea,*** PAUL S. BRUCKMAN, ***Everett, WA,*** BILL CORRELL, JR., ***Cincinnati, OH,*** GEORGE P. EVANOVICH, ***Saint Peter's College, Jersey City, NJ, MARK*** EVANS, ***Louisville, KY,*** STEPHEN I. GENDLER, ***Clarion University of Pennsylvania,*** RICHARD I. HESS, ***Rancho Palos Verdes, CA,*** MURRAY S. **KLAMKIN,** ***University of Alberta, Canada,*** PETER A. LINDSTROM, ***North Lake College, Irving, TX,*** DAVID E. MANES, ***SUNY at Oneonta,*** JOHN D. MOORES, ***Westbrook, ME,*** OXFORD RUNNING CLUB, ***University of Mississippi, University,*** WILLIAM H. PEIRCE, ***Delray Beach, FL,*** BOB **PRIELIPP,** ***University of Wisconsin-Oshkosh,*** HARRY SEDINGER, ***St. Bonaventure University, NY, KENNETH*** M. **WILKE,** ***Topeka, KS,*** J. ERNEST WILKINS, JR., ***Clark Atlanta University, GA. and the*** PROPOSER.

796. [Fall **1992]** ***Proposed by Michael W. Ecker, Clarks Summit, Pennsylvania.***

a) ***A*** die is thrown until a prescribed face **(e.g.** say 3) shows. What is the mathematically expected number of throws required for this to occur?



b) Same question, but suppose a throw now consists of rolling 2 dice. In particular, should we expect this expectation to be half that of part (a)?

c) What is the smallest whole number of dice needed to constitute one throw, if we wish to have the mathematically expected number of throws required to roll our prescribed number not exceed 2?

Solution by Charles Ashbacher, Cedar Rapids, Iowa.

a) If an event has probabilityp and all trials are equally likely, then the expected number E of trials until the event occurs is given by

$$E = \sum_{k=1}^{\infty} kq^{k-1}p = \frac{1}{p},$$

where $q = 1 - p$. Hence the expected number of trials for a six-sided die to

show a 3 is **1/(1/6)** = 6.

b) With two dice the probability of throwing at least one 3 is $1 - (5/6)^2$ = 11/36, so ***E*** = **1/(11/36)** = 36/11 ≈ 3.3 trials.

c) For three dice the probability is $1 - (5/6)^3$ = **91/216,** so ***E*** = **216/91** ≈ 2.37. For four dice, $p = 1 - (5/6)^4$ = 67111296, so ***E*** = 12961671 ≈ 1.93 < 2, so the answer is that four dice are necessary.

Note that the die you have in the sketch is wrong! You have 3 and 4 both showing on one die, but 3 and 4 are ***always*** on opposite sides of a die.

Also solved by ALMA COLLEGE PROBLEM SOLVING GROUP, ***MI,*** PAUL S. BRUCKMAN, ***Everett, WA,*** GEORGE P. EVANOVICH, ***Saint Peter's College, Jersey City, NJ, MARK*** EVANS, ***Louisville, KY,*** STEPHEN I. GENDLER, ***Clarion University of Pennsylvania,*** RICHARD I. HESS, ***Rancho Palos Verdes, CA,*** HENRY S. LIEBERMAN, ***Waban,*** MA, JOHN ***D.*** MOORES, ***Westbrook, ME,*** OXFORD RUNNING CLUB, ***University of Mississippi, University,*** HARRY SEDINGER, ***St. Bonaventure University, NY, and the*** PROPOSER. ***Only Ashbacher noticed the error in the die on the left: the top face should be a six, not a four. The right die is correct.***

Anecdotes Wanted

Mathematics is full of peoples' names: **Fermat's** theorem, Newton's method, the Euclidean algorithm, Bernoulli numbers, Euler's $-function, **L'Hôpital's** rule, **Pell's** equation, Gauss's lemma, **... .**

However, the people behind the names are hardly known at all. Writers of mathematical biography tend to concentrate on their subjects' mathematical life and ignore the rest. Pick up a book of mathematical history at random and see if that is not true. Here is an excerpt from the middle of page 154 of ***Hilbert,*** by Constance Reid **(Springer-Verlag,** New York, 1970):

> The next summer **Hilbert** lectured on relativity theory as part of a **University** series for all of the faculties.

In the middle of page 154 of ***Niels Henrik Abel,*** by Oystein Ore (U. of Minnesota Press, 1957) we find

> With very few acquaintances and low ebb in his purse Abel could do little else than write mathematics, and the last months in Paris turned out to be extremely fruitful. A few days after his great memoir had been submitted to the Institute, he completed a lesser paper on

equations, which he presented to Gergonne's Annals.

The middle of page 154 of Carl Friedrich Gauss, by G. Waldo Dunnington (Hafner, New York, 1955):

> By January, 1832, he had thrown himself with all force into the investigation of magnetism, and by February of that year had succeeded in reducing the intensity of terrestrial magnetism to absolute units.

