PROBLEM DEPARTMENT

ASHLEY AHLIN AND HAROLD REITER*

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

Solutions should be addressed to Harold Reiter, Department of Mathematics, University of North Carolina Charlotte, 9201 University City Boulevard, Charlotte, NC 28223-0001 or sent by email to `hbreiter@uncc.edu`*. Electronic submissions using LaTeX are encouraged. New proposals should be sent to Steven J. Miller, Department of Mathematics, Williams College, 880 Main St Williamstown, MA 01267 or sent to* `Steven.J.Miller@williams.edu`*. Beginning with the spring issue of 2014, Steven J. Miller will be the problems section editor. The outgoing editors wish him the best. Other electronic submissions are also encouraged. Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly identified with name, affiliation, and address. Solutions to problems in this issue should be mailed to arrive by October 1, 2013. Solutions identified as by students are given preference.*

Problems for Solution.

1274. *Proposed by Matthew McMullen, Otterbein University, Westerville, OH.*

Classify all functions $f(x) = ax + b$ such that a and b are rational, $f(x) \geq 0$ on $[0, 1]$, and

$$\int_0^1 f(x)\,dx \;=\; \int_0^1 (f(x))^2\,dx.$$

1275. *Proposed by Richard Stephens, Columbus State University, Columbus, GA.*

Let b, c, n, k, and m have integer values. Find all values of k such that any function of the form $f(x) = x^2 + bx + c$ has the following property: For each value of n there exists a value of m such that $f(n) \cdot f(n+k) = f(m)$. Furthermore, express m in terms of b, c, n, and k.

1276. *Proposed by Mihaly Bencze, Brasov, Romania.*

Let $a, b, c > 0$ be real numbers satisfying $\lambda abc > a^3 + b^3 + c^3$. Prove that if $0 < \lambda \leq 5$, then a, b, and c are the sides of a triangle.

1277. *Proposed by Steven Miller, Williams College, Williamstown, MA.*

A graph G is a collection of vertices V and edges E connecting pairs of vertices. Consider the following graph. The vertices are the integers $\{2, 3, 4, \ldots, 2012\}$. Two vertices are connected by an edge if they share a divisor greater than 1; thus 30 and 1593 are connected by an edge as 3 divides each, but 30 and 49 are not. The coloring number of a graph is the smallest number of colors needed so that each vertex is colored and if two vertices are connected by an edge, then those two vertices are not

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colored the same. The Green Chicken says the coloring number of this graph is at most 9. Prove she is wrong, and find the correct coloring number. *This problem is from the 2012 math competition between Middlebury and Williams Colleges, known as the Green Chicken contest.*

1278. *Proposed by Kenny Davenport, Dallas, PA.*

Prove that

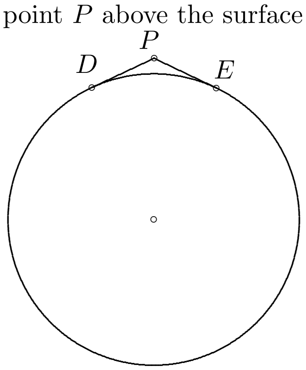
$$(a^2-1)\cos((n+3)\theta) - 2a^{1/2}\cos(n\theta) \;=\; (a-1)^2\cos((n+1)\theta)\,,$$

where a is the real number satisfying $a^3 = a^2 + a + 1$ and θ satisfies

$$\cos\theta = (1-a)a^{1/2}/2.$$

1279. *Proposed by Stas Molchanov, University of North Carolina Charlotte*

A spherical planet has a radius of 6400 kilometers. A rope that is one centimeter longer than the circumference is stretched tight as shown, so the length of PD equals that of PE. How high is the point P above the surface of the planet?



1280. *Proposed by Richard Stephens, Columbus State University, Columbus, GA.* and *Moti Levy, Rehovot, Israel.*

In Problem 1265 we asked for a proof that the series below converges. Now we ask for the sum of these integrals. For each nonnegative integer n and for $a < b$, let

$$x_n \;=\; \frac{1}{(b-a)^{2n+1}}\int_a^b (x-a)^n\,(b-x)^n\,dx.$$

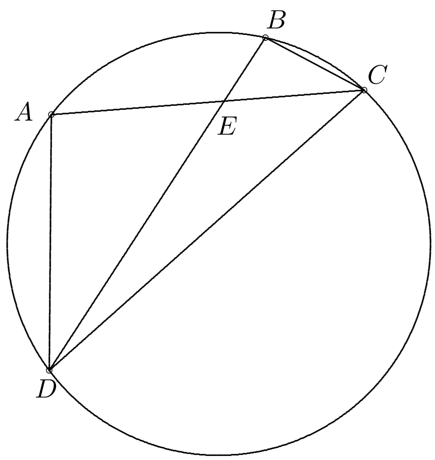
Prove that

$$\sum_{n=1}^{\infty} x_n \;=\; \frac{2\sqrt{3}\pi}{9} - 1.$$

1281. *Proposed by Abdilkadir Altintas*, Afyonkarahisar, Turkey.

In the circle shown, chords BD and AC intersect at E. The areas of triangles AED, BEC and DEC are, respectively S_1, S_2 and $S_1 + S_2$. Also, $BE = 4$ and

$DE = 13$. What is the length of segment AE?



1282. *Proposed by Paul S. Bruckman, Nanaimo, BC.*

Find closed forms for the following constants. The products are over all primes p:

(a) $$\prod_p \left(1+\frac{1}{p^2}\right)^{-1}$$

(b) $$\prod_p \left(1+\frac{1}{p^2}+\frac{1}{p^4}\right)^{-1}$$

(c) $$\prod_p \left(1+\frac{1}{p^2}+\frac{1}{p^4}+\frac{1}{p^6}\right)^{-1}$$

(d) $$\prod_p \left(1+\frac{1}{p^4}\right)^{-1}.$$