Page 154 (middle) of Joseph Fourier, by I. Grattan-Guinness (MIT Press, Cambridge, 1972):

> On calculera de même la valeur de d pour le cas de quatre inconnues et on multiplera cette valeur par
>
> $$9^2/(9^2 - 7^2),\ 11^2/(11^2 - 7^2),\ 13^2/(13^2 - 7^2),\ \ldots$$

It is hard to make mathematics human. Mathematicians mostly live dull lives, so colorful anecdotes bringing them alive as people are rare. It is a shame that mathematicians as people have been so neglected. (If you doubt that they have been neglected, can you tell which of Lagrange, Laplace, Legendre, L'Hôpital, and Lhulier was the tallest, or had the most children? Can you even tell them apart?) The names of mathematicians are not the names of people, they are the names of gods who produce theorems.

This is too bad because mathematicians were and are people, and mathematics is a human activity. When we cannot make a connection between our subject and the humans who were and are responsible for it, mathematics can be viewed as inhuman, artificial, sterile, and boring. In fact, it is so viewed by a rather large number of people.

To fix this, we need a supply of anecdotes about mathematicians. The state of mathematical anecdotes is now so bad that even the false anecdotes are no good. For example, you can find in print in more than one place the story of how, when he was old, De Moivre each night slept for fifteen minutes more than the previous night until he slept the clock around; then he died. It was clearly made up by a non-mathematician, and not by a clever non-mathematician. It is obviously false and pointless even if true. The often-printed anecdote about Euler's algebraic proof of the existence of God is another example. Why would Euler deliver such nonsense as "$(a^n + b)/c = n$; hence God exists."? Why would Diderot who according to the anecdote knew no mathematics (not true) have consented to listen? Ridiculous! That such feeble stories should gain acceptance and be constantly repeated shows how easy it is for counterfeit anecdotes to get into circulation and stay there.

We need good anecdotes about mathematicians, illustrating their human qualities. We have George Washington chopping down the cherry tree, but we have nothing similar for Euler. Benjamin Franklin flew a kite: what did Gauss do?

Since the historians of mathematics are not going to supply us with what we need, we must turn elsewhere. Let us turn to the readers of the Pi *Mu* Epsilon Journal. This is an appeal for you to make up anecdotes about mathematicians. Length and subject are immaterial as long as the anecdote is ***good.*** It should be memorable, it should illustrate something, it can be funny, touching, sentimental, stirring: anything as long as it is ***good.*** It does not have to be true. Many of the best anecdotes, for example G. W. and the cherry tree, are not. Nevertheless, they serve valuable purposes.

The best anecdotes received will be printed in a future issue of the Journal, either anonymously or with attribution, as their authors choose.

The 1993 National Pi Mu Epsilon Meeting

The meeting took place at the summer meeting of the American Mathematical Society, the Mathematical Association of America, and the Canadian Mathematical Society, in Vancouver, British Columbia, from August 15 to 19, 1993.

There were thirty-three student papers delivered in five sessions:

Upper chromatic numbers, by Aaron Abrams (University of California)

Optimal material layout in a problem of heat transfer, by Ray V. Adams (Worcester Polytechnic Institute)

Math anxiety, by Dawn Boyung (St. Norbert College)

The analytic hierarchy process with Bayes' Theorem, by Frank Castro (Youngstown State University)

An introduction to the theory of $K_2(R)$, by John Davenport (Miami University)

Functional integrals in a theory of absolute integration, by Anthony F. De Lia (University of Central Florida)

Riemann Zeta Function on the distribution of prime numbers, by Rondel DeLong (Marshall University)

A marble drop method for solving linear programs, by Vladimir Dimitrijevic (Youngstown State University)

Math methods down under, by Sandra S. Gestl (St. Norbert College)

Sailing down the river of $3x^2 + 6xy - 5y^2$, by Francis Fung (Kansas State University)

Bayesian probability and credibility theory in insurance ratemaking, by

Jennifer Garrett (Miami University)

Differential hyperbolic geometry, by Lauren D. **Hartman** (Washington and Lee University)

Data structures in the implementation of the **Huffman** algorithm, by Jon Hester (Hendrix College)

Allocations for matching games on weighted graphs, by Jennifer Howes (Drew University)

The economics of exhaustible resources, by Benjamin Keen (Miami University)

Stokes' Theorem and its application, by Deborah Kellogg (East Carolina University)

A matrix-balancing problem, by Julie Labbiento (Youngstown State University)

Hilbert's seventeenth problem, by Pasquale **Lapomarda III** (College of the Holy Cross)

Some proofs without words, by Cheryl **McClellan** (Youngstown State University)

One sample study of variance focusing on type I error, by Julie Mullett (Miami University)

The life table approach in determining actuarial mortality, by Umagasen C. Naidoo (Miami University)

Automatic differentiation, by Mai Nguyen (Miami University)

Dead horses in the desert, by Kathryn Nyman (Carthage College)

A combinatorial queuing model related to the ballot problem, by David C. Ogden (Wichita State University)

Three proofs of **Kaplansky's** Theorem, by Thomas **Peppard** (John Carroll University)

Jentzsch's Theorem in two complex variables, by **Xiaoling** Qian (University of Illinois)

The Shapley value and partially defined games, by Jennifer **Rich** (Drew University)

A generalization of triangular numbers, by Bonnie A. Sadler (East Carolina University)

Implementation of diva procedure calls on a ring of processors, by Scott Shauf (University of Richmond)

Continued fractions, by **Traca** Slusher (Youngstown State University)

Values of games in partition function form, by Maria Theoharidas (Drew University)

Matrices and AIDS, by Jeffrey A. Wallace (University of West Florida)

A study of the representations of even numbers as the sum of two primes, by Joel M. Wisdom (University of Tennessee, Chattanooga).

Some of these papers, the editor hopes, will appear in forthcoming issues of the ***Journal.***

Five prizes, for papers of unusual merit, were awarded to Vladimir Dimitrijevic, Jennifer Garrett, Lauren D. **Hartman,** Jon Hester, and Joel M. Wisdom.

At the meeting of the Pi Mu Epsilon Council, it was announced that the National Security Agency had again granted Pi Mu Epsilon $5000 for the support and encouragement of student speakers and that the American Mathematical Society had contributed $1000 towards prize awards. In addition, a donor who wished to remain anonymous has made a contribution that will more than double the fund for the Richard V. **Andree** Awards, given to the best student papers that appear in the ***Journal.*** The Council expressed gratification and thanks for the support. In other business, the Council decided to increase the cost of Pi Mu Epsilon pins from $8 to $12, but to leave all other fees unchanged.

After the annual Pi Mu Epsilon banquet, an inexpensive and **well-**attended event, the J. Sutherland Frame lecture was delivered by Professor George E. Andrews of Pennsylvania State University. His topic was "Ramanujan for students" and with energy and clarity he went from how Fibonacci numbers could aid travelers in Canada to properties of continued fractions, giving the impression, as skilled lecturers can, that mathematics is really a simple subject.

The next meeting of Pi Mu Epsilon will take place in conjunction with the summer meeting of the MAA and **AMS** in Minneapolis, August 15-17, 1994.

Editorial Statement

The ***Journal*** is always seeking manuscripts from student members of Pi Mu Epsilon.

The main purpose of the ***Journal*** is to interest and inform its readers, who are mostly undergraduates or recent graduates. Thus, the results of specialized research, of interest to experts in a field, are not in general appropriate for the readers of the ***Journal*** who, along with its editor, are experts in ***no*** Geld. Papers should give background information and place the results of research in context. The audience that authors should keep in mind is a group of bright young mathematicians who know next to nothing about your area of expertise, but who are able and willing to learn.

Miscellany

Those "problems" of the form "Find the next number in the following sequence" are not problems in the usual mathematical sense at all. You may think that the next number in the sequence 2, 4, 6, 8, ... is 10, but you are wrong. It is π, because the formula that I had in mind when writing the terms for n = 1, 2, 3, 4 was

$$2n + \frac{\pi - 10}{24}(n-1)(n-2)(n-3)(n-4).$$

The problems would be better stated as, "What was in the mind of the author as the following sequence was being written?" This of course is a problem of psychology and not of mathematics.

Nevertheless, such problems can be entertaining, especially if you can guess the next number. The British periodical ***Eureka***, a publication of the undergraduate mathematical society at Cambridge University, had a tradition of posing such problems. Here is a selection of 40-year-old sequences, which incidentally illustrates the timeless and eternal nature of mathematics. They range from the fairly obvious to the completely impossible, which is why solutions will be given in the next issue of the Journal.

1, 3, 6, 10, 15, ...	(#17, 1954)
1, 2, 4, 8, 1, 6, 3, 2, 6, ...	(#11, 1949)
0, 1, 3, 7, 15, 31, ...	(#17, 1954)
3, 5, 11, 13, 17, 19, 29, ...	(#18, 1955)
3, 4, 6, 8, 12, 14, 18, ...	(#19, 1957)
4, 6, 9, 10, 14, 15, 21, 22, 25, 26, 33, ...	(#17, 1954)
3, 2, 1, 7, 4, 1, 1, 8, 5, 2, 9, 8, ...	(#13, 1950)
1, 15, 29, 12, 26, 12, 26, 9, 23, ...	(#13, 1952)
5, 11, 15, 16, 17, 18, 23, 25, ...	(#18, 1955)

While looking through old issues of Eureka, I discovered another reason why students should publish papers in the Journal. In #17 (1954) there was a problem by Roger Penrose, then (I think) a student, now eminent in the extreme (deviser of Penrose tiles, author of the recent best-seller The Emperor's ***New Mind***, Rouse Ball Professor of Mathematics at Oxford, etc.). Here it is (it is not trivial): in a semigroup, where multiplication is associative but there are no inverses, you are given that aba = a, ***bab*** = b, ab = ba, and ac = ca. Show that bc = cb. Though the problem may be difficult, the conclusion to be drawn from its existence is easy: if Roger Penrose published while a student, and Roger Penrose has gone on to impressive achievements, then if you are a student, it follows that ... .

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